Limit theorems for quantum Markov chains

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In a quantum Markov chain, the temporal succession of states is modeled by the repeated action of a "bistochastic quantum operation" on the density matrix of a quantum system. Based on this conceptual framework, we derive some new results concerning the evolution of a quantum system, including its long-term behavior. Our treatment of this subject includes two theorems which serve to describe, in general, the limiting behavior of any quantum Markov chain governed by any bistochastic quantum operation. These results amount to substantial, if not complete, progress toward settling an important open question in this line of research. In particular, our findings encompass, as special cases, a variety of results published by previous authors.

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I. INTRODUCTION

The theory of Markov chains, when appropriately generalized, provides a potent paradigm for analyzing the stochastic evolution of quantum systems. Over the past decade, motivated largely by the prospect of super-efficient algorithms, the theory of so-called *quantum Markov chains*, especially in the guise of quantum walks, has generated a huge volume of research, including, for instance, numerous discoveries of fundamental importance [1–5].

In the context of quantum walks, the itinerary of the walker is confined to a particular topological network. The walker's every move, from node to adjacent node, is governed by a set of local rules. When applied repeatedly to a given initial state of the system (represented by a superposition of basis states), these rules yield a succession of new states, reflecting, ad infinitum, the evolution of the system. A transition rule can be either unitary (closed) or non-unitary (open), depending, respectively, on whether it is intrinsic to the system or exposes the system to external influences such as decoherence, noise or measurement.

In this paper, we adopt the formalism of "quantum operations", whereby both unitary and non-unitary rules of state transition, as well as various combinations thereof, are treated under a unified mathematical model. In this framework, the "transition matrix" of a classical Markov chain is replaced by a "bistochastic quantum operation" and the "state distribution vector" of the classical Markov chain is replaced by a "density matrix". The resulting description of quantum state evolution, known as a *quantum Markov chain* [6, 7], turns out to resemble very closely the evolution of a classical Markov chain. In particular, the same kind of questions pertain, including questions about the long-term evolution of the process and the possible existence of a limiting state.

Among our findings is the fact that the Cesàro limit of any quantum Markov chain converges always to a stationary "state", regardless of the initial state. As a noteworthy special case of this result, we remark that for any unitary quantum walk on a graph, as in [6], the limit of the time-averaged probability distribution always exists.

To complete the picture, we specify conditions for the existence of a limiting state in the strict (non-Cesàro) sense of the word "limiting". In the strict sense, it turns out that the limiting behavior depends only the disposition on the unit disc of the eigenvalues of the bistochastic quantum operation. Specifically, if $\lambda = 1$ is the only eigenvalue on the unit circle, then, for any given initial state, the associated quantum Markov chain converges to a stationary state. Moreover, if the eigenspace of $\lambda = 1$ is one-dimensional or contains only a single density operator, then the associated Markov chain converges to the maximally mixed state, irrespective of the initial state. Otherwise, if the bistochastic quantum operation possesses any eigenvalue on the unit circle other than $\lambda = 1$, then a limiting state is not guaranteed to exist except in the generalized sense of Cesàro. These findings are seen to be analogous, in a very natural way, to the fundamental properties of classical Markov chains (e.g. in [8], Chapter 8).

Our results represent substantial progress toward answering the first of a set of "open questions" posed by Ambanis in 2005 [7]. We would be remiss not to acknowledge our indebtedness to [9], which treats the special case of a quantum Markov chain generated by a *random unitary operation*.

In what follows, after a brief review of some preliminaries (Section II), we proceed (Section III) to present our main findings on the question of the limiting behavior of a quantum Markov chain, followed (Section IV) by a classification of quantum operations according to their limiting behavior. Finally, in Section V, we offer some concluding remarks, including some relevant questions

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for further investigation.

