An operatorial approach to stock markets

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Abstract

We propose and discuss some toy models of stock markets using the same operatorial approach adopted in quantum mechanics. Our models are suggested by the discrete nature of the number of shares and of the cash which are exchanged in a real market, and by the existence of conserved quantities, like the total number of shares or some linear combination of cash and shares. The same framework as the one used in the description of a gas of interacting bosons is adopted.

I Introduction

A huge literature exists concerning the time behavior of financial markets, most of which is based on statistical methods, see [1] and references therein. In recent years a strategy somehow different has also been considered. This strategy takes inspiration on the *many-body* nature of a stock market, nature which suggests the use of tools naturally related to quantum mechanics and, in particular, to QM_∞, i.e. quantum mechanics for systems with infinite degrees of freedom. Examples of this approach can be found, for instance, in [2] and [3], where the concepts of hamiltonian, phase transition, symmetry breaking and so on are introduced. However, in none of these papers, and in our knowledge not even in other existing literature, the analysis of the time evolution of the portfolio of each single trader has been undertaken. It should be mentioned, however, that a point of view not very different from the one adopted here is discussed, for instance in [4] and [5].

In this paper we use quantum mechanical ideas to construct some toy models which should mimic a simplified stock market. In all our models, where for simplicity a single kind of share is exchanged, the total number of shares does not change in time. This reminds of what happens in a totally different context, i.e. in a gas of elementary particles which can interact among them but without changing their total number. Also, the price of a single share does not change continuously, since any variation is necessarily an integer multiple of a certain minimal quantity, the *monetary unit*, which can be seen, using our quantum mechanical analogy, as a sort of *quantum of cash.* QM_∞ provides a natural framework in which these features can be taken into account. It also provides some natural tools to discuss the existence of conserved quantities and to find the differential equations of motion which drive the portfolio of each single trader, as we will see.

The paper is organized as follows:

in the next section we discuss a first easy model and we give an interpretation to the quantities used to define the model. This oversimplified model will be useful to fix some general ideas.

In Section III we improve the model introducing the *cash*, the *price* of the share and the *supply* of the market. We prove that many integrals of motion exist. The equations of motion are solved using a perturbative expansion, well known in QM_∞.

In Section IV we consider a particular version of this model which we completely solve using the so-called *mean-field* approximation. We also discuss the role of KMS-like states in our framework.

Section V contains our conclusions and plans for the future, while in the Appendix 1 we give few definitions and results concerning the mathematical framework used along the paper, which we have included here for those readers who are not familiar with quantum mechanics. In Appendix 2 we discuss some more results related to the mean-field model.

II A first model

The model we discuss in this section is really an oversimplified toy model of a stock market based on the following assumptions:

1. Our market consists of L traders exchanging a single kind of share;
2. the total number of shares, N, is fixed in time;
3. a trader can only interact with a single other trader: i.e. the traders feel only a *two-body interaction*;
4. the traders can only buy or sell one share in any single transaction;
5. there exists an *unique* price for the share, fixed by the market. In other words, we are not considering any difference between the *put* and the *buy* prices;
6. the price of the share changes with discrete steps, multiples of a given monetary unit;
7. each trader has a huge quantities of cash that he can use to buy shares but which does not enter, in the present model, in the definition of his portfolio whose value is fixed only by the number of shares.

Let us briefly comment the above assumptions: of course assuming that there is only a single kind of share may appear rather restrictive but we believe that more species of shares can be introduced without major changes. However, along all this paper we only work in this hypothesis just to simplify the treatment. The third assumption above simply means that it is not possible for, say, the traders t_1, t_2 and t_3 to interact with each other at the same time: however t_1 can interact directly with t_3 or via its interaction with t_2: t_1 interacts with t_2 and t_2 interacts with t_3. This is a typical simplification in many-body theory where often all the $N-$body interactions, $N \geq 3$, are assumed to be negligible with respect to the $2-$body interaction. Assumptions 4, 5 and 7 are useful to simplify the model and to allow us to extract some driving ideas to construct more realistic models. Finally, as we have seen, assumption 6 is a natural one. Most of these assumptions will be relaxed in the next section.

As we discuss in the Appendix, the time behavior of this model can be described by an operator called the *hamiltonian* of the model, which describes the free evolution of the model plus the effects due to the interaction between the traders. The hamiltonian of this simple model is the following:

$$H = H_0 + H_{price}, \quad H_0 = \sum_{l=1}^{L} \alpha_l a_l^\dagger a_l + \sum_{i,j=1}^{L} p_{ij} a_i a_j^\dagger, \qquad H_{price} = \epsilon p^\dagger p \tag{2.1}$$

where the following commutation rules are assumed:

$$[a_l, a_n^\dagger] = \delta_{ln} \mathbb{I}, \qquad [p, p^\dagger] = \mathbb{I}, \tag{2.2}$$

while all the other commutators are zero. The meaning of these operators is discussed in more details in Appendix 1. Here we just recall that a_l and $a_l^\dagger$ respectively destroys and creates a share in the portfolio of t_l, while the operators p and $p^\dagger$ modify the price of the share: p makes the price decrease of ϵ, while $p^\dagger$ makes it increase of the same quantity. The coefficients p_{ij}'s take value 1 or 0 depending on the fact that t_i interacts with t_j or not. We also assume that $p_{ii} = 0$ for all i, which simply means that t_i does not interact with himself. For those who are familiar with second quantization, there is an easy interpretation for the hamiltonian above, which can be deduced also from what is discussed in Appendix 1: while $\epsilon p^\dagger p + \sum_{l=1}^{L} \alpha_l a_l^\dagger a_l$ describes the free evolution of the operators $\{a_l\}$ and p, whose physical meaning will be considered again later on in this section, the single contribution $a_i a_j^\dagger$ of the interaction hamiltonian $\sum_{i,j=1}^L p_{ij} a_i a_j^\dagger$ *destroys* a share belonging to the trader t_i and *creates* a share in the portfolio of the trader t_j. In other words: if $p_{ij} = 1$ then the trader t_i sells a share to t_j. However, since H must be self-adjoint (for mathematical and physical reasons), then $p_{ij} = 1$ also implies $p_{ji} = 1$. This means that the interaction hamiltonian contains both the possibility that t_i sells a share to t_j and the possibility that t_j sells a share to t_i. Different values of α_i in the free hamiltonian are then used to introduce an *ability* of the trader, which will make more likely that the *most expert* trader sells or buys his shares to the other traders so to increase the value of his portfolio.

As we will discuss in Appendix 1, the time evolution of an operator X of the model is $X(t) = e^{iHt} X e^{-iHt}$ and it satisfies the following Heisenberg differential equation: $\frac{dX(t)}{dt} = ie^{iHt}[H, X]e^{-iHt} = i[H, X(t)]$. The only observables whose time evolution we are interested in are, clearly, the price of the share and the number of shares of each traders. Indeed, as we have already remarked, within our simplified scheme there is no room for the cash of the trader! The *price operator* $\hat{P}$ is $\hat{P} = \epsilon p^\dagger p$, while the *j-number operator* is $\hat{n}_j = a_j^\dagger a_j$, which represents the number of shares that t_j possesses. The operator *total number of shares* is finally $\hat{N} = \sum_{j=1}^L \hat{n}_j = \sum_{j=1}^L a_j^\dagger a_j$. The choice of H in (2.1) is suggested by the requirement 2) above. Indeed it is easy to check, using (2.2), that $[H, \hat{N}] = 0$. This implies that the time evolution of $\hat{N}$, $\hat{N}(t) = e^{iHt} \hat{N} e^{-iHt}$ is trivial: $\hat{N}(t) = \hat{N}$ for all t. However this does not imply also that $[H, \hat{n}_j] = 0$, which, as a matter of fact, is not true in general. This is clear from the definition of H: the term $\sum_{l=1}^L \alpha_l a_l^\dagger a_l$ does not change the number of shares of the different traders, but only counts this number. On the contrary, $\sum_{i,j=1}^L p_{ij} a_i a_j^\dagger$ destroys a share belonging to t_i but, at the same time, creates another share in the portfolio of the trader t_j. In this operation, the number of the shares of the single traders are changed, but the total number of shares remains constant! It may be worth noticing that if all the p_{ij} are zero, i.e. if there is no interaction between the traders, then we also get $[H, \hat{n}_j] = 0$: our model produce a completely stationary market, as it is expected.

We implement assumptions 5) and 6) by requiring that the price operator $\hat{P}$ has the form given above, $\hat{P} = \epsilon p^\dagger p$, where ϵ is the monetary unit. Such an operator is assumed to be part of H, see (2.1). Also, because of the simplifications which are assumed in this toy model, $\hat{P}$ is clearly a constant of motion: $[H,\hat{P}]=0$. This is not a realistic assumption, and will be relaxed in the next sections. However, it is assumed here since it allows us a better understanding of the meaning of the α_l's, as we will discuss later.

