Manifolds with boundary and of bounded geometry

Thomas Schick*
FB Mathematik — Universität Münster
Einsteinstr. 62 — 48159 Münster
Germany

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Abstract

For non-compact manifolds with boundary we prove that bounded geometry defined by coordinate-free curvature bounds is equivalent to bounded geometry defined using bounds on the metric tensor in geodesic coordinates.

We produce a nice atlas with subordinate partition of unity on manifolds with boundary of bounded geometry, and we study the change of geodesic coordinate maps.

1 Introduction

Manifolds of bounded geometry arise naturally when one deals with non-compact Riemannian manifolds, and are studied extensively in the literature. So far, the focus was on manifolds without boundary.

One main source of examples are coverings of compact manifolds, which are particularly important in the context of L^2-cohomology and other L^2-invariants. These invariants are studied frequently also for manifolds with boundary. Therefore, it is natural to look at more general manifolds with boundary and bounded geometry.

There are mainly two ways to define manifolds of bounded geometry: either one uses bounds on the curvature (and its covariant derivatives) —this is the coordinate-free description— or one uses geodesic charts and bounds on the metric tensor and its derivatives in these coordinates —the coordinate approach. A proof of the equivalence of these two definitions for manifolds without boundary can be found in Eichhorn [4], using Jacobi fields. Related but different

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[*] e-mail: thomas.schick@math.uni-muenster.de
www: http://wwwmath.uni-muenster.de/u/lueck/
Fax: ++49 -251/83 38370
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results are obtained in [1] using synchronous frames. The case of manifolds with boundary causes additional technical difficulties and seems not to be covered in the literature. Therefore we give a proof here, using synchronous frames.

Dealing with manifolds with boundary, in addition to the usual requirements in the interior we must impose boundary regularity conditions. These involve the second fundamental form (in the coordinate-free description) or special charts for the boundary (in the coordinate description).

In the last section, we show that the functions given by a change of geodesic coordinates and their derivatives admit uniform bounds on manifolds of bounded geometry. And we provide one technical tool, namely a nice atlas with subordinate nice partition of unity. This was introduced and used by Shubin [9, 1.2 and 1.3] if the boundary is empty.

This paper grew out of part of the Dissertation [7] of the author, and the results obtained here are used in [8]. I thank my advisor, Prof. Wolfgang Lück, for his constant support and encouragement. I also thank the referees for valuable comments and suggestions concerning the exposition of the paper.

2 Coordinate-free versus coordinate-wise curvature bounds

2.1. Definition. On a Riemannian manifold (M^m, g) with boundary ∂M, R denotes the curvature tensor of M, l the second fundamental form of ∂M, and $\bar{R}$ the curvature tensor of ∂M (with its induced metric). The (Levi-Civita)-covariant derivative of M is denoted with ∇, the one of ∂M with $\bar{\nabla}$. We use ν for the unit inward normal vector field at ∂M.

If not stated otherwise, a manifold M will always have dimension m.

Given an open subset $U \subset M$ and a chart $x = (x_1, \dots, x_m) : U \to \mathbb{R}^m$, we consider the corresponding derivations $\frac{\partial}{\partial x_i}$ as derivations on U, or as elements in the tangent bundle TM. We abbreviate $\partial_i := \frac{\partial}{\partial x_i}$. We let $g_{ij} := g(\partial_i, \partial_j)$ be the metric tensor in the given coordinates and g^{ij} be the coefficients of the inverse matrix.

We use the notation of multi-indices throughout: Let $\alpha = (\alpha_1, \dots, \alpha_m), \beta = (\beta_1, \dots, \beta_m)$ be multi-indices (with $\alpha_i, \beta_i \in \mathbb{N} \cup \{0\}$). Then

$$D^\alpha := D_x^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_m}}{\partial x_m^{\alpha_m}};$$

and we set $\beta \leq \alpha$ if and only if $\beta_i \leq \alpha_i$ for $i = 1, \dots, n$. Define $|\alpha| := \sum_{i=1}^m \alpha_i$.

For $V \subset M$ and $r > 0$ set $U_r(V) := \{x \in M |\ d(x, V) < r\}$. For $p \in M$ set $B(p, r) := U_r(\{p\})$. If $p \in \partial M$, $(B(p, r) \subset \partial M)$ means the corresponding set for ∂M with the induced Riemannian metric.

We use the normal geodesic flow $K : \partial M \times [0, \infty) \to M : (x', t) \mapsto \exp_{x'}^M(t\nu_{x'})$. For $p \in \partial M$ set $Z(p, r_1, r_2) := K((B(p, r_1) \subset \partial M) \times [0, r_2)) \subset M$.

Set $N(s) := K(\partial M \times [0, s])$ if $s \geq 0$.

2.2. Definition. Suppose M is a manifold with boundary ∂M (possibly empty). It is of *(coordinate-free defined) bounded geometry* if the following holds:

(N) *Normal collar:* there exists $r_C > 0$ so that the *geodesic collar*

$$\partial M \times [0, r_C) \to M : (t, x) \mapsto \exp_x(t\nu_x)$$

is a diffeomorphism onto its image (ν_x is the unit inward normal vector).

(TIC) The *injectivity radius* $r_{inj}(\partial M)$ *of* ∂M is positive.

(I) *Injectivity radius of* M*:* There is $r_i > 0$ so that if $r \leq r_i$ then for $x \in M - N(r)$ the exponential map is a diffeomorphism on $B(0,r) \subset T_xM$. Hence, if we identify T_xM with $\mathbb{R}^m$ via an orthonormal frame we have *Gaussian coordinates* $\mathbb{R}^m \supset B(0,r) \overset{\exp_x^M}{\to} M$ around every point in $M - N(r)$.

(B) *Curvature bounds:* For every $k \in \mathbb{N}$ there is $C_k > 0$ so that $\left|\nabla^i R\right| \leq C_k$ and $\left|\bar{\nabla}^i l\right| \leq C_k$ for $0 \leq i \leq k$.

The injectivity radius and curvature bounds are what one is used to for manifolds without boundary (compare e.g. [3, Section 3]). The embedding of the boundary is described by the second fundamental form. Because the injectivity radius does not make sense near the boundary, we replace it by the geodesic collar.

To give the coordinate-wise definition of bounded geometry, we have to explain which charts we want to use:

2.3. Definition. Let M be a Riemannian manifold with boundary ∂M. Fix $x' \in \partial M$ and an orthonormal basis of $T_{x'}\partial M$ to identify $T_{x'}\partial M$ with $\mathbb{R}^{m-1}$. For $r_1, r_2 > 0$ sufficiently small (such that the following map is injective) define *normal collar coordinates*

$$\kappa_{x'} : \underbrace{B(0, r_1)}_{\subset \mathbb{R}^{m-1}} \times [0, r_2) \to M : (v, t) \mapsto \exp^M_{\exp^{\partial M}_{x'}(v)}(t\nu).$$

(We compose the exponential maps of ∂M and of M, and ν is the inward unit normal vector field). The tuple (r_1, r_2) is called the *width of the normal collar chart* $\kappa_{x'}$.

We adopt the convention that the boundary defining coordinate is the last (i.e. m^{th}) coordinate.

For $x \in M - \partial M$ and $r_3 > 0$ sufficiently small the exponential map yields *Gaussian coordinates* (identifying T_xM with $\mathbb{R}^m$ via an orthonormal base)

$$\kappa_x : B(0, r_3) \to M : v \mapsto \exp_x^M(v).$$

We call r_3 the *radius of the Gaussian chart* κ_x.

We use the common name *normal coordinates* for normal collar coordinates as well as Gaussian coordinates.