II. BISTOCHASTIC QUANTUM OPERATIONS

Given a Hilbert space $\mathcal{H}$ of finite dimension N, let $\mathfrak{B}(\mathcal{H})$ denote the set of all linear operators on $\mathcal{H}$, with inner product defined by

$$\langle X, Y\rangle \equiv \mathrm{tr}(X^\dagger Y). \tag{1}$$

The corresponding norm, called *Frobenius norm* or *Schatten* 2-*norm*, is defined by

$$\|X\| \equiv [\mathrm{tr}(X^\dagger X)]^{1/2} = \sqrt{\langle X, X\rangle}. \tag{2}$$

This choice of norm on $\mathfrak{B}(\mathcal{H})$ will remain in effect throughout this paper.

Let $\mathfrak{D}(\mathcal{H}) \subset \mathfrak{B}(\mathcal{H})$ denote the set of *positive* operators $\rho : \mathcal{H} \to \mathcal{H}$ with $\mathrm{Tr}(\rho) = 1$. The operators $\rho \in \mathfrak{D}(\mathcal{H})$ are the so-called "density operators". They serve to model, as faithfully as do the "state vectors" themselves, the possible states of a quantum system whose state vectors reside in $\mathcal{H}$.

By a *super-operator* $\mathbf{\Phi}$ on $\mathfrak{B}(\mathcal{H})$, we mean a linear mapping $\mathbf{\Phi} : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$, with norm defined by

$$\|\mathbf{\Phi}\| \equiv \mathrm{Sup}_{X\in\mathfrak{B}(\mathcal{H})} \frac{\|\mathbf{\Phi}(X)\|}{\|X\|}. \tag{3}$$

Note that $\dim\mathfrak{B}(\mathcal{H}) = N^2$, where $N = \dim(\mathcal{H})$. Thus, any super-operator $\mathbf{\Phi}$ on $\mathfrak{B}(\mathcal{H})$ can be represented, relative to a given basis for $\mathfrak{B}(\mathcal{H})$, by an $N^2 \times N^2$ matrix. In the sequel, this matrix will be denoted by the symbol $[\mathbf{\Phi}]$. In particular, relative to a special basis consisting of eigenvectors and generalized eigenvectors of $\mathbf{\Phi}$, the shape of the matrix $[\mathbf{\Phi}]$ conforms to a special quasi-diagonal lay-out called the Jordan canonical form. The details can be found in any one of a number of sources, including [9].

Among the set of super-operators, we distinguish a special subset called "quantum operations". By definition, to qualify as a quantum operation, the super-operator $\mathbf{\Phi}$ must be *completely positive*, meaning that the extended map $\mathbf{\Phi} \otimes \mathbb{I}_n$ is positive for all $n \geq 1$.

The formalism of quantum operations is flexible enough to handle both unitary (closed) and non-unitary (open), or a mixture thereof, of discrete transitions of state of a quantum system. For a good introductory exposition of this subject, see [10, 11].

By Choi's Theorem [12] and [10, 11, 13], any completely positive linear operator, including any quantum operation $\mathbf{\Phi} : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$, can be represented in terms of a set $\mathcal{A} = \{A_i \mid i = 1, 2, ..., N^2\}$ of "Kraus operators", as follows:

$$\mathbf{\Phi}_{\mathcal{A}}(X) = \sum_i A_i X A_i^\dagger. \tag{4}$$

In this expression, which we call the "Choi expansion" of $\mathbf{\Phi}$, the symbol $A_i^\dagger$ denotes $\bar{A}_i{}^T$ (transpose of the complex conjugate of A_i).

In terms of the Choi expansion, the condition of being *trace-preserving*, meaning that $\mathrm{Tr}(\mathbf{\Phi}_{\mathcal{A}}(X)) = \mathrm{Tr}(X)$ for all $X \in \mathfrak{B}(\mathcal{H})$, is equivalent to the condition:

$$\sum_i A_i^\dagger A_i = \mathbb{I}_N. \tag{5}$$

On the other hand, if the Kraus operators of $\mathbf{\Phi}_{\mathcal{A}}$ satisfy the dual condition:

$$\sum_i A_i A_i^\dagger = \mathbb{I}_N, \tag{6}$$

then $\mathbf{\Phi}_{\mathcal{A}}$ is said to be *unital*. Note that (6) is equivalent to the simple statement that $\mathbf{\Phi}_{\mathcal{A}}(\mathbb{I}_N) = \mathbb{I}_N$.

