SUPERBRIDGE INDEX OF COMPOSITE KNOTS

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Abstract. An upper bound of the superbridge index of the connected sum of two knots is given in terms of the braid index of the summands. Using this upper bound and minimal polygonal presentations, we give an upper bound in terms of the superbridge index and the bridge index of the summands when they are torus knots. In contrast to the fact that the difference between the sum of bridge indices of two knots and the bridge index of their connected sum is always one, the corresponding difference for the superbridge index can be arbitrarily large.

1. Introduction

Throughout this article a *knot* is a piecewise smooth simple closed curve embedded in the three dimensional Euclidean space $\mathbb{R}^3$. Two knots are *equivalent* if there is a piecewise smooth autohomeomorphism of $\mathbb{R}^3$ mapping one knot onto the other. The equivalence class of a knot K will be called the *knot type* of K and denoted by $[K]$.

The *crookedness* of a knot K embedded in $\mathbb{R}^3$ with respect to a unit vector $\vec{v}$ is the number of connected components of the preimage of the set of local maximum values of the orthogonal projection $K \to \mathbb{R}\vec{v}$, denoted by $b_{\vec{v}}(K)$. Figure 1 illustrates an example. For any open subarc S of a knot K, the crookedness of S with respect to $\vec{v}$, denoted by $b_{\vec{v}}(K \mid S)$, can be defined similarly using the projection $S \to \mathbb{R}\vec{v}$. The *superbridge number* and the *superbridge index* of K, denoted by $s(K)$ and $s[K]$, are defined to be "$\max b_{\vec{v}}(K)$" and "$\min \max b_{\vec{v}}(K)$", respectively, where the maximum is taken over all unit vectors and the minimum taken over all equivalent embeddings of K. This invariant was introduced by Kuiper [9] who computed the superbridge index for all torus knots.

Theorem A (Kuiper). *For any two coprime integers p and q, satisfying $2 \leq p < q$, the torus knot of type (p, q) has superbridge index $\min\{2p, q\}$.*

The *bridge index* $b[K]$ can be defined in a similar way by "$\min \min b_{\vec{v}}(K)$." One of the most well-known theorem about bridge index is

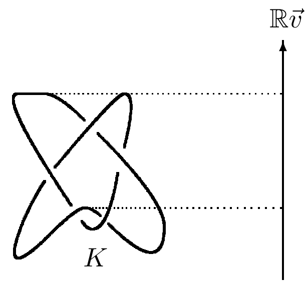

Figure 1. $b_{\vec{v}}(K) = 3$

Theorem B (Schubert). *Given two knots K_1 and K_2, any connected sum*[1] $K_1 \sharp K_2$ *satisfies*

$$b[K_1 \sharp K_2] = b[K_1] + b[K_2] - 1.$$

This work is an attempt to find a similar formula for the superbridge index. A proof of Schubert's theorem in a more generalized context can be found in [4].

Let $\beta[K]$ denote the *braid index*, i.e., the minimal number of strings among all braids whose closures are equivalent to K. According to Kuiper, the superbridge index of a nontrivial knot is always greater than the bridge index and not greater than twice the braid index [9].

Theorem C (Kuiper). *If K is a nontrivial knot, then*

$$b[K] < s[K] \leq 2\beta[K].$$

Kuiper used Milnor's total curvature to prove the first inequality [12]. The closed braid constructed by Kuiper used to prove the second inequality is discussed in Section 3. From Theorem B and Theorem C, we obtain

Corollary 1. *If K_1 and K_2 are nontrivial knots, any connected sum $K_1 \sharp K_2$ satisfies the inequality*

$$s[K_1 \sharp K_2] \geq 4.$$

2. Theorems and Conjectures

Theorem 1. *If K_1 and K_2 are nontrivial knots, any connected sum $K_1 \sharp K_2$ satisfies the inequality*

$$s[K_1 \sharp K_2] \leq \max\{2\beta[K_1] + \beta[K_2], \beta[K_1] + 2\beta[K_2]\} - 1.$$

Theorem 2. *If K_1, K_2 are torus knots, any connected sum $K_1 \sharp K_2$ satisfies the inequality*

$$s[K_1 \sharp K_2] \leq \max\{s[K_1] + b[K_2], b[K_1] + s[K_2]\} - 1.$$

The next corollary shows that the equality in Theorem 2 holds in infinitely many cases.

[1] Since K_1 and K_2 are not oriented, their (unoriented) connected sum may not be unique.

Corollary 2. *Let $p_i \geq 2$ and let K_i be the torus knot of type (p_i, p_i+1), for $i = 1, 2$. Then*

$$s[K_1 \sharp K_2] = p_1 + p_2.$$

Proof: By Theorem A, $s[K_i] = p_i + 1$. Since $b[K_i] = p_i$, from Theorem B, Theorem C and Theorem 2, we obtain $p_1 + p_2 - 1 < s[K_1 \sharp K_2] \leq p_1 + p_2$. □

Using the first inequality in Theorem C, we obtain the following generalization of [7, Corollary 11].

Corollary 3. *If K_1, K_2 are torus knots, any connected sum $K_1 \sharp K_2$ satisfies the inequality*

$$s[K_1 \sharp K_2] \leq s[K_1] + s[K_2] - 2.$$

The inequality in Theorem 2 is equivalent to

$$s[K_1] + s[K_2] - s[K_1 \sharp K_2] \geq \min\{s[K_1] - b[K_1], s[K_2] - b[K_2]\} + 1.$$

If K_i is a torus knot of type (p_i, q_i) with $2 \leq p_i < q_i$, the right hand side of the above inequality is equal to $\min\{p_1, p_2, q_1 - p_1, q_2 - p_2\} + 1$, which can be arbitrarily large. Therefore we have

Corollary 4. *The difference $s[K_1] + s[K_2] - s[K_1 \sharp K_2]$ can be arbitrarily large.*

We conjecture that Theorem 2 and Corollary 3 are true for any knots:

Conjecture 1. *Any connected sum of two knots K_1 and K_2 satisfies the inequality*

$$s[K_1 \sharp K_2] \leq \max\{s[K_1] + b[K_2], b[K_1] + s[K_2]\} - 1.$$

Conjecture 2. *If K_1 and K_2 are nontrivial knots, any connected sum $K_1 \sharp K_2$ satisfies the inequality*

$$s[K_1 \sharp K_2] \leq s[K_1] + s[K_2] - 2.$$

As Corollary 3 follows from Theorem 2, Conjecture 2 follows from Conjecture 1. The readers may wonder whether the inequality

$$s[K_1 \sharp K_2] \geq \max\{s[K_1] + b[K_2], b[K_1] + s[K_2]\} - 1$$

would be true. So far no reasonable lower bound formula for $s[K_1 \sharp K_2]$ has been found. We do not even know if the following is true.