2.4. Definition. A Riemannian manifold M with boundary ∂M has *(coordinatewise defined) bounded geometry* if and only if (N), (IC), (I) of Definition 2.2 hold and (instead of (B))

(B1) There exist $0 < R_1 \leq r_{inj}(\partial M)$, $0 < R_2 \leq r_C$ and $0 < R_3 \leq r_i$ and constants $C_K > 0$ (for each $K \in \mathbb{N}$) such that whenever we have normal boundary coordinates of width (r_1, r_2) with $r_1 \leq R_1$ and $r_2 \leq R_2$, or Gaussian coordinates of radius $r_3 \leq R_3$ then in these coordinates

$$|D^\alpha g_{ij}| \leq C_K \quad \text{and} \quad \left|D^\alpha g^{ij}\right| \leq C_K \qquad \forall\, |\alpha| \leq K.$$

The numbers R_1, R_2, R_3 and C_K are called the bounded geometry constants of M.

The main result of the paper is the following:

2.5. Theorem. *Let (M^m, g) be a Riemannian manifold with boundary ∂M. To given $C > 0$, $k \in \mathbb{N}$, and dimension m there are $R_1, R_2, R_3 > 0$ and $D > 0$ such that the following holds:*

(a1) If $x \in M - \partial M$, $0 < r_3 \leq R_3$ and $\kappa_x : B(0, r_3) \to (M - \partial M)$ is a Gaussian chart, and if $\left|\nabla^i R\right| \leq C$ for $i = 0, \ldots, k$ on the image of κ_x then in these coordinates

$$|D^\alpha g_{ij}| \leq D \quad \textit{and} \quad \left|D^\alpha g^{ij}\right| \leq D \qquad \textit{whenever } |\alpha| \leq k.$$

(a2) If on the other hand

$$|D^\alpha g_{ij}| \leq C \quad \textit{and} \quad \left|D^\alpha g^{ij}\right| \leq C \qquad \textit{for } |\alpha| \leq k+2$$

then on the image of κ_x we have

$$\left|\nabla^i R\right| \leq D \qquad \textit{for } i = 0, \ldots, k.$$

(b1) If $x' \in \partial M$, $0 < r_1 \leq R_1$, $0 < r_2 \leq R_2$ and $\kappa_{x'} : B(0, r_1) \times [0, r_2) \to M$ is a normal boundary chart, and if $\left|\nabla^i R\right| \leq C$ and $\left|\bar\nabla^i l\right| \leq C$ for $i = 0, \ldots, k$ on the image of $\kappa_{x'}$, then in these coordinates we get

$$|D^\alpha g_{ij}| \leq D \quad \textit{and} \quad \left|D^\alpha g^{ij}\right| \leq D \qquad \textit{whenever } |\alpha| \leq k.$$

(b2) If, on the other hand,

$$|D^\alpha g_{ij}| \leq C \quad \textit{and} \quad \left|D^\alpha g^{ij}\right| \leq C \qquad \textit{for } |\alpha| \leq k+2$$

then, on the image of $\kappa_{x'}$,

$$\left|\nabla^i R\right| \leq D \textit{ and } \left|\bar\nabla^i l\right| \leq D \qquad \textit{for } i = 0, \ldots, k.$$

(c) M has (coordinate-wise defined) bounded geometry if and only if it has (coordinate-free defined) bounded geometry. In particular, we can drop the prefix in notation. The bounded geometry constants of Definition 2.4 can be chosen to depend only on r_i, r_C, $r_{inj}(\partial M)$ and C_k of Definition 2.2.

Observe that (c) follows from (a1)-(b2). Moreover, (a2) and (b2) are immediate consequences of the formulas for R and l (and their covariant derivatives) in local coordinates in terms of g_{ij}, g^{ij} and their partial derivatives (compare 2.54, 3.16 and 5.1 of [5] — note that our charts near the boundary are adapted to the embedding $\partial M \hookrightarrow M$). The statement (a1) about internal points is already included in [4, Theorem A and Proposition 2.3]. It remains to establish (b1). Since in the course of this proof we have to set up most of the notation necessary for the synchronous-frame-proof of (a1), we include a complete proof also of (a1).

The proof is done in four steps. First, we give the argument for $k = 0$, using the Rauch comparison theorem. Secondly, we prove (a1). In the third step, we establish bounds on the curvature tensor of the boundary. Last, we derive (b1).

Step 1: Proof of Theorem 2.5(a1) and 2.5(b1) for $k = 0$

2.6. Proposition. *Suppose we are in the situation of Theorem 2.5(a1) or (b1) and $k = 0$. Suppose $(x_i) := \kappa^{-1} : U \subset M \to \mathbb{R}^m$ is the normal coordinate system. There are $R_1, R_2, R_3 > 0$ and $C_1, C_2 > 0$ (depending only on C and m) such that if for width or radius we have $(r_1, r_2) \leq (R_1, R_2)$ or $r_3 \leq R_3$, respectively, then*

$$C_1 \leq \left| \sum \lambda_i \frac{\partial}{\partial x_i} \right|_{TM} \leq C_2, \qquad \text{if } \sum \lambda_i^2 = 1, \tag{2.7}$$

where $|v|_{TM} := \sqrt{g(v,v)}$ for $v \in TM$.

Moreover, g_{ij} and g^{ij} are bounded with a bound depending only on C and the dimension.

The numbers R_1, R_2 and R_3 of Theorem 2.5 are determined by Proposition 2.6 and (IC), (I), (N).

Proof. The last statement is a reformulation of Inequality (2.7). To prove (2.7), we apply Warner's generalization of the Rauch comparison theorem [10, 4.3]. We compare with two complete manifolds of constant sectional curvature $-C$ and C, respectively. To compare with normal collar coordinates, choose a hypersurface in this manifold so that all the eigenvalues of its second fundamental form at one (comparison) point are equal to C in the first case and to $-C$ in the second case. Inequality (2.7) for vectors orthogonal to $\mathcal{R} := \sum x_i \partial_i$ (in Gaussian coordinates), or orthogonal to ∂_m (in normal boundary coordinates) is just the statement of the comparison theorem, with C_1 and C_2 depending only on the manifold we compare with (i.e. on C and on m). Here, r_1, r_2 and r_3 must be sufficiently small (again depending only on the manifolds we compare with).

The comparison theorem says nothing about $\mathcal{R}$ or about ∂_m, respectively. But for these vectors Euclidean length and length in TM as well as the orthogonal complements coincide by the following Proposition 2.8. Therefore, the inequality is true in general. ☐

In the proof of Proposition 2.6 we used the Gauss lemma:

2.8. Proposition. *Let* (M, g) *be a Riemannian manifold and* $\exp : B(0, R) \to M$ *a Gaussian chart. Pull the metric* g *back to* $B(0, R)$. *Then* $g(\mathcal{R}, \mathcal{R}) = r^2$, *(*$\mathcal{R} = \sum_i x_i\partial_i$*), and* $g(\mathcal{R}, v) = 0$ *if and only if* v *is a tangent vector to a sphere with center the origin* 0.
Let $K : \partial M \times [0, r_C) \to M$ *be the geodesic collar and pull* g *back to* $\partial M \times [0, r_C)$. *Then* $g(\partial_m, \partial_m) = 1$ *and* $g(\partial_m, v) = 0$ *if and only if* v *is tangent to a translate* $\partial M \times \{t\}$.

Proof. Compare [5, 2.93] — the proof there works also for the collar. ☐

Step 2: Proof of 2.5(a1).

Suppose we are in the situation of 2.5(a1) with $p \in M - \partial M$ and Gaussian coordinates $x = (x_1, \dots, x_m) = \kappa_p^{-1} : B(p, r_3) \to \mathbb{R}^m$. We will state a (differential) equation for g_{ij} in terms of the curvature tensor, so that a bound on partial derivatives of the components of the curvature tensor will give corresponding bounds for the metric. Partial and covariant derivative are related by the Christoffel symbols, so we will compute them, too.