A quantum operation which is both unital and trace-preserving is called *bistochastic*. Note that a bistochastic quantum operation transforms elements of $\mathfrak{D}(\mathcal{H})$ into elements of $\mathfrak{D}(\mathcal{H})$. In other words, since the elements of $\mathfrak{D}(\mathcal{H})$ represent the states of a quantum system, a bistochastic quantum operation transforms states of a quantum system into other legitimate states of that system.

III. LIMIT THEOREMS FOR BISTOCHASTIC QUANTUM OPERATIONS

By [14], a bistochastic quantum operation $\mathbf{\Phi}_{\mathcal{A}}$ must satisfy the condition $\|\mathbf{\Phi}_{\mathcal{A}}\| \leq 1$. Thus the spectrum of $\mathbf{\Phi}_{\mathcal{A}}$ is confined to the unit disk. Also, since $\mathbf{\Phi}_{\mathcal{A}}(\mathbb{I}_N) = \mathbb{I}_N$, we see that $\lambda = 1$ is an eigenvalue of $\mathbf{\Phi}_{\mathcal{A}}$ and $\|\mathbf{\Phi}_{\mathcal{A}}\| = 1$. For future reference, we record these observations in the form of a lemma:

Lemma 1. Let $\mathbf{\Phi}_{\mathcal{A}}$ be a bistochastic quantum operation on the Hilbert space $\mathfrak{B}(\mathcal{H})$. Then

1. $\|\mathbf{\Phi}_{\mathcal{A}}\| = 1$
2. If λ is an eigenvalue of $\mathbf{\Phi}_{\mathcal{A}}$, then $|\lambda| \leq 1$.
3. The value $\lambda = 1$ is an eigenvalue of $\mathbf{\Phi}_{\mathcal{A}}$.

The observations recorded in Lemma 1 are new by no means. See, for instance, [15].

For an eigenvalue λ of $\mathbf{\Phi}_{\mathcal{A}}$, let $\mathsf{Ker}(\mathbf{\Phi}_{\mathcal{A}} - \lambda\mathbb{I})$ and $\mathsf{Ran}(\mathbf{\Phi}_{\mathcal{A}} - \lambda\mathbb{I})$ denote, respectively, the kernel and range of the operator $\mathbf{\Phi}_{\mathcal{A}} - \lambda\mathbb{I}$ on the Hilbert space $\mathfrak{B}(\mathcal{H})$.

Lemma 2. Let $\mathbf{\Phi}_{\mathcal{A}}$ be a bistochastic quantum operation on $\mathfrak{B}(\mathcal{H})$ and let λ be an eigenvalue of $\mathbf{\Phi}_{\mathcal{A}}$ with $|\lambda| = 1$. Then $\mathsf{Ker}(\mathbf{\Phi}_{\mathcal{A}} - \lambda\mathbb{I}) \cap \mathsf{Ran}(\mathbf{\Phi}_{\mathcal{A}} - \lambda\mathbb{I}) = \{0\}$.

Proof. Suppose $X \in \mathsf{Ker}(\mathbf{\Phi}_{\mathcal{A}} - \lambda\mathbb{I}) \cap \mathsf{Ran}(\mathbf{\Phi}_{\mathcal{A}} - \lambda\mathbb{I})$. Then $\mathbf{\Phi}_{\mathcal{A}}(X) = \lambda X$ and $\mathbf{\Phi}_{\mathcal{A}}(Y) - \lambda Y = X$ for some $Y \in \mathfrak{B}(\mathcal{H})$. Applying the linearity of $\mathbf{\Phi}_{\mathcal{A}}$, we infer that $\mathbf{\Phi}_{\mathcal{A}}^n(Y) = \lambda^n Y + n\lambda^{n-1}X$ for all $n \geq 1$. Consequently $\|n\lambda^{n-1}X\| = \|\mathbf{\Phi}_{\mathcal{A}}^n(Y) - \lambda^n Y\|$, which implies

that $n\|X\| \leq \|\mathbf{\Phi}_{\mathcal{A}}\|^n\|Y\|+\|Y\|$. But since $\|\mathbf{\Phi}_{\mathcal{A}}\| = 1$ (see Lemma 1), we have $n\|X\| \leq 2\|Y\|$. Since this inequality must hold for all $n \geq 1$, we conclude that $\|X\| = 0$, whence $X = 0$. End of proof.

