Reexamination of determinant-based separability test for two qubits

Maciej Demianowicz*
Faculty of Applied Physics and Mathematics, Gdańsk University of Technology, PL-80-952 Gdańsk, Poland and National Quantum Information Center of Gdansk, PL-81-824 Sopot, Poland

It was shown in [Augusiak *et al.*, Phys. Rev. A **77**, 030301(R) (2008)] that discrimination between entanglement and separability in a two qubit state can be achieved by a measurement of a single observable on four copies of it. Moreover, a pseudo entanglement monotone π was proposed to quantify entanglement in such states. The main goal of the present paper is to show that close relationship between π and concurrence reported there is a result of sharing the same underlying construction of a spin flipped matrix. We also show that monogamy of entanglement can be rephrased in terms of π. The results of the paper allow us also for an immediate proof of the factorization law for π.

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Entanglement, first recognized by Schrodinger, Einstein, Podolsky, and Rosen [1], lies at the heart of quantum information theory. With no doubt it is the most important resource of this rapidly developing branch of science and serves as the building brick for the huge number of information tasks, just to mention *e.g.* teleportation [2] and dense coding [3]. From this point of view full recognition of this "spooky action at a distance" [4] is fundamental to our understanding of quantum mechanics. Much effort has been put to recognize its nature and not surprisingly the major progress has been achieved in the case of the simplest bipartite quantum states- states of two qubits. One of the most important qualitative results concerning such systems is the necessary and sufficient condition for inseparability- the celebrated Peres-Horodeckis criterion of non positive partial transposition [5]. On the other side research towards quantitative description of entanglement of two qubit states has culminated in the introduction of entanglement measures, among which the most notable are entanglement of formation [6] and concurrence for whom closed expressions has been found [7]. Unfortunately, both of them has still not been shown to be directly measurable and it is reasonable to conjecture that they are not in general. However, very recently it has been demonstrated that *single* collective measurement of the specially prepared observable on four copies of an unknown two qubit state can *unambiguously* discriminate between entanglement and separability, additionally *quantifying* to some extent entanglement contained in the system by providing sharp lower and upper bounds on concurrence [11].

In the present paper we continue research on the pseudo entanglement monotone π, which was introduced in Ref. [11] for entanglement quantification purposes.

Let us start with the an introduction of necessary concepts. Consider a two qubits mixed state ρ_{AB}. Define (conjugation in a standard basis) a spin flipped state [7]

$$\tilde{\rho}_{AB} = \sigma_y \otimes \sigma_y \rho^*_{AB} \sigma_y \otimes \sigma_y. \tag{1}$$

Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ be the square roots of the eigenvalues of $\rho_{AB}\tilde{\rho}_{AB} := M_{AB}$. Note that we can safely write inequalities since they are real (moreover they are nonnegative). We define concurrence to be

$$C(\rho_{AB}) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} =: C_{AB}. \tag{2}$$

Eligibility of such constructed quantity for being a good measure of entanglement is justified by the invariance of the eigenvalues λ_i under local unitary operations and by the fact that $0 \leq C_{AB} \leq 1$. When ρ_{AB} is a partial trace over C from the tripartite pure qubit state $|\psi_{ABC}\rangle\langle\psi_{ABC}| =: \psi_{ABC}$ there are only two nonzero eigenvalues thus we have just $C_{AB} = \lambda_1 - \lambda_2$. We then also define tangle [8] to be

$$\tau_{ABC} = 4\lambda_1\lambda_2. \tag{3}$$

It was shown that it does not depend on eigenvalues of which of the matrices M_{AB}, M_{BC}, M_{AC} we take. These quantities can be combined to give the so called monogamy relation [8, 9]

$$C^2_{AB} + C^2_{BC} + \tau_{ABC} = C^2_{B(AC)} = 4\det\rho_B, \rho_B = \mathrm{tr}_A\, \rho_{AB}. \tag{4}$$

Concurrence $C_{B(AC)}$ is the meaningful quantity since we consider pure state of three qubits thus effectively $B(AC)$ is a two-qubit-like state. This relations provides an interpretation for tangle as a measure of tripartite correlations.