Choose an orthonormal base $\{s_i\}$ for T_pM. Using parallel transport along geodesics emanating from p, construct a synchronous orthonormal frame $\{s_i(x)\}$ of the tangent space restricted to $B(p, r_3)$. Let $\{\theta^i\}$ be the frame of 1-forms dual to $\{s_i\}$ (therefore orthonormal). The connection forms θ^i_j for this frame are defined by

$$\nabla s_j = \sum_i \theta^i_j s_i,$$

with associated Christoffel symbols Γ^i_{jk} and curvature tensor R^i_{jkl} given by

$$\theta^i_j = \sum_k \Gamma^i_{jk} dx_k; \qquad d\theta^i_j - \sum_k \theta^i_k \wedge \theta^k_j = \sum_{k,l} R^i_{jkl} dx_k \wedge dx_l.$$

We can express the curvature entirely in terms of s_i and θ^i, which defines K^i_{jkl}:

$$d\theta^i_j - \sum_k \theta^i_k \wedge \theta^k_j = \sum_{k,l} K^i_{jkl} \theta^k \wedge \theta^l.$$

Define functions a^i_j and b^i_j via the equations

$$\theta^i = \sum_j a^i_j dx_j; \qquad dx_i = \sum_j b^i_j \theta^j. \tag{2.9}$$

$$\text{Then}\quad R^i_{jkl}=\sum_{\alpha,\beta}K^i_{j\alpha\beta}a^\alpha_k a^\beta_l\quad\text{and}\quad g_{ij}=\sum_\alpha a^\alpha_i a^\alpha_j;\quad g^{ij}=\sum_\alpha b^i_\alpha b^j_\alpha. \tag{2.10}$$

As matrix, (g_{ij}) is the product of (a^i_j) and its adjoint, and accordingly for (g^{ij}) and (b^i_j). Hence

2.11. Lemma. *There are bounds on a^i_j and b^i_j corresponding to the bounds on g_{ij} and g^{ij} given by Proposition 2.6.*

The Christoffel symbols $\tilde\Gamma^i_{jk}$ of the covariant differentials of ∂_i are given by

$$\nabla_{\partial_k}\partial_j=\sum_i\tilde\Gamma^i_{jk}s_i.$$

Dualizing (2.9) we see that $\partial_j=\sum_\alpha a^\alpha_j s_\alpha$, hence

$$\tilde\Gamma^i_{jk}=\partial_k a^i_j+\sum_\alpha a^\alpha_j\Gamma^i_{\alpha k}. \tag{2.12}$$

Atiyah, Bott, and Patodi [1, a6 and a10] derive the following equations (note that our definition of R^i_{jkl} takes care of the problems described in [2]), where $\mathcal{R}=\sum_i x_i\partial_i$:

$$\mathcal{R}\Gamma^i_{jk}+\Gamma^i_{jk}=\sum_l 2x_lR^i_{jkl}\qquad\forall i,j,k; \tag{2.13}$$

$$(\mathcal{R}^2+\mathcal{R})a^i_l=-2\sum_{j,k}R^i_{jkl}x_jx_k\qquad\forall i,l. \tag{2.14}$$

Set $f_x(t):=t\Gamma^i_{jk}(tx)$. Let $'$ denote differentiation with respect to t. Then

$$f'_x(t)=\Gamma^i_{jk}(tx)+t\sum_l x_l\partial_l\Gamma^i_{jk}(tx)\overset{(2.13)}{=}\sum_l 2t\cdot x_lR^i_{jkl}(tx).$$

$$\implies\quad \Gamma^i_{jk}(x)=f_x(1)=\int_0^1\sum_l 2\tau x_lR^i_{jkl}(\tau x)\,d\tau\qquad\text{and}$$

$$D^\alpha_x\Gamma^i_{jk}(x)=\int_0^1\tau^{|\alpha|}\left(D^\alpha_x(x\mapsto\sum_l x_lR^i_{jkl}(x))\right)(\tau x)\,d\tau. \tag{2.15}$$

Set $f_{il}(t,x):=a^i_l(tx)$. Then $tf_{ij}{}'(t,x)=\mathcal{R}a^i_j(tx)$ and $t^2f''_{il}(t,x)+tf'_{il}(t,x)=\mathcal{R}^2a^i_l(tx)$. By (2.14)

$$t^2f''_{il}+2tf'_{il}=-2t^2\sum_{k,j}R^i_{jkl}(tx)x_jx_k.$$

With $w_{il}(t,x):=t^2f'_{il}(t,x)$ we get $w'_{il}=t^2f''_{il}+2tf'_{il}$. Since $w_{il}(0)=0$,

$$t^2f'_{il}(t,x)=-2\int_0^t\tau^2\sum_{j,k}R^i_{jkl}(\tau x)x_jx_k\,d\tau\qquad\overset{\tau=tu}{\Longrightarrow}$$

$$f'_{il}(t,x) = -2t \int_0^1 u^2 \sum_{j,k,\alpha,\beta} K^i_{j\alpha\beta}(tux) a^\alpha_k(tux) a^\beta_l(tux) x_j x_k \, du. \tag{2.16}$$

Now we are in the position to explain how the bounds on R and its covariant derivatives up to order k give rise to bounds on g_{ij}, g^{ij} and their partial derivatives up to order k. Because of (2.10) we can consider a^i_j and b^i_j instead of the metric tensor. Moreover, the case $k=0$ is done by Proposition 2.6.

2.17. Lemma. *Let A, B be matrix valued functions which are inverse to each other. Then*

$$\frac{\partial}{\partial x_i} B = \frac{\partial}{\partial x_i}(A^{-1}) = -A^{-1}(\frac{\partial}{\partial x_i} A) A^{-1} = -B(\frac{\partial}{\partial x_i} A) B.$$

Iterated application of this and of the product rule yields

$$D^\alpha_x B = P_\alpha(B, D^\beta_x A;\ \beta \le \alpha),$$

where P_α is a fixed polynomial in non-commuting variables. Bounds for the partial derivatives of A up to order k and on B yield bounds for the partial derivatives of B.

Lemma 2.17 applies to the matrices $A = (a^i_j)$ and $B = (b^i_j)$. Moreover, by Proposition 2.6, we have a bound for (b_{ij}). Hence it remains to find bounds for the derivatives of (a^i_j).

2.18. Lemma. *For $\alpha = (\alpha_1, \dots, \alpha_n)$ there is a polynomial $P_{\alpha,ijkl}$ (only depending on α, i, j, k, l) in partial derivatives up to order $(|\alpha| - 1)$ of K^*_{***}, Γ^*_{**}, and a^*_* such that as functions on the set $B(p, r_3)$*

$$(\nabla_{\partial_1})^{\alpha_1} \dots (\nabla_{\partial_n})^{\alpha_n} R(s_i, s_j, \partial_k, \partial_l) = D^\alpha_x K^i_{jkl} + P_{\alpha,ijkl}.$$

Proof. This follows from the formula for covariant differentials in coordinates. Note that for $|\alpha| = 1$ only Γ^i_{jk} shows up (since K^i_{jkl} is defined entirely in terms of s_i). But if we iterate the covariant differentials, we have to take into account that we contracted ∇R with ∂_i and not with s_i. This yields (via $\nabla \partial_i$) $\tilde\Gamma^i_{jk}$ and, since we iterate the covariant differentials, their partial derivatives up to order $|\alpha| - 2$. Since $\tilde\Gamma^i_{jk} = \partial_k a^i_j + \sum_\alpha a^\alpha_j \Gamma^i_{\alpha k}$, the result follows. □

Now we proceed by induction on the order of derivatives $|\alpha|$. For $|\alpha| = 0$ observe that by assumption we have a bound on the curvature. Since $\{s_i\}$ is orthonormal this gives bounds on K^i_{jkl}. By Proposition 2.6 the same is true for a^i_j.