In order to describe a *state of the system* in which at time $t=0$ the portfolio of the first trader consists of n_1 shares, the one of t_2 of n_2 shares, and so on, and the price of the share is $P=M\epsilon$, we should impose that the market is in a vector state $\omega_{n_1,n_2,\ldots,n_L;M}$, see Appendix 1, defined by the vector

$$\varphi_{n_1,n_2,\ldots,n_L;M}:=\frac{1}{\sqrt{n_1!\,n_2!\ldots n_L!M!}}(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}\cdots(a_L^\dagger)^{n_L}(p^\dagger)^M\varphi_0, \tag{2.3}$$

where φ_0 is the *vacuum* of the model: $a_j\varphi_0=p\varphi_0=0$ for all $j=1,2,\ldots,L$. If $X\in\mathfrak{A}$, $\mathfrak{A}$ being the *algebra* of the observables of our market, then we put

$$\omega_{n_1,n_2,\ldots,n_L;M}(X)=<\varphi_{n_1,n_2,\ldots,n_L;M},X\varphi_{n_1,n_2,\ldots,n_L;M}>, \tag{2.4}$$

and $<,>$ is the scalar product in the Hilbert space of the theory, see again the Appendix. The Heisenberg equations of motion (A.2) for the annihilation operators $a_l(t)$ produce the following very simple differential equation:

$$i\dot{a}(t)=Xa(t), \tag{2.5}$$

where we have introduced the matrix X, independent of time, and the vector $a(t)$ as follows

$$X\equiv\begin{pmatrix}\alpha_1 & p_{2\,1} & p_{3\,1} & \cdot & \cdot & p_{L-1\,1} & p_{L\,1}\\ p_{1\,2} & \alpha_2 & p_{3\,2} & \cdot & \cdot & \cdot & p_{L\,2}\\ p_{1\,3} & p_{2\,3} & \alpha_3 & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ p_{1\,L-1} & p_{2\,L-1} & p_{3\,L-1} & \cdot & \cdot & \alpha_{L-1} & p_{L\,L-1}\\ p_{1\,L} & p_{2\,L} & p_{3\,L} & \cdot & \cdot & p_{L-1\,L} & \alpha_L\end{pmatrix},\quad a(t)\equiv\begin{pmatrix}a_1(t)\\ a_2(t)\\ a_3(t)\\ \cdot\\ \cdot\\ a_{L-1}(t)\\ a_L(t)\end{pmatrix}.$$

Notice that, due to the conditions on the p_{ij}'s, and since all the α_l's are real, the matrix X is self-adjoint. Equation (2.5) can now be solved as follows: let V be the (unitary) matrix which diagonalizes X: $V^\dagger X V = diag\{x_1, x_2, \ldots, x_L\} =: X_d$, x_j, being its eigenvalues, $j = 1, 2, \ldots, L$. Notice that, of course, V does not depend on time. Then, putting

$$U(t) = \begin{pmatrix} e^{ix_1 t} & 0 & 0 & . & . & . & 0 \\ 0 & e^{ix_2 t} & 0 & . & . & . & 0 \\ 0 & 0 & e^{ix_3 t} & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & e^{ix_L t} \end{pmatrix},$$

we get

$$a(t) = VU(t)V^\dagger\, a(0), \tag{2.6}$$

where, as it is clear, $a(0)^T = (a_1, a_2, \ldots, a_L)$. If we further introduce the *adjoint* of the vector $a(t)$, $a^\dagger(t) = (a_1^\dagger(t), a_2^\dagger(t), \ldots, a_L^\dagger(t)) = a^\dagger(0) V U^\dagger(t) V^\dagger$, we can explicitly check that $\hat{N}$ is a constant of motion. Indeed we have

$$\hat{N}(t) = a_1^\dagger(t) a_1(t) + a_2^\dagger(t) a_2(t) + \ldots + a_L^\dagger(t) a_L(t) = a^\dagger(t) \cdot a(t) =$$

$$= (a^\dagger(0) V U^\dagger(t) V^\dagger) \cdot (VU(t)V^\dagger\, a(0)) = a^\dagger(0) \cdot a^\dagger(0) = \hat{N}(0).$$

In order to analyze the time behavior of the different $\hat{n}_j(t)$, we simply have to compute the mean value $n_j(t) = \omega_{n_1, n_2, \ldots, n_L; M}(\hat{n}_j(t))$. This means that, for $t = 0$, the first trader possesses n_1 shares, the second trader possesses n_2 shares, and so on, and that the price of the share is $M\epsilon$. It should be mentioned that the only way in which a matrix element like $\omega_{n_1, n_2, \ldots, n_L; M}(a_j^k\, (a_l^\dagger)^m)$, can be different from zero is when $j = l$ and $k = m$. This follows from the orthonormality of the set $\{\varphi_{n_1, n_2, \ldots, n_L; M}\}$. which is a direct consequence of the canonical commutation relations.

The easiest way to get the analytic expression for $n_j(t)$ is to fix the number of the traders, starting with the simplest situation: $L = 2$. In this case we find that

$$\begin{cases} n_1(t) = \frac{1}{\Omega^2} \left\{ n_1 \left(\alpha^2 + 2p^2 (1 + \cos(\Omega t)) \right) + 2p^2 n_2 \left(1 - \cos(\Omega t) \right) \right\} \\ n_2(t) = \frac{2p^2 n_1}{\Omega^2} \left(1 - \cos(\Omega t) \right) + n_2 \left(1 + \frac{2p^2}{\Omega^2} \left(\cos(\Omega t) - 1 \right) \right) \end{cases} \tag{2.7}$$

where we have defined $\Omega^2 = \alpha^2 + 4p^2$, with $\alpha = \alpha_2 - \alpha_1$ and $p = p_{12} = p_{21}$.

It is not hard to check that $n_1(t)+n_2(t)=n_1+n_2$, as expected. Also, if $p=0$ then we find $n_1(t)=n_1$ and $n_2(t)=n_2$ for all t. This is natural and expected, since when $p=0$ there is no interaction at all between the traders, so that there is no reason for $n_1(t)$ and $n_2(t)$ to change in time. Another interesting consequence of (2.7) is that, if $n_1=n_2=n$, that is if the two traders start with the same number of shares, they do not change this equilibrium during the time: we find again $n_1(t)=n_2(t)=n$. Also this result is expected, since both t_1 and t_2 possess the same amount of money (their huge sources!) and the same number of shares. The role of α_1 and α_2, in this case, is unessential. It is further clear that $n_j(t)$ is a periodic function whose period, $T=\frac{2\pi}{\Omega}$, decreases for $|\alpha|=|\alpha_1-\alpha_2|$ and p increasing. Finally, if we call $\Delta n_j=\max_{t\in[0,T]}|n_j(t)-n_j(0)|$, which represents the highest variation of $n_j(t)$ in a period, we can easily check that Δn_j increases when $|n_1(0)-n_2(0)|$ increases and when Ω decreases.

Remark:– It is worth remarking that, since the number of shares should be integer, while the functions $n_1(t)$ and $n_2(t)$ are not integers for general values of t, we could introduce a sort of *time per the m-th transaction*, τ_m, chosen in such a way that $n_j(\tau_1)$, $n_j(\tau_2),\ldots$ are all integers, $j=1,2$.

Let us now consider a market with three traders. In Figure 1 we plot $n_3(t)$ with the initial conditions $n_1=40$, $n_2=n_3=0$, with $p_{12}=p_{13}=p_{23}=1$ and different values of α_1, α_2 and α_3. In the figure on the left we have $(\alpha_1,\alpha_2,\alpha_3)=(1,2,3)$, in the one in the middle $(\alpha_1,\alpha_2,\alpha_3)=(1,2,10)$ and in the one in the right $(\alpha_1,\alpha_2,\alpha_3)=(1,2,100)$