One also defines [10] concurrence of assistance C^a, which is the maximum over ensembles of average concurrence of pure states in the ensemble. In case of two qubits we have just $C^a = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$.

In Ref. [11] it was shown that the separability of an unknown two qubit state ρ can be unambiguously settled in a single collective measurement on four copies of this state, *i.e.* one needs at one time $\rho^{\otimes 4}$. This was obtained on the basis of two facts: (i) partially transposed density matrix ρ^Γ of an entangled two qubit state ρ is full rank (has four non zero eigenvalues), (ii) there can be only one negative eigenvalue of ρ^Γ. The above led to the conclusion that it is sufficient to measure $\det\rho^\Gamma$ and

*Electronic address: maciej@mif.pg.gda.pl

the strict negativity of the latter indicates entanglement. The authors of the mentioned paper showed that indeed such a measurement is possible using a noiseless circuit [12]. They also proposed a simple alternative scheme to measure this determinant. The question of the usage of $\det \rho^\Gamma$ for quantitative description was further addressed. It was shown that the quantity, which we will call in this paper the *det-measure*,

$$\pi(\rho) = \begin{cases} 0 & \text{for } \det \rho^\Gamma \geq 0 \\ 2\sqrt[4]{|\det \rho^\Gamma|} & \text{for } \det \rho^\Gamma < 0 \end{cases}. \quad (5)$$

is a monotone under pure local operations preserving dimensions and classical communication and provides tight upper and lower bounds on concurrence as follows

$$C(\rho) \leq \pi(\rho) \leq \sqrt[4]{C(\rho)\left(\frac{C(\rho)+2}{3}\right)}. \quad (6)$$

Normalization in Eq. (5) is chosen to impose agreement of det-measure and concurrence on pure states. From the first inequality we also have the bound for entanglement of formation E_f [7] as follows $E_f(\rho) \leq E(\pi(\rho))$, where $E(x) = H(\frac{1+\sqrt{1-x^2}}{2})$ with $H(y)$ being the Shannon entropy of a probability distribution $(y, 1-y)$. One can also prove that π shares the nice property of being continuous in the input density operator [13].

For the purpose of the present paper we propose the extension of our definition for entanglement between qubit A and qubits BC in a pure state ψ_{ABC} to $\pi_{A(BC)} \equiv 2\sqrt{\det \rho_A}$, *i.e.* we define it to be equal to $C_{A(BC)}$ on such states. Such extension is the most natural since we keep two most important properties of π: mentioned equality with concurrence and possibility of direct measurement.

Let us now turn to the main body of the paper. We start with considerations analogous to the one from Ref. [14], where the local unitary interaction of one part of maximally entangled state ψ^+_{AB} with two level environment E was considered, *i.e.* the global state after the evolution was $|\psi_{ABE}\rangle = \mathbb{I}_A \otimes U_{BE}(|\psi^+\rangle_{AB} \otimes |0\rangle_E)$.

Its bipartite reductions of interest will be denoted ρ_{AB} and ρ_{AE}. In the mentioned paper the author showed by random sampling that there is no correlation between singlet fraction [15] $F(\rho_{AB}) := F_{AB}$ after the action of the channel and concurrence C_{AB} of the decohered state and showed analytically for the chosen class of channels that $F_{AB} = \frac{1}{4}(1 + C_{AB})(1 + \sqrt{1 - C^2_{AE}})$. It turned out that this relation holds true for all channels, which was shown by random generation of channels.

We pursue the same approach using det-measure instead of concurrence. First let us consider relation of F_{AB} and π_{AB} after the action of the random channel. The result is shown in the Fig.1., which was obtained by random generation of 10000 U_{BE}'s [16].

One can see that in our case there is some connection between these two quantities, however, still there is no analytical formula linking them. We will comment on this connection later when monogamy equality will be obtained.