Assume by induction that for $r \ge 0$ we have found bounds on the partial derivatives up to order r of K^i_{jkl} and a^i_j and on the derivatives up to order $(r-1)$ of Γ^i_{jk}. The assumptions of the Theorem give bounds on $|R|, \dots, |\nabla^{r+1} R|$.

From equation (2.10), relating K^i_{jkl} and R^i_{jkl}, we get bounds on the partial derivatives up to order r of R^i_{jkl}. Then Equation (2.15) yields bounds for the derivatives of order r of Γ^i_{jk}. Lemma 2.18 and the bound on $\nabla^{r+1}R$ yield bounds on $(r+1)$-order partial derivatives of K^i_{jkl} (since by Proposition 2.6 the length of ∂_i is controlled). In all instances the new bounds are given in terms of the old ones.

It remains to deal with the derivatives of order $(r+1)$ of a^i_j. Remember Equation (2.16) for $f_{il}(t,x) = a^i_l(tx)$:

$$f'_{il}(t,x) = -2t\int_0^1 u^2 \sum_{j,k,\alpha,\beta} \left(K^i_{j\alpha\beta} a^\alpha_k a^\beta_l\right)(tux) x_j x_k \; du.$$

Let α be a multi-index with $|\alpha| = r+1$. We differentiate the equation with respect to x to get an equation for $D^\alpha_x f_{il}(t,x) = t^{|\alpha|}(D^\alpha a^i_l)(tx)$. This yields

$$\begin{aligned}(D^\alpha_x f_{il})'(t,x) = -2t\int_0^1 \big(u^2 \sum_{j,k,\beta,\gamma} K^i_{j\beta\gamma}(tux)\,\cdot \\ ((D^\alpha_x f_{\beta k}) f_{\gamma l} + +(D^\alpha f_{\gamma l}) f_{\beta k})(tu,x) x_j x_k \; du\big) - 2t\int_0^1 P_\alpha \; du. \qquad (2.19)\end{aligned}$$

Here P_α is a polynomial in t, u, x, partial derivatives up to order $(r+1)$ of K^*_{***} at tux, and partial derivatives up to order r of f_{**} at (tu,x). The left and right hand side of (2.19) are equal as function of x and t. The induction hypothesis implies for $0 \le t \le 1$ with suitable $C_1, C_2 > 0$ the inequality

$$|(D^\alpha_x f_{ij})'(t,x)| \le C_1 \sup_{0\le\tau\le t}\{|D^\alpha_x f_{ij}(\tau,x)|\} + C_2, \qquad (2.20)$$

Moreover, $D^\alpha f(0,x) = 0$ since $|\alpha| \ge 1$.

Let $h(t) := C_2(\exp(C_1 t) - 1)/C_1$ be the unique solution of $h'(t) = C_1 h(t) + C_2$ with $h(0) = 0$. This is a positive monotonous increasing function, with an explicit bound $h(t) \le C := C_2(\exp(C_1) - 1)/C_1$ for $0 \le t \le 1$.
Abbreviate $u^i_j(t) := D^\alpha_x f_{ij}(t,x)$. We will prove $\left|u^i_j(t)\right| \le h(t)$ and therefore

$$\left|D^\alpha a^i_j(x)\right| = \left|u^i_j(1)\right| \le h(1) \le C. \qquad (2.21)$$

This then finishes the induction step. To show $\left|u^i_j(t)\right| \le h(t)$, let h_n be the unique solution of

$$h'_n(t) = C_1 h_n(t) + C_2 + 1/n \quad \text{with } h_n(0) = 0.$$

Then $h_n(t) \xrightarrow{n\to\infty} h(t)$ uniformly for $0 \le t \le 1$. Therefore, it suffices to show $\left|u^i_j(t)\right| \le h_n(t)$. For a contradiction, assume $\left|u^i_j(t)\right| > h_n(t)$ for some n and t. Set $t_0 := \inf_{0\le t}\{\left|u^i_j(t)\right| > h_n(t)\}$. Then $\left|u^i_j(t_0)\right| = h(t_0)$, since $u^j_i(0) = 0 = h(0)$, h_n is monotonous, and $\left|u^i_j(t)\right| \le h_n(t)$ $\forall t \le t_0$. Consequently, $\sup_{t\le t_0}\left|u^i_j(t)\right| = \left|u^i_j(t_0)\right|$. Then (2.20) shows

$$\left|(u^i_j)'(t_0)\right| \le C_1\left|u^i_j(t_0)\right| + C_2 < h'_n(t_0).$$

Moreover, $d/dt\left|u^i_j(t_0)\right| \leq \left|(u^i_j)'(t_0)\right|$ (compare [6, III.3.2] for the difficult case $u^i_j(t_0)=0$ —d/dt is understood to be the right derivative). It follows $\left|u^i_j(t)\right| < h_n(t)$ for $t \in [t_0, t_0+\epsilon)$ and $\epsilon > 0$ sufficiently small. But this contradicts the choice of t_0.

Step 3: Curvature of ∂M.

We adopt the notation of Definition 2.1.

In the following we consider $(0,p)$-tensors T on M and their restriction to ∂M, given by the inclusion $T\partial M \hookrightarrow TM$. We will use the same notation for T and its restriction, the meaning will be clear from the context.

We compute the covariant derivatives $\bar\nabla^k \bar R$ using the following rules:

2.22. Lemma. *Suppose T is a $(0,q)$-tensor on M, S a $(0,p)$-tensor on ∂M, and S^{*_1} the $(1,p-1)$-tensor on ∂M given by $g(S^{*_1}_x(v_2,\dots,v_p),v_1) = S_x(v_1,\dots,v_p)$ for $v_1,\dots,v_p \in T_x\partial M$, where $x\in\partial M$ and S_x, $S^{*_1}_x$ are the values of S and S^{*_1}, respectively, at x. Let σ be a permutation (operating on a multiple tensor product by permutation of the factors) with $\sigma^{-1}(1)\leq p$.*
Let c denote the contraction of a $(0,r)$-tensor with a $(1,s)$-tensor which contracts the r-th entry of the $(0,r)$-tensor. The covariant derivative is understood to be a map $\nabla : C^\infty(E) \to C^\infty(E\otimes T^*M)$. Then the following holds:

1. $\bar\nabla T = \nabla T - \sum_i c(T\otimes l\circ\sigma_i,\nu)$*, where σ_i are appropriate permutations.*
2. $\bar\nabla((T\otimes S)\circ\sigma) = ((\bar\nabla T)\otimes S)\circ\sigma + (T\otimes\bar\nabla S)\circ\sigma$*.*
3. $\bar\nabla c((T\otimes S)\circ\sigma,\nu) = c((\nabla T)\otimes S\circ\sigma',\nu) + c(T\otimes\bar\nabla S\circ\sigma'',\nu) + \sum_i c(c(T\otimes S\circ\sigma,\nu)\otimes l\circ\sigma_i,\nu) + c(T\otimes S\circ\sigma, l^{*_1})$*, with σ', σ'', and σ_i appropriate permutations.*
4. $\bar\nabla c(T,(\bar\nabla^k l)^{*_1}) = c(\bar\nabla T,(\bar\nabla^k l)^{*_1}) + c(T,(\bar\nabla^{k+1} l)^{*_1})$*.*

Proof. Formulas 2. and 4. are well known. Let $v_1,\dots,v_p$ and X be vector fields on ∂M For 1. we compute:

$$\begin{aligned}
&\bar\nabla T(v_1,\dots,v_p,X)\\
&= X.T(v_1,\dots,v_p) - T(\bar\nabla_X v_1,\dots,v_p) - \dots - T(v_1,\dots,\bar\nabla_X v_p)\\
&\overset{\bar\nabla_X Y=\nabla_X Y - l(X,Y)\nu}{=} X.T(v_1,\dots,v_p) - T(\nabla_X v_1,\dots) - \dots\\
&\quad + T(\nu,v_2,\dots)l(v_1,X) + \dots + T(v_1,\dots,v_{p-1},\nu)l(v_p,X)\\
&= \nabla T(v_1,\dots,v_p,X) + \sum_i c(T\otimes l\circ\sigma_i,\nu)(v_1,\dots,v_p,X).
\end{aligned}$$