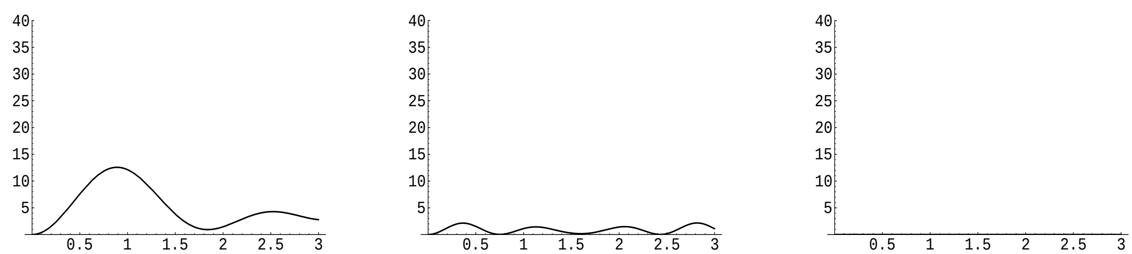


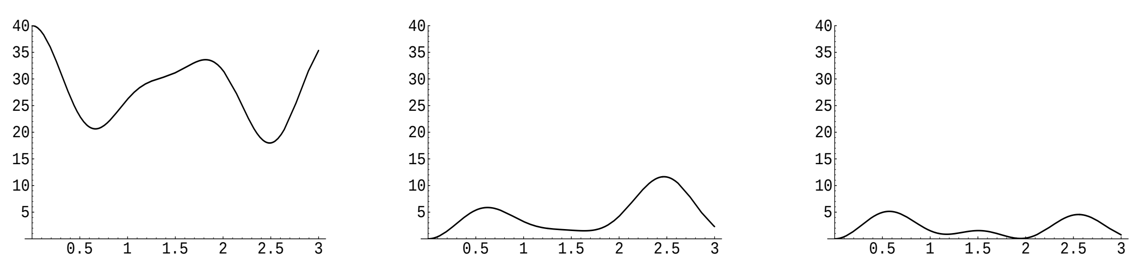
Figure 1: $n_3(t)$ for $\alpha_3=3$ (left), $\alpha_3=10$ (middle), $\alpha_3=100$ (right)

This plot, together with many others which can be obtained, e.g., considering different initial conditions, suggests to interpret α_j as a sort of *inertia*: the larger the value of α_j, the bigger the tendency of t_j of keeping the number of his shares constant in time! We could also think of α_j^{-1} as a sort of *information* reaching t_j (but not the other traders): if α_j is large then not much information reaches t_j which has therefore no input to optimize his interaction with the other traders.

In this case, and also for more traders, it is not evident from our plots if a periodic behavior is again recovered. In any case, at least a quasi-periodic behavior is observed with a quasi-period which is compatible with the same T found in the case of the two traders.

As for the $L = 2$ situation, we recover that if $n_1 = n_2 = n_3 = n$, then $n_1(t) = n_2(t) = n_3(t) = n$, for all t. Moreover, if $n_1 \simeq n_2 \simeq n_3 \simeq n$, then $n_j(t)$ have all small oscillations around n. But, if $n_1 \simeq n_2 \neq n_3$, and if $p_{ij} = 1$ for all i, j with $i \neq j$, then all the functions $n_j(t)$ change *considerably* with time. The reason is the following: since n_3 differs from n_1 and n_2, it is natural to expect that $n_3(t)$ changes with time. But, since $N = n_1(t) + n_2(t) + n_3(t)$ must be constant, both $n_2(t)$ and $n_1(t)$ must change in time as well. The same conclusion can be deduced also if $p_{23} = 0$ while all the other p_{ij}'s are equal to 1: even if t_2 does not interact with t_3, the fact that t_1 interacts with both t_2 and t_3, together with the fact that N must be constant, implies again that all the $n_j(t)$'s need to change in time. Finally, it is clear that if $p_{13} = p_{23} = 0$, then t_3 interact neither with t_1 nor with t_2 and, indeed, we find that $n_3(t)$ does not change with time: this is a consequence of the fact that, in this case, $[H, \hat{n}_3] = 0$.

Analogous conclusions can be deduced also for five (or more) traders. In particular Figure 2 shows that there is no need for all the traders to interact among them to have a non trivial time behavior. Indeed, even if $p_{15} = p_{25} = 0$, which means that t_5 may only interact directly with t_3 and t_4, we get the following plots for $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (1, 2, 3, 4, 5)$ and $(n_1, n_2, n_3, n_4, n_5) = (40, 0, 0, 0, 0)$. We see that the number of shares of each trader changes in time with the same order of magnitude.



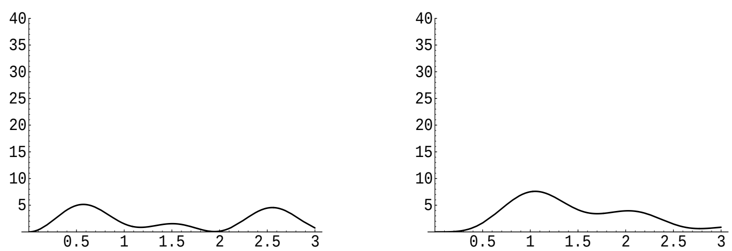


Figure 2: $n_1(t)$, $n_2(t)$, $n_3(t)$ (first row) and $n_4(t)$, $n_5(t)$ (second row) for α_j and n_j as above

We end this section stressing that the interpretation of α_j as a sort of inertia is suggested also by the analysis of this larger number of traders.

III A different model

We consider now another model, which differs from the previous one since the *cash*, the *price* of the share and the *supply* of the market are introduced in a non trivial way. In particular we require that assumptions 1, 2, 3, 4 and 6 of the previous section still hold. Moreover we require that

a. when the tendency of the market to buy a share, i.e. the *market demand*, increases then the price of the share increases as well. Equivalently, when the tendency of the market to sell a share, i.e. the *market supply*, increases then the price of the share decreases;

b. for our convenience the demand and the supply are expressed in term of natural numbers;

c. we take $\epsilon = 1$ in the following: 1 is therefore the unit of money.

The *formal* hamiltonian of the model is the following operator:

$$\begin{cases} \tilde{H} = H_0 + \tilde{H}_I, \text{ where} \\ H_0 = \sum_{l=1}^{L} \alpha_l a_l^\dagger a_l + \sum_{l=1}^{L} \beta_l c_l^\dagger c_l + o^\dagger o + p^\dagger p \\ \tilde{H}_I = \sum_{i,j=1}^{L} p_{ij} \left(a_i^\dagger a_j (c_i \, c_j^\dagger)^{\hat{P}} + a_i \, a_j^\dagger (c_j \, c_i^\dagger)^{\hat{P}} \right) + (o^\dagger p + p^\dagger o), \end{cases} \tag{3.1}$$

where, as before, $\hat{P}=p^\dagger p$. Here the following commutation rules are assumed:

$$[a_l,a_n^\dagger]=[c_l,c_n^\dagger]=\delta_{ln}\mathbb{I},\qquad [p,p^\dagger]=[o,o^\dagger]=\mathbb{I}, \tag{3.2}$$

while all the other commutators are zero. As for the previous model we assume that $p_{ii}=0$. Here the operators $a_l^\sharp$ and $p^\sharp$ have the same meaning as in the previous section, while $c_l^\sharp$ and $o^\sharp$ are respectively the *cash* and the *supply* operators. The states in (2.4) must be replaced by the states

$$\omega_{\{n\};\{k\};O;M}(\,.\,)=<\varphi_{\{n\};\{k\};O;M},\,.\,\varphi_{\{n\};\{k\};O;M}>, \tag{3.3}$$

where $\{n\}=n_1,n_2,\ldots,n_L$, $\{k\}=k_1,k_2,\ldots,k_L$ and

$$\varphi_{\{n\};\{k\};O;M}:=\frac{(a_1^\dagger)^{n_1}\cdots(a_L^\dagger)^{n_L}(c_1^\dagger)^{k_1}\cdots(c_L^\dagger)^{k_L}(o^\dagger)^O(p^\dagger)^M}{\sqrt{n_1!\ldots n_L!\,k_1!\ldots k_L!\,O!\,M!}}\,\varphi_0. \tag{3.4}$$

Here φ_0 is the *vacuum* of the model: $a_j\varphi_0=c_j\varphi_0=p\varphi_0=o\varphi_0=0$, for $j=1,2,\ldots,L$.

Let us now see what is the meaning of the hamiltonian above and for which reason we call it *formal*.

H_0 contains all that is related to the free dynamics of the model.

$\tilde{H}_I$ is the interaction hamiltonian, whose terms have a natural interpretation:

the presence of $o^\dagger\,p$ implies that when the supply increases then the price must decrease. Of course $p^\dagger\,o$ produces exactly the opposite effect;

the presence of $a_i^\dagger a_j(c_i\,c_j^\dagger)^{\hat{P}}$ implies that t_i increases of one unit the number of shares in his portfolio but, at the same time, his cash decreases because of $c_i^{\hat{P}}$, that is it must decrease of as many units of cash as the price operator $\hat{P}$ demands. Moreover, the trader t_j behaves exactly in the opposite way: he has one share less because of a_j but his cash increases because of $(c_j^\dagger)^{\hat{P}}$. Of course, the hermitian conjugate term $a_i\,a_j^\dagger(c_j\,c_i^\dagger)^{\hat{P}}$ in $\tilde{H}_I$ produces a specular effect for the two traders.

As in the previous section, if $\tilde{H}_I=0$, then there is no nontrivial dynamics of the relevant *observables* of the system, like the $c_j^\dagger c_j$ and $n_j^\dagger n_j$. This can also be seen as a criterium to fix the free hamiltonian of the system: it is only the interaction between the traders which may modify their status!