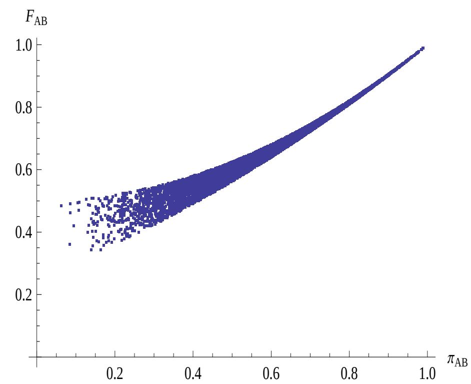

FIG. 1: Singlet fraction F_{AB} *vs.* det-measure π_{AB} after action of the local channel

Following Ref. [14] let us consider a class of local channels implemented by unitaries defined by:

$$|00\rangle_{BE} \to \sqrt{1-q}|00\rangle_{BE} + \sqrt{q}|11\rangle_{BE} \quad (7)$$
$$|10\rangle_{BE} \to \sqrt{1-p}|10\rangle_{BE} + \sqrt{p}|01\rangle_{BE}. \quad (8)$$

For such channels we obtain the following (with previously established notation):

$$\pi_{AB} = \sqrt{|p+q-1|} \quad (9)$$
$$\pi_{AE} = \sqrt{|p-q|} \quad (10)$$
$$F_{AB} = \begin{cases} \frac{2-p-q+2\sqrt{(1-p)(1-q)}}{4} & \text{for } p+q-1 < 0 \\ \frac{p+q+2\sqrt{pq}}{4} & \text{for } p+q-1 \geq 0 \end{cases}. \quad (11)$$

Direct calculation reveals that

$$F_{AB} = \frac{1}{4}(1+\pi^2_{AB})\left(1 + \sqrt{1 - \frac{\pi^4_{AE}}{(\pi^2_{AB}+1)^2}}\right). \quad (12)$$

As it was in the case of concurrence, the relation we obtained can be shown to hold for all channels and is independent of which maximally entangled state we choose to be the input. Note the close resemblance of both forms.

Closed formula for singlet fraction using det-measure π opens hope for a monogamy relation of entanglement in terms of it. In what follows we prove the existence of such equation.

Consider a pure state of three qubits ψ_{ABC}. As it was shown [17], as far as the entanglement properties are considered, such state can be parameterized by five real numbers as

$$|\psi_{ABC}\rangle = p|001\rangle + q|010\rangle + r|001\rangle + s|111\rangle + te^{i\theta}|000\rangle \quad (13)$$

with normalization condition $p^2 + q^2 + r^2 + s^2 + t^2 = 1$. From this we obtain eigenvalues of the matrix M_{AB} and determinant of the partially transposed matrix of the reduced state of qubits A and B

$$\lambda_1^2 = 2p^2q^2 + 2r^2s^2 + s^2t^2 + 2\sqrt{p^4q^4 + p^2s^2\left(t^2 - 2r^2\right)q^2 - 2prs^3t^2\cos(2\theta)q + r^2s^4\left(r^2 + t^2\right)} \quad (14)$$

$$\lambda_2^2 = 2p^2q^2 + 2r^2s^2 + s^2t^2 - 2\sqrt{p^4q^4 + p^2s^2\left(t^2 - 2r^2\right)q^2 - 2prs^3t^2\cos(2\theta)q + r^2s^4\left(r^2 + t^2\right)} \quad (15)$$

$$\det \rho_{AB}^{\Gamma} = -p^4q^4 - p^2s^2\left(t^2 - 2r^2\right)q^2 + 2prs^3t^2\cos(2\theta)q - r^2s^4\left(r^2 + t^2\right), \quad (16)$$

which immediately yields

$$\pi_{AB} = \sqrt{\lambda_1^2 - \lambda_2^2}. \quad (17)$$

Recalling Eqs. (2) and (3) we obtain analytical relationship between π_{AB}, C_{AB}, and τ_{ABC} in a pure three qubit state

$$\pi_{AB} = \sqrt{C_{AB}\sqrt{C_{AB}^2 + \tau_{ABC}}}. \quad (18)$$

This can be put into a nice compact form

$$\pi_{AB} = \sqrt{C_{AB}C_{AB}^a}\ , \quad (19)$$

which means that in case of rank two states the det-measure is the geometric mean value of concurrence and concurrence of assistance. We also conclude that the bound in Eq. (6) can be tightened for such states to obtain

$$\pi_{AB} \le \sqrt{C_{AB}}. \quad (20)$$

Eq. (18) leads us to a simple corollary showing that the only states on which π and C agree are pure states and separable states.