For 3. set $v_1 := \nu$ and calculate:

$$
\begin{aligned}
&\bar\nabla_X c(T\otimes S\circ\sigma,\nu)(v_2,\dots,v_{p+q}) \overset{v_1=\nu}{=}\\
&= (X.T(v_{\sigma 1},\dots,v_{\sigma p}))S(v_{\sigma(p+1)},\dots,v_{\sigma(p+q)})\\
&\quad + T(v_{\sigma 1},\dots,v_{\sigma p})(X.S(v_{\sigma(p+1)},\dots,v_{\sigma(p+q)}))\\
&\quad - \sum_{\substack{i=1\\ \sigma i\neq 1}}^{p+q} T\otimes S(v_{\sigma 1},\dots,\bar\nabla_X v_{\sigma i},\dots,v_{\sigma(p+q)})\\
&= c(T\otimes\bar\nabla_X S\circ\sigma,\nu)(v_2,\dots,v_{p+q})\\
&\quad + \left(X.T(v_{\sigma 1},\dots,v_{\sigma p}) - \sum_{i=1}^{p} T(v_{\sigma 1},\dots,\nabla_X v_{\sigma i},\dots,v_{\sigma p})\right) S(\dots)\\
&\quad - \sum_{\substack{i=1\\ \sigma(i)\neq 1}}^{p} \underbrace{T(v_{\sigma 1},\dots,l(X,v_{\sigma i})\nu,\dots,v_{\sigma p})}_{=T(\cdots)l(X,v_{\sigma i})} S(\dots)\\
&\quad + T(v_{\sigma 1},\dots,\nabla_X\nu,\dots,v_{\sigma p})S(\dots)\\
&= c(T\otimes\bar\nabla_X S\circ\sigma,\nu)(\dots) + c((\nabla_X T)\otimes S\circ\sigma,\nu)(\dots)\\
&\quad - \sum_i c(c(T\otimes S\circ\sigma,\nu)\otimes l\circ\sigma_i,\nu)(\dots,X) + c(T\otimes S\circ\sigma,\nabla_X\nu)(\dots).
\end{aligned}
$$

If $Y\in C^\infty(T\partial M)$ then

$$
\begin{aligned}
&0 = X.g(\nu,Y) = g(\nabla_X\nu,Y) + g(\nu,\nabla_X Y)\\
&\quad\implies g(\nabla_X\nu,Y) = l(X,Y) = l(Y,X)\\
&0 = X.g(\nu,\nu) = 2g(\nabla_X\nu,\nu).\\
\implies\quad &\nabla_X\nu = l^{*_1}(X) \implies \nabla\nu = l^{*_1}.
\end{aligned}
$$

This finishes the proof. ☐

2.23. Corollary. *$\bar\nabla^k\bar R$ is a finite sum of tensor products and possibly iterated contractions, composed with permutations, involving (i) $\nabla^j R$ for $j\le k$; (ii) $\bar\nabla^j l$ for $j<k$; (iii) $(\bar\nabla^j l)^{*_1}$ for $j<k-1$; and (iv) ν.*
Bounds for the building blocks (i) and (ii) yield a bound for $\bar\nabla^k\bar R$.

Proof. The first statement follows by iterated application of Lemma 2.22. The last statement follows since tensor products and contractions of tensors are bounded in terms of the bounds on the factors, and because permutations are isometric. Note that $|\nu|=1$ and $|S^{*_1}|=|S|$ for an arbitrary tensor S. Moreover, restriction to the boundary only decreases the norm of a tensor. ☐

2.24. Corollary. *If M is a Riemannian manifold of (coordinate-free defined) bounded geometry, the same is true for its boundary.*

Step 4: Proof of Theorem 2.5(b1).

Suppose we are in the situation of 2.5(b1) with $p \in \partial M$ and normal collar coordinates $(x_1, \dots, x_m) = \kappa_p^{-1} : U \to \mathbb{R}^m$ around p. By our convention x_m is the boundary defining coordinate, i.e. $\partial_m|_{\partial M} = \nu$.

First consider ∂M as a Riemannian $(m-1)$-dimensional manifold of its own. Corollary 2.23 shows that bounds on the covariant derivatives $\nabla^j R$ and $\bar{\nabla}^j l$ give rise to bounds on $\bar{\nabla}^j \bar{R}$ $(0 \leq j \leq k)$. As in Step 2 (applied to ∂M) construct the orthonormal frame $\{s_i\}_{1 \leq i \leq m-1}$ of $T\partial M$. Extend this to an orthonormal frame of $TM|_{\partial M}$ by setting $s_m := \nu$. By parallel transport along geodesics with initial speed ν we get a synchronous orthonormal frame of TM on the normal collar neighborhood. Define the dual frame $\{\theta^i\}$, the Christoffel symbols Γ^i_{jk} and $\tilde{\Gamma}^i_{jk}$, the curvature coefficients R^i_{jkl} and K^i_{jkl}, and a^i_j and b^i_j in exactly the same way as in Step 2. Note that (2.10) and (2.12) remain true.

Now we come to the differential equations which relate these quantities. By construction, $\{s_i\}$ is parallel to ∂_m. This translates to

$$c(\partial_m)\theta^i_j = 0, \quad \text{i.e.} \quad \Gamma^i_{jm} = 0 \qquad \forall i, j \tag{2.25}$$

($c(\partial_m)$ denotes contraction with ∂_m). The Lie derivative along ∂_m (denoted by ∂_m) acts on differential forms via $\partial_m = c(\partial_m)d + dc(\partial_m)$. Hence

$$\begin{aligned} \partial_m \theta^i_j = c(\partial_m) d\theta^i_j &\overset{(2.25)}{=} c(\partial_m)(d\theta^i_j - \sum_{k=1}^m \theta^i_k \wedge \theta^k_j) \\ &= c(\partial_m)(\sum_{k,l} R^i_{jkl}) dx_k \wedge dx_l = 2 \sum_k R^i_{jmk} dx_k. \end{aligned}$$

On the other hand, $\partial_m \theta^i_j = \sum_k \partial_m(\Gamma^i_{jk}) dx_k$. Hence, applying D^α yields

$$\partial_m(D^\alpha \Gamma^i_{jk}) = 2 D^\alpha R^i_{jmk} \quad \forall i, j, k. \tag{2.26}$$

Additionally, we need an equation for a^i_j. We apply ∂_m twice to the dual frame. Since $\partial_m = s_m$ we have $c(\partial_m)\theta^i = \delta_{im}$ (δ the Kronecker symbol). Then

$$\partial_m \theta^i = c(\partial_m) d\theta^i + dc(\partial_m)\theta^i = c(\partial_m) d\theta^i.$$

The connection is torsion free. This means

$$\begin{aligned} & d\theta^i = \sum_j \theta^i_j \wedge \theta^j \\ \Longrightarrow\ & \partial_m(\theta^i) = \sum_j c(\partial_m)(\theta^i_j \wedge \theta^j) \overset{(2.25)}{=} -\theta^i_m \\ \Longrightarrow\ & \partial_m^2(\theta^i) = -\partial_m(\theta^i_m) = -2 \sum_k R^i_{mmk} dx_k. \end{aligned} \tag{2.27}$$

The left hand side can be computed in terms of a^i_j

$$\partial_m(\theta^i) = \sum_j \partial_m(a^i_j) dx_j \qquad \Longrightarrow \qquad \partial_m^2(\theta^i) = \sum_j \partial_m^2(a^i_j) dx_j. \tag{2.28}$$

Equating coefficients, applying D^α, and expressing R^i_{jkl} in terms of K^i_{jkl} yields

$$\partial_m^2 D^\alpha a^i_j = -2\sum_{k,l} D^\alpha(K^i_{mkl}a^k_m a^l_j) \qquad \forall i,j. \tag{2.29}$$

For $|\alpha| > 0$ this is (for each point in the boundary) a system of inhomogeneous linear ordinary differential equations for $D^\alpha a^i_j$, with coefficients given by partial derivatives of K^i_{mkl} up to order $|\alpha|$ and of a^i_j up to order $|\alpha| - 1$.