However, despite of this clear physical interpretation, the hamiltonian in (3.1) suffers of a technical problem: since c_j and $c_j^\dagger$ are not self-adjoint operators, it is not obvious how to define, for instance, the operator $c_j^{\hat{P}}$. Indeed, if we formally write $c_j^{\hat{P}}$ as $e^{\hat{P}\,\log c_j}$, then we cannot use functional calculus to define $\log c_j$. Also, we cannot use a simple series expansion since the operators involved are all unbounded so that the series we get is surely not norm convergent and many domain problems appear. For this reason, we find more convenient to replace $\tilde{H}$ with an *effective* hamiltonian, H, defined as

$$\left\{\begin{array}{l} H = H_0 + H_I, \text{ where} \\ H_0 = \sum_{l=1}^{L}\alpha_l a_l^\dagger a_l + \sum_{l=1}^{L}\beta_l c_l^\dagger c_l + o^\dagger\, o + p^\dagger\, p \\ H_I = \sum_{i,j=1}^{L} p_{ij}\left(a_i^\dagger a_j (c_i\, c_j^\dagger)^M + \,a_i\, a_j^\dagger (c_j\, c_i^\dagger)^M\right) + (o^\dagger\, p + p^\dagger\, o), \end{array}\right. \tag{3.5}$$

where $M = \omega_{\{n\};\{k\};O;M}(\hat{P})$. Notice that, because of the fact that $p_{ii} = 0$, there is no difference in H_I above if we write $(c_i\, c_j^\dagger)^M$ or $(c_i)^M (c_j^\dagger)^M$ even if the two operator c_j and $c_j^\dagger$ do not commute. Notice also that if we consider a state ω over $\mathfrak{A}$ different from $\omega_{\{n\};\{k\};O;M}$, as we will do in the next section, then $\omega(\hat{P})$ could be different from M.

Three integrals of motion for our model trivially exist:

$$\hat{N} = \sum_{i=1}^{L} a_i^\dagger\, a_i, \quad \hat{K} = \sum_{i=1}^{L} c_i^\dagger c_i \quad \text{and} \quad \hat{\Gamma} = o^\dagger o + p^\dagger p. \tag{3.6}$$

This can be easily checked since the canonical commutation relations in (3.2) imply that $[H, \hat{N}] = [H, \hat{\Gamma}] = [H, \hat{K}] = 0$.

The fact that $\hat{N}$ is conserved clearly means that no new share is introduced in the market. Of course, also the total amount of money must be a constant of motion since the cash is assumed to be used only to buy shares. Since also $\hat{\Gamma}$ commutes with H, moreover, if the mean value of $o^\dagger o$ increases with time then necessarily the mean value of the price operator must decrease and vice-versa. This is exactly the mechanism assumed in point **a.** at the beginning of this section.

Remark:– it may be worth noticing that this is not the only way in which Requirement **a.** could be implemented, but it is surely the simplest one. Just to give few other examples, we could ask for one the following combinations to remain constant in time: $(o^\dagger o)^2 + (p^\dagger p)^2$, $o^\dagger o p^\dagger p$ or many others.

Another consequence of the definition of H is that L other constants of motion also exist. They are the following operators:

$$\hat{Q}_j = a_j^\dagger\, a_j + \frac{1}{M}\, c_j^\dagger\, c_j, \tag{3.7}$$

for $j = 1, 2, \ldots, L$. This can be checked explicitly computing $[H, \hat{Q}_j]$ and proving that all these commutators are zero. But we can also understand this feature simply noticing that: (i) $\hat{Q}_j$ commutes trivially with H_0 and (ii) the term $a_i^\dagger a_j (c_i\, c_j^\dagger)^M$ in H_I obviously preserves not only the total number of shares and the total amount of cash, but also a certain combination of the shares and the cash: as far as t_i is concerned, $a_i^\dagger$ increases of one unit the number of shares while c_i^M decreases of M units the amount of cash. This means that if a certain vector Ψ represents n_i shares and k_i units of cash, then $a_i^\dagger\, c_i^M \Psi$ describes n_i+1 shares and $k_i - M$ units of cash. Therefore we have $\hat{Q}_i \Psi = (n_i + \frac{1}{M} k_i)\Psi$ and $\hat{Q}_i(a_i^\dagger\, c_i^M \Psi) = (n_i + 1 + \frac{1}{M}(k_i - M))(a_i^\dagger\, c_i^M \Psi) = (n_i + \frac{1}{M} k_i)(a_i^\dagger\, c_i^M \Psi)$. So, it is not surprising that $[\hat{Q}_i, a_i^\dagger\, c_i^M] = 0$ and, as a consequence, that $[\hat{Q}_i, H] = 0$.

The hamiltonian (3.5) contains a contribution, $h_{po} = o^\dagger\, o + p^\dagger\, p + (o^\dagger\, p + p^\dagger\, o)$, which is decoupled from the other terms. This means that, within our model, the time evolution of the supply and the price operators do not depend on the number of shares or on the cash, and can be deduced referring only to h_{po}. The Heisenberg equations of motion are the following:

$$\begin{cases} i\dot{o}(t) = o(t) + p(t) \\ i\dot{p}(t) = o(t) + p(t), \end{cases} \tag{3.8}$$

which shows that $o(t) - p(t)$ is constant in t, so that $\hat{\Delta} = o - p$ is still another integral of motion. Solving this system we get $o(t) = \frac{1}{2}\{o(e^{-2it} + 1) + p(e^{-2it} - 1)\}$ and $p(t) = \frac{1}{2}\{p(e^{-2it} + 1) + o(e^{-2it} - 1)\}$. It is now trivial to check explicitly that both $\hat{\Delta}(t) = o(t) - p(t)$ and $\hat{\Gamma}(t) = o^\dagger(t) o(t) + p^\dagger(t) p(t)$ do not depend on time. If we now compute the mean value of the price and supply operators on a state number we get

$$\begin{cases} P_r(t) = \frac{1}{2}\{P_r + O_f + (P_r - O_f)\cos(2t)\} \\ O_f(t) = \frac{1}{2}\{P_r + O_f - (P_r - O_f)\cos(2t)\}, \end{cases} \tag{3.9}$$

where we have called $O_f(t) = \omega_{\{n\};\{k\};O;M}(o^\dagger(t) o(t))$ and $P_r(t) = \omega_{\{n\};\{k\};O;M}(p^\dagger(t) p(t))$. Recall that $P_r = P_r(0) = M$. Equations (3.9) show that, if $O_f = P_r$ then $O_f(t) =$

$P_r(t) = O_f$ for all t while, if $O_f \simeq P_r$ then $O_f(t)$ and $P_r(t)$ are *almost* constant. In the following we will replace $P_r(t)$ with an integer value, the value M which appears in the hamiltonian (3.5), which is therefore fixed after the solution (3.9) is found. This value is obtained by taking a suitable mean of $P_r(t)$ or working in one of the following assumptions: (i) $O_f = P_r$; or (ii) $O_f \simeq P_r$ or yet (iii) $|O_f + P_r| \gg |P_r - O_f|$. In these last two situations we may replace $P_r(t)$, with a temporal mean, $< P_r(t) >$, since there is not much difference between these two quantities.

Let us now recall that the main aim of each trader is to improve the total value of his portfolio, which we define as follows:

$$\hat{\Pi}_j(t) = \gamma \hat{n}_j(t) + \hat{k}_j(t). \tag{3.10}$$

Here we have introduced the value of the share γ *as decided by the market*, which does not necessarily coincides with the amount of money which is payed to buy the share. As it is clear, $\hat{\Pi}_j(t)$ is the sum of the complete value of the shares, plus the cash. The fact that for each j the operator Q_j is an integral of motion allows us to rewrite the operator $\hat{\Pi}_j(t)$ only in terms of $\hat{n}_j(t)$ and of the initial conditions. We find:

$$\hat{\Pi}_j(t) = \hat{\Pi}_j(0) + (\gamma - M)(\hat{n}_j(t) - \hat{n}_j(0)), \tag{3.11}$$

