NUMÉRAIRE-INVARIANT PREFERENCES IN FINANCIAL MODELING

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Abstract. We provide an axiomatic foundation for the representation of numéraire-invariant preferences of economic agents acting in a financial market. In a static environment, the simple axioms turn out to be equivalent to the following choice rule: the agent prefers one outcome over another if and only if the expected (under the agent's subjective probability) relative rate of return of the latter outcome with respect to the former is nonpositive. With the addition of a transitivity requirement, this last preference relation has an extension that can be numerically represented by expected logarithmic utility. We also treat the case of a dynamic environment, where consumption streams are the objects of choice. There, a novel result concerning a canonical representation of unit-mass optional measures enables to explicitly solve the investment-consumption problem by separating the two aspects of investment and consumption. Finally, we give an application to the problem of optimal numéraire investment with a random time-horizon.

0. Introduction

Within the class of expected utility maximization problems in economic theory, the special case of maximizing expected *logarithmic* utility has undoubtedly attracted considerable attention. The major reason for its celebrity is the computational advantage it offers: the use of the logarithmic function allows for explicit solutions of the optimal investment-consumption problem in general semimartingale models; see [10]. Furthermore, in many diverse applications, optimal portfolios stemming from expected log-utility maximization are crucial. We mention, for example, the problem of quantifying the additional utility of a trader using insider information (see [1] and the references therein), as well as the use of the log-optimal portfolios as benchmarks in financial theory, as is presented in [19].

The emergence of expected log-utility maximization dates as back as 1738, when Daniel Bernoulli offered a solution to the St. Petersburg paradox, which can be found in the translated manuscript [4]. Bernoulli's use of the logarithmic (and, indeed, of any other increasing and concave) utility function was ad-hoc and lacked any axiomatization based on rational agent's choice behavior. In the context of financial choice, [25] seems to be the first work that has proposed maximizing growth as a reasonable optimization criterion, which is exactly consistent with expected log-utility optimization. After Kelly's information-theoretical justification of using growth-optimal strategies in [14], there had been further attempts to justify maximizing expected log-utility, for example in [15]. Along came heavy criticism by distinguished scholars, notably by P. Samuelson; see [21] and [22]. However, the interest in log-optimality has not ceased, and is even growing. Statistical or behavioral tests do

Date: December 3, 2009.

2000 *Mathematics Subject Classification.* 91B08 91B16 91B28 60G07.

not seem to uniformly favor one side or the other; for example, J. B. Long's work [16], which has inspired some of the recent development, fails to answer with statistical significance the question whether the log-optimal portfolio coincides with the market portfolio.

In spite of all the debate that has prolonged over the years, there has been no attempt in the realms of the theory of choice to investigate the exact behavioral axioms that, when imposed, would explain the cases where agents act as if they are maximizing expected logarithmic utility under a subjective probability measure. Of course, there has immense work on axiomatizing agent's preferences, with [24] being the first example where axioms were imposed ensuring that agents act like they are maximizing expected utility over lotteries, with a known statistical nature of the uncertain environment. Savage's work [23] provided an axiomatic framework where both the statistical views and the utility function came as a byproduct. Since then, there have been numerous successful efforts in relaxing in some direction the axioms in order to explain agents' behavior in more depth. In all these works, the representation of preferences via utilities of logarithmic shape does not appear to have any form of significance. Naturally, there are descriptive characterizations aplenty; for example, one could argue that agents that act consistently with maximizing expected log-utility have constant, and equal to unit, relative risk aversion. However, a normative characterization seems to be absent in the literature.

The purpose of this paper is to address the aforementioned issue. Certain axioms are proposed on the choice of agents amongst random outcomes that result in the following preference representation: agents act as if they were making choices based on the *expected relative rate of return* of an outcome with respect to some alternative based on a *subjective* probability measure. In particular, an outcome will be preferred over another if the expected relative rate of return of the latter with respect to the former outcome is nonpositive. Choices based on the previous rule are closely connected to preferences stemming from a numerical representation of expected logarithmic utility, as can be seen using first-order conditions for optimality. Actually, we shall discuss how one can extend preferences based on expected relative rates of return to preferences that have a numerical representation of expected logarithmic utility, by imposing an extra transitivity axiom. However, working with expected relative rate of return is far more appealing, as the agent is not forced to express a preference between all pairs of alternatives; in other words, the preference relation will not be complete. The agent is only required to be able to make choices from certain convex *bundle sets*; in this respect, we take a more behavioral route in formulating preferences via *choice rules*.

The key axiom that is imposed to ensure that an agent makes choices according to the intuitive way described above is the *numéraire invariance* of preferences — this simply means that the agent's comparison of one outcome to another does not depend on the units that these outcomes are denominated. This is clearly necessary if we are using expected relative rate of return as a means of comparison, as relative rates of return do not depend on the denomination. Furthermore, preferences with expected logarithmic utility representation are also numéraire invariant, as follows from the simple fact that the logarithmic function transforms multiplication to addition.

We also consider the extension of the preferences in a dynamic environment, where agents make choices over *consumption streams*. The theory regarding choice is more or less a straightforward extension of the previous static case; "subjective probabilities" are now defined on a product space of states and time. The novel element is a decomposition of unit-mass optional measures on the last product space in two parts: one that has the interpretation of subjective views on the state space (the interpretation being somewhat loose, since it might involve density processes that are local martingales instead of martingales) and another that acts as an agent-specific consumption clock. This decomposition, a result that sharpens Doléans' characterization of optional measures, allows for a solution of the investment-consumption problem for an agent with numéraire-invariant preferences that separates the investment and consumption parts of Merton's problem in a general semimartingale-asset-price setting. A further application discussed in the text is a solution to the pure investment log-utility maximization problem with a time-horizon that is random but not necessarily a stopping time with respect to the agent's information flow. Such problems have lately been discussed in the context of credit risk and defaults; see for example [6] and [5].

From a mathematical point of view, the results of the present paper concern geometric and topological properties of $\mathbb{L}^0_+$, and, in the dynamic case, of the space of adapted, right-continuous, nonnegative and nondecreasing processes. The rich structure of the previous very important spaces is still the subject of scrutinized study (see [7], [27]); this work contributes to this line of research.

The structure of the paper is simple. Section 1 contains all the foundational results for the static case, which includes in particular the axiomatization of numéraire-invariant preferences. The dynamic case is treated in Section 2, where the main focus is on a canonical representation of unit-mass optional measures and the applications it has for the numéraire-invariant investment-consumption problem, as well as for the numéraire property under random sampling.

1. Numéraire-Invariant Preferences: the Static Case

1.1. **Definitions and notation.**

Throughout, $\mathbb{R}_+$ denotes the nonnegative real numbers and $\mathbb{R}_{++}$ denotes the strictly positive real numbers. For $x \in \mathbb{R}_+$ and $y \in \mathbb{R}_+$, x/y is defined as usual when $y \in \mathbb{R}_{++}$. When $x \in \mathbb{R}_{++}$ and $y = 0$, we set $x/y = \infty$. Finally, if $x = y = 0$, we set $x/y = 1$. This last non-conventional definition will allow for good bookkeeping in the sequel.

On the probability space $(\Omega, \mathcal{F})$ we consider a family Π of all probabilities that are equivalent to some baseline probability $\overline{\mathbb{P}}$. All probabilities in Π have the same sets of zero measure, which we shall be calling Π-*null*. A set will be called Π-*full* if its complement is Π-null. We write "Π-a.s." to mean $\mathbb{P}$-a.s. with respect to any, and then all, $\mathbb{P} \in \Pi$. All relationships between random variables are to be understood in the Π-a.s. sense: for example, $f \leq g$ means that $\{f \leq g\}$ is Π-full. The indicator function of $A \in \mathcal{F}$ is denoted by $\mathbb{I}_A$; we use simply 1 for $\mathbb{I}_\Omega$. Also, "$\mathbb{E}_\mathbb{P}$" denotes expectation under the probability $\mathbb{P} \in \Pi$.

The vector space of equivalence classes of random variables under Π-a.s. equality is denoted by $\mathbb{L}^0$. Following standard practice, we do not distinguish between a random variable and the equivalence class it generates. We endow $\mathbb{L}^0$ with the usual metric topology: a sequence $(f^n)_{n \in \mathbb{N}}$ in $\mathbb{L}^0$ converges to $f \in \mathbb{L}^0$ if and only if for all $\epsilon > 0$ we have $\lim_{n\to\infty} \mathbb{P}[|f^n - f| > \epsilon] = 0$, where $\mathbb{P}$ is any probability in Π. Thus, $\mathbb{L}^0$ becomes a topological vector space. Whenever we consider a topological property (for example, limits or closedness), it will be understood under the aforementioned metric topology, unless explicitly noted otherwise. A set $\mathcal{C} \subseteq \mathbb{L}^0$ is called *bounded* if $\lim_{\ell\to\infty} \left(\sup_{f\in\mathcal{C}} \mathbb{P}[|f| > \ell] \right) = 0$ holds for some, and then for all, $\mathbb{P} \in \Pi$. Furthermore, a set $\mathcal{C} \subseteq \mathbb{L}^0$ will be called *convexly compact* if it is convex, closed and bounded. The last terminology is borrowed from [27], where one can find more information, particularly on explaining the appellation; convexly compact sets share lots of properties of convex and compact sets of Euclidean spaces.

We define $\mathbb{L}^0_+ := \left\{ f \in \mathbb{L}^0_+ \mid f \geq 0,\ \Pi\text{-a.s.} \right\}$ and $\mathbb{L}^0_{++} = \left\{ f \in \mathbb{L}^0_+ \mid f > 0,\ \Pi\text{-a.s.} \right\}$. Note that $\mathbb{L}^0_{++}$ is the subset of Π-a.s. *strictly* positive random variables and is *not* equal to $\mathbb{L}^0_+ \setminus \{0\}$. A set $\mathcal{C} \subseteq \mathbb{L}^0_+$ is called *solid* if the conditions $0 \leq f \leq g$ and $g \in \mathcal{C}$ imply that $f \in \mathcal{C}$ as well. The set $\mathcal{C} \subseteq \mathbb{L}^0_+$ will be called *log-convex* if for all $f \in \mathcal{C}$, all $g \in \mathcal{C}$ and all $\alpha \in [0,1]$, the geometric mean $f^\alpha g^{1-\alpha}$ belongs to $\mathcal{C}$ as well.

1.2. **Preferences induced by expected relative rates of return.** In (1.1) below and all that follows we are using the division conventions explained in the first paragraph of §1.1.

Fix $\mathbb{P} \in \Pi$ and set

$$\mathsf{rel}_{\mathbb{P}}(f \,|\, g) := \mathbb{E}_{\mathbb{P}}\left[f/g \right] - 1, \text{ for all } f \in \mathbb{L}^0_+ \text{ and } g \in \mathbb{L}^0_+. \tag{1.1}$$

In words, $\mathsf{rel}_{\mathbb{P}}(f \,|\, g)$ is the expected, under $\mathbb{P}$, rate of return of f in units of g; we therefore call $\mathsf{rel}_{\mathbb{P}}(f \,|\, g)$ the *expected relative rate of return of f with respect to g under $\mathbb{P}$*. Unless $f = g$, in which case $\mathsf{rel}_{\mathbb{P}}(g \,|\, f) = \mathsf{rel}_{\mathbb{P}}(f \,|\, g) = 0$, it is straightforward to see that the strict inequality $\mathsf{rel}_{\mathbb{P}}(g \,|\, f) > -\mathsf{rel}_{\mathbb{P}}(f \,|\, g)$ holds. Also, if $h \in \mathbb{L}^0_{++}$, $\mathsf{rel}_{\mathbb{P}}(f/h \,|\, g/h) = \mathsf{rel}_{\mathbb{P}}(f \,|\, g)$; the expected relative rate of return operation is numéraire-invariant.

For $\mathbb{P} \in \Pi$, the *preference relation* $\preccurlyeq_{\mathbb{P}}$ is defined to be the following binary relation on $\mathbb{L}^0_+$:

$$\text{for } f \in \mathbb{L}^0_+ \text{ and } g \in \mathbb{L}^0_+, \quad f \preccurlyeq_{\mathbb{P}} g \Longleftrightarrow \mathsf{rel}_{\mathbb{P}}(f \,|\, g) \leq 0. \tag{1.2}$$

By our division conventions, $f \preccurlyeq_{\mathbb{P}} g$ holds if and only if $\{f > 0\} \subseteq \{g > 0\}$ and $\mathbb{E}_{\mathbb{P}}[f/g \mid g > 0] \leq 1$.

Given the preference relation $\preccurlyeq_{\mathbb{P}}$, the *strict* preference relation $\prec_{\mathbb{P}}$ is defined by requiring that $f \prec_{\mathbb{P}} g$ if and only if $f \preccurlyeq_{\mathbb{P}} g$ holds and $g \preccurlyeq_{\mathbb{P}} f$ fails. It is straightforward to check that $f \prec_{\mathbb{P}} g \Longleftrightarrow \mathsf{rel}_{\mathbb{P}}(f \,|\, g) < 0$. Note also that if $f \preccurlyeq_{\mathbb{P}} g$ and $g \preccurlyeq_{\mathbb{P}} f$, then $f = g$, i.e., the equivalence classes for $\preccurlyeq_{\mathbb{P}}$ are singletons. (Indeed, if $\{f \neq g\}$ were not Π-null, then $0 \leq -\mathsf{rel}_{\mathbb{P}}(f \,|\, g) < \mathsf{rel}_{\mathbb{P}}(g \,|\, f) \leq 0$, which is impossible.)

We list some important properties of the preference relation of (1.2).

Theorem 1.1. *Fix $\mathbb{P} \in \Pi$ and simply write $\preccurlyeq$ and $\prec$ for the preference relation $\preccurlyeq_{\mathbb{P}}$ on $\mathbb{L}^0_+$ of* (1.2) *and the induced strict preference relation $\prec_{\mathbb{P}}$. Then:*

(1) *$f \preccurlyeq g$ holds if and only if $\{f > 0\} \subseteq \{g > 0\}$ and $(f/g)\mathbb{I}_{\{g>0\}} + \mathbb{I}_{\{g=0\}} \preccurlyeq 1$.*
(2) *If $f \leq g$, then $f \preccurlyeq g$. Furthermore, if $f \leq g$ and $\{f = g\}$ is not Π-full, then $f \prec g$.*
(3) *If $h \in \mathbb{L}^0_+$, $\left\{ f \in \mathbb{L}^0_+ \mid f \preccurlyeq h \right\}$ is convexly compact and log-convex, and $\left\{ f \in \mathbb{L}^0_+ \mid h \preccurlyeq f \right\}$ is convex and log-convex. If actually $h \in \mathbb{L}^0_{++}$, $\left\{ f \in \mathbb{L}^0_+ \mid h \preccurlyeq f \right\}$ is further closed.*

(4) *If $\mathcal{C} \subseteq \mathbb{L}^0_+$ is convexly compact, there exists a unique $\widehat{f} \in \mathcal{C}$ such that $f \preccurlyeq \widehat{f}$ holds for all $f \in \mathcal{C}$.*

Proof. The proofs of (1) and (2) are straightforward, so we shall focus on proving (3) and (4). We hold $\mathbb{P} \in \Pi$ fixed and drop any subscripts "$\mathbb{P}$" in the sequel.

(3) Call $\mathcal{C}^h_{\preccurlyeq} := \left\{ f \in \mathbb{L}^0_+ \mid f \preccurlyeq h \right\}$. From the definition (1.1) of rel, it is clear that $\mathcal{C}^h_{\preccurlyeq}$ is convex. Let $(f^n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{C}^h_{\preccurlyeq}$ such that $\lim_{n\to\infty} f^n = f$. Since $f^n \preccurlyeq h$ for all $n \in \mathbb{N}$, property (1) implies that $\{h = 0\} \subseteq \{f^n = 0\}$ for all $n \in \mathbb{N}$. Then, $\{h = 0\} \subseteq \bigcap_{n \in \mathbb{N}} \{f^n = 0\} \subseteq \{f = 0\}$. An application of Fatou's lemma gives $\mathbb{E}\left[f/h \mid h > 0\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[f^n/h \mid h > 0\right] \leq 1$, which, in view of $\{f > 0\} \subseteq \{h > 0\}$, is equivalent to $\mathsf{rel}(f \,|\, h) \leq 0$. Therefore, $\mathcal{C}^h_{\preccurlyeq}$ is closed. Now, $\mathbb{E}[f/h \mid h > 0] \leq 1$ for all $f \in \mathcal{C}^h_{\preccurlyeq}$ gives $\sup_{f \in \mathcal{C}^h_{\preccurlyeq}} \mathbb{P}[f/h > \ell \mid h > 0] \leq 1/\ell$ for all $\ell \in \mathbb{R}_+$. In other words, $\left\{ f \mathbb{I}_{\{h>0\}} \mid f \in \mathcal{C}^h_{\preccurlyeq} \right\} \subseteq \mathbb{L}^0_+$ is bounded. Since $f = f \mathbb{I}_{\{h>0\}}$ holds for all $f \in \mathcal{C}^h_{\preccurlyeq}$, we get that $\mathcal{C}^h_{\preccurlyeq}$ is bounded. We have therefore established the convex compactness of $\mathcal{C}^h_{\preccurlyeq}$. It remains to establish log-convexity, which is an easy application of Hölder's inequality: for $f \in \mathcal{C}^h_{\preccurlyeq}$, $g \in \mathcal{C}^h_{\preccurlyeq}$, and $\alpha \in [0, 1]$,

$$\mathbb{E}\left[\frac{f^\alpha g^{1-\alpha}}{h} \;\Big|\; h > 0 \right] = \mathbb{E}\left[\left(\frac{f}{h}\right)^\alpha \left(\frac{g}{h}\right)^{1-\alpha} \;\Big|\; h > 0 \right] \leq \left(\mathbb{E}\left[\frac{f}{h} \;\Big|\; h > 0 \right] \right)^\alpha \left(\mathbb{E}\left[\frac{g}{h} \;\Big|\; h > 0 \right] \right)^{1-\alpha} \leq 1,$$

which shows that $(f^\alpha g^{1-\alpha}) \in \mathcal{C}^h_{\preccurlyeq}$.