Corollary. Given pure three qubit state ABC one has $\pi_{AB} = C_{AB}$ iff $\tau_{ABC} = 0$.

One can argue that the pattern in the plot of singlet fraction (Fig.1.) is the result of quantifying to some extent tripartite correlations by the det-measure.

Now let us reverse (18) to get

$$C_{AB}^2 = \frac{-\tau_{ABC} + \sqrt{\tau_{ABC}^2 + 4\pi_{AB}^4}}{2}. \quad (21)$$

Inserting this into Eq. (4) one obtains the advertised elegant monogamy relation in terms of the det-measure

$$\sqrt{(\frac{\tau_{ABC}}{2})^2 + \pi_{AB}^4} + \sqrt{(\frac{\tau_{ABC}}{2})^2 + \pi_{BC}^4} = \pi_{B(AC)}^2. \quad (22)$$

This gives also the recipe to measure tangle on ten copies of the state relying directly on the measurements of determinants of two partially transposed density matrices (four plus four copies) and the determinant of the reduced qubit density matrix (two copies). The question of optimality of such measurement is beyond the scope of this paper (see [18]).

Now we will argue that the det-measure π in a general case of mixed states of arbitrary rank is an analytical function of the eigenvalues of the matrix M. Consider states

$$\rho_{Bdiag} = p_1\psi_+ + p_2\psi_- + p_3\phi_+ + p_4\phi_-, \quad (23)$$

which are diagonal in the Bell basis $|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$, $|\phi_\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$. Such states are entangled iff one of the probabilities is larger than $\frac{1}{2}$. W.l.o.g. assume that it holds for p_1. One has then

$$\pi(\rho_{Bdiag}) = \sqrt[4]{|-p_1 + p_2 + p_3 + p_4|(p_1 - p_2 + p_3 + p_4)(p_1 + p_2 - p_3 + p_4)(p_1 + p_2 + p_3 - p_4)}. \quad (24)$$

We also easily compute that $\lambda_i = p_i$. Motivated by the form of π for ρ_{Bdiag} we further define

$$C_1 = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \equiv C, \quad (25)$$

$$C_2 = \lambda_1 - \lambda_2 + \lambda_3 + \lambda_4, \quad (26)$$

$$C_3 = \lambda_1 + \lambda_2 - \lambda_3 + \lambda_4, \quad (27)$$

$$C_4 = \lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 \quad (28)$$

and

$$\hat{\pi}(\rho) = \sqrt[4]{C_1C_2C_3C_4}. \quad (29)$$

The main result of this part of the paper is the following.

Claim. For any two qubit state ρ one has

$$\pi(\rho) = \hat{\pi}(\rho). \quad (30)$$

As shown above this certainly holds for rank two states and Bell diagonal states providing the first evidence of correctness of the postulated form. We have also verified the relation for some number of different classes of states and found full agreement. Confirmation for all states, however, comes from random generation [19] of two qubit states, which gave no counterexample.

We see that, provided validity of Eq. (30) for which we gave an evidence, π can be regarded as some kind of symmetrization of concurrence allowing for experimental direct accessibility. Natural question is to what extent det-measure quantifies also tripartite correlations in the general case. Unfortunately we have not been able to find definite answer so far.

Representation (30) allows us to prove a remarkable factorization law which was originally stated for concurrence [20]

Fact. Det-measure π obeys the factorization law *i.e.* for an arbitrary channel Λ, pure state ϕ, and a Bell state ψ_+ it holds

$$\pi(\mathcal{I}\otimes\Lambda(\phi)) = \pi(\mathcal{I}\otimes\Lambda(\psi_+))\pi(\phi). \qquad (31)$$

Proof. The proof with no changes follows [20].

Thus the det-measure provides *factorizable* measurable bound on concurrence (see [21] for recent effort in this direction).

As no surprise comes now the fact, noted in Ref. [11], that under local filtering both concurrence and det-measure change by the same multiplicative factor.