Conjecture 3. *If K_1 and K_2 are nontrivial knots, any connected sum $K_1 \sharp K_2$ satisfies the inequality*

$$s[K_1 \sharp K_2] > \max\{s[K_1], s[K_2]\}.$$

In Table 1, the symbols used for factors of K indicate the prime knots as in the knot tables of [1, 14]. The knots $3_1, 5_1, 7_1, 8_{19}, 9_1$ are torus knots of type $(2,3), (2,5), (2,7), (3,4), (2,9)$, respectively. Theorem 2 is used to find upper bounds of superbridge index for the connected sums of pairs of these knots. There are three among them for which Corollary 3 also applies. For the others, we used the inequality

$$2\, s[K] \leq p[K] \qquad (1)$$

factors of K	$s[K]$	lower bound	upper bound
3_1 3_1	4	$b[K]=3$	Corollary 3
3_1 4_1	4	$b[K]=3$	$p[K]=9$
3_1 5_1	$5\star$	$s[5_1]=4$	Theorem 2
3_1 7_1	$5\star$	$s[7_1]=4$	Theorem 2
3_1 7_6	$5\star$	$s[7_6]=4$	$p[K]\leq 11$
3_1 7_7	$5\star$	$s[7_7]=4$	$p[K]\leq 11$
3_1 8_{16}	5	$b[K]=4$	$p[K]\leq 11$
3_1 8_{17}	5	$b[K]=4$	$p[K]\leq 11$
3_1 8_{18}	5	$b[K]=4$	$p[K]\leq 11$
3_1 8_{19}	5	$b[K]=4$	Corollary 3
3_1 8_{20}	5	$b[K]=4$	$p[K]\leq 10$
3_1 8_{21}	5	$b[K]=4$	$p[K]\leq 11$
3_1 9_1	$5\star$	$s[9_1]=4$	Theorem 2
3_1 9_{40}	5	$b[K]=4$	$p[K]\leq 11$
3_1 9_{41}	5	$b[K]=4$	$p[K]\leq 11$
3_1 9_{44}	5	$b[K]=4$	$p[K]\leq 11$
3_1 9_{46}	5	$b[K]=4$	$p[K]\leq 11$
4_1 5_1	$5\star$	$s[5_1]=4$	$p[K]\leq 11$
4_1 8_{19}	5	$b[K]=4$	$p[K]\leq 11$
4_1 8_{20}	5	$b[K]=4$	$p[K]\leq 11$
5_1 5_1	$5\star$	$s[5_1]=4$	Theorem 2
5_1 7_1	$5\star$	$s[7_1]=4$	Theorem 2
7_1 7_1	$5\star$	$s[7_1]=4$	Theorem 2
8_{19} 8_{19}	6	$b[K]=5$	Corollary 3
3_1 3_1 3_1	5	$b[K]=4$	$p[K]\leq 10$
3_1 3_1 4_1	5	$b[K]=4$	$p[K]\leq 11$

Table 1.

to find upper bounds, where $p[K]$ is the *polygon index* [7, 8], i.e., the minimal number of straight edges required to present the knot type of K. Using the polygonal knots given in [10, 11, 15], we verified that the inequality

$$p[K_1\sharp K_2]\leq p[K_1]+p[K_2]-4$$

of [7, Theorem 8] can be applied to find upper bounds of $p[K]$ as given in the table. The nine-edged polygonal knot[2] of Figure 2 is a connected sum of a trefoil knot and a figure eight knot. It has polygon index 9 because it does not appear in the list of [2] containing all eight-edged knots.

The values marked with $\star$ are conjectured using Theorem A, Conjecture 3 and [5, Table 1]. If Conjecture 3 is not true for any of them, the correct value will be one less than as given in the table. For all others, Theorem B and Theorem C are used to determine strict lower bounds.

[2] It has vertices at $(-30,0,-10)$, $(10,20,30)$, $(-27,-35,-70)$, $(0,30,10)$, $(0,-40,10)$, $(-4,-7,8)$, $(16,6,-21)$, $(-18,-32,36)$, $(30,0,-10)$. Figure 2 is its projection into the xy-plane.

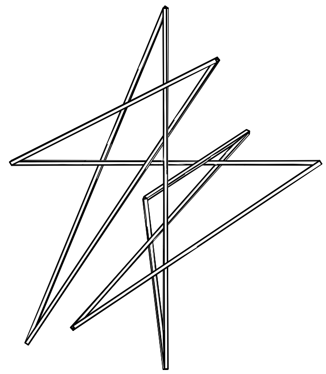

FIGURE 2. A minimal polygonal connected sum of 3_1 and 4_1.

The next two sections describe the constructions and their properties required to prove Theorem 1 and Theorem 2. Section 5 contains the proofs.

3. CLOSED BRAIDS

Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ denote the standard basis vectors of $\mathbb{R}^3$ and let η be the trivial knot given by the embedding $(x, y) \mapsto (x, y, x^2)$ of the circle $x^2 + y^2 = 1$. By [9, Lemma 4.1], we know that $s(\eta) = 2$. Therefore, for any unit vector $\vec{v}$, either $b_{\vec{v}}(\eta) = 1$ or $b_{\vec{v}}(\eta) = 2$. Let $N = \{\vec{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} \in S^2 \mid v_3 > 0, b_{\vec{v}}(\eta) = 2\}$. This is an open subset of S^2 satisfying the condition that $b_{\vec{v}}(\eta) = 2$ if and only if $\vec{v} \in N \cup (-N)$. Two projections of N and $-N$ are shown in Figure 3.

Lemma 1. *Let* $G_{\rho,\alpha}(t) = -\rho \sin(t - \alpha) - (1 - \rho^2)^{1/2} \sin 2t$.

(a) *For any* α*, there is a unique positive number* $\xi(\alpha) \in \left[1/\sqrt{2}, 2/\sqrt{5}\,\right]$ *such that the function* $G_{\xi(\alpha),\alpha}(t)$ *has a multiple root.*

(b) ∂N *has a parametrization* $\alpha \mapsto \big(\xi(\alpha)\cos\alpha, \xi(\alpha)\sin\alpha, (1 - \xi(\alpha)^2)^{1/2}\big)$.