Continuing, fix $h \in \mathbb{L}^0_+$ and let $\mathcal{C}^h_{\succcurlyeq} := \left\{ f \in \mathbb{L}^0_+ \mid h \preccurlyeq f \right\}$. The convexity of $\mathcal{C}^h_{\succcurlyeq}$ follows from the definition of rel and the convexity of the mapping $\mathbb{R}_+ \ni x \mapsto 1/x \in \mathbb{R}_+ \cup \{\infty\}$. Also, log-convexity of $\mathcal{C}^h_{\succcurlyeq}$ follows similarly as log-convexity of $\mathcal{C}^h_{\preccurlyeq}$. If, furthermore, $h \in \mathbb{L}^0_{++}$, closedness of $\mathcal{C}^h_{\succcurlyeq}$ follows directly by noticing that $\mathcal{C}^h_{\succcurlyeq} = \{ f \in \mathbb{L}^0_{++} \mid (1/f) \in \mathcal{C}^{1/h}_{\preccurlyeq} \}$ and that $\mathcal{C}^{1/h}_{\preccurlyeq}$ is closed.

(4) We shall be assuming throughout that $\mathcal{C} \neq \{0\}$; otherwise, trivially, $\widehat{f} = 0$.

We begin by showing there exists $g \in \mathcal{C}$ such that $\{f > 0\} \subseteq \{g > 0\}$ holds for all $f \in \mathcal{C}$. Indeed, let $p := \sup \{ \mathbb{P}[f > 0] \mid f \in \mathcal{C} \} > 0$. Using the convexity and closedness of $\mathcal{C}$, a standard exhaustion argument shows that there exists $g \in \mathcal{C}$ such that $\mathbb{P}[g > 0] = p$. If $\{f > 0\} \cap \{g = 0\}$ were not Π-null for some $f \in \mathcal{C}$, then, with $h = (f+g)/2 \in \mathcal{C}$, we have $\mathbb{P}[h > 0] = \mathbb{P}[g > 0] + \mathbb{P}[\{f > 0\} \cap \{g = 0\}] > p$, which is impossible.

We claim that, in order to show (4), we may assume that $\mathcal{C} \cap \mathbb{L}^0_{++} \neq \emptyset$. Indeed, with $g \in \mathcal{C}$ as above, let $\widetilde{\mathcal{C}} := \left\{ f + \mathbb{I}_{\{g=0\}} \mid f \in \mathcal{C} \right\}$. It is straightforward that $\widetilde{\mathcal{C}}$ is convexly compact, as well as that $\widetilde{\mathcal{C}} \cap \mathbb{L}^0_{++} \neq \emptyset$. Furthermore, $f \preccurlyeq \widehat{f}$ holds for all $f \in \mathcal{C}$ if and only if $\widetilde{f} \preccurlyeq \widehat{f} + \mathbb{I}_{\{g=0\}}$ holds for all $\widetilde{f} \in \widetilde{\mathcal{C}}$. Therefore, changing from $\mathcal{C}$ to $\widetilde{\mathcal{C}}$ if necessary, we may assume that $\mathcal{C} \cap \mathbb{L}^0_{++} \neq \emptyset$.

Since we can assume that $\mathcal{C} \cap \mathbb{L}^0_{++} \neq \emptyset$, we may additionally assume that $1 \in \mathcal{C}$. Indeed, otherwise, we consider $\widetilde{\mathcal{C}} := (1/g)\mathcal{C}$ for some $g \in \mathcal{C} \cap \mathbb{L}^0_{++}$. Then, $1 \in \widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{C}}$ is still convexly compact. Furthermore, $f \preccurlyeq \widetilde{f}$ holds for $f \in \widetilde{\mathcal{C}}$, then $\widehat{f} := g\widetilde{f} \in \mathcal{C}$ satisfies $f \preccurlyeq \widehat{f}$ for all $f \in \mathcal{C}$ by the numéraire-invariance property (1).

In the sequel, assume that $1 \in \mathcal{C}$ and that $\mathcal{C}$ is convexly compact. We claim that we can further assume without loss of generality that $\mathcal{C}$ is solid. Indeed, let $\mathcal{C}'$ be the *solid hull* of $\mathcal{C}$, i.e., $\mathcal{C}' := \left\{ f \in \mathbb{L}^0_+ \mid 0 \leq f \leq h \text{ holds for some } h \in \mathcal{C} \right\}$. Then, it is straightforward that $1 \in \mathcal{C}'$, as well as that $\mathcal{C}'$ is still convex and bounded. It is also true that $\mathcal{C}'$ is still closed. (To see the last fact, pick a $\mathcal{C}'$-valued sequence $(f^n)_{n\in\mathbb{N}}$ that converges $\mathbb{P}$-a.s. to $f \in \mathbb{L}^0_+$. Let $(h^n)_{n\in\mathbb{N}}$ be a $\mathcal{C}$-valued sequence with $f^n \leq h^n$ for all $n \in \mathbb{N}$. By Lemma A.1 from [8], we can extract a sequence $(\widetilde{h}^n)_{n\in\mathbb{N}}$ such that, for each $n \in \mathbb{N}$, $\widetilde{h}^n$ is a convex combination of $h^n, h^{n+1}, \ldots$, and such that $h := \lim_{n\to\infty} \widetilde{h}^n$ exists. Of course, $h \in \mathcal{C}$ and it is easy to see that $f \leq h$. We then conclude that $f \in \mathcal{C}'$.) Suppose that there exists $\widehat{f} \in \mathcal{C}'$ such that $f \preccurlyeq \widehat{f}$ holds for all $f \in \mathcal{C}'$. Then, $\widehat{f} \in \mathcal{C}$ (since $\widehat{f}$ has to be a *maximal* element of $\mathcal{C}'$ with respect to the order structure of $\mathbb{L}^0$), and that $f \preccurlyeq \widehat{f}$ holds for all $f \in \mathcal{C}$ (simply because $\mathcal{C} \subseteq \mathcal{C}'$).

To recapitulate, in the course of the proof of (4), we shall be assuming without loss of generality that $\mathcal{C} \subseteq \mathbb{L}^0_+$ is solid, convexly compact, as well as that $1 \in \mathcal{C}$.

For all $n \in \mathbb{N}$, let $\mathcal{C}^n := \{f \in \mathcal{C} \mid f \leq n\}$, which is convexly compact and satisfies $\mathcal{C}^n \subseteq \mathcal{C}$. Consider the following optimization problem:

$$\text{find } f^n_* \in \mathcal{C}^n \text{ such that } \mathbb{E}[\log(f^n_*)] = \sup_{f\in\mathcal{C}^n} \mathbb{E}[\log(f)]. \tag{1.3}$$

The fact that $1 \in \mathcal{C}^n$ implies that the value of the above problem is not $-\infty$. Further, since $f \leq n$ for all $f \in \mathcal{C}^n$, one can use of Lemma A.1 from [8] in conjunction with the inverse Fatou's lemma and obtain the existence of the optimizer f^n_* of (1.3). For all $f \in \mathcal{C}^n$ and $\epsilon \in]0, 1/2]$, one has

$$\mathbb{E}\left[\Delta_\epsilon(f \mid f^n_*)\right] \leq 0, \text{ where } \Delta_\epsilon(f \mid f^n_*) := \frac{\log\left((1-\epsilon)f^n_* + \epsilon f\right) - \log\left(f^n_*\right)}{\epsilon}. \tag{1.4}$$

Fatou's lemma will be used on (1.4) as $\epsilon \downarrow 0$. For this, observe that $\Delta_\epsilon(f \mid f^n_*) \geq 0$ on the event $\{f > f^n_*\}$. Also, the inequality $\log(y) - \log(x) \leq (y-x)/x$, valid for $0 < x < y$, gives that, on $\{f \leq f^n_*\}$, the following lower bound holds (remember that $\epsilon \leq 1/2$):

$$\Delta_\epsilon(f \mid f^n_*) \geq -\frac{f^n_* - f}{f^n_* - \epsilon(f^n_* - f)} \geq -\frac{f^n_* - f}{f^n_* - (f^n_* - f)/2} = -2\frac{f^n_* - f}{f^n_* + f} \geq -2.$$

Using Fatou's Lemma on (1.4) gives $\mathbb{E}\left[(f - f^n_*)/f^n_*\right] \leq 0$, or, equivalently, that $f \preccurlyeq f^n_*$, for all $f \in \mathcal{C}^n$.

Lemma A.1 from [8] again gives the existence of a sequence $(\widehat{f}^n)_{n\in\mathbb{N}}$ such that each $\widehat{f}^n$ is a finite convex combination of $f^n_*, f^{n+1}_*, \ldots$, and $\widehat{f} := \lim_{n\to\infty} \widehat{f}^n$ exists. Since $\mathcal{C}$ is convex, $\widehat{f}^n \in \mathcal{C}$ for all $n \in \mathbb{N}$; therefore, since $\mathcal{C}$ is closed, $\widehat{f} \in \mathcal{C}$ as well. Fix $n \in \mathbb{N}$ and some $f \in \mathcal{C}^n$. For all $k \in \mathbb{N}$ with $k \geq n$, we have $f \in \mathcal{C}^k$. Therefore, $f \preccurlyeq f^k_*$, for all $k \geq n$. Since $\widehat{f}^n$ is a finite convex combination of $f^n_*, f^{n+1}_*, \ldots$, by part (3) of Theorem 1.1 which we already established, we have $f \preccurlyeq \widehat{f}^n$, i.e., $\mathbb{E}[f/\widehat{f}^n] \leq 1$. Then, Fatou's lemma implies that for all $f \in \bigcup_{k\in\mathbb{N}} \mathcal{C}^k$ one has $\mathbb{E}[f/\widehat{f}] \leq 1$. The extension of the last inequality to all $f \in \mathcal{C}$ follows from the solidity of $\mathcal{C}$ by an application of the monotone convergence theorem. □

Our main point will be to give certain axioms on a preference relation $\preccurlyeq$ on $\mathbb{L}^0_+$ that will imply the representation given by (1.2) for some "subjective" probability $\mathbb{P} \in \Pi$. This will eventually be achieved in Theorem 1.5, and the properties obtained in Theorem 1.1 above will serve as guidelines. Before that, we slightly digress in order to better understand the preference relation given by (1.2), as well as to discuss a class of subsets of $\mathbb{L}^0_+$ with a special structure that will prove important.

1.3. **On the relation $\preccurlyeq_{\mathbb{P}}$ of (1.2).** For the purposes of §1.3, fix $\mathbb{P} \in \Pi$ and let $\preccurlyeq$ denote the binary relation of (1.2), dropping the subscript "$\mathbb{P}$" from $\preccurlyeq_{\mathbb{P}}$. We also simply use "$\mathsf{rel}$" to denote "$\mathsf{rel}_{\mathbb{P}}$" and "$\mathbb{E}$" to denote expectation under $\mathbb{P}$. Also, throughout §1.3, we tacitly preclude the uninteresting case where $\mathbb{L}^0_+$ is isomorphic to the nonnegative real line, i.e., when $\mathcal{F}$ is trivial modulo Π.

As shall soon be revealed, the relation $\preccurlyeq$ fails to satisfy the fundamental tenets of a *rational* preference relation, namely, completeness and transitivity. We shall try nevertheless to argue that this failure is natural in the present setting.

1.3.1. *Quasi-convexity.* The convexity of the upper-contour set $\{f \in \mathbb{L}^0_+ \mid h \preccurlyeq f\}$, where $h \in \mathbb{L}^0_+$, makes $\preccurlyeq$ a so-called *quasi-convex* preference relation. If $\preccurlyeq$ were complete, the lower-contour sets $\{f \in \mathbb{L}^0_+ \mid f \preccurlyeq h\}$ would fail to be convex in general. However, lower-contour sets *are* convex, according to property (3) of Theorem 1.6 — this already points out that $\preccurlyeq$ is not complete. The convexity of $\{f \in \mathbb{L}^0_+ \mid f \preccurlyeq h\}$ is natural when one recalls the definition of the preference relation: if both $f \in \mathbb{L}^0_+$ and $g \in \mathbb{L}^0_+$ have nonpositive expected relative rate of return with respect to h, so does any convex combination of f and g.

1.3.2. *The relation $\preccurlyeq$ is not complete.* Pick $A \in \mathcal{F}$ with $0 < \mathbb{P}[A] < 1$. With $f = \mathbb{I}_{\Omega \setminus A}$ and $g = \mathbb{I}_A$, we have $\mathsf{rel}(f \,|\, g) = \infty = \mathsf{rel}(g \,|\, f)$, therefore neither $f \preccurlyeq g$ nor $g \preccurlyeq f$ holds. One can find more interesting examples involving elements of $\mathbb{L}^0_{++}$. Let $p := \mathbb{P}[A]$, $f := (1/p)\mathbb{I}_A + (1-p)\mathbb{I}_{\Omega \setminus A}$ and $g := 1$. Then, $\mathsf{rel}(f \,|\, g) = (1-p)^2 > 0$ and $\mathsf{rel}(g \,|\, f) = p^2 > 0$, i.e., neither $f \preccurlyeq g$ nor $g \preccurlyeq f$ holds.

The relation $\preccurlyeq$ is really too strong: $f \preccurlyeq g$ implies that g is preferred over *any* convex combination of f and g. More precisely, statement (3) of Theorem 1.1 implies that, if $f \preccurlyeq g$ then, for all $\alpha \in [0,1]$ and $\beta \in [0,1]$ with $\alpha \le \beta$, we have $(1-\alpha)f + \alpha g \preccurlyeq (1-\beta)f + \beta g$. A pair of $f \in \mathbb{L}^0_+$ and $g \in \mathbb{L}^0_+$ will be comparable if and only if one of $f \in \mathbb{L}^0_+$ or $g \in \mathbb{L}^0_+$ is preferable over the whole set $\mathsf{conv}(f,g) := \{(1-\alpha)f + \alpha g \mid \alpha \in [0,1]\}$. The equivalent of the completeness property here is the following: if $f \in \mathbb{L}^0_+$ and $g \in \mathbb{L}^0_+$, there exists $h \in \mathsf{conv}(f,g)$ that dominates all elements in $\mathsf{conv}(f,g)$. In both examples that were given above ($f = \mathbb{I}_{\Omega \setminus A}$ and $g = \mathbb{I}_A$, as well as $f = (1/p)\mathbb{I}_A + (1-p)\mathbb{I}_{\Omega \setminus A}$ and $g = 1$), one can actually check that $h = (1-p)f + pg$.

1.3.3. *The relation $\preccurlyeq$ is not transitive.* Pick $A \in \mathcal{F}$ with $0 < \mathbb{P}[A] < 1$. With $p := \mathbb{P}[A]$, let $f := (1/p)\mathbb{I}_A$, $g := 1$ and $h := \big(2p/(1+p)\big)\mathbb{I}_A + 2\mathbb{I}_{\Omega \setminus A}$. It is straightforward to check that $\mathsf{rel}(f \,|\, g) = 0$, $\mathsf{rel}(g \,|\, h) = 0$, as well as $\mathsf{rel}(f \,|\, h) = (1-p)/(2p) > 0$. In other words, we have $f \preccurlyeq g$ and $g \preccurlyeq h$, but $f \preccurlyeq h$ fails.

Whereas failure of completeness of preference relations is not considered dramatic, and is indeed welcome in certain cases, transitivity is a more or less unquestionable requirement. The reason for its failure in the present context does not have to do with irrationality of agents making choices according to $\preccurlyeq$. Recall that $f \preccurlyeq g$ and $g \preccurlyeq h$ mean that g is best choice from the set $\mathsf{conv}(f,g)$ and h is best choice amongst $\mathsf{conv}(g,h)$. However, when an agent is presented with the set of alternatives $\mathsf{conv}(f,h)$, some strict convex combination of f and h might be preferable to h, especially when f pays off considerably better on an event where h does not.

Although $f \preccurlyeq h$ fails in the example above, one expects that $h \preccurlyeq f$ fails as well, and this is indeed the case. In general, even though transitivity does not hold, we have a weaker "chain" property holding. For $n \in \mathbb{N}$, let $f^0, \ldots, f^n$ be elements of $\mathbb{L}^0_+$ satisfying $f^{i-1} \preccurlyeq f^i$ for $i \in \{1, \ldots, n\}$ and $f^0 = f^n$. Then, actually, $f^i = f^0$ holds for all $i \in \{1, \ldots, n\}$. Indeed, let $\phi^i := f^{i-1}/f^i$ for $i \in \{1, \ldots, n\}$. We wish to show that $\phi^i = 1$ for all $i \in \{1, \ldots, n\}$. Suppose the contrary. Since $\mathbb{E}[\phi^i] \leq 1$ holds for all $i \in \{1, \ldots, n\}$, the strict convexity of the mapping $\mathbb{R}^n_{++} \ni (x^1, \ldots, x^n) \mapsto \prod_{i=1}^n (1/x^i)$, combined with the fact that $\mathbb{P}[\phi^i = 1] < 1$ holds for some $i \in \{1, \ldots, n\}$ and a use of Jensen's inequality gives $\mathbb{E}\big[\prod_{i=1}^n (1/\phi^i)\big] > 1$. However, $\prod_{i=1}^n (1/\phi^i) = 1$, which is a contradiction.