In the above proof, we have borrowed liberally from the reasoning employed in [9], which treats the special case of a *random unitary operation*.

From the preceding lemma, we can derive an important inference concerning the algebraic and geometric multiplicities of the eigenvalues of $\mathbf{\Phi}_{\mathcal{A}}$. For an eigenvalue λ of $\mathbf{\Phi}_{\mathcal{A}}$, let its algebraic multiplicity be denoted by $m(\lambda)$ and lets its geometric multiplicity be denoted by $g(\lambda)$. Recall that $g(\lambda) = \mathrm{dim}\mathsf{Ker}(\mathbf{\Phi}_{\mathcal{A}} - \lambda\mathbb{I})$. In addition, let the spectrum of $\mathbf{\Phi}_{\mathcal{A}}$ be denoted by $\Lambda(\mathbf{\Phi}_{\mathcal{A}})$ and let $\Lambda_1(\mathbf{\Phi}_{\mathcal{A}})$ denote the subset of $\Lambda(\mathbf{\Phi}_{\mathcal{A}})$ consisting of $\lambda \in \Lambda(\mathbf{\Phi}_{\mathcal{A}})$ with $|\lambda| = 1$.

Lemma 3. If $\lambda \in \Lambda_1(\mathbf{\Phi}_{\mathcal{A}})$, where $\mathbf{\Phi}_{\mathcal{A}}$ is a bistochastic quantum operation on $\mathfrak{B}(\mathcal{H})$, then $m(\lambda) = g(\lambda)$.

Proof. We proceed by contradiction. Suppose $m(\lambda) \neq g(\lambda)$. Let $[\mathbf{\Phi}_{\mathcal{A}}]$ denote the $N^2 \times N^2$ matrix representation of $\mathbf{\Phi}_{\mathcal{A}}$ in Jordan canonical form. On the one hand, $[\mathbf{\Phi}_{\mathcal{A}}]$ must contain a Jordan block J belonging to λ of size > 1. On the other hand, by standard matrix theory, there must exist a generalized eigenvector, say v, of $\mathbf{\Phi}_{\mathcal{A}}$ such that $(\mathbf{\Phi}_{\mathcal{A}} - \lambda\mathbb{I})v$ is itself an eigenvector of $\mathbf{\Phi}_{\mathcal{A}}$. This implies that $\mathsf{Ker}(\mathbf{\Phi}_{\mathcal{A}} - \lambda\mathbb{I}) \cap \mathsf{Ran}(\mathbf{\Phi}_{\mathcal{A}} - \lambda\mathbb{I}) \neq \{0\}$, which contradicts Lemma 2.

As in the proof of Lemma 3, let $[\mathbf{\Phi}_{\mathcal{A}}]$ denote the $N^2 \times N^2$ Jordan canonical matrix representation of $\mathbf{\Phi}_{\mathcal{A}}$. To conserve type-set space, and without undue risk of confusion to the reader, we prefer to display the matrix $[\mathbf{\Phi}_{\mathcal{A}}]$ in the following self-explanatory format:

$$[\mathbf{\Phi}_{\mathcal{A}}] = \mathrm{diag}\left(\lambda_1, \lambda_2, ..., \lambda_k, J_1, J_2, ..., J_p\right), \tag{7}$$

where $\lambda_i \in \Lambda_1(\mathbf{\Phi}_{\mathcal{A}})$ and J_r, $r = 1, 2, ..., p$, denote the Jordan blocks corresponding to eigenvalues whose norms are strictly less than unity.