To make use of the differential equation (2.26) and (2.29) we have to determine the initial values (at $x_m = 0$).

If $i, j < m$, $a^i_j|_{x_m=0}$ is given by application of Step 2 to ∂M, which is possible because of Corollary 2.23. In particular, we get bounds for these functions. And by construction $a^i_m = a^m_i = \delta_{im}$. For the first derivative we have $\partial_m a^i_j = -\Gamma^i_{mj}$ (this follows from (2.27) and (2.28)).

Next we compute Γ^i_{jk} on ∂M. By definition $\bar\nabla_{\partial_i} s_j = \nabla_{\partial_i} s_j - l(\partial_i, s_j)\nu$ for $i, j < m$. If we define $l_{ij} := l(\partial_i, s_j)$ then for $j, k < m$

$$\Gamma^i_{jk}|_{x_m=0} = \begin{cases} \bar\Gamma^i_{jk}|_{x_m=0}; & i < m \\ l_{kj}; & i = m. \end{cases}$$

By (2.25) $\Gamma^i_{jm} = 0$ $\forall i, j$. To compute Γ^i_{mj} we use for $j < m$

$$\begin{aligned} g(\nabla_{\partial_j}\partial_m, s_i) = \partial_j g(\partial_m, s_i) - g(\partial_m, \nabla_{\partial_j} s_i) &\overset{\partial_m|_{\partial M}=\nu}{=} -l(\partial_j, s_i) \quad \text{for } i < m \\ 2g(\nabla_{\partial_j}\partial_m, \partial_m) = \partial_j g(\partial_m, \partial_m) &= 0 \\ \implies \nabla_{\partial_j}\partial_m &= -\sum_i l_{ji} s_i \qquad \text{on } \partial M. \end{aligned}$$

It follows for $i < m$

$$\Gamma^i_{mj}|_{x_m=0} = \begin{cases} -l_{ji}; & j < m \\ 0; & j = m. \end{cases}$$

The arguments given in the proof of Step 2 show that if bounds exist on the covariant derivatives up to order k of the second fundamental form and of R (hence by Corollary 2.23 also on $\bar R$) then the initial values of (2.26) and (2.29), namely $D^\alpha\Gamma^i_{jk}|_{x_m=0}$, $D^\alpha a^i_j|_{x_m=0}$ and $\partial_m D^\alpha a^i_j|_{x_m=0}$ are bounded, as long as $\alpha_m = 0$ (if $\alpha = (\alpha_1, \dots, \alpha_m)$). Later, we will by induction on $|\alpha|$ get bounds on the right had sides of (2.26) and (2.29) (using the bounds on the initial values), giving in particular bounds on $\partial_m D^\alpha \Gamma^i_{jk}|_{x_m=0}$ and $\partial_m^2 D^\alpha a^i_j|_{x_m=0}$ which are the initial values of the equation for $D^\beta\Gamma^i_{jk}$ and $D^\beta a^i_j$ where $\beta = (\alpha_1, \dots, \alpha_m + 1)$. We will therefore, inductively, get the required bounds

$$\left|D^\gamma\Gamma^i_{jk}|_{x_m=0}\right| \le C, \quad \left|D^\gamma a^i_j|_{x_m=0}\right| \le C, \quad \left|\partial_m D^\gamma a^i_j|_{x_m=0}\right| \le C. \tag{2.30}$$

We proceed with a bootstrap argument similar to the one in Step 2. We have to find bounds on g_{ij}, g^{ij} and their derivatives. Because of (2.9) it suffices to look at a^i_j and b^i_j and their derivatives. Bounds on a^i_j and b^i_j are given by Lemma 2.11. As in the proof of Step 2, because of Lemma 2.17 we only have to control the derivatives of a^i_j. We do this by induction on the order of these derivatives. To carry out the induction step, we also have to control the derivatives of K^i_{jkl}, R^i_{jkl} and Γ^i_{jk} (and the initial values in (2.30)).

To conclude the start of the induction, Lemma 2.18 and the assumptions give bounds on K^i_{jkl} and (since the length of ∂_i is bounded by Proposition 2.6) on R^i_{jkl}. Integrating Equation (2.26), we find bounds for Γ^i_{jk} (depending also on the given width R_1 of the normal boundary charts).

Assume now by induction that we have bounds on $D^\alpha a_{ij}$, $D^\alpha \Gamma^k_{ij}$, $D^\alpha K^i_{jkl}$, and $D^\alpha R^i_{jkl}$ for $|\alpha| \le r$. By Lemma 2.18, the assumed bound on $\nabla^{r+1} R$ therefore gives bounds on $\nabla^\gamma K^i_{jkl}$ ($|\gamma| = r+1$). Because of these and the bounds on $\nabla^\alpha K^i_{jkl}$, $\nabla^\alpha a^i_j$ ($|\alpha| \le r$) and on the initial values (2.30) we can apply [6, IV.4.2] to (2.29) to obtain bounds on $\nabla^\gamma a^i_j$ ($|\gamma| = r+1$). This in turn, together with the relation (2.10) between R^i_{jkl} and K^i_{jkl} yields bounds on $\nabla^\gamma R^i_{jkl}$ ($|\gamma| = r+1$).

Now we integrate (2.26) to get bounds on $\nabla^\gamma \Gamma^i_{jk}$ ($|\gamma| = r+1$) to finish the induction step and to conclude the proof of Theorem 2.5.

The bounds we obtain, inductively, depend only on the bounds we started with.

3 Technical properties of manifolds of bounded geometry

We use the notation of Definition 2.1.

3.1. Lemma. *Let* (M^n, g) *be a Riemannian manifold with boundary and with bounded geometry as in Definition 2.4. We find* $r_0 > 0$ *such that for all* $r, s \le r_0$ *the following holds:*

1. *If* $x, x' \in \partial M$ *and* $Z(x', s, \frac{2R_2}{3}) \cap U_r(Z(x, \frac{R_1}{2}, \frac{2R_2}{3})) \neq \emptyset$ *then* $Z(x', s, \frac{2R_2}{3}) \subset Z(x, \frac{9R_1}{10}, \frac{2R_2}{3})$.
2. *We find* $0 < D_1(r) < D_2(r)$ *for* $r \ge 0$ *such that*

$$D_1(r) \le \operatorname{vol}(B(x,r)) \le D_2(r) \qquad \forall x \in M - N(\frac{R_2}{3}),\ \text{if } r < R_3$$
$$D_1(r) \le \operatorname{vol}(Z(x', r, \frac{2R_2}{3})) \le D_2(r) \qquad \forall x' \in \partial M, \quad \text{if } r < R_1.$$

$D_1(r)$, $D_2(r)$ and r_0 can be chosen to depend only on the bounded geometry constants.