1.3.4. *The relation $\preccurlyeq$ does not respect addition.* Pick $A \in \mathcal{F}$ such that $0 < \mathbb{P}[A] \leq 1/2$. With $p := \mathbb{P}[A]$, let $f := p^2 \mathbb{I}_A + (1+p)^2 \mathbb{I}_{\Omega \setminus A}$ and $g := p \mathbb{I}_A + (1+p) \mathbb{I}_{\Omega \setminus A}$. Observe that $f = g^2$, $f \neq g$ and $\mathbb{E}[g] = 1$. Then, $\mathsf{rel}(f \,|\, g) = 0$, so $f \prec g$. However,

$$\mathsf{rel}(1+g \,|\, 1+f) = \mathbb{E}\left[\frac{g(1-g)}{1+g^2}\right] = \frac{p(1-p)}{1+p^2} p + \frac{(1+p)(-p)}{1+(1+p)^2}(1-p) = \frac{p(1-p)(p^2+p-1)}{(1+p^2)(1+(1+p)^2)} < 0,$$

the last fact following from $p^2 + p - 1 < 0$, which holds in view of $p \leq 1/2$. Therefore, $1 + g \prec 1 + f$. Even though initially g was preferred to f, as soon as the agent is endowed with an extra unit of account, the choice completely changes. Note that f pays off very close to zero on A; even though f pays off more than g on $\Omega \setminus A$, a risk-averse agent will prefer g. However, once the risk associated with the outcome A is reduced by the assurance that a unit of account will be received in any state of the world, f is preferred.

In fact, regardless of whether $f \prec g$ holds or not, if the event $\{g < f\}$ is not Π-null, one can find $h \in \mathbb{L}^0_+$ such that $g + h \prec f + h$. The proof of this is based on the aforementioned simple idea: a sufficiently large "insurance" h on $\{f \leq g\}$ will make $f + h$ better than $g + h$. Indeed, for $n \in \mathbb{N}$,

$$\mathsf{rel}(g + ng\mathbb{I}_{\{f \leq g\}} \,|\, f + ng\mathbb{I}_{\{f \leq g\}}) = \mathbb{E}\left[\frac{g-f}{f}\mathbb{I}_{\{g<f\}}\right] + \mathbb{E}\left[\frac{g-f}{f+ng}\mathbb{I}_{\{f \leq g\}}\right]$$

The first summand of the right-hand-side is strictly negative and the second one tends to zero as $n \to \infty$ by the monotone convergence theorem. Therefore, there exists some large enough $N \in \mathbb{N}$ such that, with $h := Ng\mathbb{I}_{\{f \leq g\}}$, $\mathsf{rel}(g + h \,|\, f + h) < 0$, which completes the argument.

1.4. **Full simplices in $\mathbb{L}^0_+$.** We shall describe here a special class of convexly compact sets, which are the equivalents of simplices with non-empty interior in finite-dimensional spaces. These sets will turn out crucial in our statement of Theorem 1.5 on the axiomatic definition of numéraire-invariant preferences. The results presented here concern the structure of $\mathbb{L}^0_+$; as such, they are of independent interest.

For $\mathcal{C} \subseteq \mathbb{L}^0_+$, define $\mathcal{C}^{\max}$ to be the subset of $\mathcal{C}$ containing all the *maximal elements* of $\mathcal{C}$, i.e., $f \in \mathcal{C}^{\max}$ if and only if $f \in \mathcal{C}$ and the relationships $f \leq g$ and $g \in \mathcal{C}$ imply that $f = g$.

For a measure μ on $(\Omega, \mathcal{F})$, we shall write $\mu \sim \Pi$ if $\mu[A] = 0$ holds for all Π-null $A \in \mathcal{F}$.

Theorem 1.2. *Let $\mathcal{B} \subseteq \mathbb{L}^0_+$. Then, the following statements are equivalent:*

(1) *$\mathcal{B}$ is closed and solid, $\mathcal{B} \cap \mathbb{L}^0_{++} \neq \emptyset$, $\mathcal{B}^{\max}$ is convex, and $\mathcal{B} = \bigcup_{a \in [0,1]} a\mathcal{B}^{\max}$.*

(2) *For any* $\mathbb{P} \in \Pi$, *there exists* $\widehat{f} = \widehat{f}(\mathbb{P}) \in \mathcal{B} \cap \mathbb{L}^0_{++}$ *such that* $\mathcal{B} = \{ f \in \mathbb{L}^0_+ \mid f \preccurlyeq_{\mathbb{P}} \widehat{f} \}$.

(3) *There exists a* σ*-finite measure* $\mu \sim \Pi$ *such that* $\mathcal{B} = \{ f \in \mathbb{L}^0_+ \mid \int_\Omega f \mathrm{d}\mu \leq 1 \}$.

Proof. We first prove the easy implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1); then, (1) $\Rightarrow$ (2) will be tackled.

(2) $\Rightarrow$ (3). Let $\mathbb{P} \in \Pi$ and $\widehat{f} \in \mathcal{B} \cap \mathbb{L}^0_{++}$ be such that $\mathcal{B} = \{ f \in \mathbb{L}^0_+ \mid \mathbb{E}_\mathbb{P}[f/\widehat{f}] \leq 1 \}$. Define μ via $\mu[A] = \mathbb{E}_\mathbb{P}[\widehat{f}\mathbb{I}_A]$ for all $A \in \mathcal{F}$. With $A^n := \{ \widehat{f} \leq n \}$ for $n \in \mathbb{N}$ we have $\mu[A^n] < \infty$ and $\lim_{n\to\infty} \mathbb{P}[A^n] = 1$; therefore, μ is σ-finite. Furthermore, $\widehat{f} \in \mathbb{L}^0_{++}$ implies that $\mu \sim \Pi$. The equality $\mathcal{B} = \{ f \in \mathbb{L}^0_+ \mid \int_\Omega f \mathrm{d}\mu \leq 1 \}$ holds by definition.

(3) $\Rightarrow$ (1). Suppose that $\mathcal{B} = \{ f \in \mathbb{L}^0_+ \mid \int_\Omega f \mathrm{d}\mu \leq 1 \}$ for some σ-finite $\mu \sim \Pi$. Closedness of $\mathcal{B}$ follows from Fatou's lemma and solidity is obvious from the monotonicity of the Lebesgue integral. As μ is σ-finite, there exists $f \in \mathbb{L}^0_{++}$ such that $\int_\Omega f \mathrm{d}\mu < \infty$; therefore, $\left(1/\int_\Omega f \mathrm{d}\mu\right) f \in \mathcal{B} \cap \mathbb{L}^0_{++}$, which shows that $\mathcal{B} \cap \mathbb{L}^0_{++} \neq \emptyset$. It is straightforward that $\mathcal{B}^{\max} = \{ f \in \mathbb{L}^0_+ \mid \int_\Omega f \mathrm{d}\mu = 1 \}$, which implies that $\mathcal{B}^{\max}$ is convex by the linearity of Lebesgue integral. For $f \in \mathcal{B} \setminus \{0\}$, set $a := \int_\Omega f \mathrm{d}\mu \in (0,1]$. Then, $f = ag$, where $g := (1/a) f \in \mathcal{B}^{\max}$. Therefore, $\mathcal{B} = \bigcup_{a \in [0,1]} a\mathcal{B}^{\max}$.

(1) $\Rightarrow$ (2). We start by showing that *any* $\mathcal{B} \subseteq \mathbb{L}^0_+$ *satisfying the requirements of statement (1) of Theorem 1.2 is convexly compact.* Since $\mathcal{B}$ is closed, only convexity and boundedness of $\mathcal{B}$ have to be established. We start with convexity. Let $f \in \mathcal{B}$, $g \in \mathcal{B}$, and $\lambda \in [0,1]$. We know that there exist $a \in [0,1]$, $b \in [0,1]$, $f' \in \mathcal{B}^{\max}$ and $g' \in \mathcal{B}^{\max}$ such that $f = af'$ and $g = bg'$. Then,

$$(1-\lambda) f + \lambda g = ((1-\lambda)a + \lambda b) \left(\frac{(1-\lambda)a}{(1-\lambda)a + \lambda b} f' + \frac{\lambda b}{(1-\lambda)a + \lambda b} g' \right)$$

and the last element belongs to $\mathcal{B}$ due to the fact that $\mathcal{B}^{\max}$ is convex and $((1-\lambda)a + \lambda b) \in [0,1]$. We have shown that $\mathcal{B} \subseteq \mathbb{L}^0_+$ is convex, solid and closed. If it were not bounded, it would follow from Lemma 2.3 in [7] that there existed a non-Π-null $A \in \mathcal{F}$ such that $\{x\mathbb{I}_A \mid x \in \mathbb{R}_+\} \subseteq \mathcal{B}$. But in that case $\mathcal{B}^{\max}$ would not contain any element of $\{x\mathbb{I}_A \mid x \in \mathbb{R}_+\}$, and therefore the property $\mathcal{B} = \bigcup_{a \in [0,1]} a\mathcal{B}^{\max}$ would be violated. It follows then that $\mathcal{B}$ has to be bounded.

Continuing, fix $\mathbb{P} \in \Pi$. Since $\mathcal{B}$ is convexly compact and $\mathcal{B} \cap \mathbb{L}^0_{++} \neq \emptyset$, by Theorem 1.1(4) there exists $\widehat{f} \in \mathcal{B} \cap \mathbb{L}^0_{++}$ such that $\mathbb{E}_\mathbb{P}[f/\widehat{f}] \leq 1$ holds for all $f \in \mathcal{B}$. Let $\widehat{\mathcal{B}} := (1/\widehat{f})\mathcal{B}$. Then, $\widehat{\mathcal{B}}$ also satisfies the requirements of statement (1) of Theorem 1.2, $1 \in \widehat{\mathcal{B}}^{\max}$ and $\widehat{\mathcal{B}} \subseteq \{ f \in \mathbb{L}^0_+ \mid \mathbb{E}_\mathbb{P}[f] \leq 1 \} =: \mathcal{B}^1_\mathbb{P}$. We shall argue that $\mathcal{B}^1_\mathbb{P} \subseteq \widehat{\mathcal{B}}$, therefore establishing that $\widehat{\mathcal{B}} = \mathcal{B}^1_\mathbb{P}$ and completing the proof. Assume by way of contradiction that there exists $g \in \mathcal{B}^1_\mathbb{P} \setminus \widehat{\mathcal{B}}$. Since $\widehat{\mathcal{B}}$ is closed and solid, it follows that $(g \wedge M) \notin \widehat{\mathcal{B}}$ for large enough $M \in \mathbb{R}_+$; of course, $(g \wedge M) \in \mathcal{B}^1_\mathbb{P}$ also holds, since $\mathcal{B}^1_\mathbb{P}$ is solid. In other words, we may suppose that there exists $g \in (\mathcal{B}^1_\mathbb{P} \setminus \widehat{\mathcal{B}}) \cap \mathbb{L}^\infty_+$. Since $\widehat{\mathcal{B}} = \bigcup_{a \in [0,1]} a\widehat{\mathcal{B}}^{\max}$, $1 \in \widehat{\mathcal{B}}$, $\widehat{\mathcal{B}}$ is solid and $g \in \mathbb{L}^\infty_+$ does *not* belong to $\widehat{\mathcal{B}}$, there exists $a \in (0,1)$ such that $\widetilde{g} := ag \in \widehat{\mathcal{B}}^{\max}$. We shall now establish the following claim (we use $|\cdot|_{\mathbb{L}^\infty}$ will denote the usual $\mathbb{L}^\infty$-norm): $(1 + \epsilon - \epsilon\widetilde{g}) \in \widehat{\mathcal{B}}^{\max}$ *holds whenever* $0 < \epsilon < 1/|\widetilde{g}|_{\mathbb{L}^\infty}$. First of all, observe that $(1 + \epsilon - \epsilon\widetilde{g}) \in \mathbb{L}^\infty_+$ whenever $0 < \epsilon < 1/|\widetilde{g}|_{\mathbb{L}^\infty}$. Therefore, since $\widehat{\mathcal{B}} = \bigcup_{a \in [0,1]} a\widehat{\mathcal{B}}^{\max}$, $1 \in \widehat{\mathcal{B}}$, and $\widehat{\mathcal{B}}$ is solid, there exists $b \in \mathbb{R}_+$ such that $b(1 + \epsilon - \epsilon\widetilde{g}) \in \widehat{\mathcal{B}}^{\max}$. Since $\widehat{\mathcal{B}}^{\max}$ is convex and $\widetilde{g} \in \widehat{\mathcal{B}}^{\max}$, we have

$$\widehat{\mathcal{B}}^{\max} \ni \left(\frac{b\epsilon}{1 + b\epsilon} \widetilde{g} + \frac{1}{1 + b\epsilon} b(1 + \epsilon - \epsilon\widetilde{g}) \right) = \frac{b + b\epsilon}{1 + b\epsilon}.$$

The last element is a real multiple of $1 \in \widehat{\mathcal{B}}^{\max}$. Therefore, $1 = (b + b\epsilon)/(1 + b\epsilon)$, which gives $b = 1$ and establishes that $(1 + \epsilon - \epsilon\widetilde{g}) \in \widehat{\mathcal{B}}^{\max}$ whenever $0 < \epsilon < 1/|\widetilde{g}|_{\mathbb{L}^\infty}$. But then, with fixed $\epsilon \in \mathbb{R}_+$ such that $0 < \epsilon < 1/|\widetilde{g}|_{\mathbb{L}^\infty}$, we have $\mathbb{E}_\mathbb{P}[1 + \epsilon - \epsilon\widetilde{g}] = 1 + \epsilon(1 - a\mathbb{E}_\mathbb{P}[g]) > 1$, the last strict inequality holding because $a \in (0, 1)$ and $\mathbb{E}_\mathbb{P}[g] \leq 1$. In other words, $(1 + \epsilon - \epsilon\widetilde{g}) \notin \mathcal{B}^1_\mathbb{P}$, which is a contradiction to $\widehat{\mathcal{B}} \subseteq \mathcal{B}^1_\mathbb{P}$. We conclude that $\widehat{\mathcal{B}} = \mathcal{B}^1_\mathbb{P}$, which finishes our argument. □

Definition 1.3. A set $\mathcal{B} \subseteq \mathbb{L}^0_+$ satisfying any of the equivalent statements of Theorem 1.2 will be called a *full simplex in* $\mathbb{L}^0_+$.

The description of a full simplex $\mathcal{B}$ of $\mathbb{L}^0_+$ given by (1) in Theorem 1.2 is structural. The convex set $\mathcal{B}^{\max}$ is the "outer face" of $\mathcal{B}$ and one can create the whole set $\mathcal{B}$ by contracting this face "inwards" towards zero. This way one actually obtains a convexly compact set, though this is not completely trivial to show. Note that the idea of maximality in $\mathbb{L}^0_+$ was utilized in order to describe the "outer face" $\mathcal{B}^{\max}$ of $\mathcal{B}$. Theorem 1.2 shows immediately why characterizations using topological boundaries would be useless. Indeed, consider the σ-finite measure $\mu \sim \Pi$ such that $\mathcal{B} = \left\{ f \in \mathbb{L}^0_+ \mid \int_\Omega f \mathrm{d}\mu \leq 1 \right\}$. Suppose that $\mathbb{L}^0$ is infinite-dimensional, which is equivalent to the existence of a sequence $(h^n)_{n \in \mathbb{N}}$ of elements of $\mathbb{L}^0_+$ with $\int_\Omega h^n \mathrm{d}\mu > 1$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} h^n = 0$. Then, the closure of $\mathbb{L}^0_+ \setminus \mathcal{B} = \left\{ f \in \mathbb{L}^0_+ \mid \int_\Omega f \mathrm{d}\mu > 1 \right\}$ is actually equal to $\mathbb{L}^0_+$; this is straightforward once one notices that $f = 0$ belongs in this closure. Therefore, the topological boundary of the closed set $\mathcal{B}$ is $\mathcal{B}$ itself.

A preference-theoretic characterization of a full simplex in $\mathbb{L}^0_+$ is provided in statement (2) of Theorem 1.2. For any probability $\mathbb{P} \in \Pi$, there exists an optimal choice $\widehat{f} \in \mathcal{B}$ for $\preccurlyeq_\mathbb{P}$, depending on $\mathbb{P}$, that makes $\mathcal{B}$ exactly equal to the lower contour set of $\widehat{f}$.

Statement (3) of Theorem 1.2 describes a full simplex $\mathcal{B}$ of $\mathbb{L}^0_+$ in a geometric way, loosely as the intersection of $\mathbb{L}^0_+$ with a half-space. Observe however that the mappings $\mathbb{L}^0_+ \ni f \mapsto \int_\Omega f \mathrm{d}\mu$ for a σ-finite measure $\mu \sim \Pi$ are in general extended-real-valued and not continuous in $\mathbb{L}^0_+$. From the perspective of economic theory, $\mathcal{B}$ is the budget set associated with an agent with unit endowment, when prices of bundles in $\mathbb{L}^0_+$ are given in a linear way by μ: the price of $f \in \mathbb{L}^0_+$ is simply $\int_\Omega f \mathrm{d}\mu$.