In conclusion, we have provided monogamy relation for entanglement quantified by determinant based measure π. As a byproduct we obtained explicit formulas for the latter in terms of other entanglement quantities. We showed that close relation with concurrence is the result of bearing the similar construction in its roots. We also provided evidence that the disagreement of π and C on general mixed states stems from the fact that π quantifies to some extent both bipartite and tripartite correlations. The natural question motivated by the result of the present paper is about possibility of constructing other measurable quantifiers of entanglement, which are based on the analogous procedure and provide better bounds on concurrence of an unknown state. We hope our results will stimulate research on this topic and will provide some tools for improved understanding of two qubits entanglement. The issue of using the det-measure for detecting and quantifying entanglement in higher dimensional systems is the subject of the ongoing research [22].

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[1] E. Schrödinger, Proc. Cambridge Philos. Soc. **31**, 555 (1935); A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).

[2] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).

[3] C. Bennett and S.J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).

[4] A. Einstein, *The Born-Einstein Letters; Correspondence between Albert Einstein and Max and Hedwig Born from 1916 to 1955*, Walker, New York, 1971.

[5] A. Peres, Phys. Rev. Lett. **77** 1413 (1996); M. Horodecki, P. Horodecki, R. Horodecki, Phys. Lett. A **223**, 1 (1996).

[6] C. H. Bennett, H. J. Bernstein, S. Popescu, B. Schumacher Phys. Rev. A **53**, 2046 (1996); C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).

[7] S. Hill and W. K. Wootters, Phys. Rev. Lett **78**, 5022 (1997); W.K.Wootters, Phys. Rev. Lett **80**, 2245 (1998).

[8] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A **61**, 052306 (2000).

[9] T. J. Osborne and F. Verstraete, Phys. Rev. Lett. **96**, 220503 (2006).

[10] T. Lautsen, F. Verstraete, and S. J. Van Enk, Quant. Inf. Comp **3**, 64 (2003).

[11] R.Augusiak, M. Demianowicz, and P. Horodecki, Phys. Rev. A **77**, 030301(R) (2008).

[12] P. Horodecki, R. Augusiak, and M. Demianowicz, Phys. Rev. A **74**, 052323 (2006).

[13] Assume $||\rho-\sigma||_1 \leq \epsilon$, where $||\cdot||_1$ denotes the trace norm. Recall that for any hermitian operator on $d\otimes d$ it holds $||M^{\Gamma_A}||_1 \leq d||M||_1$. From this relation we immediately obtain that $||\rho^{\Gamma_A} - \sigma^{\Gamma_A}||_1 \leq 2\epsilon$. Let $\lambda_i^{\downarrow}$ and $\sigma_j^{\downarrow}$ be the eigenvalues of ρ^{Γ_A} and σ^{Γ_A} ordered decreasingly. Then obviously $\forall_i|\lambda_i^{\downarrow}-\sigma_i^{\downarrow}| \leq 2\epsilon$. From this we obtain $|\det\rho^{\Gamma_A} - \det\sigma^{\Gamma_A}| \leq \eta$ with $\eta \to 0$ when $\epsilon \to 0$.

[14] S. Ishizaka, Phys. Rev. A **63**, 034301 (2001).

[15] It is defined by $F(\rho) = \max_{\psi_m}\langle\psi_m|\rho|\psi_m\rangle$ with ψ_m maximally entangled state.

[16] K. Życzkowski and M. Kuś, J. Phys. A **27**, 4235 (1994).

[17] A. Acin, A. Andrianov, L. Costa, E. Jane, J.I. Latorre, and R. Tarrach, Phys. Rev. Lett. **85**, 1560 (2000).

[18] H. Carteret, e-print arXiv.org: quant-ph/0309212.

[19] K. Życzkowski, P. Horodecki, M. Lewenstein, and A. Sanpera, Phys. Rev. A **58**, 883 (1998).

[20] T. Konrad, F. de Melo, M. Tiersch, C. Kasztelan, A. Aragao, and A. Buchleitner, Nat. Phys. **4**, 99 (2007).

[21] Z. Liu and H. Fan, Phys. Rev. A **79**, 032306 (2009); S. Y. Mirafzali, I. Sargolzahi, A. Ahanj, K. Javidan, and M. Sarbishaei, e-print, arXiv:1007.1294

[22] R. Augusiak and M. Demianowicz, in preparation.