In order to get the time behavior of the portfolio, therefore, it is enough to obtain $\hat{n}_j(t)$. If we write the Heisenberg equation for $\hat{n}_j(t)$, $\dot{\hat{n}}_j(t) = i[H, \hat{n}_j(t)]$, we see that this equation involves the time evolution of a_j, c_j and their adjoint. The equations of motion for these operators should be added to close the system, and the final system of differential equations cannot be solved exactly. The easiest way to proceed is to develop the following simple perturbative expansion, well known in quantum mechanics:

$$\hat{n}_j(t) = e^{iHt} \hat{n}_j e^{-iHt} = \hat{n}_j + it[H, \hat{n}_j] + \frac{(it)^2}{2!}[H, \hat{n}_j]_2 + \frac{(it)^3}{3!}[H, \hat{n}_j]_3 + \dots, \tag{3.12}$$

where $[H, \hat{n}_j]_1 = [H, \hat{n}_j] = H\hat{n}_j - \hat{n}_j H$ and $[H, \hat{n}_j]_{n+1} = [H, [H, \hat{n}_j]_n]$ for $n \geq 1$, and then to take its mean value on a state $\omega_{\{n\};\{k\};O;M}$ up to the desired order of accuracy. Of course, we can compute as many contributions of the above expansion as we want. However, the expression for $[H, \hat{n}_j]_n$ becomes more and more involved as n and L increase. Just as an example, we consider here the case $L = 2$: up to the third order in time we find

$$n_1(t) = \omega_{\{n\};\{k\};O;M}(\hat{n}_1(t)) = n_1 + t^2 p_{12}^2(\epsilon_+^2 - \epsilon_-^2) + O(t^4), \tag{3.13}$$

where $\epsilon_\pm$ are related to the state $\omega_{\{n\};\{k\};O;M}$ as follows:

$$\epsilon_+ = \sqrt{(n_1+1)\, n_2 \frac{k_1!}{(k_1-M)!}\,\frac{(k_2+M)!}{k_2!}}, \qquad \epsilon_- = \sqrt{(n_2+1)\, n_1\, \frac{(k_1+M)!}{k_1!}\,\frac{k_2!}{(k_2-M)!}}.$$

Of course, in order to have all the above quantities well defined, we need to have both $k_2 - M \geq 0$ and $k_1 - M \geq 0$. This is a natural requirement since it simply states that a trader can buy a share only if he has the money to pay for it!

Remarks:– (1) This solution has some analogies with that given in (2.7). Indeed, if we expand $n_1(t)$ in (2.7) as a power of t, we find an expression which is very close to equation (3.13). In particular, we find that for both models there is no contribution coming from α_j (and from β_j here) up to the order t^3. Also, for this model, if t_1 and t_2 possess the same amount of cash for $t=0$, $k_1 = k_2$, then, since $\epsilon_+^2 - \epsilon_-^2$ turns out to be proportional to $n_2 - n_1$, we deduce that $n_1(t) = n_2(t)$ if $n_1 = n_2$. This is again exactly the same conclusion we have obtained in Section II: $n_1 = n_2$ is a *stability* condition.

(2) Using the fact that Q_1 is constant we can also find the value of the cash of t_1 as a function of time: $k_1(t) = k_1 - Mt^2\, p_{12}^2(\epsilon_+^2 - \epsilon_-^2) + O(t^4)$ while its portfolio evolves like

$$\Pi_1(t) = \Pi_1(0) + (\gamma - M)t^2\, p_{12}^2(\epsilon_+^2 - \epsilon_-^2) + O(t^4) \tag{3.14}$$

(3) This formula allows us to get some conclusions concerning the time behavior of the portfolio of t_1 for small time. In particular we can deduce that:

if $k_1 = k_2$ and $n_1 = n_2$ then $k_j(t) = k_j$ and $n_j(t) = n_j$, $j = 1, 2$. The two traders are already in an equilibrium state and there is no way to let them change their state;

if $k_1 = k_2 =: k$ but $n_1 \neq n_2$ then, since $\epsilon_+^2 - \epsilon_-^2 = \frac{(k+M)!}{(k-M)!}(n_2 - n_1)$, it follows that $\epsilon_+^2 - \epsilon_-^2 > 0$ if $n_2 > n_1$ and it is negative otherwise. This implies that, for small t, $n_1(t)$ increases with t if $n_2 > n_1$ and decreases if $n_2 < n_1$. This means that the trader with more shares tends to sell some of his shares to the other trader, to increase his liquidity. Moreover, since $k_1(t) = Q_1 - n_1(t)$, $k_1(t)$ decreases when $n_1(t)$ increases and viceversa.

We also find

$$\Pi_1(t) \simeq \Pi_1(0) + (\gamma - M)t^2\, p_{12}^2\, \frac{(k+M)!}{(k-M)!}\,(n_2 - n_1), \tag{3.15}$$

which shows that, if $\gamma > M$, $\Pi_1(t)$ increases with t if $n_2 > n_1$ and decreases if $n_1 > n_2$. This can be understood as follows: if $\gamma > M$, then the market is giving to the shares a larger value than the amount of cash used to buy them. Therefore, if $n_2 > n_1$, since as we have seen $n_1(t)$ increases for small t, the first trader is paying M for a share whose value is $\gamma > M$. That's way the value of his portfolio increases!

Let now take $n_1 = n_2 =: n$ and $k_1 \neq k_2$. In this case after few algebraic computations we see that $\epsilon_+^2 - \epsilon_-^2 > 0$ if $k_1 > k_2$ while $\epsilon_+^2 - \epsilon_-^2 < 0$ if $k_1 < k_2$. This implies that, if $k_1 > k_2$, then $n_1(t)$ increases while $k_1(t)$ decreases as t increases. Moreover, if $\gamma > M$, then $\Pi_1(t)$ increases its value with t. This can be understood again as before: since $n_1(t)$ is an increasing function, for $t > 0$ small enough, and since the market price of the share γ is larger than M, t_1 improves his portfolio since he spends M to get γ.

(4) As for the second trader, we can easily find $n_2(t)$, $k_2(t)$ and $\Pi_2(t)$ simply recalling that $N = n_1(t) + n_2(t)$ is constant in time.

(5) The case in which $\gamma < M$ can be analyzed in the very same way as before.

IV Mean-field approximation

It is clear that the results of the previous section suffer of the two major approximations: first of all our final considerations have been obtained only in the case of two traders. Considering more traders is technically much harder and goes beyond the real aims of this paper. Secondly, the perturbation expansion we have introduced in (3.12), gives only an approximated version of the exact solution. In this section we propose a particular version of the model considered before which, under a sufficiently general assumption on α_j and β_j, can be explicitly solved in the so-called mean-field approximation. This different version of our model is relevant since it is related to a market in which the number of traders is very large, virtually divergent. In other words, while in the previous section we have considered a stock market with very few traders, using the mean-field approximation we will be able to analyze a different market, namely one with a very large number of traders.

Our model is defined by the same hamiltonian as in (3.5) but with $M = 1$. This is not a major requirement since it corresponds to a renormalization of the price of the share, which we take equal to 1: if you buy a share, then your liquidity decreases of one unit while it increases, again of one unit, if you sell a share. It is clear that all the same integrals of motion as before exist: $\hat{N}$, $\hat{K}$, $\hat{\Gamma}$, $\hat{\Delta}$ and $Q_j = \hat{n}_j + \hat{k}_j$, $j = 1, 2, \ldots, L$. They all commute with H, which we now write as

$$\begin{cases} H = h + h_{po}, \text{ where} \\ h = \sum_{l=1}^{L} \alpha_l \hat{n}_l + \sum_{l=1}^{L} \beta_l \hat{k}_l + \sum_{i,j=1}^{L} p_{ij} \left(a_i^\dagger a_j c_i \, c_j^\dagger + \, a_i \, a_j^\dagger c_j \, c_i^\dagger \right) \\ h_{po} = o^\dagger \, o + p^\dagger \, p + (o^\dagger \, p + p^\dagger \, o), \end{cases} \tag{4.1}$$

For h_{po} we can repeat the same argument as in the previous section and an explicit solution can be found which is completely independent of h. In particular we have $\omega_{\{n\};\{k\};O;M}(\hat{P}) = 1$. For this reason, from now on, we will identify H only with h and work only with this hamiltonian. Let us introduce the operators

$$X_i = a_i \, c_i^\dagger, \tag{4.2}$$

$i = 1, 2, \ldots, L$. The hamiltonian h can be rewritten as

$$h = \sum_{l=1}^{L} \left(\alpha_l \hat{n}_l + \beta_l \hat{k}_l \right) + \sum_{i,j=1}^{L} p_{ij} \left(X_i^\dagger \, X_j + X_j^\dagger \, X_i \right). \tag{4.3}$$

The following commutation relations hold:

$$[X_i, X_j^\dagger] = \delta_{ij} (\hat{k}_i - \hat{n}_i), \quad [X_i, \hat{n}_j] = \delta_{ij} \, X_i \quad [X_i, \hat{k}_j] = -\delta_{ij} \, X_i, \tag{4.4}$$

which show how the operators $\{\{X_i, \, X_i^\dagger, \, \hat{n}_i, \, \hat{k}_i\}, \, i = 1, 2, \ldots, L\}$ are closed under commutation relations. This is quite important, since it produces the following system of differential equations:

$$\begin{cases} \dot{X}_l = i(\beta_l - \alpha_l) X_l + 2i X_l^{(L)} (\hat{n}_l - \hat{k}_l) \\ \dot{\hat{n}}_l = 2i \left(X_l \, {X_l^{(L)}}^\dagger - X_l^{(L)} \, X_l^\dagger \right) \\ \dot{\hat{k}}_l = -2i \left(X_l \, {X_l^{(L)}}^\dagger - X_l^{(L)} \, X_l^\dagger \right) \end{cases}$$

whose first obvious consequence is that $\frac{d}{dt}(\hat{n}_l+\hat{k}_l)=0$, as we already knew from the general analysis of the integrals of motion for our model. Here we have introduced the following *mean* operators: $X_l^{(L)}=\sum_{i=1}^L p_{li}X_i$, $l=1,2,\ldots,L$. Using the constant $Q_l=\hat{n}_l+\hat{k}_l$ and considering only the relevant equations, the above system simplifies and becomes

$$\left\{\begin{array}{l} \dot{X}_l=i(\beta_l-\alpha_l)X_l+2iX_l^{(L)}(2\hat{n}_l-Q_l)\\ \dot{\hat{n}}_l=2i\left(X_l\,X_l^{(L)\dagger}-X_l^{(L)}\,X_l^\dagger\right)\end{array}\right. \tag{4.5}$$