The concept of a full simplex naturally incorporates numéraire-invariance. If $\mathcal{B}$ is a full simplex in $\mathbb{L}^0_+$ and $f \in \mathbb{L}^0_{++}$, then $(1/f)\mathcal{B}$ is also a full simplex in $\mathbb{L}^0_+$. In fact, and in view of the characterization given in statement (3) of Theorem 1.2, starting from a full simplex $\mathcal{B}$ in $\mathbb{L}^0_+$, the class of sets of the form $(1/f)\mathcal{B}$, where f ranges in $\mathbb{L}^0_{++}$, coincides with the class of *all* the full simplices in $\mathbb{L}^0_+$. Therefore, the class of full simplices in $\mathbb{L}^0_+$ has the same cardinality as $\mathbb{L}^0_{++}$.

To further get a feeling for the "fullness" of full simplices, we mention the following result. Apart from its independent interest, it will be crucial in proving the axiomatic characterization of numéraire-invariant choices given in Theorem 1.5.

Proposition 1.4. *Let $\mathcal{B}$ be a full simplex in $\mathbb{L}^0_+$ and $\mathcal{C}$ be a convex subset of $\mathbb{L}^0_+$ such that $\mathcal{B} \subseteq \mathcal{C}$ and $\mathcal{B}^{\max} \cap \mathcal{C}^{\max} \cap \mathbb{L}^0_{++} \neq \emptyset$. Then, actually, $\mathcal{B} = \mathcal{C}$.*

Proof. Pick $h \in \mathcal{B}^{\max} \cap \mathcal{C}^{\max} \cap \mathbb{L}^0_{++}$. Replacing $\mathcal{B}$ and $\mathcal{C}$ with $(1/h)\mathcal{B}$ and $(1/h)\mathcal{C}$ respectively, we may assume that $\mathcal{C} \subseteq \mathbb{L}^0_+$ is convex, $\mathcal{B} \subseteq \mathcal{C}$, $\mathcal{B}$ is a full simplex in $\mathbb{L}^0_+$, and $1 \in \mathcal{B}^{\max} \cap \mathcal{C}^{\max}$. Furthermore, we can assume that $\mathcal{C}$ is solid, replacing it if necessary with $\{f \in \mathbb{L}^0_+ \mid f \leq g \text{ for some } g \in \mathcal{C}\}$, since all the above properties will still hold. By Theorem 1.2, there exists a σ-finite measure $\mu \sim \Pi$ such that $\mathcal{B} = \{f \in \mathbb{L}^0_+ \mid \int_\Omega f \mathrm{d}\mu \leq 1\}$. As $1 \in \mathcal{B}^{\max}$, it is easy to see that μ has to actually be a probability, which we then denote by $\mathbb{P}$; that is, $\mathcal{B} = \{f \in \mathbb{L}^0_+ \mid \mathbb{E}_\mathbb{P}[f] \leq 1\}$. All the previous assumptions and notation will be in force in the sequel. We have to show that $\mathcal{B} = \mathcal{C}$.

For $n \in \mathbb{N}$, define a convexly compact set E^n as the closure of $\mathcal{C} \cap \{f \in \mathbb{L}^0_+ \mid f \leq n\}$. With $\preccurlyeq_\mathbb{P}$ defined via (1.2), for each $n \in \mathbb{N}$ let $h^n \in E^n$ satisfy $f \preccurlyeq_\mathbb{P} h^n$ for all $f \in E^n$. If $h^n = 1$ for all $n \in \mathbb{N}$, then $\mathbb{E}_\mathbb{P}[f] \leq 1$ for all $\mathcal{C} \cap \mathbb{L}^\infty_+$ and, by Fatou's lemma and the solidity of $\mathcal{C}$, $\mathbb{E}_\mathbb{P}[f] \leq 1$ for all $\mathcal{C}$; therefore, $\mathcal{C} \subseteq \mathcal{B}$ and there is nothing left to prove. By way of contradiction, assume that $\mathbb{P}[h^n = 1] < 1$ for some $n \in \mathbb{N}$; then, a fortiori, $n \geq 2$. Note then that $\mathbb{E}_\mathbb{P}[h^n] > 1$, i.e., $h^n \notin \mathcal{B}$, which follows from the facts that $\mathbb{E}_\mathbb{P}[1/h^n] \leq 1$ (since $1 \in E^n$) and $\mathbb{P}[h^n = 1] < 1$. From now onwards, fix $n \in \mathbb{N}$ with $n \geq 2$ such that h^n has the previous property, and we drop the superscript "n" from everywhere for typographical convenience. Let also $D := \mathcal{B} \cap \{f \in \mathbb{L}^0_+ \mid f \leq n\}$. Remember throughout that the elements of D and E are included in the $\mathbb{L}^\infty$-ball of radius n, that $D \subseteq E$, and that $h \in E \setminus D$.

Let π be the $\mathbb{L}^2(\mathbb{P})$-projection of h on D — observe that this is well defined since all elements of E (and therefore also of $D \subseteq E$) belong to $\mathbb{L}^\infty \subseteq \mathbb{L}^2(\mathbb{P})$ and D is convex and $\mathbb{L}^2(\mathbb{P})$-closed. Also, let $\nu := h - \pi$. Since $h \notin D$, $\mathbb{P}[\nu = 0] < 1$. Define $\pi' := \pi \mathbb{I}_{\{\nu \geq 0\}} + h \mathbb{I}_{\{\nu < 0\}}$. Since $h < \pi$ on $\{\nu < 0\}$, we have $\pi' \leq \pi$, which implies in particular that $\pi' \in D$. Also, since $\{\pi' < \pi\} = \{\nu < 0\}$, $\mathbb{P}[\nu < 0] > 0$ would imply $\mathbb{E}_\mathbb{P}\left[|\pi' - h|^2\right] = \mathbb{E}_\mathbb{P}\left[|\pi - h|^2 \mathbb{I}_{\{\nu \geq 0\}}\right] < \mathbb{E}_\mathbb{P}\left[|\pi - h|^2\right]$, which contradicts the fact that π is the $\mathbb{L}^2(\mathbb{P})$-projection of h on D. Therefore, $\nu \in \mathbb{L}^\infty_+$.

Define

$$\delta := \min\left\{\frac{\mathbb{E}_\mathbb{P}[h] - 1}{\mathbb{E}_\mathbb{P}[\nu]}, 1\right\} \in (0, 1], \text{ as well as } \zeta := 1 + \frac{1}{n} - \frac{1}{n}\big(h - \delta\nu\big) = 1 + \frac{1}{n} - \frac{1}{n}\big(\pi + (1 - \delta)\nu\big).$$

The above definition of δ ensures that $\mathbb{E}_\mathbb{P}[\zeta] \leq 1$. Also, $0 \leq \pi = h - \nu \leq h - \delta\nu \leq h \leq n$, which implies that $\mathbb{P}[1/n \leq \zeta \leq 1 + 1/n] = 1$, and, therefore, that $\zeta \in D$, since $n \geq 2$. If $\zeta \in E$, then also $1 + \delta\nu/(n+1) = ((n/(n+1))\, \zeta + (1/(n+1))\, h) \in E$, which is impossible in view of $1 \in E^{\max}$ ($1 \in E \subseteq \mathcal{C}$ and $1 \in \mathcal{C}^{\max}$). We obtain that $\zeta \in D \setminus E$, which is a contradiction to the fact that $D \subseteq E$. The last contradiction implies that $\mathbb{P}[h \neq 1] > 0$ is impossible, which concludes the proof. □

1.5. **Axiomatic characterization of numéraire-invariant choices.**

1.5.1. *The characterization result.* We are ready to give the main result of this section.

Theorem 1.5. *Let $\preccurlyeq$ be a binary relation on $\mathbb{L}^0_+$ that satisfies the following properties:*

(A1) *$f \preccurlyeq g$ holds if and only if $\{f > 0\} \subseteq \{g > 0\}$ and $(f/g)\mathbb{I}_{\{g>0\}} + \mathbb{I}_{\{g=0\}} \preccurlyeq 1$.*
(A2) *If $f \leq 1$, then $f \preccurlyeq 1$. Furthermore, if $f \leq 1$ and $\{f < 1\}$ is not Π-null, then $f \prec 1$.*
(A3) *The lower-contour set $\{f \in \mathbb{L}^0_+ \mid f \preccurlyeq 1\}$ is convex.*

(A4) *For some full simplex* $\mathcal{B}$ *of* $\mathbb{L}^0_+$, *there exists* $\widehat{f} \in \mathcal{B}$ *such that* $f \preccurlyeq \widehat{f}$ *holds for all* $f \in \mathcal{B}$.

Then, there exists a unique $\mathbb{P} \in \Pi$ *that* generates $\preccurlyeq$, *in the sense that* $\preccurlyeq$ *is exactly the relation* $\preccurlyeq_{\mathbb{P}}$ *of* (1.2).

Proof. For *any* $\mathbb{Q} \in \Pi$, let $\mathcal{B}^1_{\mathbb{Q}} := \left\{ f \in \mathbb{L}^0_+ \mid \mathbb{E}_{\mathbb{Q}}[f] \leq 1 \right\}$. Also let $\mathcal{C}^1_{\preccurlyeq} := \left\{ f \in \mathbb{L}^0_+ \mid f \preccurlyeq 1 \right\}$. By the numéraire-invariance axiom A1, proving Theorem 1.5 amounts to finding $\mathbb{P} \in \Pi$ such that $\mathcal{B}^1_{\mathbb{P}} = \mathcal{C}^1_{\preccurlyeq}$.

A combination of (A1) and (A4) imply that for *any* full simplex $\mathcal{B}$ of $\mathbb{L}^0_+$, there exists $\widehat{f} \in \mathcal{B}$ such that $f \preccurlyeq \widehat{f}$ holds for all $f \in \mathcal{B}$. Fix $\mathbb{Q} \in \Pi$. By Theorem 1.2, $\mathcal{B}^1_{\mathbb{Q}}$ is a full simplex in $\mathbb{L}^0_+$; therefore, there exists $g \in \mathcal{B}^1_{\mathbb{Q}}$ such that $f \preccurlyeq g$ holds for all $f \in \mathcal{B}^1_{\mathbb{Q}}$. We claim that $g \in \mathbb{L}^0_{++}$, as well as $\mathbb{E}_{\mathbb{Q}}[g] = 1$. Indeed, $g \in \mathbb{L}^0_{++}$ follows from the fact $\mathcal{B}^1_{\mathbb{Q}} \ni 1 \preccurlyeq g$, since (A1) implies that in this case $\Omega = \{1 > 0\} \subseteq \{g > 0\}$. Also, if $\mathbb{E}_{\mathbb{Q}}[g] < 1$, then $h := (\mathbb{E}_{\mathbb{Q}}[g])^{-1} g \in \mathcal{B}^1_{\mathbb{Q}}$ with $\mathbb{P}[g < h] = 1$, which means that $g \prec h$ by (A2) and contradicts the fact that $h \preccurlyeq g$ for $h \in \mathcal{B}^1_{\mathbb{Q}}$.

Define $\mathbb{P} \in \Pi$ via $\mathbb{P}[A] := \mathbb{E}_{\mathbb{Q}}[g \mathbb{I}_A]$ for all $A \in \mathcal{F}$. Observe that $f \in \mathcal{B}^1_{\mathbb{P}}$ if and only if $(fg) \in \mathcal{B}^1_{\mathbb{Q}}$, and in that case we have $fg \preccurlyeq g$, or $f \preccurlyeq 1$ in view of axiom A1. In other words, $\mathcal{B}^1_{\mathbb{P}} \subseteq \mathcal{C}^1_{\preccurlyeq}$. Since $\mathcal{C}^1_{\preccurlyeq}$ is convex by (A3), and $1 \in (\mathcal{B}^1_{\mathbb{P}})^{\max} \cap (\mathcal{C}^1_{\preccurlyeq})^{\max} \cap \mathbb{L}^0_{++}$, where $1 \in (\mathcal{C}^1_{\preccurlyeq})^{\max}$ follows from (A2), an application of Proposition 1.4 gives $\mathcal{B}^1_{\mathbb{P}} = \mathcal{C}^1_{\preccurlyeq}$.

We finally discuss the uniqueness of the representative $\mathbb{P} \in \Pi$. If $\mathbb{P}' \in \Pi$ also generates $\preccurlyeq$, then $\mathcal{B}^1_{\mathbb{P}} = \mathcal{C}^1_{\preccurlyeq} = \mathcal{B}^1_{\mathbb{P}'}$ should hold, which implies that $\mathbb{P} = \mathbb{P}'$, and completes the proof. □

A comparison with the statement of Theorem 1.1 is in order. Axioms (A1) and (A2) of Theorem 1.5 are really the same as statements (1) and (2) of Theorem 1.1 — it is enough to deal with the case $g = 1$ in axiom (A2) of Theorem 1.5 because of the numéraire-invariance axiom (A1). The first surprise comes from the simplicity of axiom (A3) of Theorem 1.5, where we *only* require convexity of the lower contour set. This should be compared to the very rich structure that is given in statement (3) of Theorem 1.1 for both the lower-contour and upper-contour sets. The numéraire-invariance axiom (A1) is strong enough so that *no* closedness or even risk-aversion axiom is needed. Also, axiom (A4) of Theorem 1.5 is significantly weaker than statement (4) of Theorem 1.1, as it only asks that an optimal choice exists for *some* full simplex of $\mathbb{L}^0_+$, and not for *all* convexly compact subsets of $\mathbb{L}^0_+$. Although, in view of (A1), (A4) actually implies that an optimal choice exists for *all* full simplices of $\mathbb{L}^0_+$, this class is still much smaller than the class of all convexly compact sets.

1.5.2. *Subjective probability and risk aversion.* The probability $\mathbb{P} \in \Pi$ that generates the relation $\preccurlyeq$ satisfying the axioms of Theorem 1.5 should be thought as the subjective probability of the agent whose choices are represented by $\preccurlyeq$, as it corresponds to the idea of "agent risk aversion". If the agent's subjective probability is $\mathbb{Q} \in \Pi$, risk aversion would translate into $f \preccurlyeq \mathbb{E}_{\mathbb{Q}}[f]$ holding for all $f \in \mathbb{L}^\infty_+$. Let $\mathbb{P} \in \Pi$ generate $\preccurlyeq$. Then, $\mathbb{P}[A]/\mathbb{Q}[A] = \mathbb{E}_{\mathbb{P}}[(1/\mathbb{E}_{\mathbb{Q}}[\mathbb{I}_A])\mathbb{I}_A] \leq 1$, i.e., $\mathbb{P}[A] \leq \mathbb{Q}[A]$, holds for all non-null $A \in \mathcal{F}$. Therefore, $\mathbb{Q} = \mathbb{P}$.

1.5.3. *Choice rules.* A more behavioral-based alternative to modeling preferences via binary relations is to model the *choice rules* of an agent; for a quick introduction and the material we shall need here, see Chapter 1 of [17]. For all $\mathcal{C} \subseteq \mathbb{L}^0_+$, define $\varepsilon_{\preccurlyeq}(\mathcal{C}) := \{g \in \mathcal{C} \mid f \preccurlyeq g, \text{ for all } f \in \mathcal{C}\}$.

This way we get a *choice function* $\varepsilon = \varepsilon_{\preccurlyeq}$. Forgetting that ε came from $\preccurlyeq$, we can define the *revealed* preference $\preccurlyeq_\varepsilon$ from ε as follows: $f \preccurlyeq_\varepsilon g$ if and only if there exists $\mathcal{C} \subseteq \mathbb{L}^0_+$ such that $f \in \mathcal{C}$ and $g \in \varepsilon(\mathcal{C})$. Then, it can be shown that $\preccurlyeq_\varepsilon$ coincides with $\preccurlyeq$ on $\mathbb{L}^0_+$. Furthermore, the axioms of Theorem 1.5 can be expressed directly in terms of the choice rule ε; therefore, this can be viewed as as the starting point of axiomatization, which will then induce the preference structure $\preccurlyeq$.

1.6. **Extending the preference structure.** As noted in §1.3.3, one of the "drawbacks" of a preference relation that satisfies the axioms of Theorem 1.5 is that it fails to be transitive. We shall extend $\preccurlyeq$ to a preference relation $\trianglelefteq$ that is transitive and satisfies some extremely weak continuity properties. To avoid unnecessary technicalities, we shall work on $\mathbb{L}^0_{++}$. As it will turn out, $\trianglelefteq$ *almost* has a numerical representation given by expected logarithmic utility under the probability $\mathbb{P} \in \Pi$ that generates $\preccurlyeq$. We shall discuss the previous use of the word "almost" after stating and proving Theorem 1.6 below.

As with any preference relation, $f \triangleleft g$ will mean that $f \trianglelefteq g$ holds, whereas $g \trianglelefteq f$ fails to hold. Also, for $x \in \mathbb{R}_{++}$, we set $\log_+(x) = \max\{\log(x), 0\}$.