Proof: (a) If t_0 is a multiple root of $G_{\rho,\alpha}(t)$, then

$$\begin{aligned} G_{\rho,\alpha}(t_0) &= -\rho \sin(t_0 - \alpha) - (1 - \rho^2)^{1/2} \sin 2t_0 = 0, \\ G'_{\rho,\alpha}(t_0) &= -\rho \cos(t_0 - \alpha) - 2(1 - \rho^2)^{1/2} \cos 2t_0 = 0. \end{aligned}$$

Eliminating α, we get $\rho = ((1 + 3\cos^2 2t_0)/(2 + 3\cos^2 2t_0))^{1/2}$. Therefore the inequality $1/\sqrt{2} \leq \rho \leq 2/\sqrt{5}$ holds.

Suppose $1/\sqrt{2} < \rho < 2/\sqrt{5}$, then $1/2 < (1/\rho^2 - 1)^{1/2} < 1$. As illustrated in Figure 4, there are eight distinct values of α modulo 2π, such that the graphs of $p(t) = -\sin(t - \alpha)$ and $q(t) = (1/\rho^2 - 1)^{1/2} \sin 2t$ are tangent at some point. For these values of α, the function $G_{\rho,\alpha}(t)$ has double roots.

If $\alpha = k\pi \pm \pi/4$, $k = 0, 1$, then $\rho = 1/\sqrt{2}$. In these cases, the graphs of $p(t)$ and $q(t)$ are tangent at $t_0 = \pi - \alpha$, where $G_{\rho,\alpha}(t)$ has a double root. If $\alpha = k\pi/2$, $k = 0, 1, 2, 3$, then $\rho = 2/\sqrt{5}$. In these cases, the graphs of $p(t)$ and $q(t)$ are tangent at $t_0 = \pi - \alpha$, where $G_{\rho,\alpha}(t)$ has a triple root. This finishes the proof of part (a) except the uniqueness which we omit.

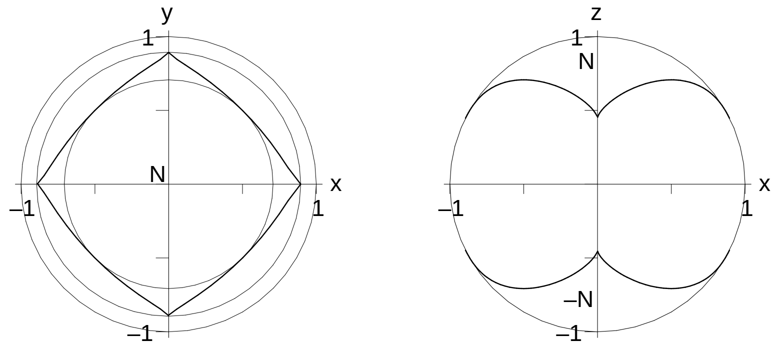

FIGURE 3. Projections of N and $-N$

(b) For a unit vector $\vec{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, the projection $\eta \to \mathbb{R}\vec{v}$ is parametrized by

$$f_{\vec{v}}(t) = v_1 \cos t + v_2 \sin t + v_3 \cos^2 t. \tag{2}$$

Suppose $0 < v_3 < 1$, then there is a unique number $\alpha_{\vec{v}}$ modulo 2π such that $\cos\alpha_{\vec{v}} = v_1(1-v_3^2)^{-1/2}$ and $\sin\alpha_{\vec{v}} = v_2(1-v_3^2)^{-1/2}$. Substituting $\rho = (1-v_3^2)^{1/2}$, we get

$$f_{\vec{v}}(t) = \rho\cos(t-\alpha_{\vec{v}}) + (1-\rho^2)^{1/2}\cos^2 t. \tag{3}$$

If $\vec{v} \in \partial N$, then $f'_{\vec{v}}(t) = 0$ has a multiple root. Since $f'_{\vec{v}}(t) = G_{\rho,\alpha_{\vec{v}}}(t)$, we know that ∂N has the required parametrization. The projection of ∂N into the xy-plane in Figure 3 is the graph of the polar equation $\rho = \xi(\alpha)$. □

Lemma 2. *Let $\vec{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be a unit vector. Then*

$$b_{\vec{v}}(\eta \mid \eta_+) = \begin{cases} 1 & \text{if } \vec{v} \in N \text{ or } \min\{v_1, v_3\} > 0 \\ 0 & \text{if } \vec{v} \notin N, v_1 < 0 \text{ and } v_3 > 0, \end{cases}$$

where $\eta_+ = \eta \cap \{(x,y,z) \mid x > 0\}$.

Proof: Again we use the parametrizations (2) and (3) for η. We have

$$\begin{aligned} f'_{\vec{v}}(t) &= -v_1 \sin t + v_2 \cos t - v_3 \sin 2t \\ &= -\rho\sin(t-\alpha_{\vec{v}}) - (1-\rho^2)^{1/2}\sin 2t. \end{aligned}$$

CASE 1. Suppose $\vec{v} \notin N$ and $v_1 > 0$. Then $f'_{\vec{v}}(\pi/2) = -v_1 < 0 < v_1 = f'_{\vec{v}}(-\pi/2)$. Therefore $b_{\vec{v}}(\eta \mid \eta_+) = 1$.
CASE 2. Suppose $v_3 > 1/\sqrt{2}$. Since $0 \le \rho < (1-\rho^2)^{1/2}$, we have

$$f'_{\vec{v}}(\pi/4) < 0 < f'_{\vec{v}}(-\pi/4) \text{ and } f'_{\vec{v}}(5\pi/4) < 0 < f'_{\vec{v}}(3\pi/4). \tag{4}$$

Therefore there are two local maximum points, one in each of the two intervals $(-\pi/4, \pi/4)$ and $(3\pi/4, 5\pi/4)$. Therefore $\vec{v} \in N$ and $b_{\vec{v}}(\eta \mid \eta_+) = 1$.
CASE 3. Suppose that $\vec{v} \in N$ and $v_3 = 1/\sqrt{2}$, then $\rho = (1-\rho^2)^{1/2} = 1/\sqrt{2}$ and $\alpha_{\vec{v}} \ne k\pi/2 + \pi/4$ for any integer k. Therefore condition (4) holds, and again we have $b_{\vec{v}}(\eta \mid \eta_+) = 1$.