This system, as l takes all the values $1,2,\ldots,L$, is a closed system of differential equations for which an unique solution surely exists. However, in order to find explicitly this solution, it is convenient to introduce now the mean-field approximation which essentially consists in replacing the two-traders interaction p_{ij} with a sort of global interaction (meaning with this that all the traders may *speak* among them) whose strength is inversely proportional to the number of traders: this concretely means that we have to replace p_{ij} with $\frac{\tilde{p}}{L}$, with $\tilde{p}\geq 0$. After this replacement we have that

$$X_l^{(L)}=\sum_{i=1}^L p_{li}X_i\longrightarrow\frac{\tilde{p}}{L}\sum_{i=1}^L X_i,$$

whose limit, for L diverging, only exists in suitable topologies, [6, 7], like, for instance, the strong one restricted to a set of relevant states. Let τ be such a topology. We define

$$X^\infty=\tau-\lim_{L\to\infty}\frac{\tilde{p}}{L}\sum_{i=1}^L X_i, \tag{4.6}$$

where, as it is clear, the dependence on the index l is lost because of the replacement $p_{li}\to\frac{\tilde{p}}{L}$. This is a typical behavior of transactionally invariant quantum systems, where $p_{l,i}=p_{l-i}$. The operator X^∞ belongs to the center of the algebra $\mathfrak{A}$, that is it commutes with all the elements of $\mathfrak{A}$: $[X^\infty,A]=0$ for all $A\in\mathfrak{A}$. In this limit system (4.5) above becomes

$$\left\{\begin{array}{l} \dot{X}_l=i(\beta_l-\alpha_l)X_l+2iX^\infty(2\hat{n}_l-Q_l)\\ \dot{\hat{n}}_l=2i\left(X_l\,X^{\infty\dagger}-X^\infty\,X_l^\dagger\right),\end{array}\right. \tag{4.7}$$

which, following the notation introduced in [8] in a different context, can be called *the semiclassical approximation* of (4.5). This system can now be solved if we assume that

$$\beta_l-\alpha_l=:\Phi \tag{4.8}$$

for all $l = 1, 2, \ldots, L$. Under this assumption, in fact, we can deduce the time dependence of $X^{\infty}(t)$ and, as a consequence, we can completely solve system (4.7). The procedure is as follows:

(i) using (4.7) we construct the following means: $\frac{1}{L}\sum_{l=1}^{L}\dot{X}_l = \frac{d}{dt}X_l^{(L)}$ and $\frac{1}{L}\sum_{l=1}^{L}\dot{\hat{n}}_l$.

(ii) Then we take the $\tau - \lim_{L\to\infty}$ of the system we have obtained in this way. Introducing the following intensive operators

$$\eta = \tau - \lim_{L\to\infty}\frac{1}{L}\sum_{l=1}^{L}\hat{n}_l, \quad Q = \tau - \lim_{L\to\infty}\frac{1}{L}\sum_{l=1}^{L}Q_l, \tag{4.9}$$

which again belong to the center of the algebra, we find that

$$\left\{\begin{array}{l} \dot{X}^{\infty} = i\Phi X^{\infty} + 2iX^{\infty}(2\eta - Q) \\ \dot{\eta} = 2i\left(X^{\infty}\,X^{\infty\dagger} - X^{\infty}\,X^{\infty\dagger}\right) = 0. \end{array}\right. \tag{4.10}$$

This system can be easily solved: $\eta(t) = \eta$ and $X^{\infty}(t) = e^{i\nu t}X_0^{\infty}$, where $\nu = \Phi + 4\eta - 2Q$. Notice that equation $\eta(t) = \eta$ has an obvious interpretation: the various $\hat{n}_l(t)$ change in time in such a way that their mean does not change, see (4.9). This is again a consequence of $[H, \hat{N}] = 0$.

(iii) This solution must be now replaced in (4.7). It is convenient to consider two different situations: $\Phi = \nu$ and $\Phi \neq \nu$. We begin with this last case. With the change of variable $X_l(t) = e^{it\nu}\left\{Z_l(t) + \frac{2}{\Phi-\nu}X_0^{\infty}\,Q_l\right\}$, since both Q_l and X_0^{∞} do not depend on time, we deduce the following system:

$$\left\{\begin{array}{l} \dot{Z}_l = i(\Phi - \nu)Z_l + 4iX_0^{\infty}\hat{n}_l \\ \dot{\hat{n}}_l = 2i\left(Z_l\,X_0^{\infty\dagger} - X_0^{\infty}\,Z_l^{\dagger}\right), \end{array}\right. \tag{4.11}$$

which becomes closed if we also add the differential equation for $Z_l^{\dagger}$. Then we have

$$\dot{\Theta}_l(t) = i\Delta\,\Theta_l(t), \tag{4.12}$$

where we have introduced

$$\Delta \equiv \begin{pmatrix} \Phi - \nu & 4X_0^{\infty} & 0 \\ 2X_0^{\infty} & 0 & -2X_0^{\infty} \\ 0 & -4X_0^{\infty} & -(\Phi - \nu) \end{pmatrix}, \quad \Theta_l(t) \equiv \begin{pmatrix} Z_l(t) \\ \hat{n}_l(t) \\ Z_l^{\dagger}(t) \end{pmatrix}.$$

Remark:– Notice that the procedure developed here implies, as a consequence, that the dynamical behavior of all the traders is driven by the same differential equations. This is a consequence of condition (4.8), which introduce the same quantity Φ for all the traders. Possible differences in the time evolution of the portfolios may arise therefore only because of different initial conditions. We discuss in Appendix 2 a different approximation, which produce different equations of motion for different traders.

The solution of equation (4.12) can be written as

$$\Theta_l(t) = V\, e^{i\Delta_d t}\, V^{-1}\Theta_l(0), \tag{4.13}$$

where V is the matrix which diagonalizes the matrix Δ in the following sense:

$$V^{-1}\Delta V = \Delta_d := \begin{pmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{pmatrix}$$

Remark:– Notice that V needs not to be unitary since Δ is not hermitian.

It is clear that we are only interested in the second component of the vector $\Theta_l(t)$, which is exactly $\hat{n}_l(t)$. Carrying out all the computations and computing the mean value of $\hat{n}_l(t)$ on a state number $\omega_{\{n\};\{k\};O;M}$, we find that

$$n_l(t) = \frac{1}{\omega^2}\left\{n_l(\Phi-\nu)^2 - 8|X_0^\infty|^2\left(k_l(\cos(\omega t)-1) - n_l(\cos(\omega t)+1)\right)\right\}, \tag{4.14}$$

where we have introduced $\omega = \sqrt{(\Phi-\nu)^2 + 16|X_0^\infty|^2}$. This formula shows that $n_l(t)$ is a periodic function whose period, $T = \frac{2\pi}{\omega}$, increases when Φ approaches ν and when $|X_0^\infty|$ approaches zero. It is also interesting to remark that, since $\dot{n}_l(0) = 0$ and $\ddot{n}_l(0) = 8|X_0^\infty|^2(k_l - n_l)$, then $n_l(t)$ is an increasing function for t in a right neighborhood of 0 if $k_l > n_l$, while it is decreasing if $k_l < n_l$. This means that if t_l has a large liquidity, then he spends money to buy shares. On the contrary, if t_l has a lot of shares, then he tends to sell shares and to increase his liquidity, until the situation changes again.

As for the portfolio, its behavior is the following: since $\Pi_l(t) = \Pi_l(0) + (\gamma-1)(n_l(t) - n_l(0))$, it is clear that $\dot{\Pi}_l(0) = 0$ and, if $\gamma > 1$ and $k_l > n_l$, $\ddot{\Pi}_l(0) > 0$. This means that, in a right neighborhood of $t = 0$, $\Pi_l(t)$ increases as we expect, because of the same arguments discussed in the previous section.