Proof. Bounded geometry implies the existence of $C_1, C_2 > 0$ so that in normal coordinates

$$\left\|(g^{ij})_{i,j}\right\| < C_1, \quad \|(g_{ij})_{i,j}\| < C_1, \quad \text{and } C_2 \le \sqrt{|\det(g_{ij})|} \le C_1.$$

Observe that $d(Z(x, \frac{R_1}{2}, \frac{2R_2}{3}), M - Z(x, \frac{3R_1}{4}, \frac{9R_2}{10}))$ is bounded independent of $x \in \partial M$, using the bounds on the metric tensor. With all sets and distances in ∂M, $d(B(x, \frac{3R_1}{4}), \partial M - B(x, \frac{9R_1}{10})) \leq R_1/10$. Choose r_0 smaller than half the minimum of these two bounds. If $r, s < r_0$ and $Z(x', s, \frac{2R_2}{3}) \cap U_r(Z(x, \frac{R_1}{2}, \frac{R_2}{2})) \neq \emptyset$ for $x, x' \in \partial M$ then $Z(x', s, \frac{2R_2}{3}) \cap Z(x, \frac{3R_1}{4}, \frac{9R_2}{10}) \neq \emptyset$ which in turn implies $Z(x', s, \frac{2R_2}{3}) \subset Z(x, \frac{9R_1}{10}, \frac{2R_2}{3})$ by the choice of r_0. This proves the first assertion.

The assertion about the volume bounds follows immediately from the upper and lower bounds of $\sqrt{|\det(g_{ij})|}$.

We can choose all constants to depend only on the bounded geometry constants. □

The following is important to do analysis on manifolds of bounded geometry. The corresponding result for empty boundary is due to Shubin [9, A1.2 and A1.3].

3.2. Proposition. *(Partition of unity)*
Let M be a manifold with boundary and of bounded geometry as in Definition 2.4. There are $r_m > 0$ and, for $0 < r < r_m$ constants $C_K > 0$ ($K \in \mathbb{N}$), $M_f \in \mathbb{N}$, all depending only on the bounded geometry constants (and r) such that a covering of M exists by sets $\{U(x_i, r)\}_{i \in I \subset \mathbb{Z}}$ which has the following properties:

1. *$x_i \in \partial M$ for $i \geq 0$ and $U(x_i, r) = Z(x_i, r, \frac{2R_2}{3})$;*
   $x_i \in M - N(\frac{R_2}{2})$ for $i < 0$ and $U(x_i, r) = B(x_i, r)$.
2. *If $s < r_m$ and $x \in M$ then $B(x, s) \cap U(x_i, r) \neq \emptyset$ for at most M_f of the x_i.*
3. *$\{U(x_i, r/2)\}_{i \in I}$ is a covering of M, too.*

Denote with $\kappa_i : B(0, r) \to U(x_i, r)$ ($i < 0$) and $\kappa_i : B(0, r) \times [0, \frac{2R_2}{3}) \to U(x_i, r)$ for $i \geq 0$ the corresponding normal charts.

To this covering, a subordinate partition of unity $\{\varphi_i\}$ exists such that

$$|D^\alpha \varphi_i| \leq C_K \qquad \forall i \in \mathbb{Z} \quad \forall\, |\alpha| \leq K \qquad \textit{(in normal coordinates).}$$

Proof. Set $r_m := \min\{R_1/2, R_2/12, R_3, r_0/2\}$, where r_0 is given by Lemma 3.1. Let $0 < r < r_m$. First choose a maximal set of points $\{x_i \in \partial M;\ i = 0, 1, 2, \dots\}$ such that all $(B(x_i, r/4) \in \partial M)$ are disjoint. Next, choose a maximal set of points $\{x_i \in M - N(\frac{R_2}{2});\ i = -1, -2, \dots\}$ such that all $B(x_i, r/4)$ $(i < 0)$ are disjoint. Note that the set I of i obtained this way may be a proper subset of $\mathbb{Z}$. For $0 < s \leq r_0$ set $U(x_i, s) := B(x_i, s)$ if $i < 0$ and $U(x_i, s) := Z(x_i, s, \frac{2R_2}{3})$ if $i \geq 0$. Then

$$\bigcup_{i<0} U(x_i, r/2) = \bigcup_{i<0} B(x_i, r/2)\} \quad \text{covers } M - N(\frac{R_2}{2}).$$

This is true because else we find $z \in M - N(\frac{R_2}{2})$ which has distance $\geq r/2$ to all of the x_i. Then $B(z, r/4) \cap B(x_i, r/4) = \emptyset\ \forall i < 0$, violating the maximality

of $\{x_i\}_{i<0}$. Similarly, $\{B(x_i, r/2) \subset \partial M\}_{i \geq 0}$ covers $\partial M \implies \{U(x_i, r/2)\}_{i \geq 0}$ covers $N(\frac{2R_2}{3})$.
Now we have to show that the covering $\{U(x_i, r)\}_{i \in I}$ has Property 2. So fix $0 < s < r_m$ and $x \in M$.

- If $x \in N(\frac{R_2}{3})$ and $i < 0$ then $B(x, s) \cap U(x_i, r) = \emptyset$ since $d(N(\frac{R_2}{3}), M - N(\frac{R_2}{2})) = \frac{R_2}{6} > r + s$.

- If $x \in M - N(\frac{R_2}{3})$ then the number N_1 of x_i $(i < 0)$ with $U(x_i, r) \cap B(x, s) \neq \emptyset$ is by Lemma 3.1 bounded by

  $$N_1 \leq \frac{\operatorname{vol}(B(x, s + r))}{\inf_{x_i \in M - N(\frac{R_2}{2})} \operatorname{vol}(B(x_i, r/4))} \leq \frac{D_2(2r_m)}{D_1(r/4)}$$

  since for such x_i we have $B(x_i, r/4) \subset B(x, s + r)$ and all of these are disjoint.

- If $x \in M - N(R_2)$ then $B(x, s) \cap U(x_i, r) = \emptyset$ for $i \geq 0$ since $d(N(\frac{2R_2}{3}), M - N(R_2)) = \frac{R_2}{3} > s$.

- If $x \in N(R_2)$ then the number N_2 of x_i $(i \geq 0)$ with $B(x, s) \cap U(x_i, r) \neq \emptyset$ is bounded by

  $$N_2 \leq \frac{\sup_{x_i \in \partial M} \operatorname{vol}(Z(x_i, \frac{9R_1}{10}, \frac{2R_2}{3}))}{\inf_{x_i \in \partial M} \operatorname{vol}(Z(x_i, \frac{r}{4}, \frac{2R_2}{3}))} \leq \frac{D_2(9R_1/10)}{D_1(r/4)}$$

  since if there is one such i_0 then for all other such i by Lemma 3.1 $Z(x_i, \frac{r}{4}, \frac{2R_2}{3}) \subset Z(x_{i_0}, 9R_1/10, \frac{2R_2}{3})$, and all the $Z(x_i, \frac{r}{4}, \frac{2R_2}{3})$ are disjoint.

  It follows in all cases

  $$M_f(r) \leq \frac{D_2(9R_1/10) + D_2(2r_0)}{D_1(r/4)} < \infty.$$

It remains to construct the subordinate partition of unity. Choose a smooth cutoff function $\varphi : \mathbb{R}^m \to [0, 1]$ with $\varphi(x) = 1$ if $|x| \leq r/2$ and $\varphi(x) = 0$ if $|x| \geq r$. Denote the restriction to $\mathbb{R}^{m-1}$ also with φ. Choose smooth $\psi : \mathbb{R} \to [0, 1]$ with $\psi(x) = 0$ if $x \geq 2R_2/3$ and $\psi(x) = 1$ if $x \leq R_2/2$. Via the normal coordinates this yields cutoff functions f_i on $U(x_i, r)$ with $f_i \circ \kappa_i(y', t) = \varphi(y')\psi(t)$ if $i \geq 0$ and $f_i \circ \kappa_i = \varphi$ if $i < 0$. Therefore, if κ is any normal chart, $f_i \circ \kappa = \varphi \circ (\kappa_i^{-1} \circ \kappa)$ $(i < 0)$ and $f_i \circ \kappa = (\varphi \cdot \psi) \circ (\kappa_i^{-1} \circ \kappa)$ $(i \geq 0)$. The chain rule shows that the bounds on derivatives up to order K of the coordinate changes (Proposition 3.3) yield bounds on the partial derivatives up to order K of f_i in normal coordinates. To construct the partition of unity, set $F = \sum_{i \in I} f_i$ (at each point there are at most M_f non-zero summands).