Theorem 1.6. *Let $\preccurlyeq$ denote a binary relation on $\mathbb{L}^0_+$ satisfying the axioms of Theorem 1.5. Then, there exists a (not necessarily unique) binary relation $\trianglelefteq$ on $\mathbb{L}^0_{++}$ such that:*

(1) *If $f \in \mathbb{L}^0_{++}$ and $g \in \mathbb{L}^0_{++}$, $f \trianglelefteq g$ holds if and only if $(f/g) \trianglelefteq 1$.*
(2) *For $f \in \mathbb{L}^0_{++}$, $f \prec 1$ implies $f \triangleleft 1$.*
(3) *$\trianglelefteq$ is transitive.*
(4) *For $f \in \mathbb{L}^0_{++}$, $f \trianglelefteq 1$ is implied by either of the conditions below:*
    (a) *$af \trianglelefteq 1$ holds for all $a \in (0,1)$.*
    (b) *$f \geq \epsilon$ for some $\epsilon \in \mathbb{R}_{++}$, and $f \wedge n \trianglelefteq 1$ holds for all $n \in \mathbb{N}$.*

In this case, and with $\mathbb{P} \in \Pi$ generating $\preccurlyeq$, the following holds: For any $f \in \mathbb{L}^0_{++}$ and $g \in \mathbb{L}^0_{++}$ with $\mathbb{E}_\mathbb{P}\left[\log_+(f/g)\right] < \infty$, we have

$$f \trianglelefteq g \iff \mathbb{E}_\mathbb{P}\left[\log\left(\frac{f}{g}\right)\right] \leq 0. \tag{1.5}$$

As a corollary, the restriction of any binary relation $\trianglelefteq$ satisfying (1), (2), (3) and (4) above on $\mathcal{L}_\mathbb{P} := \left\{f \in \mathbb{L}^0_{++} \mid \mathbb{E}_\mathbb{P}[|\log f|] < \infty\right\}$ is uniquely defined via the numerical representation:

$$\text{For } f \in \mathcal{L}_\mathbb{P} \text{ and } g \in \mathcal{L}_\mathbb{P}, \quad f \trianglelefteq g \iff \mathbb{E}_\mathbb{P}[\log(f)] \leq \mathbb{E}_\mathbb{P}[\log(g)].$$

In particular, $\trianglelefteq$ is complete on $\mathcal{L}_\mathbb{P}$.

Proof. We shall first establish the existence of a binary relation $\trianglelefteq$ on $\mathbb{L}^0_{++}$ that satisfies the requirements (1), (2), (3) and (4) of Theorem 1.6. We use the following definition: for $f \in \mathbb{L}^0_{++}$ and $g \in \mathbb{L}^0_{++}$, we *set $f \trianglelefteq g$ if and only if* $\mathbb{E}_\mathbb{P}[\log_+(f/g)] < \infty$ *and* $\mathbb{E}_\mathbb{P}[\log(f/g)] \leq 0$ *hold.* The numéraire-invariance property (1) and the transitivity property (3) are straightforward. For property (2), note that if $f \prec 1$, i.e., $\mathbb{E}_\mathbb{P}[f] < 1$, for $f \in \mathbb{L}^0_{++}$, Jensen's inequality implies that $\mathbb{E}_\mathbb{P}[\log(f)] < 0 = \mathbb{E}_\mathbb{P}[\log(1)]$, i.e., $f \triangleleft 1$. Finally, property (4a) is trivial to check, while property (4b) follows from the monotone convergence theorem.

Conversely, consider *any* binary relation that satisfies all the requirements of Theorem 1.6. First of all, we claim that $f \trianglelefteq 1$ and $g \trianglelefteq 1$ imply that $fg \trianglelefteq 1$. Indeed, $g \trianglelefteq 1$ is equivalent to $1 \trianglelefteq 1/g$ by the numéraire-invariance property (1), and then the transitivity property (3) gives $f \trianglelefteq 1/g$. The numéraire-invariance property (1) applied once again gives $fg \trianglelefteq 1$.

We now show that $f \vartriangleleft g$ and $g \trianglelefteq h$ imply $f \vartriangleleft h$. We already know that $f \trianglelefteq h$ from the transitivity property (3). If $h \trianglelefteq f$, then $h/f \trianglelefteq 1$ and $g/h \trianglelefteq 1$ would imply $(h/f)(g/h) \trianglelefteq 1$, or $g/f \trianglelefteq 1$, or again equivalently that $g \trianglelefteq f$, which is false. Therefore, $f \vartriangleleft h$.

Pick $f \in \mathbb{L}^0_{++}$ such that $f \leq M$ for some $M \in \mathbb{R}_+$ and $\mathbb{E}_\mathbb{P}[\log(f)] < 0$. Define $\ell^n := n(f^{1/n} - 1)$ for all $n \in \mathbb{N}$. Then, $\downarrow \lim_{n\to\infty} \ell^n = \log(f)$ and $\ell^n \leq \ell^1 \leq M - 1$ for all $n \in \mathbb{N}$. Therefore, the monotone convergence theorem gives that $\mathbb{E}_\mathbb{P}[\ell^n] < 0$ for some large enough $n \in \mathbb{N}$. This means that $\mathbb{E}_\mathbb{P}[f^{1/n}] \leq 1$. As $f \neq 1$ (which follows from $\mathbb{E}_\mathbb{P}[\log(f)] < 0$), we have $f^{1/n} \prec 1$, i.e., $f^{1/n} \vartriangleleft 1$ by the extension property (2), and therefore $f \vartriangleleft 1$ by the results of the preceding paragraphs.

Pick $f \in \mathbb{L}^0_{++}$ with $\mathbb{E}_\mathbb{P}[\log(f)] < 0$. Choose $\epsilon \in \mathbb{R}_{++}$ such that $\mathbb{E}_\mathbb{P}[\log(f + \epsilon)] < 0$. Then, $\mathbb{E}_\mathbb{P}[\log((f + \epsilon) \wedge M)] < 0$ holds for all $M \in \mathbb{R}_{++}$; therefore, $(f + \epsilon) \wedge M \trianglelefteq 1$ holds for all $M \in \mathbb{R}_{++}$ by the result of the preceding paragraph. Since $f + \epsilon \geq \epsilon$, the weak continuity property (4b) gives $(f + \epsilon) \trianglelefteq 1$. Finally, since $f \prec f + \epsilon$, we have $f \vartriangleleft f + \epsilon$ by the extension property (2), which combined with $(f + \epsilon) \trianglelefteq 1$ gives $f \vartriangleleft 1$.

Up to now, we have shown that $f \in \mathbb{L}^0_{++}$ with $\mathbb{E}_\mathbb{P}[\log(f)] < 0$ implies $f \vartriangleleft 1$. Pick $f \in \mathbb{L}^0_{++}$ with $\mathbb{E}_\mathbb{P}[\log(f)] \leq 0$. Then, for all $a \in (0, 1)$ we have $\mathbb{E}_\mathbb{P}[\log(af)] < 0$, therefore $af \trianglelefteq 1$. The continuity property (4a) gives $f \trianglelefteq 1$. Therefore, $f \in \mathbb{L}^0_{++}$ with $\mathbb{E}_\mathbb{P}[\log(f)] \leq 0$ implies $f \trianglelefteq 1$.

Finally, pick $f \in \mathbb{L}^0_{++}$ with $\mathbb{E}_\mathbb{P}[\log_+(f)] < \infty$ and assume that $f \trianglelefteq 1$. Then, we claim that we must have $\mathbb{E}_\mathbb{P}[\log(f)] \leq 0$. Suppose on the contrary that $\mathbb{E}_\mathbb{P}[\log(f)] > 0$; this would imply that $1 \vartriangleleft f$, which is impossible. Therefore, for $f \in \mathbb{L}^0_{++}$ with $\mathbb{E}_\mathbb{P}[\log_+(f)] < \infty$ we have that $f \trianglelefteq 1$ if and only if $\mathbb{E}_\mathbb{P}[\log(f)] \leq 0$, which is exactly what we needed to show. □

The special relation $\trianglelefteq$ constructed in the first paragraph of the proof of Theorem 1.6 is the *minimal* way to construct a binary relation on $\mathbb{L}^0_{++}$ that satisfies the requirements (1), (2), (3) and (4) of Theorem 1.6; any other such relation has to be an extension of the one described there. Observe that if $\mathbb{L}^0$ is finite-dimensional, $\mathcal{L}_\mathbb{P} = \mathbb{L}^0_{++}$ and therefore in this case we obtain the uniqueness of $\trianglelefteq$ that satisfies the requirements (1), (2), (3) and (4) of Theorem 1.6.

Theorem 1.6 remains silent on how to define the relation between $f \in \mathbb{L}^0_{++}$ and $g \in \mathbb{L}^0_{++}$ when both $\mathbb{E}_\mathbb{P}[\log_+(f/g)] = \infty$ and $\mathbb{E}_\mathbb{P}[\log_+(g/f)] = \infty$ hold. (When $\mathbb{L}^0$ is infinite-dimensional, one can always find pairs like this.) Note that, for $f \in \mathbb{L}^0_{++}$ such that $\mathbb{E}_\mathbb{P}[\log_+(f)] < \infty$, $\mathbb{E}_\mathbb{P}[\log_+(1/f)] = \infty$ implies $f \trianglelefteq 1$ by (1.5). One would be tempted to define $f \trianglelefteq 1$ whenever $\mathbb{E}_\mathbb{P}[\log_+(1/f)] = \infty$, claiming that there is too much "downside risk" in f. However, with this understanding, if $f \in \mathbb{L}^0_{++}$ is such that $\mathbb{E}_\mathbb{P}[\log_+(f)] = \mathbb{E}_\mathbb{P}[\log_+(1/f)] = \infty$, we would get $f \trianglelefteq 1$ and $1/f \trianglelefteq 1$, or equivalently that $f \trianglelefteq 1$ and $1 \trianglelefteq f$, which would make all $f \in \mathbb{L}^0_{++}$ such that $\mathbb{E}_\mathbb{P}[\log_+(f)] = \mathbb{E}_\mathbb{P}[\log_+(1/f)] = \infty$ belong to the same equivalence class. This is impossible: if $f \in \mathbb{L}^0_{++}$ is such that $\mathbb{E}_\mathbb{P}[\log_+(f)] = \mathbb{E}_\mathbb{P}[\log_+(1/f)] = \infty$, then $2f$ has the same property, but $f \vartriangleleft 2f$. We may simply opt to leave the relation of f and g when $\mathbb{E}_{\mathbb{P}}[\log_+(f/g)] = \mathbb{E}_{\mathbb{P}}[\log_+(g/f)] = \infty$ undefined, implicitly claiming that they are too risky relatively to each other to be compared. It remains an *open question* whether one can extend $\trianglelefteq$ to make it complete on $\mathbb{L}^0_{++}$, still having the properties of Theorem 1.6 holding, when $\mathbb{L}^0$ is infinite-dimensional.

2. Numéraire-Invariant Preferences in a Dynamic Environment

2.1. **Notation and terminology.** All stochastic processes in the sequel are defined on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$. Here, the probability $\mathbb{P}$ on $(\Omega, \mathcal{F}_\infty)$, where $\mathcal{F}_\infty := \bigvee_{t\in\mathbb{R}_+} \mathcal{F}_t$ will be fixed and we shall be using "$\mathbb{E}$" for the expectation of $\mathcal{F}_\infty$-measurable random variables under $\mathbb{P}$. The filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is assumed to be right-continuous and $\mathcal{F}_0$ is assumed $\mathbb{P}$-trivial. The optional σ-algebra on $\Omega \times \mathbb{R}_+$ is denoted by $\mathcal{O}$. A set $A \in \mathcal{O}$ is called *evanescent* if the random set $\Omega \ni \omega \mapsto \{t \in \mathbb{R}_+ \mid (t,\omega) \in A\}$ is $\mathbb{P}$-a.s. empty; an optional process V is evanescent if $\{V \neq 0\} \in \mathcal{O}$ is an evanescent set. For $A \in \mathcal{O}$ and $t \in \mathbb{R}_+$, we set $A_t := \{\omega \in \Omega \mid (\omega, t) \in A\} \in \mathcal{F}_t$.

For a càdlàg process X we define the process $X_- = (X_{t-})_{t\in\mathbb{R}_+}$ by $X_{0-} = 0$, and X_t being the left-limit of X at $t \in \mathbb{R}_{++}$. Also, we let $\Delta X := X - X_-$. Every predictable process H is supposed to satisfy $H_0 = 0$. Whenever H and X are d-dimensional processes such that X is a semimartingale to be used as an integrator and H can be used as integrand with respect to X, we denote by $\int_{[0,\cdot]} \langle H_t, \mathrm{d}X_t \rangle$ the integral process, where "$\langle \cdot, \cdot \rangle$" is used to (sometimes, formally) denote the usual inner product in $\mathbb{R}^d$. We assume vector stochastic integration — see for example [12]. Note that $\int_{\{0\}} \langle H_t, \mathrm{d}X_t \rangle = \langle H_0, \Delta X_0 \rangle = \langle H_0, X_0 \rangle$; therefore, if H is predictable, $\int_{\{0\}} \langle H_t, \mathrm{d}X_t \rangle = 0$. We also define $\int_{(0,\cdot]} \langle H_t, \mathrm{d}X_t \rangle := \int_{[0,\cdot]} \langle H_t, \mathrm{d}X_t \rangle - \int_{\{0\}} \langle H_t, \mathrm{d}X_t \rangle = \int_{[0,\cdot]} \langle H_t, \mathrm{d}X_t \rangle - \langle H_0, X_0 \rangle$.

2.2. **A canonical representation of unit-mass optional measures.** The natural space to define "subjective probabilities" of agents in the dynamic case is $(\Omega \times \mathbb{R}_+, \mathcal{O})$. We begin with a result regarding the structure of nonnegative measures on $(\Omega \times \mathbb{R}_+, \mathcal{O})$ with unit total mass.

Theorem 2.1. *On $(\Omega \times \mathbb{R}_+, \mathcal{O})$, consider a measure p such that $p[\Omega \times \mathbb{R}_+] = 1$ and $p[A] = 0$ for every evanescent set $A \in \mathcal{O}$. Then, there exists a pair of processes (L, K) such that:*

(1) *L is a nonnegative local martingale with $L_0 = 1$.*
(2) *K is adapted, right-continuous, nondecreasing, and $0 \leq K \leq 1$.*
(3) *$\int_{\Omega\times\mathbb{R}_+} V \mathrm{d}p = \mathbb{E}\left[\int_{\mathbb{R}_+} V_t L_t \mathrm{d}K_t\right]$ holds for all nonnegative optional process V.*
(4) *$L = \int_{[0,\cdot]} \mathbb{I}_{\{K_{t-}<1\}} \mathrm{d}L_t$ and $K = \int_{[0,\cdot]} \mathbb{I}_{\{L_t>0\}} \mathrm{d}K_t$.*

Furthermore, $\{L_\infty > 0\} \subseteq \{K_\infty = 1\}$ holds.

A pair (L, K) that satisfies the above requirements is essentially unique, in the following sense: if (K', L') is another pair that satisfies the above requirements, then $K = K'$ up to evanescence, while $L_t = L'_t$ for all $t \in \mathbb{R}_+$ holds on $\{K_\infty > 0\}$.

Definition 2.2. For a measure p on $(\Omega \times \mathbb{R}_+, \mathcal{O})$ with $p[\Omega \times \mathbb{R}_+] = 1$ and $p[A] = 0$ holding for every evanescent set $A \in \mathcal{O}$, a pair of processes (L, K) that satisfies requirements (1), (2), (3) and (4) of Theorem 2.1 will be called a *canonical representation pair for p*.

Remark 2.3. Let $p \in \Pi$ with canonical representation pair (L, K), and suppose that L is the density process of a probability $\mathbb{Q}$ with respect to $\mathbb{P}$; for this, it is necessary that L is a martingale and sufficient that L is a uniformly integrable martingale. For all $t \in \mathbb{R}_+$ and $A \in \mathcal{F}_t$, $\mathbb{Q}[A] = \mathbb{E}[L_t \mathbb{I}_A]$, i.e., $\mathbb{Q}$ is locally absolutely continuous with respect to $\mathbb{P}$. Furthermore, using integration-by-parts and a standard localization argument, it is straightforward to check that $\int_{\Omega \times \mathbb{R}_+} V \mathrm{d}p = \mathbb{E}_{\mathbb{Q}} \left[\int_{\mathbb{R}_+} V_t \mathrm{d}K_t \right]$ holds for all nonnegative optional process V. Since $p[\Omega \times \mathbb{R}_+] = 1$ and $\mathbb{Q}[K_\infty \leq 1] = 1$ hold, it must be the case that $\mathbb{Q}[K_\infty = 1] = 1$.

As it turns out, however, the above special case is not exhaustive. It may happen that L is a *strict* local martingale in the sense of [9], which precludes it from being a density process of some probability $\mathbb{Q}$ with respect to $\mathbb{P}$. (Nevertheless, at least in the case of finite time-horizon, one is able to interpret L as the density process of a *finitely* additive probability with respect to $\mathbb{P}$, that is only locally countably additive; for more information, see [26].) It might also happen that $\{K_\infty < 1\}$ is not $\mathbb{P}$-null; actually, it can even happen that $\mathbb{P}[K_\infty < 1] = 1$. The previous are illustrated in Example 2.5 later on in the text.

2.3. **Existence of a canonical representation pair in Theorem 2.1.** Doléans' representation of optional measures — see, for example, §VI.20 of [20] — implies the existence of an adapted, right-continuous, nonnegative and nondecreasing process H such that $\int_{\Omega \times \mathbb{R}_+} V \mathrm{d}p = \mathbb{E} \left[\int_{\mathbb{R}_+} V_t \mathrm{d}H_t \right]$ for all nonnegative optional processes V. We shall establish below that any adapted, right-continuous, nonnegative and nondecreasing process H with $\mathbb{E}[H_\infty] = 1$ can be decomposed as $H = \int_{[0,\cdot]} L_t \mathrm{d}K_t$ for a pair (L, K) satisfying (1), (2), and (4) of Theorem 2.1. The question of essential uniqueness of the pair (L, K) satisfying properties (1), (2), (3) and (4) of Theorem 2.1 will be tackled in §2.6.