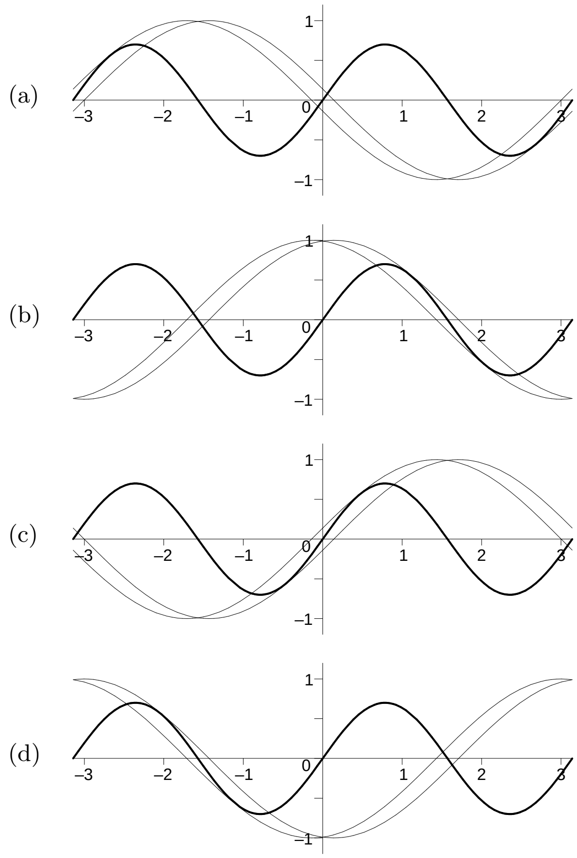

FIGURE 4. $p(t)$'s and $q(t)$ with $1/2 < (1/\rho^2 - 1)^{1/2} < 1$

CASE 4. Suppose that $\vec{v} \in N$ and $v_3 < 1/\sqrt{2}$. Then $1/\sqrt{5} < v_3 < 1/\sqrt{2}$, hence $1/\sqrt{2} < \rho < 2/\sqrt{5}$ and $1/2 < (1/\rho^2 - 1)^{1/2} < 1$. The circle $x^2 + y^2 = \rho^2$ on the unit sphere meets ∂N at eight distinct points as shown in Figure 5. Let α_0 be the smallest positive number that $G_{\rho,\alpha_0}(t)$ has double roots. Since $\vec{v} \in N$, it is on one of the four open arcs of the circle inside N. These arcs correspond to the four intervals for $\alpha_{\vec{v}}$ given in the table below.

(a)	(b)	(c)	(d)
$\lvert\alpha_{\vec{v}}\rvert < \alpha_0$	$\lvert\alpha_{\vec{v}} - \pi/2\rvert < \alpha_0$	$\lvert\alpha_{\vec{v}} - \pi\rvert < \alpha_0$	$\lvert\alpha_{\vec{v}} - 3\pi/2\rvert < \alpha_0$

The four pairs of $p(t)$'s in Figure 4 correspond to the endpoints of these intervals. From Figure 4, we easily see that the sign of $f'_{\vec{v}}(t) = \rho(p(t) - q(t))$ changes from positive to negative once in each of the intervals $(-\pi/2, \pi/2)$ and $(\pi/2, 3\pi/2)$. Therefore $b_{\vec{v}}(\eta \mid \eta_+) = 1$.

CASE 5. Suppose $\vec{v} \notin N$ and $v_1 \leq 0$. If $v_1 = 0$, any local extremum of $f_{\vec{v}}$ occurs only at $(0, 1, 0)$ or $(0, -1, 0)$. If $v_1 < 0$, then $f'_{\vec{v}}(3\pi/2) = v_1 < 0 < -v_1 = f'_{\vec{v}}(\pi/2)$.

Therefore $b_{\vec{v}}(\eta \mid \eta_-) = 1$ where $\eta_- = \eta \cap \{(x, y, z) \mid x < 0\}$. Since $b_{\vec{v}}(\eta) = 1$, we obtain $b_{\vec{v}}(\eta \mid \eta_+) = 0$. □

Suppose n is a positive integer and K is a knot parametrized by

$$K(t) = ((1 + \lambda_1(t)) \cos nt, (1 + \lambda_1(t)) \sin nt, \lambda_2(t) + \cos^2 nt)$$

over any interval of length 2π, for some smooth periodic functions λ_1 and λ_2 with period 2π satisfying the conditions

$$\lambda_1(t)^2 + \lambda_2(t)^2 < 1, \tag{5}$$

$$\lambda_1(t) = \lambda_2(t) = 0 \text{ if } |t| \leq 3\pi/4n, \tag{6}$$

$$\begin{aligned} &\lambda_1(t),\ \lambda_2(t) \text{ are locally constant and negative} \\ &\qquad\qquad \text{if } 5\pi/4n \leq |t| \leq \pi \text{ and } \cos nt \geq -1/\sqrt{2}. \end{aligned} \tag{7}$$

For any ε with $0 \leq \varepsilon \leq 1$, we define

$$K^\varepsilon(t) = ((1 + \varepsilon\lambda_1(t)) \cos nt, (1 + \varepsilon\lambda_1(t)) \sin nt, \varepsilon\lambda_2(t) + \cos^2 nt). \tag{8}$$

Then K^ε is a knot isotopic to K and is the closure of the n-braid $K^\varepsilon \cap \{(x, y, z) \mid x \leq y \leq -x\}$ when $0 < \varepsilon \leq 1$. When $\varepsilon = 0$, K^ε is an n-fold covering of η. Since $K^\varepsilon_+ = K^\varepsilon \cap \{(x, y, z) \mid x > 0\}$ is the union of n disjoint parallel copies of η_+ up to radial scaling about the z-axis, we have $b_{\vec{v}}(K^\varepsilon \mid K^\varepsilon_+) = n \cdot b_{\vec{v}}(\eta \mid \eta_+)$, hence by Lemma 2, we obtain

$$b_{\vec{v}}(K^\varepsilon \mid K^\varepsilon_+) = \begin{cases} n & \text{if } \vec{v} \in N \text{ or } \min\{v_1, v_3\} > 0 \\ 0 & \text{if } \vec{v} \notin N, v_1 < 0 \text{ and } v_3 > 0 \end{cases} \tag{9}$$

for any unit vector $\vec{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$.

By [9], we know that there is a number $\varepsilon' > 0$ such that $s(K^\varepsilon) = 2n$ whenever $0 < \varepsilon \leq \varepsilon'$. Let $0 < \varepsilon \leq \varepsilon'$ and let

$$N^\varepsilon = \{\vec{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} \in S^2 \mid v_3 > 0, b_{\vec{v}}(K^\varepsilon) = 2n\}.$$

For any ε, N^ε is an open set intersecting N in a neighborhood of $\mathbf{k}$. Since $N^0 = N$ and is connected, N^ε is also connected whenever $0 < \varepsilon \leq \varepsilon''$ for some $\varepsilon'' \in (0, \varepsilon']$.