Remark:– It may be worth noticing that if $X_0^\infty = 0$ then the number of shares does not change with time. This is a trivial consequence of (4.14) and of the definition of ω, but can also be deduced directly from (4.12) and from the extremely simple expression of Δ in this case. From the first equation in (4.7) and from the definition of Φ we can also deduce that, in this case, $X_l(t) = e^{i\Phi t} X_l(0)$.

Let us finally consider what happens if $\Phi = \nu$. In this case the system (4.7) takes a simpler expression and, again, the solution can be found explicitly. Without going in many details we find $n_l(t) = \frac{Q_l}{2} + (n_l - \frac{Q_l}{2})\cos(\omega t) + B\sin(\omega t)$, where $\omega = 4|X_0^\infty|$ (since $\Phi = \nu$), and $B = \frac{2i}{\omega}(X_0^{\infty\dagger} X_l - X_0^\infty X_l^\dagger)$, which is again periodic with the same period as before.

Let us now briefly consider what happens on states of a different nature. In particular we want to understand if any meaning can be given to a KMS-state, that is, see Appendix 1, to an equilibrium state for a non-zero temperature.

Suppose that this is so, that is that a state ω_β satisfying condition (A.6) can be used to deduce the existence of an equilibrium for the system under consideration. It is well known that $\omega_\beta \neq \omega_{\{n\};\{k\};O;M}$, so that our previous conclusions do not necessarily hold. However, if we consider the easiest non-trivial situation, $X_0^\infty = 0$, it is still true that $X_l(t) = e^{i\Phi t} X_l = e^{i\Phi t} a_l\, c_l^\dagger$. If we now take $A = B^\dagger = X_l$ in (A.6), we find that $e^{\beta\Phi}\omega_\beta(a_l a_l^\dagger c_l^\dagger c_l) = \omega_\beta(a_l^\dagger a_l c_l c_l^\dagger)$. Assume now that ω_β can be factorized as follows, $\omega_\beta = \omega_\beta^{(a)} \otimes \omega_\beta^{(c)}$, with $\omega_\beta^{(a)}$ related to the number of shares and $\omega_\beta^{(c)}$ to the cash, and let us put $m_l^{(a)} = \omega_\beta^{(a)}(a_l\, a_l^\dagger)$, $n_l^{(a)} = \omega_\beta^{(a)}(a_l^\dagger\, a_l)$, $m_l^{(c)} = \omega_\beta^{(c)}(c_l\, c_l^\dagger)$ and $n_l^{(c)} = \omega_\beta^{(c)}(c_l^\dagger\, c_l)$. Then the KMS condition becomes $e^{\beta\Phi} m_l^{(a)}\, n_l^{(c)} = n_l^{(a)}\, m_l^{(c)}$. Since the commutation relations also imply that $m_l^{(a)} = 1 + n_l^{(a)}$ and $m_l^{(c)} = 1 + n_l^{(c)}$, this equality produces the following condition:

$$e^{\beta\Phi} = \frac{n_l^{(a)}(1 + n_l^{(c)})}{n_l^{(c)}(1 + n_l^{(a)})}, \tag{4.15}$$

at least if the denominator is different from zero. A first obvious remark is that, even if the single *two-particles states* may depend on l, the combination in the rhs of equation

(4.15) must not.

In order to analyze condition (4.15), it is convenient to consider three different conditions: (i) $\Phi > 0$, (ii) $\Phi = 0$ and (iii) $\Phi < 0$, and, for each of these situations, the following cases: (a) $n_l^{(a)} > n_l^{(c)}$, (b) $n_l^{(a)} = n_l^{(c)}$ or (c) $n_l^{(a)} < n_l^{(c)}$.

Case (ia): In this case, for all values of $\Phi > 0$, it is not hard to check that an unique pair $(\beta_0, n_{(o)}^{(c)})$ exists such that (4.15) can be verified. It is worth remarking that this also fixes the value of $n_{(o)}^{(a)}$, since Q_l is a constant of motion. It is also possible to check that the smaller $\beta\,\Phi$, the larger the value of $n_{(o)}^{(c)}$, so that $n_{(o)}^{(a)}$ turns out to be smaller.

Case (ib): In this case (4.15) can be verified if and only if $\beta = 0$ independently of the value of $n_l^{(c)}$.

Case (ic): In this case no solution of (4.15) exists.

Case (ii): In this case a solution of (4.15) exists only if $n_l^{(a)} = n_l^{(c)}$.

Finally, if $\Phi < 0$, our conclusions are exactly specular to those in (i): no solution exists for (iiia), $\beta = 0$ is the only possibility for equation (4.15) to hold in case (iiib) and, finally, an unique pair $(\beta_0, n_{(o)}^{(c)})$ exists which verifies (4.15) in case (iiic).

Since $\Phi > 0$ implies that $\beta_l > \alpha_l$ for all l, using the interpretation discussed in Section II we could say that *the inertia of the cash is larger than that of shares.*

Exactly the opposite happens when $\Phi < 0$, since in this case the inertia of the shares is larger than that of cash.

In $\Phi = 0$ an equilibrium can be reached only if the system was already in an equilibrium state for $t = 0$, i.e. if $n_l^{(a)} = n_l^{(c)}$, that is if t_l has the same amount of cash and shares for $t = 0$.

Also, if $\Phi \neq 0$ and if, for $t = 0$, $n_l^{(a)} = n_l^{(c)}$, then an equilibrium can be reached only if $\beta = 0$.

For what concerns the value of the portfolio at the time $\tilde{t}$ in which the equilibrium is reached, we get

$$\Pi_l(\tilde{t}) = \Pi_l(0) + (\gamma - 1)(k_l(0) - n_{(o)}^{(c)})$$

From this we deduce that, when $\gamma > 1$, t_l increments the value of his portfolio if $k_l(0) > n_{(o)}^{(c)}$. But, for this to be possible, the value of β_o (for fixed $\Phi > 0$) must be sufficiently high. If $\gamma < 1$ the trader t_l increments the value of his portfolio if $k_l(0) < n_{(o)}^{(c)}$. In this case the value of β_o (again for fixed $\Phi > 0$) must be sufficiently low.

These considerations suggest therefore to interpret $\beta^{\gamma-1}$ as a kind of *information* reaching the trader t_l, which should be considered together with the information already arising because of α_l and β_l. This is again because we are assuming that a larger amount of information produces a larger increment of the portfolio.

Remark:– It must be observed, however, that in this procedure all the traders receive the same information, since $\beta^{\gamma-1}$ is the same for all t_l. What can make the difference between the traders is the information coming from α_j^{-1} and β_j^{-1}, as suggested in Section II. So we can distinguish between a *global* information, reaching all the traders in the same way, and a *local* information, which may be different from trader to trader.

V Conclusions and outcome

In this paper we have proposed an operator approach to the analysis of some toy models of a stock market. We have shown that non trivial results concerning the dynamical behavior of the portfolio of each trader can be obtained, even using the existence of conserved quantities, i.e., of some operators commuting with the hamiltonian. We have also discussed a possible use of the KMS-states within this contest.

Many things are still to be done. Among these, first of all we should introduce more than a single kind of shares. Then a different, and more realistic, mechanism to determine the price of the shares should be considered. Also, the role of condition (4.8) should be better understood, and a deeper analysis and understanding of KMS-states has to be carried out.

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Appendix 1: Mathematical Background

This Appendix, which is meant only for those who are not familiar with operator algebras and their applications to QM_∞, is essentially based on known results discussed in [9] and [10], for instance. We want to stress that only few useful facts will be discussed here, paying no particular care about the mathematical rigor. In particular we will not insist on the unbounded nature of the operators involved in the game. This is possible since the relevant spectrum of all the operators relevant for our discussion are usually bounded subsets of $\mathbb{R}$.

Let $\mathcal{H}$ be an Hilbert space and $B(\mathcal{H})$ the set of all the bounded operators on $\mathcal{H}$. $B(\mathcal{H})$ is a C*-algebra, that is an algebra with involution which is complete under a norm $\| \,.\, \|$ satisfying the so-called C*-property: $\|A^*A\| = \|A\|^2$, for all $A \in B(\mathcal{H})$. As a matter of fact $B(\mathcal{H})$ is usually seen as a *concrete realization* of an abstract C*-algebra. It has been widely discussed in literature that, as far as physical applications are concerned, it is convenient to assume that the relevant observables of a certain system generate a von Neumann algebra, i.e. a closed subset of $B(\mathcal{H})$, or a topological quasi *-algebra. Let $\mathcal{S}$ be our physical system and $\mathfrak{A}$ the set of all the operators useful for a complete description of $\mathcal{S}$ (sometimes called the *observables* of $\mathcal{S}$). For simplicity reasons it is convenient to assume that $\mathfrak{A}$ is a C* or a von Neumann-algebra, even if this is not always possible. The description of the time evolution of $\mathcal{S}$ is related to a self-adjoint operator $H = H^\dagger$, which will be assumed not to depend explicitly on time, which is called *the hamiltonian* of $\mathcal{S}$. Several equivalent descriptions are possible: the *Schrödinger* or the *interaction* representation, which will not be used here, or the *Heisenberg* representation, in which the time evolution of an observable $X \in \mathfrak{A}$ is given by

$$X(t) = e^{iHt} X e^{-iHt} \tag{A.1}$$

or, equivalently, by the solution of the differential equation

$$\frac{dX(t)}{dt} = ie^{iHt}[H, X]e^{-iHt} = i[H, X(t)], \tag{A.2}$$

where $[A, B] := AB - BA$ is the *commutator* between A and B. The time evolution defined in this way is usually a one parameter group of automorphisms of $\mathfrak{A}$.