Theorem 4. Let $\mathbf{\Phi}_{\mathcal{A}}$ be a bistochastic quantum operation on the Hilbert space $\mathfrak{B}(\mathcal{H})$ and let $\rho(0) \in \mathfrak{D}(\mathcal{H})$ denote the density matrix representing the initial state of a quantum system. Then, for every $\rho(0) \in \mathfrak{D}(\mathcal{H})$, the iterated succession of quantum states $\rho(t) = \mathbf{\Phi}_{\mathcal{A}}^t\rho(0)$ converges to $\bar{\mathbf{\Phi}}_{\mathcal{A}}^{\infty}\rho(0)$ if and only if $\lambda = 1$ is the only eigenvalue of $\mathbf{\Phi}_{\mathcal{A}}$ on the unit circle.

where

$$\bar{\mathbf{\Phi}}_{\mathcal{A}}^{\infty} = \begin{bmatrix} \mathbb{I}_{g(1)} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } g(1) = \mathrm{dim}\mathsf{Ker}(\mathbf{\Phi}_{\mathcal{A}} - \mathbb{I}). \tag{8}$$

In particular, if the eigenspace of the eigenvalue $\lambda = 1$ is one-dimensional or contains only a single density operator $\frac{1}{N}\mathbb{I}$, then $\lim_{t\to\infty} \mathbf{\Phi}_{\mathcal{A}}^t\rho(0) = \frac{1}{N}\mathbb{I}$, independently of the initial state $\rho(0)$.

Proof. Consider what becomes of the Jordan blocks of the powers $[\mathbf{\Phi}_{\mathcal{A}}]^t$ as $t \to \infty$. Since each of the Jordan blocks J_r is an upper triangular matrix whose diagonal is populated by a single eigenvalue of modulus strictly less than unity, it is a simple exercise in elementary algebra to show that $\lim_{t\to\infty} J_r^t = O_r$ (zero matrix of same size as J_r). Thus, if we define

$$[\mathbf{\Phi}_{\mathcal{A}}^{\infty}(t)] = \mathrm{diag}\left({\lambda_1}^t, {\lambda_2}^t, ..., {\lambda_k}^t, O_1, O_2, ..., O_p\right), \tag{9}$$

then $\|\mathbf{\Phi}_{\mathcal{A}}^t - \mathbf{\Phi}_{\mathcal{A}}^{\infty}(t)\| \to 0$. Plainly, as $t \to \infty$, the expression $\mathbf{\Phi}_{\mathcal{A}}^{\infty}(t)\rho(0)$, whose behavior mimics the behavior of $\mathbf{\Phi}_{\mathcal{A}}^t\rho(0)$, converges for every choice of $\rho(0) \in \mathfrak{D}(\mathcal{H})$ if and only if $\lambda_1 = \lambda_2 = ... = \lambda_k = 1$. The last assertion of the theorem follows from the fact that $\frac{1}{N}\mathbb{I}$ is the only density operator contained in the space $\mathsf{Ker}(\mathbf{\Phi}_{\mathcal{A}} - \mathbb{I})$.

The statement of Theorem 4 pertains only to bistochastic quantum operations $\mathbf{\Phi}_{\mathcal{A}}$ all of whose eigenvalues on the unit circle are equal to unity. In general, when this condition is relaxed to allow for other eigenvalues on the unit circle, the expression $\rho(t) = \mathbf{\Phi}_{\mathcal{A}}^t\rho(0)$ no longer necessarily converges to a stationary state. Examples can be found in the literature, including the case of all (unitary) quantum walk on finite graphs [6].

However, even if a limiting state fails to exist in the usual sense, we still might want to probe the possibility of a "limiting state" in the sense of Cesàro:

$$\lim_{t\to\infty} \frac{\mathbf{\Phi}_{\mathcal{A}}\rho(0) + \mathbf{\Phi}_{\mathcal{A}}^2\rho(0) + ... + \mathbf{\Phi}_{\mathcal{A}}^t\rho(0)}{t}. \tag{10}$$

In terms of this generalized sense of "limiting state", it turns that every quantum Markov chain converges.