Suppose $N^\varepsilon \cap \{(x, y, z) \mid x < 0\} \not\subset N \cap \{(x, y, z) \mid x < 0\}$. Then there exists a unit vector $\vec{v} \in \partial N \cap N^\varepsilon \cap \{(x, y, z) \mid x < 0\}$. Since the projection $K^\varepsilon \to \mathbb{R}\vec{v}$

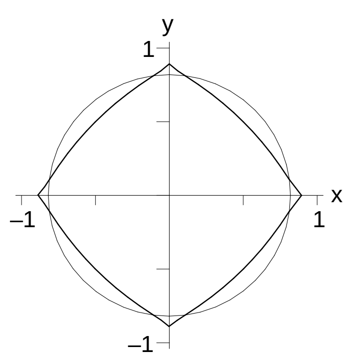

FIGURE 5. ∂N and the circle $x^2 + y^2 = \rho^2$

assumes no local maximum in $K^\varepsilon \cap \{(x,y,z) \mid x=0\}$, and by the equation (9), we have

$$b_{\vec{v}}(K^\varepsilon \mid K^\varepsilon_-) = b_{\vec{v}}(K^\varepsilon \mid K^\varepsilon_+) + b_{\vec{v}}(K^\varepsilon \mid K^\varepsilon_-) = b_{\vec{v}}(K^\varepsilon) = 2n.$$

There exists an open neighborhood V of $\vec{v}$ contained in $N^\varepsilon \cap \{(x,y,z) \mid x<0\}$ such that

$$b_{\vec{u}}(K^\varepsilon \mid K^\varepsilon_-) = 2n \tag{10}$$

for any $\vec{u} \in V$. For any $\vec{u} \in V \cap N$, we obtain the following contradiction from (9) and (10):

$$2n = b_{\vec{u}}(K^\varepsilon) = b_{\vec{u}}(K^\varepsilon \mid K^\varepsilon_+) + b_{\vec{u}}(K^\varepsilon \mid K^\varepsilon_-) = 3n.$$

Proposition 1. *There exist positive numbers ε_0 and δ_0 such that the following conditions hold for any $\varepsilon \in (0, \varepsilon_0]$.*

(a) $s(K^\varepsilon) = 2n$,
(b) $N^\varepsilon \cap \{(x,y,z) \mid x<0\} \subset N \cap \{(x,y,z) \mid x<0\}$,
(c) $b_{\vec{v}}(K^\varepsilon) = n$, *for any unit vector* $\vec{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ *with* $|v_3| < \delta_0$.

Proof: It remains to prove the part (c). As Kuiper did to prove part (a), we investigate the number of real roots of the function $t \mapsto (d/dt)K^\varepsilon(t)\cdot\vec{v}$ for a unit vector $\vec{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Approximating $\lambda_1(t)$ and $\lambda_2(t)$ by finite linear combinations of powers of $\sin t$ and $\cos t$, we get a curve $\tilde{K}^\varepsilon$ which is C^1-close to K^ε. We then substitute

$$\cos t = \frac{2w}{1+w^2}, \quad \sin t = \frac{1-w^2}{1+w^2}$$

to have

$$\frac{d}{dt}\tilde{K}^\varepsilon(t)\cdot\vec{v} = \frac{A^{2n}(w)}{(1+w^2)^n} + v_3 \cdot \frac{B^{4n}(w)}{(1+w^2)^{2n}} + \varepsilon \cdot \frac{C^{2N}(w)}{(1+w^2)^N}$$

where A^{2n}, B^{4n} and C^{2N} are polynomials of degree $2n$, $4n$ and $2N$, respectively, for some possibly large N. The real roots of this function are the same as those of the polynomial

$$P(w) = A^{2n}(w)\cdot(1+w^2)^{N-n} + v_3 \cdot B^{4n}(w)\cdot(1+w^2)^{N-2n} + \varepsilon \cdot C^{2N}(w).$$

Since $A^{2n}(w) = -n\, v_1 \sin nt + n\, v_2 \cos nt = -n\,(v_1^2+v_2^2)^{1/2}\sin(nt-\alpha)$, it has $2n$ real roots. If $\varepsilon = v_3 = 0$, they are the real roots of $P(w) = A^{2n}(w)\cdot(1+w^2)^{N-n}$, each of which is at least one unit away from the remaining roots $\pm\sqrt{-1}$ of multiplicity $N-n$. Since the roots of $P(w)$ depend continuously on ε and v_3, $P(w)$ has exactly $2n$ real roots, when ε and v_3 are sufficiently small. One half of them correspond to the local maxima of the projection $\tilde{K}^\varepsilon \to \mathbb{R}\vec{v}$ and the other half to local minima. Since K^ε is C^1-close to $\tilde{K}^\varepsilon$, part (c) is proved. □

4. Deformations of knots

In this section, we describe two kinds of deformations which do not increase the superbridge number. One is a local deformation and the other is a global one.

Lemma 3. *Given a knot K, let $\bar{K}$ be a knot obtained by replacing a subarc of K with a straight line segment joining the end points of the subarc. Then $s(K) \geq s(\bar{K})$.*

Proof: Given a unit vector $\vec{v}$, let $g\colon (-1,2) \to \mathbb{R}\vec{v}$ be a parametrization of the orthogonal projection of an open neighborhood of the subarc into $\mathbb{R}\vec{v}$, where the subarc corresponds to the closed interval $[0,1]$. Then the projection of a neighborhood of the straight line segment in $\bar{K}$ can be parametrized by

$$\bar{g}(t) = \begin{cases} (1-t)g(0) + tg(1) & \text{if } t \in [0,1] \\ g(t) & \text{if } t \in (-1,0] \cup [1,2). \end{cases}$$

Since $\bar{g}$ has no more local maxima than g, we have $b_{\vec{v}}(K) \geq b_{\vec{v}}(\bar{K})$ for any $\vec{v}$. Therefore $s(K) \geq s(\bar{K})$. □

For a unit vector $\vec{v}$ and a non-singular linear transformation $\phi\colon \mathbb{R}^3 \to \mathbb{R}^3$, let $\vec{v}^\phi$ denote the unit vector contained in the one-dimensional subspace $(\phi(\vec{v}^\perp))^\perp$ satisfying $\phi(\vec{v}) \cdot \vec{v}^\phi > 0$. For any subset $A \subset S^2$, we define

$$A^\phi = \{\vec{v}^\phi \mid \vec{v} \in A\}.$$

Lemma 4. *Given a unit vector $\vec{v} \in \mathbb{R}^3$ and a nonsingular linear transformation ϕ of $\mathbb{R}^3$, the formulas*

$$b_{\vec{v}^\phi}(\phi(K)) = b_{\vec{v}}(K)$$
$$b_{\vec{v}^\phi}(\phi(K) \mid \phi(S)) = b_{\vec{v}}(K \mid S)$$

hold for any knot K and any open subarc S of K.