Consider the nonnegative càdlàg martingale M that satisfies $M_t = \mathbb{E}[H_\infty \mid \mathcal{F}_t]$ for all $t \in \mathbb{R}_+$. Then, define the supermartingale $Z := M - H$; Z is nonnegative since $Z_t = \mathbb{E}[H_\infty - H_t \mid \mathcal{F}_t]$ holds for all $t \in \mathbb{R}_+$. The expected total mass of H over $\mathbb{R}_+$ is $M_0 = \mathbb{E}[H_\infty] = 1$. If $\mathbb{P}[H_\infty > 1] = 0$, in which case $\mathbb{P}[H_\infty = 1] = 1$, defining $K := H$ and $L := 1$ would suffice for the purposes of Theorem 2.1. However, it might happen that $\mathbb{P}[H_\infty > 1] > 0$ as is illustrated in Example 2.5. In this case, we shall construct the pair (K, L) from H. Before going to the technical details, we shall provide some intuition on the definition of (K, L). For $t \in \mathbb{R}_+$, $Z_t + \Delta H_t = \mathbb{E}[H_\infty - H_{t-} \mid \mathcal{F}_t]$ is the expected total remaining "life" of H on $[t, \infty[$, conditional on $\mathcal{F}_t$; then, formally, $\mathrm{d}H_t / (Z_t + \Delta H_t)$ is the "fraction of remaining life spent" at t. The equivalent "fraction of remaining life spent" for K, assuming that $K_\infty = 1$, would be $\mathrm{d}K_t / (1 - K_{t-})$. We shall ask that K formally satisfies $\mathrm{d}K_t / (1 - K_{t-}) = \mathrm{d}H_t / (Z_t + \Delta H_t)$ for $t \in \mathbb{R}_+$. To get a feeling of how L should be defined, observe that $\Delta K = (1 - K_-) \, \Delta H / (Z + \Delta H)$ implies that $(1 - K)/Z = (1 - K_-)/(Z + \Delta H)$; therefore, formally, $\mathrm{d}K_t / (1 - K_t) = \mathrm{d}H_t / Z_t$ holds for $t \in \mathbb{R}_+$. Since $H = \int_{[0,\cdot]} L_t \mathrm{d}K_t$ has to hold in view of property (3) in Theorem 2.1, we obtain $L(1 - K) = Z$, which will be the defining equation for L as long as $K < 1$. We shall use the previous intuition to define the pair (K, L) rigorously below.

We proceed with our development, first assuming that $\mathbb{P}[H_t < H_\infty \mid \mathcal{F}_t] = 1$ holds for all $t \in \mathbb{R}_+$ — later, this assumption will be removed. Under the previous assumption on H, it is straightforward to see that $Z > 0$ (and, since Z is a supermartingale, also $Z_- > 0$) holds. We define K as the unique solution of the stochastic integral equation

$$K = H_0 + \int_{(0,\cdot]} \left(\frac{1 - K_{t-}}{Z_t + \Delta H_t} \right) \mathrm{d}H_t,$$

the latter being the rigorous equivalent of "$\mathrm{d}K_t/(1-K_t) = \mathrm{d}H_t/(Z_t + \Delta H_t)$". The solution to the last equation is given by

$$K = 1 - (1 - H_0) \exp\left(- \int_{(0,\cdot]} \frac{\mathrm{d}H_t}{Z_t + \Delta H_t} \right) \prod_{t \in (0,\cdot]} \left(\left(1 - \frac{\Delta H_t}{Z_t + \Delta H_t} \right) \exp\left(\frac{\Delta H_t}{Z_t + \Delta H_t} \right) \right), \tag{2.1}$$

which is an adapted, nondecreasing process with $0 \leq K < 1$, the latter strict inequality holding due to our assumption on H. Set $L := Z/(1-K)$, which is well-defined in view of $K < 1$; L is nonnegative and $L_0 = Z_0/(1-K_0) = (1-H_0)/(1-H_0) = 1$. Actually, L is a local martingale. To see this, first observe that a use of (2.1) in reciprocal form gives

$$\frac{1}{1-K} = \frac{1}{1-H_0} + \int_{(0,\cdot]} \frac{\mathrm{d}H_t}{(1-K_{t-})\, Z_t}.$$

Then, the integration-by-parts formula gives

$$\begin{aligned} L = \frac{Z}{1-K} &= 1 + \int_{(0,\cdot]} \frac{\mathrm{d}Z_t}{1-K_{t-}} + \int_{(0,\cdot]} Z_t \mathrm{d}\left(\frac{1}{1-K_t} \right) \\ &= 1 + \int_{(0,\cdot]} \frac{\mathrm{d}Z_t}{1-K_{t-}} + \int_{(0,\cdot]} Z_t \frac{\mathrm{d}H_t}{(1-K_{t-})Z_t} \\ &= 1 + \int_{(0,\cdot]} \frac{\mathrm{d}M_t}{1-K_{t-}} \\ &= 1 + \int_{(0,\cdot]} L_{t-} \frac{\mathrm{d}M_t}{Z_{t-}}. \end{aligned}$$

The above string of equalities gives that L is a local martingale, and that it is actually equal to the stochastic exponential of the local martingale $\int_{(0,\cdot]} (\mathrm{d}M_t / Z_{t-})$.

Now, drop the simplifying assumption $\mathbb{P}[H_t < H_\infty \mid \mathcal{F}_t] = 1$ for all $t \in \mathbb{R}_+$. Then, $Z > 0$ is no longer necessarily true and more care has to be given in the definition of K and L. For each $n \in \mathbb{N}$, consider the stopping time $\tau^n := \inf\{t \in \mathbb{R}_+ \mid Z_t \leq 1/n\}$, and define the predictable set $\Theta := \bigcup_{n \in \mathbb{N}} [\![0, \tau^n]\!]$. Then, $\Theta \subseteq \{Z_- > 0\}$. Furthermore, with $\tau^\infty := \inf\{t \in \mathbb{R}_+ \mid Z_{t-} = 0 \text{ or } Z_t = 0\}$, we have $\uparrow \lim_{n \to \infty} \tau^n = \tau^\infty$, as well as $[\![\tau^\infty, \infty[\![= \{Z = 0\} \supseteq \{H = H_\infty\}$.

Define K via (2.1), and observe that K is well-defined: our division conventions imply that $Z/(Z + \Delta H) = 1$ on $\{Z = 0\}$, in view of the fact that H is constant on $\{Z = 0\}$. It is clear that K is adapted, right-continuous, nondecreasing, and $0 \leq K \leq 1$. Furthermore, $K = \int_{[0,\cdot]} \mathbb{I}_{\Theta_t} \mathrm{d}K_t$ and $\Theta \subseteq \{K_- < 1\}$. We shall also consider the nonnegative local martingale L that formally satisfies $\mathrm{d}L_t / L_{t-} = \mathrm{d}M_t / Z_{t-}$ for $t \in \mathbb{R}_+$; some care has to be given in defining L, since Z_- might become zero. Observe that $1 + \Delta M / Z_- = (Z + \Delta H)/Z_- \geq 0$ holds on $[\![0, \tau^n]\!]$ for all $n \in \mathbb{N}$. As $Z_- \geq 1/n$ on $[\![0, \tau^n]\!]$, we can define a process L^n as the stochastic exponential of $\int_{(0,\tau^n \wedge \cdot]} (\mathrm{d}M_t / Z_{t-})$. Then, L^n is a nonnegative local martingale and $L^{n+1} = L^n$ holds on $[\![0, \tau_n]\!]$ for all $n \in \mathbb{N}$. As $(L^n_{\tau^n})$ is a discrete-time nonnegative local martingale, $L_{\tau^\infty} := \lim_{n\to\infty} L^n_{\tau^n}$ $\mathbb{P}$-a.s. exists in $\mathbb{R}_+$. It follows that we can define a process L such that $L = L^n$ on $[\![0, \tau^n]\!]$ for each $n \in \mathbb{N}$ and $L = L_{\tau^\infty}$ on $[\![\tau^\infty, \infty[\![$. Note that $L = \int_{(0,\cdot]} \mathbb{I}_{\Theta_t} \mathrm{d}L_t = 1 + \int_{[0,\cdot]} \mathbb{I}_{\Theta_t} (L_{t-}/Z_{t-}) \mathrm{d}M_t$. By the Ansel-Stricker theorem (see [2]), L, being a nonnegative process that is the stochastic integral of the martingale M, is a local martingale. As $\Theta \subseteq \{K_- < 1\}$, $L = \int_{(0,\cdot]} \mathbb{I}_{\Theta_t} \mathrm{d}L_t$ implies that $L = \int_{[0,\cdot]} \mathbb{I}_{\{K_t<1\}} \mathrm{d}L_t$. Furthermore, since $[\![0, \tau^n[\![\subseteq \{L > 0\}$ and $\{L_{\tau^n} = 0\} = \{\Delta M_{\tau^n}/Z_{\tau^n -} = -1\} = \{Z_{\tau^n} + \Delta H_{\tau^n} = 0\} = \{Z_{\tau^n} = 0, \Delta H_{\tau^n} = 0\} = \{\tau^n = \tau^\infty, \Delta H_{\tau^\infty} = 0\}$ holds for all $n \in \mathbb{N}$, $K = \int_{[0,\cdot]} \mathbb{I}_{\Theta_t} \mathrm{d}K_t$ implies $K = \int_{[0,\cdot]} \mathbb{I}_{\{L_t>0\}} \mathrm{d}K_t$.

With the above definitions, we shall establish that $L(1 - K) = Z$. This result has already been obtained in a special case; we shall utilize an approximation argument to show that it holds in general. For any $\epsilon \in \mathbb{R}_+$, define the adapted, nonnegative, nondecreasing and right-continuous process H^ϵ via $H^\epsilon_t = (H_t + \epsilon(1 - \exp(-t)))/(1 + \epsilon)$ for $t \in \mathbb{R}_+$. Then, for all $\epsilon \in \mathbb{R}_{++}$, $\mathbb{E}[H^\epsilon_\infty] = 1$, as well as $\mathbb{P}[H^\epsilon_t < H^\epsilon_\infty \mid \mathcal{F}_t] = 1$ holds for all $t \in \mathbb{R}_+$. Let M^ϵ, Z^ϵ, K^ϵ and L^ϵ be the equivalents of the processes M, Z, K and L defined with H^ϵ in place of $H = H^0$. Then, $L^\epsilon(1 - K^\epsilon) = Z^\epsilon$ holds for all $\epsilon \in \mathbb{R}_{++}$. It is straightforward to check that $Z^\epsilon_t = (Z_t + \epsilon \exp(-t))/(1 + \epsilon)$, for all $t \in \mathbb{R}_+$; in particular, $|Z^\epsilon - Z| \leq \epsilon(1 + Z)/(1 + \epsilon)$. In view of $\mathbb{P}[\sup_{t\in\mathbb{R}_+} Z_t < \infty] = 1$, we obtain $\mathbb{P}[\lim_{\epsilon\downarrow 0} \sup_{t\in\mathbb{R}_+} |Z^\epsilon_t - Z_t| = 0] = 1$. We shall also show the corresponding convergence of K^ϵ to K and L^ϵ to L on every stochastic interval $[\![0, \tau^n]\!]$, $n \in \mathbb{N}$. Define a function $\lambda : \mathbb{R} \mapsto \mathbb{R}_+ \cup \{\infty\}$ via $\lambda(x) = x - \log(1 + x)$ for $x \in]-1, \infty[$ and $\lambda(x) = \infty$ for $x \in]-\infty, -1]$. Note that $0 \leq \lambda(ax) \leq \lambda(x)$ holds for all $x \in \mathbb{R}$ and $a \in [0, 1]$, which will be used in the limit theorems that will follow. Further, let μ^H be the *jump measure* of H, i.e., the random counting measure on $\mathbb{R}_+ \times \mathbb{R}$ defined via $\mu^H((0, \cdot] \times E) := \sum_{t\in(0,\cdot]} \mathbb{I}_{E\setminus\{0\}}(\Delta H_t)$ for $E \subseteq \mathbb{R}$. A use of (2.1), coupled with straightforward algebra, allows to write

$$
\begin{aligned}
1 - K^\epsilon_{\cdot\wedge\tau^n} = \frac{1 - H_0}{1 + \epsilon} \exp\left(- \int_{(0,\cdot\wedge\tau^n]} \frac{\mathrm{d}H_t}{Z_t + \Delta H_t + \epsilon \exp(-t)} - \epsilon \int_{(0,\cdot\wedge\tau^n]} \frac{\exp(-t)\mathrm{d}t}{Z_t + \Delta H_t + \epsilon \exp(-t)} \right) \\
\times \exp\left(- \int_{(0,\cdot\wedge\tau^n]\times\mathbb{R}} \lambda\left(\frac{x}{Z_t + \Delta H_t + \epsilon \exp(-t)} \right) \mu^H[\mathrm{d}t, \mathrm{d}x] \right).
\end{aligned}
$$

By straightforward applications of the monotone convergence theorem as $\epsilon \downarrow 0$ on the above equality, we obtain $\mathbb{P}[\lim_{\epsilon\downarrow 0} \sup_{t\in[0,\tau^n]} |K^\epsilon_t - K_t| = 0] = 1$ for all $n \in \mathbb{N}$. Furthermore, note that $M^\epsilon = (M + \epsilon)/(1 + \epsilon)$; therefore, L^ϵ is the stochastic exponential of

$$
\int_{(0,\cdot]} \frac{\mathrm{d}M^\epsilon_t}{Z^\epsilon_{t-}} = \int_{(0,\cdot]} \frac{\mathrm{d}M_t}{Z_{t-} + \epsilon \exp(-t)}
$$

Let ${}^{c}[M, M] := [M, M] - \sum_{t\in[0,\cdot]} |\Delta M_t|^2$ be the continuous part of the quadratic variation of M, and μ^M being the jump measure of M defined as μ^H before with "H" replaced by "M" throughout.

Using the definition of the stochastic exponential, we obtain

$$L^{\epsilon}_{\cdot\wedge\tau^n} = \exp\left(\int_{(0,\cdot\wedge\tau^n]} \frac{\mathrm{d}M_t}{Z_{t-} + \epsilon\exp(-t)} - \frac{1}{2}\int_{(0,\cdot\wedge\tau^n]} \frac{\mathrm{d}[M,M]_t}{|Z_{t-} + \epsilon\exp(-t)|^2}\right)$$
$$\times \exp\left(-\int_{\mathbb{R}\times(0,\cdot\wedge\tau^n]} \lambda\left(\frac{x}{Z_{t-} + \epsilon\exp(-t)}\right)\mu^M[\mathrm{d}x, \mathrm{d}t]\right)$$

The dominated theorem for stochastic integrals and the monotone convergence theorem for ordinary Lebesgue integrals give $\mathbb{P}[\lim_{\epsilon\downarrow 0}\sup_{t\in[0,\tau^n]}|L^{\epsilon}_t - L_t| = 0] = 1$ for all $n \in \mathbb{N}$. It follows that $L(1-K) = Z$ holds on $\Theta = \bigcup_{n\in\mathbb{N}}[\![0,\tau^n]\!]$. As $L = \int_{[0,\cdot]}\mathbb{I}_{\Theta_t}\mathrm{d}L_t$, $K = \int_{[0,\cdot]}\mathbb{I}_{\Theta_t}\mathrm{d}K_t$ and $Z = \int_{[0,\cdot]}\mathbb{I}_{\Theta_t}\mathrm{d}Z_t$, we obtain that $L(1-K) = Z$ identically holds.

We have thus established that properties (1), (2), (3) and (4) of Theorem 2.1 are satisfied by the pair (L,K) that was constructed. Since $L(1-K) = Z$ and $Z_\infty = 0$, the set-inclusion $\{L_\infty > 0\} \subseteq \{K_\infty = 1\}$ is apparent.

Remark 2.4. When H has continuous paths, K has continuous paths as well — in particular, K is predictable. The formula $Z = L(1-K)$ then implies that L coincides with the local martingale that appears in the multiplicative decomposition of the nonnegative supermartingale Z.

Example 2.5. On $(\Omega, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$, let L be any nonnegative local martingale with $L_0 = 1$, $\Delta L \leq 0$ and $L_\infty = 0$. Define $L^* = \max_{t\in[0,\cdot]} L_t$; since $\Delta L \leq 0$, L^* is continuous. Define also the nonnegative, nondecreasing, continuous and adapted process $H := \log(L^*)$, as well as p via $\int_{\Omega\times\mathbb{R}_+} V\mathrm{d}p = \mathbb{E}\left[\int_{\mathbb{R}_+} V_t\mathrm{d}H_t\right]$ for all nonnegative optional process V. It is well-known that $H_\infty = \log(L^*_\infty)$ has the standard exponential distribution (see also (2.6) later on); therefore, $\mathbb{P}[H_\infty > 1] > 0$, $\mathbb{E}[H_\infty] = 1$, and p is a unit-measure optional measure. Define $K := 1 - 1/L^*$, which is continuous, adapted, nondecreasing and satisfies $0 \leq K < 1$. Then,

$$\int_{[0,\cdot]} L_t\mathrm{d}K_t = \int_{[0,\cdot]} \frac{L_t}{|L^*_t|^2}\mathrm{d}L^*_t = \int_{[0,\cdot]} \frac{1}{L^*_t}\mathrm{d}L^*_t = \log(L^*) = H,$$

where the second equality follows from the fact that the random measure on $\mathbb{R}_+$ that is generated by the nondecreasing continuous process L^* is carried by the random set $\{t \in \mathbb{R}_+ \mid L_t = L^*_t\}$. It follows that (L,K) is actually a canonical representation pair for p. Of course, it may happen that L is a strict local martingale; for example, L could be the reciprocal of a 3-dimensional Bessel process starting from one. Observe also that $\mathbb{P}[K_\infty < 1] = \mathbb{P}[L^*_\infty < \infty] = 1$.