Theorem 5. Let $\mathbf{\Phi}_{\mathcal{A}}$ be a bistochastic quantum operation on the Hilbert space $\mathfrak{B}(\mathcal{H})$ and let $\rho(0) \in \mathfrak{D}(\mathcal{H})$ denote a density matrix representing the initial state of a quantum system. Then, for every $\rho(0) \in \mathfrak{D}(\mathcal{H})$:

$$\lim_{t\to\infty} \frac{1}{t}\sum_{n=1}^{t} \mathbf{\Phi}_{\mathcal{A}}^n\rho(0) = \bar{\mathbf{\Phi}}_{\mathcal{A}}^{\infty}\rho(0), \tag{11}$$

where

$$\bar{\mathbf{\Phi}}_{\mathcal{A}}^{\infty} = \begin{bmatrix} \mathbb{I}_{g(1)} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } g(1) = \mathrm{dim}\mathsf{Ker}(\mathbf{\Phi}_{\mathcal{A}} - \mathbb{I}). \tag{12}$$

Proof. To evaluate the limit in Eq.(10), we have only to reexamine Eq.(7). For every eigenvalue $\lambda = \lambda_i$ of $\mathbf{\Phi}_{\mathcal{A}}$, if $\lambda \neq 1$, then $\frac{1}{t}\sum_{n=1}^{t} \lambda^n = \frac{1}{t}\frac{\lambda - \lambda^{t+1}}{1-\lambda} \to 0$. Similarly, for every Jordan block $J = J_r$ in Eq.(7), we have $\frac{1}{t}\sum_{n=1}^{t} J^n = \frac{1}{t}(\mathbb{I} - J)^{-1}(J - J^{t+1}) \to 0$. Based on these two observations, we deduce that $\frac{1}{t}\sum_{n=1}^{t} \mathbf{\Phi}_{\mathcal{A}}^n$ converges to the diagonal matrix

$$\bar{\mathbf{\Phi}}_{\mathcal{A}}^{\infty} = \begin{bmatrix} \mathbb{I}_{g(1)} & 0 \\ 0 & 0 \end{bmatrix}, \tag{13}$$

where $g(1) = \mathrm{dim}\mathsf{Ker}(\mathbf{\Phi}_{\mathcal{A}} - \mathbb{I})$.

Thus, the Cesàro average, given by Eq.(10), does, in fact, converge and equals $\bar{\mathbf{\Phi}}_{\mathcal{A}}^{\infty}\rho(0)$. However, it is clear by no means how this limit should be interpreted, even in the mysterious context of quantum mechanics.

According to Theorem 3.4 in [6], the time-averaged probability distribution for any unitary quantum walk

on a graph must converge. We remark that this result follows as an immediate corollary of Theorem 5. To see this, let $\overline{P}_t(v)$ denote the time-averaged probability on the node v over the time period $[1,t]$. Then $\overline{P}_t(v|\rho(0)) = \mathrm{tr}\left(|v\rangle\langle v|\frac{1}{t}\sum_{n=1}^{t}\mathbf{\Phi}_{\mathcal{A}}^{n}\rho(0)\right)$. Since $\mathrm{tr}\left(|v\rangle\langle v|\cdot\right)$ is a continuous function of the argument, it follows, by Theorem 5, that $\overline{P}_t(v|\rho(0))$ converges to $\mathrm{tr}\left(|v\rangle\langle v|\bar{\mathbf{\Phi}}_{\mathcal{A}}^{\infty}\rho(0)\right)$.

IV. CLASSIFICATION OF BISTOCHASTIC QUANTUM OPERATIONS

Let $\mathbf{\Phi}_{\mathcal{A}} : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ be a bistochastic quantum operation and let $\rho(0) \in \mathfrak{D}(\mathcal{H})$ denote the density matrix representing the initial state of a quantum system. Depending on the limiting behavior, as $t \to \infty$, of the corresponding quantum Markov process, $\mathbf{\Phi}_{\mathcal{A}}$ must belong to one of the following four mutually exclusive categories:

(1) $\lim_{t\to\infty}\mathbf{\Phi}_{\mathcal{A}}^{t}\rho(0)$ converges to the maximally-mixed state $\frac{1}{N}\mathbb{I}$, independently of the initial state $\rho(0)$.