Proof: At each local maximum point P of the projection $S \to \mathbb{R}\vec{v}$, there is an open disk d_P perpendicular to $\vec{v}$ and tangent to S at P. Then $\phi(d_P)$ is tangent to $\phi(S)$ at $\phi(P)$ and is perpendicular to $\vec{v}^\phi$. By the definition of $\vec{v}^\phi$, $\phi(P)$ is a local maximum point of the projection $\phi(S) \to \mathbb{R}\vec{v}^\phi$ and hence $b_{\vec{v}}(K \mid S) \leq b_{\vec{v}^\phi}(\phi(K) \mid \phi(S))$. Since $(\vec{v}^\phi)^{\phi^{-1}} = \vec{v}^{(\phi^{-1}\phi)} = \vec{v}$, we also get

$$b_{\vec{v}}(K \mid S) = b_{(\vec{v}^\phi)^{\phi^{-1}}}(\phi^{-1}(\phi(K)) \mid \phi^{-1}(\phi(S))) \geq b_{\vec{v}^\phi}(\phi(K) \mid \phi(S)).$$

This proves the second formula. Setting $S = K$, the first formula is obtained. □

The next proposition easily follows from Lemma 4.

Proposition 2. *Given a knot K and a nonsingular linear transformation ϕ of $\mathbb{R}^3$, we have $s(\phi(K)) = s(K)$. In particular, if a knot K and a unit vector $\vec{v}$ satisfy $b_{\vec{v}}(K) = s(K) = s[K]$, then $b_{\vec{v}^\phi}(\phi(K)) = s(\phi(K)) = s[K]$.*

5. Proofs

For any λ with $0 < \lambda \leq 1$, let ϕ_λ, ψ_λ, ψ be the autohomeomorphisms of $\mathbb{R}^3$ defined by

$$\begin{aligned} \phi_\lambda(x,y,z) &= (x, y, \lambda z) \\ \psi_\lambda(x,y,z) &= (1+\lambda-\lambda z, -y, 1+\lambda-x) \\ \psi(x,y,z) &= (-z,-y,-x). \end{aligned}$$

The map ψ is the 180° rotations about the line $\{(x,0,z) \mid x+z=0\}$ and the map ψ_λ is the composite map ϕ_λ followed by the 180° rotations about the line $\{(x,0,z) \mid x+z = 1+\lambda\}$.

For any locally one-to-one closed parametrized path $\gamma\colon S^1 \to \mathbb{R}^3$, we extend the definition of the crookedness $b_{\vec{v}}(\gamma)$ by considering the parametrized projection $t \mapsto \gamma(t)\cdot\vec{v} : S^1 \to \mathbb{R}\vec{v}$ instead of the projection $\gamma(S^1) \to \mathbb{R}\vec{v}$. In this way we can consider the crookedness for finite-fold coverings of knots and singular knots.

Proof of Theorem 1. Throughout this proof, λ is a constant satisfying $0 < \lambda \le 1/4$, $\vec{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is a unit vector, and $i = 1$ or 2. We may assume that the knot K_i is parametrized by

$$K_i(t) = ((1 + \lambda_{i1}(t))\cos n_i t, (1 + \lambda_{i1}(t))\sin n_i t, \lambda_{i2}(t) + \cos^2 n_i t)$$

where λ_{i1} and λ_{i2} are smooth periodic functions with period 2π satisfying the conditions corresponding to (5), (6) and (7), for $i = 1, 2$. For any ε with $0 \le \varepsilon \le 1$, we define K_1^ε and K_2^ε as in (8). Then K_i^ε is a knot isotopic to K_i and is the closure of the n_i-braid $K_i^\varepsilon \cap \{(x, y, z) \mid x \le y \le -x\}$ when $0 < \varepsilon \le 1$. When $\varepsilon = 0$, K_i^ε is an n_i-fold covering of η. Since the two knots $\phi_\lambda(K_1^\varepsilon)$ and $\psi_\lambda(K_2^\varepsilon)$ are tangent at the point $(1, 0, \lambda)$, their union K_λ can be regarded as a singular knot parametrized by

$$K_\lambda(t) = \begin{cases} \phi_\lambda(K_1^\varepsilon(2t)) & \text{if } -\pi \le t \le 0 \\ \psi_\lambda(K_2^\varepsilon(-2t)) & \text{if } 0 \le t \le \pi. \end{cases}$$

Then $\bar{K}_\lambda = (K_\lambda - \phi_\lambda(\eta_+) \cup \psi_\lambda(\eta_+)) \cup S_+ \cup S_-$ is a singular knot with only one singular point at $((1+\lambda)/2, 0, (1+\lambda)/2)$ where

$$S_\pm = \{(0, \mp 1, 0) + s(1 + \lambda, \pm 2, 1 + \lambda) \mid 0 < s < 1\}.$$

By Lemma 3, $b_{\vec{v}}(\bar{K}_\lambda) \le b_{\vec{v}}(K_\lambda)$. Since $(1, 0, \lambda)$ is a local maximum point of the parametrized projection $t \mapsto K_\lambda(t)\cdot\vec{v}$ only if both of the projections $\phi_\lambda(K_1^\varepsilon) \to \mathbb{R}\vec{v}$ and $\psi_\lambda(K_2^\varepsilon) \to \mathbb{R}\vec{v}$ have local maximum at $(1, 0, \lambda)$, we have

$$b_{\vec{v}}(\bar{K}_\lambda) \le b_{\vec{v}}(\phi_\lambda(K_1^\varepsilon)) + b_{\vec{v}}(\psi_\lambda(K_2^\varepsilon)). \tag{11}$$