2.4. **Numéraire-invariant preferences on consumption streams.** Define Π to be the class of measures on $(\Omega \times \mathbb{R}_+, \mathcal{O})$ with unit mass that are equivalent to some representative $\overline{p} \in \Pi$. Then, let $\mathcal{I}$ be the class of all adapted, right-continuous, nonnegative and nondecreasing processes F satisfying the following property: if $A \in \mathcal{O}$ is Π-null, $\int_{[0,\cdot]}\mathbb{I}_{A_t}\mathrm{d}F_t$ is an evanescent process. The processes in $\mathcal{I}$ model all cumulative consumption streams that an agent could potentially choose from; if $A \in \mathcal{O}$ is Π-null, the agent gives no consumption value on A, and therefore will not consume there. The following result gives a convenient characterization of the set $\mathcal{I}$.

Proposition 2.6. *Fix $p \in \Pi$ with canonical representation pair (L, K). Then, $\mathcal{I}$ is the class of all finite processes $\int_{[0,\cdot]} a_t \mathrm{d}K_t$, where a ranges though the nonnegative optional processes.*

Proof. Let $p \in \Pi$, and let (L, K) be a pair of adapted càdlàg processes satisfying properties (1), (2), (3) and (4) of Theorem 2.1. Let also $H := \int_{[0,\cdot]} L_t \mathrm{d}K_t$, so that $\int_{\Omega \times \mathbb{R}_+} V \mathrm{d}p = \mathbb{E}\left[\int_{\mathbb{R}_+} V_t \mathrm{d}H_t\right]$ holds for all nonnegative optional process V. For $A \in \mathcal{O}$, $p[A] = 0$ if and only if $\int_{[0,\cdot]} \mathbb{I}_{A_t} \mathrm{d}H_t$ is evanescent.

By Theorem V.5.14 of [11], for all $F \in \mathcal{I}$ there exists an nonnegative optional process b such that $F = \int_{[0,\cdot]} b_t \mathrm{d}H_t$. Letting $a := bL$, we have $F = \int_{[0,\cdot]} a_t \mathrm{d}K_t$.

Now, let $A \in \mathcal{O}$. We have $p[A] = 0$ if and only if $\int_{[0,\cdot]} \mathbb{I}_{A_t} L_t \mathrm{d}K_t$ is evanescent. As L is a nonnegative local martingale, this is equivalent to saying that $\int_{[0,\cdot]} \mathbb{I}_{A_t} \mathbb{I}_{\{L_t > 0\}} \mathrm{d}K_t$ is evanescent. Since $K = \int_{[0,\cdot]} \mathbb{I}_{\{L_t > 0\}} \mathrm{d}K_t$, this is further equivalent to saying that $\int_{[0,\cdot]} \mathbb{I}_{A_t} \mathrm{d}K_t$ is evanescent. To recapitulate, $A \in \mathcal{O}$ is Π-null if and only if $\int_{[0,\cdot]} \mathbb{I}_{A_t} \mathrm{d}K_t$ is evanescent. We then have $K \in \mathcal{I}$, and therefore $\int_{[0,\cdot]} a_t \mathrm{d}K_t$ also belongs to $\mathcal{I}$ for each nonnegative optional process a such that the last integral is non-exploding in finite time. This completes the argument. □

Remark 2.7. The essential uniqueness of a canonical representation pair (L, K) for $p \in \Pi$, which has not been established yet, was *not* used in the proof above. Just the existence of a pair (L, K) that satisfies properties (1), (2), (3) and (4) of Theorem 2.1 was utilized, which was shown in §2.3.

In view of the previous result, for $p \in \Pi$, and with (L, K) a canonical representation pair for p, each $F \in \mathcal{I}$ can be written as $F = \int_{[0,\cdot]} \partial_t^{F|K} \mathrm{d}K_t$. Then, for $F \in \mathcal{I}$ and $G \in \mathcal{I}$ we define

$$\frac{\mathrm{d}F}{\mathrm{d}G} := \frac{\partial^{F|K}}{\partial^{G|K}}, \tag{2.2}$$

where once again we are using the conventions on division discussed in the first paragraph of §1.1. If $p' \in \Pi$ has canonical representation pair (L', K'), then, since $K \in \mathcal{I}$ and $K' \in \mathcal{I}$, we have $\partial^{K'|K} > 0$ and $\partial^{K|K'} > 0$ holding Π-a.e., as well as $\partial^{F|K'} = \partial^{F|K} \partial^{K|K'}$, Π-a.e., for all $F \in \mathcal{I}$. Therefore, the definition of $\partial F / \partial G$ in (2.2) does not depend on the choice of $p \in \Pi$.

For $p \in \Pi$ with canonical representation pair (L, K), and all $F \in \mathcal{I}$ and $G \in \mathcal{I}$, we define

$$\mathsf{rel}_p(F \,|\, G) := \int_{\Omega \times \mathbb{R}_+} \left(\frac{\mathrm{d}F}{\mathrm{d}G}\right) \mathrm{d}p - 1 = \mathbb{E}\left[\int_{\mathbb{R}_+} \left(\frac{\partial_t^{F|K}}{\partial_t^{G|K}}\right) L_t \mathrm{d}K_t\right] - 1. \tag{2.3}$$

and the corresponding preference relation $\preccurlyeq_p$ on $\mathcal{I}$ via $F \preccurlyeq_p G \Longleftrightarrow \mathsf{rel}_p(F \,|\, G) \leq 0$ for all $F \in \mathcal{I}$ and $G \in \mathcal{I}$.

Such preference relations can be seen to stem from axiomatic foundations, just as in the static case that is presented in Theorem 1.5. Since the details of such generalization are straightforward, we shall not delve into them here. Rather, we shall focus on novel features appearing in a dynamic environment.

Remark 2.8. Recall the discussion in Remark 2.3. Let $p \in \Pi$ with canonical representation pair (L, K), and suppose that L is the density process of a probability $\mathbb{Q}$ with respect to $\mathbb{P}$. Then,

$\mathbb{Q}[K_\infty = 1] = 1$, and

$$\mathsf{rel}_p(F \,|\, G) = \mathbb{E}_{\mathbb{Q}}\left[\int_{\mathbb{R}_+} \left(\frac{\mathrm{d}F_t - \mathrm{d}G_t}{\mathrm{d}G_t}\right) \mathrm{d}K_t\right] = \mathbb{E}_{\mathbb{Q}}\left[\int_{\mathbb{R}_+} \left(\frac{\partial_t^{F|K} - \partial_t^{G|K}}{\partial_t^{G|K}}\right) \mathrm{d}K_t\right]$$

holds for all $F \in \mathcal{I}$ and $G \in \mathcal{I}$. We interpret $\mathbb{Q}$ as the *subjective views* of an agent and K as the agent's *consumption clock*. As was described in Example 2.5, L might fail to be the density process of a probability $\mathbb{Q}$ with respect to $\mathbb{P}$, and $\mathbb{P}[K_\infty = 1] = 1$ might fail. We still "loosely" interpret L as subjective views and K as consumption clock.

2.5. **The investment-consumption problem.** The canonical representation pair for an optional measure with unit mass allows for a very satisfactory solution to an agent's investment-consumption problem.

2.5.1. *Pure investment.* Henceforth, $S = (S^i)_{i=1,\dots,d}$ will be a vector-valued semimartingale. For each $i \in \{1, \dots, d\}$, S^i should be thought as representing the discounted, with respect to some baseline security, price of a liquid asset traded in the market, satisfying $S^i > 0$ and $S^i_- > 0$.

Consider a set-valued process $\mathfrak{K} : \Omega \times \mathbb{R}_+ \mapsto 2^{\mathbb{R}^d} \setminus \{\emptyset\}$, where $2^{\mathbb{R}^d}$ denotes the powerset of $\mathbb{R}^d$, which will represent constraints imposed on the agent on the percentage of capital-at-hand invested in the liquid assets. The last set-valued process is assumed to satisfy some natural properties; namely, $\mathfrak{K}(\omega, t)$ is convex and closed for all $(\omega, t) \in \Omega \times \mathbb{R}_+$, $\mathfrak{K}$ is predictable, in the sense that the set $\{(\omega, t) \in \Omega \times \mathbb{R}_+ \mid \mathfrak{K}(\omega, t) \cap A \neq \emptyset\}$ is predictable for all closed $A \subseteq \mathbb{R}^d$, and finally $\mathfrak{K}$ large enough as to contain all investments that produce zero wealth. Under a simple non-redundancy condition on the liquid assets, the last requirement simply reads $0 \in \mathfrak{K}(\omega, t)$ for all $(\omega, t) \in \Omega \times \mathbb{R}_+$. More precise information about these requirements can be found in [13].

Starting with capital $x \in \mathbb{R}_+$, and investing according to some d-dimensional, predictable strategy θ representing the number of liquid assets held in the portfolio, an economic agent's discounted wealth is given by

$$X^{x,\theta} = x + \int_{[0,\cdot]} \langle \theta_t, \mathrm{d}S_t \rangle \tag{2.4}$$

We define

$$\mathcal{X}(x) := \left\{ X^{x,\theta} \mid X^{x,\theta} \text{ is defined in (2.4)}, \ X^{x,\theta} \geq 0, \text{ and } \left\{(\theta^i S^i_-)_{i=1,\dots,d} \in X^{x,\theta}_- \mathfrak{K}\right\} \text{ is } \Pi\text{-full.} \right\}$$

The elements of $\mathcal{X}(x)$ are *pure-investment* outcomes, starting with initial capital $x \in \mathbb{R}_+$. We also set $\mathcal{X} = \bigcup_{x \in \mathbb{R}_+} \mathcal{X}(x)$. The next result regards the viability of the market. Its validity follows from Theorem 4.12 in [13] coupled with a localization argument; its straightforward proof is omitted.

Theorem 2.9. *With the above notation, the following two conditions are equivalent:*

1. *For all $t \in \mathbb{R}_+$, the set $\{X_t \mid X \in \mathcal{X}(1)\} \subseteq \mathbb{L}^0_+$ is bounded.*
2. *For any nonnegative local martingale L with $L_0 = 1$, there exists $\widehat{X}^L \in \mathcal{X}(1)$ such that*
   - (a) *$L(X/\widehat{X}^L)$ is a supermartingale for all $X \in \mathcal{X}$.*
   - (b) *$\int_{(0,\cdot]} \mathbb{I}_{\{L_{t-}=0\}} \mathrm{d}\widehat{X}^L_t$ is an evanescent process.*

With the above specifications, $\widehat{X}^L$ is unique up to indistinguishability.

Under any of the above equivalent conditions, we have $\mathcal{X}(0) = \{0\}$.

Remark 2.10. In the spirit and notation of the discussion of Remark 2.8, and if L is the density process of a probability $\mathbb{Q}$ with respect to $\mathbb{P}$, the process $\widehat{X}^L$ of Theorem 2.9 above is simply the *numéraire portfolio* under $\mathbb{Q}$ — see [16], [3], [13]. According to Theorem 2.9, the equivalent of the numéraire portfolio when the "views" of the agent are given by L exists even in cases where L is a strict local martingale and does not stem from a change of probability.

2.5.2. *Investment and consumption.* We now introduce agent's consumption. For $x \in \mathbb{R}_+$, a *consumption stream* $C \in \mathcal{I}$ will be called *financeable* starting from capital $x \in \mathbb{R}_+$ if there exists a predictable, d-dimensional and S-integrable η with the property that $X^{x,\eta,C} := X^{x,\eta} - C$ is such that $X^{x,\eta,C} \geq 0$ and $\big\{(\eta^i S^i_-)_{i=1,\ldots,d} \in X^{x,\eta,C}_- \mathfrak{K}\big\}$ is Π-full. The class of all consumption streams that can be financed starting from $x \in \mathbb{R}_+$ will be denoted by $\mathcal{C}(x)$. It is straightforward that $\mathcal{C}(x) = x\mathcal{C}(1)$ for $x \in \mathbb{R}_{++}$. Furthermore, under any of the equivalent conditions of Theorem 2.9, $\mathcal{C}(0) = \{0\}$ holds.

For the solution to the agent's optimal investment-consumption problem that will be presented in Theorem 2.11 below, a "multiplicative" representation for elements of $\mathcal{C}(x)$, $x \in \mathbb{R}_+$ will turn out to be more appropriate. To begin with, let $\mathcal{I}(1)$ be the set of all $F \in \mathcal{I}$ with $F_\infty \leq 1$; observe that $\mathcal{I}(1)$ corresponds to the set $\mathcal{C}(1)$ if $S = 0$, i.e., if there are no investment opportunities. For $x \in \mathbb{R}_{++}$, let $C \in \mathcal{C}(x)$, and let η be a strategy that finances C. Then, we can write $X^{x,\eta,C} = X^{x,\theta}(1 - F)$, where $F \in \mathcal{I}(1)$ formally satisfies $\mathrm{d}F_t/(1 - F_t) = \mathrm{d}C_t/X^{x,\theta,C}_t$ (in other words, $\mathrm{d}F_t/(1 - F_t)$ is the rate of consumption relative to the capital-at-hand), and $\theta := (1/(1 - F_-))\eta$. Note also that $\big\{(\theta^i S^i_-)_{i=1,\ldots,d} \in X^{x,\theta}_- \mathfrak{K}\big\} = \big\{(\eta^i S^i_-)_{i=1,\ldots,d} \in X^{x,\eta,C}_- \mathfrak{K}\big\}$, which is Π-full, and therefore $X^{x,\theta} \in \mathcal{X}(x)$. Conversely, start with $X^{x,\theta} \in \mathcal{X}(x)$ and $F \in \mathcal{I}(1)$ and define $C := \int_{[0,\cdot]} X^{x,\theta}_t \mathrm{d}F_t$ and $\eta := (1 - F_-)\theta$. Then, $X^{x,\eta,C} = X^{x,\theta}(1 - F)$ and $\big\{(\eta^i S^i_-)_{i=1,\ldots,d} \in X^{x,\eta,C}_- \mathfrak{K}\big\} = \big\{(\theta^i S^i_-)_{i=1,\ldots,d} \in X^{x,\theta}_- \mathfrak{K}\big\}$, which is Π-full. Under any of the equivalent conditions of Theorem 2.9, since $\mathcal{X}(0) = \{0\} = \mathcal{C}(0)$, an alternative equivalent description the class of financeable consumption streams starting from capital $x \in \mathbb{R}_+$ is

$$\mathcal{C}(x) = \left\{ \int_0^\cdot X_t \mathrm{d}F_t \ \Big| \ X \in \mathcal{X}(x) \text{ and } F \in \mathcal{I}(1) \right\} \tag{2.5}$$

Theorem 2.11. *Let $p \in \Pi$ with canonical representation pair (L, K). Assume any of the equivalent conditions of Theorem 2.9, and let $\widehat{X}^L \in \mathcal{X}(1)$ be defined as in the latter result. Fix $x \in \mathbb{R}_+$ and define $\mathcal{C}(x)$ via (2.5). Then, with $\widehat{C} := x \int_{[0,\cdot]} \widehat{X}^L_t \mathrm{d}K_t \in \mathcal{C}(x)$, $C \preccurlyeq_p \widehat{C}$ holds for all $C \in \mathcal{C}(x)$.*

Proof. For $x \in \mathbb{R}_{++}$, fix $X \in X(x)$ and $F \in \mathcal{I}(1)$ and let $C = \int_0^\cdot X_t \mathrm{d}F_t = \int_0^\cdot X_t \partial^{F|K}_t \mathrm{d}K_t$ Let $N := (1/x)L(X/\widehat{X}^L)$. Then, recalling that $\widehat{C} := x \int_{[0,\cdot]} \widehat{X}^L_t \mathrm{d}K_t$, we have

$$\mathsf{rel}_p(C \mid \widehat{C}) = \mathbb{E}\left[\int_{\mathbb{R}_+} \frac{X_t \partial^{F|K}_t}{x \widehat{X}^L_t} L_t \mathrm{d}K_t\right] - 1 = \mathbb{E}\left[\int_{\mathbb{R}_+} N_t \mathrm{d}F_t\right] - 1.$$

For any finite stopping time τ, and in view of $N_0 = 1$, one has

$$\int_{[0,\tau]} N_t \mathrm{d}F_t - 1 = N_\tau F_\tau - N_0 - \int_{[0,\tau]} F_{t-} \mathrm{d}N_t \leq N_\tau - N_0 - \int_{[0,\tau]} F_{t-} \mathrm{d}N_t = \int_{(0,\tau]} (1 - F_{t-}) \mathrm{d}N_t.$$

Pick an increasing sequence $(\tau^n)_{n\in\mathbb{N}}$ of stopping times that $\mathbb{P}$-a.s. converges to infinity and is such that $\mathbb{E}\big[\sup_{t\in[0,\tau^n]} N_t\big] < \infty$ for all $n \in \mathbb{N}$. Then, $\mathbb{E}\big[\int_{(0,\tau^n]}(1 - F_{t-})\mathrm{d}N_t\big] \leq 0$ hold for all $n \in \mathbb{N}$, because N is a nonnegative supermartingale and $0 \leq F \leq 1$. Therefore,

$$\mathsf{rel}_p(C \,|\, \widehat{C}) = \lim_{n\to\infty} \mathbb{E}\left[\int_{[0,\tau^n]} N_t \mathrm{d}F_t\right] - 1 \leq \limsup_{n\to\infty} \mathbb{E}\left[\int_{(0,\tau^n]} (1 - F_{t-}) \mathrm{d}N_t\right] \leq 0,$$

which finishes the proof. □

The result of Theorem 2.11 describes how an agent with numéraire-invariant preferences generated by p will dynamically invest and consume in an optimal manner. The canonical representation pair (L, K) of p conveniently separates the investment and consumption problems. The optimal strategy, when described in proportions of wealth invested in the assets, is completely characterized by L; indeed, these proportions will be the same as the ones held in the portfolio that results in the pure-investment wealth $\widehat{X}^L$. On the other hand, the optimal consumption in an infinitesimal interval around $t \in \mathbb{R}_+$ relative to the capital-at-hand is $\mathrm{d}K_t/(1 - K_t)$, which solely depends on K.