(2) $\lim_{t\to\infty}\mathbf{\Phi}_{\mathcal{A}}^{t}\rho(0)$ converges, but the limit depends upon the initial state $\rho(0)$.

(3) $\lim_{t\to\infty}\mathbf{\Phi}_{\mathcal{A}}^{t}\rho(0)$ fails to converge, but the Cesàro average $\lim_{t\to\infty}\frac{1}{t}\sum_{n=1}^{t}\mathbf{\Phi}_{\mathcal{A}}^{n}\rho(0)$ exists and equals the maximally-mixed state $\frac{1}{N}\mathbb{I}$, independently of the initial state $\rho(0)$.

(4) $\lim_{t\to\infty}\mathbf{\Phi}_{\mathcal{A}}^{t}\rho(0)$ fails to converge, but the Cesàro average $\lim_{t\to\infty}\frac{1}{t}\sum_{n=1}^{t}\mathbf{\Phi}_{\mathcal{A}}^{n}\rho(0)$ exists and depends upon the initial state $\rho(0)$.

To elucidate each of the above categories, we proceed to offer some comments and examples.

In category (1), the quantum operation $\mathbf{\Phi}_{\mathcal{A}}$ possesses no eigenvalue on the unit circle other than $\lambda = 1$. Moreover, the eigenspace $\mathsf{Ker}(\mathbf{\Phi}_{\mathcal{A}} - \mathbb{I})$ of $\lambda = 1$ is one-dimensional, spanned only by the density operator $\frac{1}{N}\mathbb{I}$, or contains only a single density operator $\frac{1}{N}\mathbb{I}$.

As an example, consider the quantum operation [7] defined by

$$\mathbf{\Phi}_{\mathcal{A}}(X) = \frac{1}{2}X + \frac{1}{2}UXU^{\dagger} \tag{14}$$

where U is the unitary transformation given by

$$U = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}. \tag{15}$$

According to [9], this is an example of a so-called *random unitary operation*. The eigenspace $\mathsf{Ker}(\mathbf{\Phi}_{\mathcal{A}} - \lambda\mathbb{I})$ where $\lambda \in \Lambda_1(\mathbf{\Phi}_{\mathcal{A}})$ is equal to the set $D_\lambda := \{X \in \mathfrak{B}(\mathcal{H}) : \mathbb{I}X = \lambda X\mathbb{I} \text{ and } UX = \lambda XU\}$. Evidently, $\Lambda_1(\mathbf{\Phi}_{\mathcal{A}}) = \{1\}$. A simple calculation shows that the eigenspace $\mathsf{Ker}(\mathbf{\Phi}_{\mathcal{A}} - \mathbb{I})$ of the eigenvalue $\lambda = 1$ is $D_1 = \{k\mathbb{I} : k \in \mathbb{C}\}$, which is one-dimensional. Therefore $\lim_{t\to\infty}\mathbf{\Phi}_{\mathcal{A}}^{t}X = \frac{1}{2}\mathbb{I}_2$ for any initial state X.

It can be verified that the *bit-phase flip* channel (p377 in [10]) also belongs to category (1). A less trivial example of this sort can be found in [16].

In category (2), $\mathbf{\Phi}_{\mathcal{A}}$ possesses as its only eigenvalue on the unit circle the value $\lambda = 1$ and $\mathsf{Ker}(\mathbf{\Phi}_{\mathcal{A}} - \mathbb{I})$ is at least two-dimensional.