The vectors $\vec{w}_\pm = \pm(1+\lambda)(\mathbf{i}+\mathbf{k}) + 2\mathbf{j}$, are parallel to the segments $S_\pm$, respectively. A computation shows that

$$\begin{aligned} \vec{w}_+ \cdot \vec{v} &= (1 + \lambda)(v_1 + v_3) + 2v_2 \ge (10 - \sqrt{89})/\sqrt{80} > 0,{}^3 \\ \vec{w}_- \cdot \vec{v} &= -(1 + \lambda)(v_1 + v_3) + 2v_2 \le -(10 - \sqrt{89})/\sqrt{80} < 0,{}^4 \end{aligned}$$

whenever $v_3 \ge (4\lambda^2 + 1)^{-1/2}$. Therefore there exists a number $\delta \in (1/\sqrt{2}, (4\lambda^2 + 1)^{-1/2})$ such that $\vec{w}_- \cdot \vec{v} < 0 < \vec{w}_+ \cdot \vec{v}$ whenever $v_3 \ge \delta$. At the endpoints $(1 + \lambda, \pm 1, 1 + \lambda)$ of $S_\pm$, we have

$$\lim_{t \to \frac{4n_2 - 1}{4n_2}\pi -} \frac{d}{dt} K_\lambda(t)\cdot\vec{v} = -2n_2 v_3 < 0 < 2n_2 v_3 = \lim_{t \to \frac{\pi}{4n_2}+} \frac{d}{dt} K_\lambda(t)\cdot\vec{v}$$

if $v_3 > 0$. Therefore there exist open arcs $\tilde{S}_\pm$ of $\bar{K}_\lambda$, containing the closures of $S_\pm$, respectively, satisfying $b_{\vec{v}}(\bar{K}_\lambda \mid \tilde{S}_+ \cup \tilde{S}_-) = 0$ whenever $v_3 \ge \delta$. Similarly we also have $b_{\vec{v}}(\bar{K}_\lambda \mid \tilde{S}_+ \cup \tilde{S}_-) = 0$ whenever $v_1 \le -\delta$.

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[3] The equality holds when $\lambda = 1/4$ and $\vec{v} = -\sqrt{5/89}\mathbf{i} - 8\sqrt{445}\mathbf{j} + 2\sqrt{5}\mathbf{k}$.
[4] The equality holds when $\lambda = 1/4$ and $\vec{v} = -\sqrt{5/89}\mathbf{i} + 8\sqrt{445}\mathbf{j} + 2\sqrt{5}\mathbf{k}$.

By Lemma 2, Lemma 4, and the last two conditions, we have[5]

$$\begin{aligned} b_{\vec{v}}(\bar{K}_\lambda) &\leq b_{\vec{v}}(\phi_\lambda(K_1^\varepsilon) \mid \phi_\lambda(K_1^\varepsilon - \overline{\eta}_+)) + b_{\vec{v}}(\bar{K}_\lambda \mid \tilde{S}_+ \cup \tilde{S}_-) \\ &\qquad + b_{\vec{v}}(\psi_\lambda(K_2^\varepsilon) \mid \psi_\lambda(K_2^\varepsilon - \overline{\eta}_+)) \\ &\leq b_{\vec{v}}(\phi_\lambda(K_1^\varepsilon)) + b_{\vec{v}}(\psi_\lambda(K_2^\varepsilon)) - 1 \end{aligned} \tag{12}$$

whenever $\vec{v} \in N^{\phi_\lambda} \cup Q_\delta \cup \psi(N^{\phi_\lambda} \cup Q_\delta)$ where $Q_\delta = \{(x, y, z) \in S^2 \mid x > 0, z > \delta\}$. By Proposition 1 (a)–(b), we may assume that

$$s(K_i^\varepsilon) = 2n_i \tag{13}$$

$$(N_i^\varepsilon)^{\phi_\lambda} \subset N^{\phi_\lambda} \cup Q_\delta \tag{14}$$

where $N_i^\varepsilon = \{\vec{v} \in S^2 \mid v_3 > 0, b_{\vec{v}}(K_i^\varepsilon) = 2n_i\}$. Since

$$\begin{aligned} \vec{v}^{\phi_\lambda} \cdot \mathbf{k} &= v_3(\lambda^2(1 - v_3^2) + v_3^2)^{-1/2}, \\ \vec{v}^{\psi_\lambda} \cdot \mathbf{i} &= -v_1(\lambda^2(1 - v_1^2) + v_1^2)^{-1/2}, \end{aligned}$$

Proposition 1 (c) implies that

$$\begin{aligned} b_{\vec{v}}(\phi_\lambda(K_1^\varepsilon)) = n_1 \text{ whenever } |v_3| \leq 1/\sqrt{2} \\ b_{\vec{v}}(\psi_\lambda(K_2^\varepsilon)) = n_2 \text{ whenever } |v_1| \leq 1/\sqrt{2} \end{aligned} \tag{15}$$

provided λ is sufficiently small. By (11) and (15), we get

$$b_{\vec{v}}(\bar{K}_\lambda) \leq n_1 + n_2 \text{ if } \max\{|v_1|, |v_3|\} \leq 1/\sqrt{2}. \tag{16}$$

By (11), (13) and (15), we get

$$b_{\vec{v}}(\bar{K}_\lambda) \leq \begin{cases} 2n_1 + n_2 - 1 & \text{if } \pm\vec{v} \notin (N_1^\varepsilon)^{\phi_\lambda}, |v_3| > 1/\sqrt{2} \\ n_1 + 2n_2 - 1 & \text{if } \pm\vec{v} \notin \psi((N_2^\varepsilon)^{\phi_\lambda}), |v_1| > 1/\sqrt{2} \end{cases} \tag{17}$$

By (12), (13), (14) and (15), we get

$$b_{\vec{v}}(\bar{K}_\lambda) \leq \begin{cases} 2n_1 + n_2 - 1 & \text{if } \pm\vec{v} \in (N_1^\varepsilon)^{\phi_\lambda} \cup Q_\delta \\ n_1 + 2n_2 - 1 & \text{if } \pm\vec{v} \in \psi((N_2^\varepsilon)^{\phi_\lambda} \cup Q_\delta) \end{cases} \tag{18}$$

For the last two formulas, we used the fact $b_{-\vec{v}}(\bar{K}_\lambda) = b_{\vec{v}}(\bar{K}_\lambda)$. For a very small positive number ϵ, let $\bar{S}_+ = S_+ \cup \{(\cos t, \sin t, \lambda \cos^2 t) \mid -\pi/2 - \epsilon \leq t \leq -\pi/2\}$ and let $\check{S}_+$ be the line segment joining the endpoints of $\bar{S}_+$. By the conditions (5), (6) and (7), the knot $\check{K}_\lambda = (\bar{K}_\lambda - \bar{S}_+) \cup \check{S}_+$ is a knot representing $K_1 \sharp K_2$. By Lemma 3, (16), (17) and (18), we have

$$b_{\vec{v}}(\check{K}_\lambda) \leq b_{\vec{v}}(\bar{K}_\lambda) \leq \max\{2n_1 + n_2, n_1 + 2n_2\} - 1. \qquad \square$$

[5] $\overline{\eta}_+$ is the closure of η_+.