$$\text{Since} \qquad M - N(\frac{R_2}{2}) \subset \bigcup_{i<0} U(x_i, r/2) \quad \text{and} \quad N(\frac{R_2}{2}) \subset \bigcup_{i \geq 0} U(x_i, r/2),$$

for each $z \in M$ at least one of $f_i(z) = 1 \implies F \geq 1$. Define

$$\varphi_i := f_i/F.$$

Obviously, $\{\varphi_i\}_{i\in I}$ is a smooth partition of unity subordinate to our covering. Pick one φ_i and one normal chart κ. For partial derivatives up to order K in normal coordinates observe

$$\begin{aligned} |D^\alpha(\varphi_i \circ \kappa)| = \left| D^\alpha \frac{f_i \circ \kappa}{F \circ \kappa} \right| &= \frac{\left| P_\alpha(D^\beta(f_i \circ \kappa), D^\gamma(F \circ \kappa);\ |\beta|, |\gamma| \leq |\alpha|)\right|}{|F \circ \kappa|^{2^{|\alpha|}}} \\ &\overset{|F|\geq 1}{\leq} \left| P_\alpha(D^\beta(f_i \circ \kappa), D^\gamma(F \circ \kappa))\right|. \end{aligned}$$

P_α is a polynomial entirely determined by α. At every point $x \in M$, $D^\gamma(F \circ \kappa)|_x$ is the sum of at most M_f summands of the type $D^\gamma(f_s \circ \kappa)|_x$. Therefore, we have bounds for all the entries of P_α. This yields a bound C_K, depending only on the bounded geometry constants, for $|D^\alpha(\varphi_i \circ \kappa)|$ if $|\alpha| \leq K$. $\square$

Changes of normal coordinates

3.3. Proposition. *Suppose M is a Riemannian manifold with boundary and of bounded geometry. More precisely, suppose $C > 0$ is a bound for partial derivatives up to order $k+1$ of g^{ij} and g_{ij} in normal coordinates. Then $D > 0$ exists, depending only on C so that, if $\kappa_1 : U_1 \subset \mathbb{R}^m \to M$ and $\kappa_2 : U_2 \subset \mathbb{R}^m \to M$ are normal charts as in 2.5, the following holds for $f := \kappa_1^{-1} \circ \kappa_2 : U_0 \subset \mathbb{R}^m \to \mathbb{R}^m$ (U_0 the domain of definition of the composition):*

$$|D^\alpha f| \leq D \quad \forall\, |\alpha| \leq k.$$

Since the maps κ_i are solutions of certain ordinary differential equation, namely the equation for geodesics, we first recall a result about differential equations.

3.4. Lemma. *Let $x'(t) = F(t, x(t))$ be a system of ordinary differential equations ($t \in \mathbb{R}$, $x(t) \in \mathbb{R}^n$), $F \in C^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. Suppose $\varphi(t,x)$ is the flow of this equation. We find a universal expression $Expr_\alpha$, only depending on α such that for all $t \geq 0$ where $\phi(t, x_0)$ is defined*

$$|D_x^\alpha \phi(t, x_0)| \leq Expr_\alpha \left(\sup_{0\leq\tau\leq t} \{\left|D_x^\beta F(\tau, \phi(\tau, x_0)\right|\}|\ \beta \leq \alpha, t \right). \tag{3.5}$$

Proof. The theory of ordinary differential equations [6, V.3.1.] tells us that we have the linear differential equation

$$\alpha'(t) = \frac{\partial F}{\partial x}(t, \varphi(t,x)) \cdot \alpha(t); \quad \alpha(0) = e_k = (0, \ldots, 1, \ldots, 0)$$

for $\partial_k \varphi(t,x)$. For linear differential equations [6, IV.4.2] gives inequalities which directly imply (3.5) if $|\alpha| = 1$.

Inductively one shows that higher derivatives fulfill the linear differential equation

$$(D_x^\alpha\varphi)'(t,x) = (D_xF)(t,\varphi(t,x))\cdot D_x^\alpha\varphi(t,x) + \\ P_\alpha(D_x^\gamma\varphi,(D_x^\beta F)(t,\varphi(t,x));\ \gamma<\alpha,\ \beta\le\alpha) \quad (3.6)$$

with $D_x^\alpha\varphi(0,x)=0$ if $|\alpha|>1$. Here P_α is a polynomial matrix which depends only on α. By induction and using [6, IV.4.2] again, the proposition follows. ☐

Reduction of order implies:

3.7. Corollary. *A statement corresponding to Lemma 3.4 holds for ordinary differential equations of order k.*

3.8. Lemma. *Let $p\in U\subset\mathbb{R}^m$, g_{ij} a Riemannian metric on U and $\exp_p : B(r,0)\to U$ the exponential map at p (we identify T_pU with $\mathbb{R}^m$ via an orthonormal frame). If r is sufficiently small then $\exp_p$ is a diffeomorphism onto some open subset of U, and the derivatives up to order k of $\exp_p$ and its inverse are bounded in terms of g_{ij}, g^{ij}, their derivatives up to order $k+1$, and r.*

Proof. We have $\exp_p(x)=\varphi(x,p,1)$, where φ with $\varphi(x,q,0)=q$, $\varphi'(x,q,0)=x$ is the flow of the differential equation for geodesics. Corollary 3.7 applies to this equation $x''=F(x)$, and $F(x)=-\sum_{i,j}\Gamma_{ij}(x)x_i'x_j'$ is given by g_{ij} and its first order derivatives.

For the inverse, by Lemma 2.17 it suffices to study its first order derivatives. Bounds on these follow from Proposition 2.6. ☐

3.9. Lemma. *Suppose $U',V\subset\mathbb{R}^{m-1}$, $\kappa: U'\times[0,r_C)\to V\times[0,r_C)$ is a normal boundary chart centered at $p\in V$ on the Riemannian manifold $V\times[0,r_C)$ with metric g_{ij} ($g_{im}=\delta_{im}=g_{mi}$). Then the derivatives of κ and its inverse up to order k are bounded in terms of g_{ij}, g^{ij} and their derivatives up to order $k+1$.*

Proof. $\kappa(q,s)=\varphi_1(s\cdot\partial_m,\varphi_2(q,p,1),0,1)$, where φ_1 is the flow of the differential equation for the geodesics in $V\times[0,r_C)$ ($\varphi_1(v,p,\tau,0)=(p,\tau)$, $\varphi_1'(v,(p,\tau),0)=v$), and φ_2 is the flow of the differential equation for geodesics on V. Hence κ is the composition of two flows to which Corollary 3.7, and then Lemma 2.17 and Proposition 2.6 applies exactly as in the previous lemma. ☐

We prove Proposition 3.3 using these Lemmas as follows: By Theorem 2.5 we have bounds for g_{ij} and their derivatives up to order $k+1$ in normal coordinates. Write

$$\kappa_1^{-1}\circ\kappa_2 = (\kappa_1^{-1}\circ\kappa_0)\circ(\kappa_0^{-1}\circ\kappa_2).$$

with κ_0 either being an exponential map or a normal boundary map with suitable range, respectively. If we use κ_1 or κ_2 to pull back the given Riemannian metric to the domain of the charts, $\kappa_0^{-1}\circ\kappa_2$ and $\kappa_1^{-1}\circ\kappa_0$ each fulfill exactly the assumptions of one of the two lemmas. The conclusion of these lemmas is then true for there composition, as well, and the Proposition follows.

3.10. Corollary. *To check condition (B1) in Definition 2.4 it suffices to do this for an atlas of such charts.*

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