As can be seen from its proof, the validity of Theorem 2.9 goes well-beyond the framework of investing in a market with certain finite number of liquid assets. All that is needed is a class of nonnegative "wealth" processes $(\mathcal{X}(x))_{x\in\mathbb{R}_+}$ with $\mathcal{X}(x) = x\mathcal{X}(1)$ for $x \in \mathbb{R}_+$, such that statement (2) of Theorem 2.9 holds; in other words, the crucial element is the existence of a numéraire portfolio under the "local change in probability" with the local martingale L acting as a "density process". The computational advantage of assuming a semimartingale S that generates the wealth processes is that the process $\widehat{X}^L$ appearing in Theorem 2.9 and Theorem 2.11 can be completely described by the use of the triplet of predictable characteristics (see [12]) of the $(1 + d)$-dimensional process (L, S). The formulas appear in [10], where the closely-related problem of log-utility consumption maximization under a random clock is treated. Nevertheless, in the latter paper, the authors did not utilize the canonical representation pair in the solution; for this reason, unless the consumption clock is deterministic, it is not apparent that the two aspects of investment and consumption can be separated, as was previously pointed out.

Remark 2.12. Theorem 2.11 solves in particular the *pure consumption problem.* Assume that an agent stats with a unit of account, has no access in a market, and needs to choose how this unit of account will be consumed throughout time. This is modeled by setting $\mathcal{C}(1) = \mathcal{I}(1)$. Let $p \in \Pi$ with (L, K) be its canonical representation pair. Then, $F \preccurlyeq_p K$ holds for all $F \in \mathcal{I}(1)$. Note that the optimal solution does *not* depend on L, in par with the discussion that followed Theorem 2.11.

In fact, the same consumption stream K solves the optimization problem for more general preference structures. Let $U : \mathbb{R}_{++} \mapsto \mathbb{R}$ be a concave and nondecreasing function, and extend the definition of U by setting $U(0) = \lim_{x \downarrow 0} U(x)$. Consider a preference structure on $\mathcal{I}(1)$ with numerical representation given via the utility functional

$$\mathcal{I}(1) \ni F \mapsto \mathbb{U}(F) = \int_{\Omega \times \mathbb{R}_+} U\left(\partial^{F|K}\right) \mathrm{d}p = \mathbb{E}\left[\int_{\mathbb{R}_+} U\left(\partial_t^{F|K}\right) L_t \mathrm{d}K_t\right],$$

where we shall soon see that the above integrals are well-defined, in the sense that the positive part of the integrand is integrable. Let $(\tau^n)_{n \in \mathbb{N}}$ ba a localizing sequence such that $\mathbb{E}\left[\sup_{t \in [0,\tau^n]} L_t\right] < \infty$ for all $n \in \mathbb{N}$. Since

$$\begin{aligned} \int_{\Omega \times \mathbb{R}_+} \partial^{F|K} \mathrm{d}p &= \mathbb{E}\left[\int_{\mathbb{R}_+} L_t \mathrm{d}F_t\right] = \lim_{n \to \infty} \mathbb{E}\left[\int_{[0,\tau^n]} L_t \mathrm{d}F_t\right] \\ &= \lim_{n \to \infty} \mathbb{E}\left[L_{\tau^n} F_{\tau^n} - \int_{[0,\tau^n]} F_{t-} \mathrm{d}L_t\right] \leq 1, \end{aligned}$$

Jensen's inequality gives $\mathbb{U}(F) \leq U\left(\int_{\Omega \times \mathbb{R}_+} \partial^{F|K} \mathrm{d}p\right) \leq U(1) = \mathbb{U}(K)$. Therefore, K is the optimal consumption plan.

2.6. **Essential uniqueness of a canonical representation pair in Theorem 2.1.** Let $p \in \Pi$, and let (L, K) and (L', K') be two pairs of processes having the properties (1), (2), (3) and (4) in Theorem 2.1. The equality $\int_{[0,\cdot]} L_t \mathrm{d}K_t = \int_{[0,\cdot]} L'_t \mathrm{d}K'_t$ holds due to the uniqueness of Doléans' representation of p.

Since $K \in \mathcal{I}(1)$ and $K' \in \mathcal{I}(1)$, Theorem 2.11 implies that $K \preccurlyeq_p K'$ and $K' \preccurlyeq_p K$. (In view of Remark 2.7, the result of Theorem 2.11 does not assume uniqueness of canonical representation pairs; therefore, there is no cyclic argument.) It follows that $\partial^{K'|K} = 1$ holds Π-a.e., or, in other words, that $K = K'$ in the sense that K and K' are indistinguishable.

Since $K = K'$, the equality $\int_{[0,\cdot]} L_t \mathrm{d}K_t = \int_{[0,\cdot]} L'_t \mathrm{d}K'_t$ translates to $KL - \int_{[0,\cdot]} K_{t-} \mathrm{d}L_t = KL' - \int_{[0,\cdot]} K_{t-} \mathrm{d}L'_t$. Let $(\tau^n)_{n \in \mathbb{N}}$ be a nondecreasing sequence of stopping times such that, $\mathbb{P}$-a.s., $\uparrow \lim_{n \to \infty} \tau^n = \infty$, as well as $\mathbb{E}\big[\sup_{t \in [0,\tau^n]} L_t\big] < \infty$ and $\mathbb{E}\big[\sup_{t \in [0,\tau^n]} L'_t\big] < \infty$ holds for all $n \in \mathbb{N}$. Then, $\mathbb{E}[K_{\tau \wedge \tau^n} L_{\tau \wedge \tau^n}] = \mathbb{E}[K_{\tau \wedge \tau^n} L'_{\tau \wedge \tau^n}]$ holds for all $n \in \mathbb{N}$ and stopping times τ. Since L, L' and K are all adapted càdlàg processes, it follows that KL and KL' are indistinguishable. This, coupled with the fact that L and L' are both local martingales, gives $\{K_\infty > 0\} \subseteq \{L_t = L'_t, \ \forall t \in \mathbb{R}_+\}$.

2.7. **A random time-horizon investment problem.** We retain all the notation from §2.5.1 for the market description and the investment sets. We shall also be assuming throughout that the market satisfies the viability requirement of Theorem 2.9. In particular, recall the notation $\widehat{X}^L \in \mathcal{X}(1)$ from the last result. We are interested in characterizing the equivalent of the numéraire portfolio under $\mathbb{P}$, sampled at a random, *not necessarily stopping*, time. Here, by a *random time* we simply mean a $\mathbb{R}_+$-valued, $\mathcal{F}_\infty$-measurable random variable T.

Theorem 2.13. *For any random time T, define the measure $p = p^T$ on $(\Omega \times \mathbb{R}_+, \mathcal{O})$ via $\int_{\Omega \times \mathbb{R}_+} V \mathrm{d}p = \mathbb{E}[V_T]$ for all nonnegative optional process V. Since $p[\Omega \times \mathbb{R}_+] = 1$ and $p[A] = 0$ holds for all evanescent $A \in \mathcal{O}$, let (L, K) be the canonical representation pair for p. Then, $\mathbb{E}\left[X_T / \widehat{X}^L_T\right] \leq X_0 / \widehat{X}^L_0 = X_0$ holds for all $X \in \mathcal{X}$.*

Proof. For $X \in \mathcal{X}(1)$, define $C := \int_{[0,\cdot]} X_t \mathrm{d}K_t$. Define also $\widehat{C} := \int_{[0,\cdot]} \widehat{X}^L_t \mathrm{d}K_t$. Then, $C \in \mathcal{C}(1)$, $\widehat{C} \in \mathcal{C}(1)$ and $\mathbb{E}\left[\int_{\mathbb{R}_+} \left(\partial_t^{C|K} / \partial_t^{\widehat{C}|K}\right) L_t \mathrm{d}K_t\right] \leq 1$. Therefore,

$$\mathbb{E}\left[\frac{X_T}{\widehat{X}^L_T}\right] = \int_{\Omega \times \mathbb{R}_+} \left(\frac{X}{\widehat{X}^L}\right) \mathrm{d}p = \mathbb{E}\left[\int_{\mathbb{R}_+} \left(\frac{X_t}{\widehat{X}^L_t}\right) L_t \mathrm{d}K_t\right] = \mathbb{E}\left[\int_{\mathbb{R}_+} \left(\frac{\partial_t^{C|K}}{\partial_t^{\widehat{C}|K}}\right) L_t \mathrm{d}K_t\right] \leq 1.$$

The result follows by simply noting that $\mathcal{X}(x) = x\mathcal{X}(1)$ holds for all $x \in \mathbb{R}_+$. □

The next result is a partial converse to Theorem 2.13, in the sense that the nonnegative local martingale L will be given and the random time T will be constructed from L. Recall that the *jump process* of a process L is defined via $\Delta L_t = L_t - L_{t-}$ for all $t \in \mathbb{R}_+$.

Theorem 2.14. *Let L be a nonnegative local martingale with $L_0 = 1$, $\Delta L \leq 0$ and $L_\infty = 0$. Let T be any random time with $L_T = \max_{t \in \mathbb{R}_+} L_t$. Then, $\mathbb{E}\left[X_T / \widehat{X}^L_T\right] \leq X_0 / \widehat{X}^L_0 = X_0$ holds for all $X \in \mathcal{X}$.*

Proof. The key to proving Theorem 2.14 is the following version of Doob's maximal identity, which can be found for example in Lemma 2.1 of [18]: for all finite stopping times τ and $\mathcal{F}_\tau$-measurable and nonnegative random variables γ, one has

$$\mathbb{P}\left[\sup_{t \in [\tau, \infty)} L_t > \gamma \;\Big|\; \mathcal{F}_\tau\right] = \left(\frac{L_\tau}{\gamma}\right) \wedge 1. \tag{2.6}$$

The assumption $\Delta L \leq 0$ implies that the nondecreasing process $L^* := \max_{t \in [0,\cdot]} L_t$ is continuous. Consider the random times $T_{\sup} := \sup\{t \in \mathbb{R}_+ \mid L_t = L^*_\infty\}$ and $T_{\inf} := \inf\{t \in \mathbb{R}_+ \mid L_t = L^*_\infty\}$. Obviously, $T_{\inf} \leq T \leq T_{\sup}$. A use of (2.6) gives that for any finite stopping time τ we have $\mathbb{P}\big[T_{\sup} > \tau \mid \mathcal{F}_\tau\big] = \mathbb{P}\big[\sup_{t \in [\tau,\infty)} L_t \geq L^*_\tau \mid \mathcal{F}_\tau\big] = L_\tau / L^*_\tau$, as well as the equality $\mathbb{P}\left[T_{\inf} > \tau \mid \mathcal{F}_\tau\right] = \mathbb{P}\big[\sup_{t \in [\tau,\infty)} L_t > L^*_\tau \mid \mathcal{F}_\tau\big] = L_\tau / L^*_\tau$.

Define the measure p^T on $(\Omega \times \mathbb{R}_+, \mathcal{O})$ via $\int_{\Omega \times \mathbb{R}_+} V \mathrm{d}p^T = \mathbb{E}[V_T] = \mathbb{E}\big[\int_{\mathbb{R}_+} V_t \mathrm{d}H_t\big]$ for nonnegative optional processes V, where H is the dual optional projection of the process $\mathbb{I}_{[\![T,\infty[\![}$. Let Z be the nonnegative supermartingale such that $Z_t = \mathbb{E}[H_\infty - H_t \mid \mathcal{F}_t] = \mathbb{P}[T > t \mid \mathcal{F}_t]$ holds for all $t \in \mathbb{R}_+$. Since $T_{\inf} \leq T \leq T_{\sup}$, it follows that $Z = L/L^*$. In the notation of Theorem 2.1, and according to Remark 2.4, L is the local martingale in the canonical representation pair of p^T. Then, it follows from Theorem 2.13 that $\mathbb{E}[X_T / \widehat{X}^L_T] \leq X_0$ for all $X \in \mathcal{X}$. □

Let S be a one-dimensional semimartingale that generates the wealth-process class $\mathcal{X}$. Assume that $S > 0$, $\Delta S \geq 0$, $1/S$ is a local martingale and $\lim_{t \to \infty} S_t = \infty$. Define $L = S_0/S$, and let T be any random time such that $S_T = \min_{t \in \mathbb{R}_+} S_t$, i.e., $L_T = \max_{t \in \mathbb{R}_+} L_t$. It is straightforward to see that $\widehat{X}^L = 1$ and $\widehat{X}^1 = S/S_0 = 1/L$. In view of Theorem 2.14, it follows that $\mathbb{E}[X_T] \leq X_0$ for all $X \in \mathcal{X}$. In words, at the random time of the overall minimum of S, which is the time of the overall minimum the numéraire portfolio, the whole market is at a downturn. We shall show below that the last fact is always true, regardless of whether S is a one-dimensional semimartingale with $1/S$ is a local martingale or not. The next result adds yet one more remarkable fact to the long list of optimality properties of the numéraire portfolio, with the loose interpretation of the numéraire portfolio being an index of market status.

Theorem 2.15. *Suppose that* $\widehat{X} \equiv \widehat{X}^1 \in \mathcal{X}(1)$ *is such that* $\Delta \widehat{X} \geq 0$ *and* $\lim_{t \to \infty} \widehat{X}_t = \infty$. *Let* T *be any random time such that* $\widehat{X}_T = \min_{t \in \mathbb{R}_+} \widehat{X}_t$. *Then,* $\mathbb{E}[X_T] \leq X_0$ *holds for all* $X \in \mathcal{X}$.

Proof. Let $L := 1/\widehat{X}$. Since $\widehat{X} \in \mathcal{X}(1)$, $L_0 = 1$. Also, $\Delta \widehat{X} \geq 0$ is equivalent to $\Delta L \leq 0$, as well as $\lim_{t \to \infty} \widehat{X}_t = \infty$ is equivalent to $\lim_{t \to \infty} L_t = 0$. Therefore, in view of Theorem 2.14, Theorem 2.15 will be proved as long as L is shown to be a nonnegative local martingale. Note that we already know that L is a supermartingale with $L > 0$ and $L_- > 0$, as follows by the definition of $\widehat{X}$.

Since both $\widehat{X}_- > 0$ and $\widehat{X} > 0$ hold, we have $\widehat{X} = 1 + \int_{(0,\cdot]} \widehat{X}_{t-} \langle \rho_t, \mathrm{d} S_t \rangle$ for some d-dimensional predictable and S-integrable process ρ. A straightforward application of Lemma 3.4 in [13] shows that $L = 1 - \int_{(0,\cdot]} L_{t-} \langle \rho_t, \mathrm{d} \widehat{S}_t \rangle$, where

$$\widehat{S} := S - \left[{}^{c}S, \int_{(0,\cdot]} \langle \rho_t, \mathrm{d}\, {}^{c}S_t \rangle \right] - \sum_{t \leq \cdot} \frac{\Delta \widehat{X}_t}{\widehat{X}_t} \Delta S_t,$$

with ${}^{c}S$ denoting the uniquely defined continuous local martingale part of S (see, for example, [12]). Since $L_- > 0$ and $L > 0$, L is a local martingale if and only if $\int_{(0,\cdot]} \langle \rho_t, \mathrm{d} \widehat{S}_t \rangle$ is a local martingale. The supermartingale property of L already gives that $\int_{(0,\cdot]} \langle \rho_t, \mathrm{d} \widehat{S}_t \rangle$ is a local submartingale. We shall show that $\int_{(0,\cdot]} \langle \rho_t, \mathrm{d} \widehat{S}_t \rangle$ is also a local supermartingale. Since $\langle 2\rho, \Delta S \rangle = 2(\Delta \widehat{X} / \widehat{X}_-) \geq 0$, the process X' defined implicitly via $X' = 1 + \int_{(0,\cdot]} X'_{t-} \langle 2\rho_t, \mathrm{d} S_t \rangle$ is an element of $\mathcal{X}$with $X' > 0$ and $X'_- > 0$. Therefore, $X'/\widehat{X}$ is a nonnegative supermartingale. Again, Lemma 3.4 in [13] shows that $X'/\widehat{X} = 1 + \int_{(0,\cdot]} (X'_{t-}/\widehat{X}_{t-}) \langle \rho_t, \mathrm{d} \widehat{S}_t \rangle$. The supermartingale property of $X'/\widehat{X}$ implies that $\int_{(0,\cdot]} \langle \rho_t, \mathrm{d} \widehat{S}_t \rangle$ is a local supermartingale. As $\int_{(0,\cdot]} \langle \rho_t, \mathrm{d} \widehat{S}_t \rangle$ is a local submartingale, we conclude that $\int_{(0,\cdot]} \langle \rho_t, \mathrm{d} \widehat{S}_t \rangle$ (and, therefore, L) is a local martingale. □

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