In our paper a special role is played by the so called *canonical commutation relations* (CCR): we say that a set of operators $\{a_l,\, a_l^\dagger, l=1,2,\ldots,L\}$ satisfy the CCR if the following hold:

$$[a_l,a_n^\dagger]=\delta_{ln}\mathbb{I},\qquad [a_l,a_n]=[a_l^\dagger,a_n^\dagger]=0 \tag{A.3}$$

for all $l,n=1,2,\ldots,L$. These operators, which are widely analyzed in any textbook in quantum mechanics, see [11] for instance, are those which are used to describe L different *modes* of bosons. The operators $\hat n_l=a_l^\dagger a_l$ and $\hat N=\sum_{l=1}^L \hat n_l$ are both self-adjoint operators. In particular $\hat n_l$ is the *number operator* for the l-th mode, while $\hat N$ is the *number operator of* $\mathcal{S}$.

The Hilbert space of our system is constructed as follows: we introduce the *vacuum* of the theory, that is a vector φ_0 which is annihiled by all the *annihilation* operators a_l: $a_l\varphi_0=0$ for all $l=1,2,\ldots,L$. Then we act on φ_0 with the *creation* operators $a_l^\dagger$:

$$\varphi_{n_1,n_2,\ldots,n_L}:=\frac{1}{\sqrt{n_1!\,n_2!\ldots n_L!}}(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}\cdots(a_L^\dagger)^{n_L}\varphi_0 \tag{A.4}$$

These vectors form an orthonormal set and are eigenstates of both $\hat n_l$ and $\hat N$: $\hat n_l\varphi_{n_1,n_2,\ldots,n_L}=n_l\varphi_{n_1,n_2,\ldots,n_L}$ and $\hat N\varphi_{n_1,n_2,\ldots,n_L}=N\varphi_{n_1,n_2,\ldots,n_L}$, where $N=\sum_{l=1}^L n_l$. For this reason the following interpretation is given: if the L different modes of bosons of $\mathcal{S}$ are described by the vector $\varphi_{n_1,n_2,\ldots,n_L}$ then n_1 bosons are in the first mode, n_2 in the second mode, and so on. The operator $\hat n_l$ acts on $\varphi_{n_1,n_2,\ldots,n_L}$ and returns n_l, which is exactly the number of bosons in the l-th mode. The operator $\hat N$, finally, counts the total number of bosons.

A particle in mode l is created or annihilated by simply acting on $\varphi_{n_1,n_2,\ldots,n_L}$ respectively with $a_l^\dagger$ or a_l. Indeed we have $\hat n_l(a_l\varphi_{n_1,n_2,\ldots,n_L})=(n_l-1)(a_l\varphi_{n_1,n_2,\ldots,n_L})$ and $\hat n_l(a_l^\dagger\varphi_{n_1,n_2,\ldots,n_L})=(n_l+1)(a_l^\dagger\varphi_{n_1,n_2,\ldots,n_L})$.

The Hilbert space is obtained by taking the closure of the linear span of all these vectors.

An operator $Z\in\mathfrak{A}$ is a *constant of motion* if it commutes with H. Indeed in this case equation (A.2) implies that $\dot Z(t)=0$, so that $Z(t)=Z$ for all t.

The vector $\varphi_{n_1,n_2,\ldots,n_L}$ in (A.4) defines a *vector (or number) state* over the algebra $\mathfrak{A}$ as

$$\omega_{n_1,n_2,\ldots,n_L}(X)=<\varphi_{n_1,n_2,\ldots,n_L},X\varphi_{n_1,n_2,\ldots,n_L}>, \tag{A.5}$$

where $<\,,\,>$ is the scalar product in the Hilbert space $\mathcal{H}$ of the theory. To be more precise, we should replace (A.5) with the following formula:

$$\omega_{n_1,n_2,\ldots,n_L}(X) =< \varphi_{n_1,n_2,\ldots,n_L}, \pi(X)\varphi_{n_1,n_2,\ldots,n_L} >,$$

where π is a representation of the (abstract) algebra $\mathfrak{A}$ in the Hilbert space $\mathcal{H}$. We will avoid this unessential complication along this paper.

In general, a state ω over $\mathfrak{A}$ is a linear functional which is normalized, that is such that $\omega(\mathbb{I}) = 1$, where $\mathbb{I}$ is the identity of $\mathfrak{A}$. The states introduced above describe a situation in which the number of all the different modes of bosons is clear. But different states also exist and are relevant. In particular the so-called KMS-state, i.e. the equilibrium state for systems with infinite degrees of freedom, are usually used to prove the existence of phase transitions or to find conditions for an equilibrium to exist. Without going into the mathematical rigorous definition, see [10], a KMS-state ω with inverse temperature β satisfies the following equality:

$$\omega(A\,B(i\beta)) = \omega(B\,A), \tag{A.6}$$

where A and B are general elements of $\mathfrak{A}$ and $B(i\beta)$ is the time evolution of the operator B computed at the complex value $i\beta$ of the time.

Appendix 2: On system (4.7)

We will show now how to solve system (4.7) without using condition (4.8).

For this we introduce the following quantities: $\gamma_l = \beta_l - \alpha_l$, $X^\infty_{\gamma^k} = \tau - \lim_L \frac{1}{L}\sum_{l=1}^L \gamma_l^k X_l$, $k = 1, 2, \ldots$, $\eta_\gamma = \tau - \lim_L \frac{1}{L}\sum_{l=1}^L \gamma_l \hat{n}_l$, and $Q_\gamma = \tau - \lim_L \frac{1}{L}\sum_{l=1}^L \gamma_l Q_l$. Of course, we are assuming here that all these limits do exist. Repeating the same steps as in Section IV, we find the following system:

$$\begin{cases} \dot{X}^\infty = iX^\infty_\gamma + 2iX^\infty(2\eta - Q) \\ \dot{\eta} = 0. \end{cases}$$

To close this system, we also need the differential equation for X^∞_γ which, as it is easily understood, involves $X^\infty_{\gamma^2}$, η_γ and Q_γ. Notice that, in our previous approximation, these operators turned out to be equal respectively to $\Phi^2 X^\infty$, $\Phi\eta$ and ΦQ. Moreover, in that approximation, we also had $X^\infty_\gamma = \Phi X^\infty$, so that the system above was already closed. We improve our original approximation by taking now X^∞_γ as a new variable and replacing $X^\infty_{\gamma^2}$, η_γ and Q_γ with $\tilde\Phi^2 X^\infty$, $\tilde\Phi\eta$ and $\tilde\Phi Q$, where we have introduced $\tilde\Phi = \lim_L \frac{1}{L}\sum_{l=1}^L \gamma_l$, assuming that it exists. It should be noticed that $\tilde\Phi$ extends Φ in the sense that they coincide if $\gamma_l = \Phi$ for all l. The equation for X^∞_γ is therefore $\dot X^\infty_\gamma = i\tilde\Phi^2 X^\infty + 2iX^\infty\tilde\Phi(2\eta - Q)$. For the sake of simplicity we will work here assuming that $X^\infty_\gamma(0) = 0$ and $2\mu + \tilde\Phi = 0$, where $\mu = 2\eta - Q$. With these assumptions, which could be avoided in a more general analysis, we can repeat the same steps as in Section IV, getting the following result:

$$n_l(t) = \frac{1}{\omega_l^2}\left\{n_l(\gamma_l+\tilde\Phi)^2 - \frac{32\mu^2}{\tilde\Phi^2}|X_0^\infty|^2\left(k_l(\cos(\omega_l t)-1) - n_l(\cos(\omega_l t)+1)\right)\right\},$$

where we have introduced $\omega_l = \sqrt{(\gamma_l+\tilde\Phi)^2 + \frac{64\mu^2}{\tilde\Phi^2}|X_0^\infty|^2}$. It is clear now that different traders may have different behaviors, depending on the related value of γ_l: it is interesting to notice, for instance, that if $|\gamma_l| \to \infty$, that is when α_l and β_l are *very different* from each other, then $n_l(t) = n_l$. This is not so for zero or intermediate values of $|\gamma_l|$, for which a non trivial time evolution of $n_l(t)$ is recovered.

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