As an example, consider the quantum operation $\mathbf{\Phi}_{\mathcal{A}}$ associated with a so-called *phase flip* channel on single qubits [10], given by

$$\mathbf{\Phi}_{\mathcal{A}}(X) = pX + (1-p)ZXZ^{\dagger}, \tag{16}$$

where Z is the *Pauli matrix*:

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{17}$$

By a pattern of reasoning similar that that employed in the previous example, we infer that $\Lambda_1(\mathbf{\Phi}_{\mathcal{A}}) = \{1\}$, and the eigenspace $\mathsf{Ker}(\mathbf{\Phi}_{\mathcal{A}} - \mathbb{I})$ is spanned by the two linearly independent density operators:

$$X_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{18}$$

In this case, $\lim_{t\to\infty}\mathbf{\Phi}_{\mathcal{A}}^{t}X = X_\infty$ where X and X_∞ are given by:

$$X = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad X_\infty = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}. \tag{19}$$

It can be verified without much difficulty that the *bit flip* channel ([10], p.376) belongs also to category (2). Additional examples of this sort can be found in [17].

In category (3), the quantum operation $\mathbf{\Phi}_{\mathcal{A}}$ possesses at least two distinct eigenvalues (including $\lambda = 1$) on the unit circle. The eigenspace of $\lambda = 1$, namely $\mathsf{Ker}(\mathbf{\Phi}_{\mathcal{A}} - \mathbb{I})$, is one-dimensional and spanned by $\mathbb{I}$. An example of this type of quantum operation is provided by *quantum walks* on the N-cycle. In this scenario, N is assumed even, the Hadamard transform serves as the coin operator and the evolution of the system is subject to decoherence on both the position and the coin degrees freedom. A detailed treatment of this model is planned for our forthcoming paper.

In category (4), $\mathbf{\Phi}_{\mathcal{A}}$ has at least two distinct eigenvalues (including $\lambda = 1$) on the unit circle and $\mathsf{Ker}(\mathbf{\Phi}_{\mathcal{A}} - \mathbb{I})$ contains at least two linearly independent density operators. As an example of this type of quantum operation, we cite [9] which studies the properties so-called *two-qubit controlled-not* operators.

V. CONCLUSION AND RELATED QUESTIONS

Evidently, for a stochastic quantum operation $\mathbf{\Phi}_{\mathcal{A}}$, the eigenvalues lying on the unit circle determine the evolution of the associated quantum Markov process, including the existence or non-existence of a long-term stationary state. More precisely, the long-term behavior of the quantum Markov process is intimately linked to the

structure of the eigenspaces of eigenvalues on the unit circle.

We speculate that the eigenspace of an eigenvalue λ on the unit circle, denoted by D_λ, might conform always to a formulation in terms of a set $\mathcal{A} = \{A_i \mid i = 1, 2, ..., N^2\}$ of "Kraus operators", as follows:

$$D_\lambda = \{X \in \mathfrak{B}(\mathcal{H}) : XA_i = \lambda A_i X \text{ for } i = 1, 2, ...\}. \quad (20)$$

If valid, this formulation conceivably might provide an efficient means for identifying the eigenspaces of all eigenvalues of absolute value 1. The origin of this idea can be attributed to [9, 17–19].

The existence on the unit circle of any eigenvalue $\lambda \neq 1$ prevents the existence of a limiting state in the usual sense. However, in the sense of Cesàro, a limiting state always exists for any biostochastic quantum operation. A meaningful interpretation for this sort of limit within the context of quantum mechanics is aggressively needed.

It is a fact that $\mathfrak{D}(\mathcal{H})$ contains N^2 linearly independent density operators. However, this does *not* imply that every $k-$dimensional subspace of $\mathfrak{B}(\mathcal{H})$ contains k linearly independent density operators. For instance, consider the subspace $S = \{X : \mathrm{Tr}(X) = 0\}$, which is of dimension $N^2 - 1$. It contains no density operators at all. We thank John Watrous for bringing to our attention this example.

In view of the critical role played by $\mathsf{Ker}(\mathbf{\Phi}_\mathcal{A} - \mathbb{I})$ in determining the limiting behavior of a quantum Markov process, it would be interesting to know if this privileged subspace of $\mathfrak{B}(\mathcal{H})$ possesses always a basis in $\mathfrak{D}(\mathcal{H})$.

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