Proof of Theorem 2. Let K_i be a torus knot of type (p_i, q_i) where p_i and q_i are coprime integers satisfying $2 \le p_i < q_i$, for $i = 1, 2$. This proof breaks into three cases.

Case 1. Suppose that the inequality $2 \le p_i < q_i/2$ holds for $i = 1, 2$. In this case, we have $\beta[K_i] = b[K_i] = s[K_i]/2 = p_i$. Therefore a direct application of Theorem 1 shows that $s[K_1 \sharp K_2] \le \max\{2p_1 + p_2, p_1 + 2p_2\} - 1$.

Case 2. Suppose that the inequality $2 \le p_i < q_i < 2p_i$ holds for $i = 1, 2$. In this case, $\beta[K_i] = b[K_i] = p_i$ and $s[K_i] = q_i$. As shown in [7], K_i can be represented by a polygonal knot $\tau_i = \tau_i(\alpha_i)$ of $2q_i$ edges embedded on the torus $H_{\alpha_i} \cup H_{\beta_i}$ where

$$H_\theta = \{(x, y, z) \mid x^2 + y^2 - z^2 \sin^2 \frac{\theta}{2} = \cos^2 \frac{\theta}{2}, |z| \le 1\},$$

$\pi p_i/q_i < \alpha_i < 2\pi p_i/q_i$ and $\alpha_i + \beta_i = \pi$. The knot K_i has $2q_i$ vertices; q_i on each of the two unit circles $\{(x, y, \pm 1) \mid x^2 + y^2 = 1\}$. By (1), we know that $s(K_i) = q_i$. We may assume that K_i has a vertex at $(1, 0, 1)$. We define

$$\begin{aligned} N_i &= \{\vec{v} \in S^2 \mid \vec{v} \cdot \mathbf{k} > 0, b_{\vec{v}}(K_i) = q_i\}, \\ M_i &= \{\vec{v} \in S^2 \mid \text{The projection } K_i \to \mathbb{R}\vec{v} \text{ has a local minimum at } (1, 0, 1)\}. \end{aligned}$$

For any $\vec{v} \in N_i$, the q_i vertices of K_i on the circle $\{(x, y, 1) \mid x^2 + y^2 = 1\}$ are local maximum points of the projection $K_i \to \mathbb{R}\vec{v}$. Let $t \mapsto K_i(t)$ parametrize K_i modulo 2π with $K_i(0) = (1, 0, 1)$, as a closed p_i-braid around the z-axis. The singular knot K_λ given by the parametrization

$$K_\lambda(t) = \begin{cases} \phi_\lambda(K_1(2t)) & \text{if } -\pi \le t \le 0 \\ \psi_\lambda(K_2(-2t)) & \text{if } 0 \le t \le \pi \end{cases}$$

has only one singular point at $(1, 0, \lambda)$. Straightening an arc near the singular point, we get a knot representing $K_1 \sharp K_2$ whose crookedness is not bigger than that of K_λ in any direction. As λ approaches zero, N_1 shrinks to the north pole $(0, 0, 1)$ whereas M_1 approaches a region of positive area containing the point $(-1, 0, 0)$. Therefore, for a sufficiently small λ, we have

$$(N_1)^{\phi_\lambda} \subset \psi((M_2)^{\phi_\lambda}), \text{ and } \psi((N_2)^{\phi_\lambda}) \subset (M_1)^{\phi_\lambda}, \tag{19}$$

and as in (15), we also have

$$\begin{aligned} b_{\vec{v}}(\phi_\lambda(K_1)) &= p_1 \text{ whenever } |\vec{v} \cdot \mathbf{k}| \le 1/\sqrt{2}, \\ b_{\vec{v}}(\psi_\lambda(K_2)) &= p_2 \text{ whenever } |\vec{v} \cdot \mathbf{i}| \le 1/\sqrt{2}. \end{aligned} \tag{20}$$

By (19), if $\pm\vec{v} \in (N_1)^{\phi_\lambda} \cup \psi((N_2)^{\phi_\lambda})$, the point $(1, 0, \lambda)$ in not a local maximum point of K_λ. Therefore we have

$$b_{\vec{v}}(K_\lambda) = \begin{cases} q_1 + p_2 - 1 & \text{if } \pm\vec{v} \in (N_1)^{\phi_\lambda} \\ p_1 + q_2 - 1 & \text{if } \pm\vec{v} \in \psi((N_2)^{\phi_\lambda}). \end{cases}$$

By (19) and (20), we obtain

$$b_{\vec{v}}(K_\lambda) \le b_{\vec{v}}(\phi_\lambda(K_1)) + b_{\vec{v}}(\psi_\lambda(K_2)) < \max\{q_1 + p_2, p_1 + q_2\},$$

if $\pm\vec{v} \notin (N_1)^{\phi_\lambda} \cup \psi((N_2)^{\phi_\lambda})$. Therefore $b_{\vec{v}}(K_\lambda) \le \max\{q_1 + p_2, p_1 + q_2\} - 1$, for any unit vector $\vec{v}$.

CASE 3. Suppose that the inequalities $2 \le p_1 < q_1/2$ and $2 \le p_2 < q_2 < 2p_2$ hold. Let K_1 ($= K_1^\varepsilon$) and K_2 be embedded and parametrized as in the Proof of Theorem 1 and in CASE 2, respectively. We consider the singular knot parametrized by

$$K_\lambda(t) = \begin{cases} \phi_\lambda(K_1^\varepsilon(2t)) & \text{if } -\pi \le t \le 0 \\ \psi_\lambda(K_2(-2t)) & \text{if } 0 \le t \le \pi. \end{cases}$$

We replace the arc $\phi_\lambda(\eta_+)$ of $\phi_\lambda(K_1^\varepsilon)$ by the broken line joining the three points $(0,-1,0)$, $(1,0,\lambda)$ and $(0,1,0)$, consecutively, to get a new singular knot $\bar{K}_\lambda$. The remaining argument will be very similar to that of CASE 2. □

ACKNOWLEDGEMENT: This work was done while the author was visiting the University of British Columbia during the academic year 1998–99. He is grateful to the members of the Department of Mathematics of UBC, especially Dale Rolfsen with whom he had many helpful discussions.

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