How often is a random quantum state k-entangled?

Stanisław J. Szarek[1,2], Elisabeth Werner[1,3], and Karol Życzkowski[4,5]

[1]Université Paris Pierre et Marie Curie–Paris 6, France

[2]Case Western Reserve University, Cleveland, Ohio 44106-7058, USA

[3]Université de Lille 1, France

[4]Institute of Physics, Jagiellonian University, ul Reymonta 4, 30-059 Kraków, Poland

[5]Center for Theoretical Physics, Polish Academy of Sciences

October 7, 2010

Abstract

The set of trace preserving, positive maps acting on density matrices of size d forms a convex body. We investigate its nested subsets consisting of k-positive maps, where $k = 2, \ldots, d$. Working with the measure induced by the Hilbert-Schmidt distance we derive asymptotically tight bounds for the volumes of these sets. Our results strongly suggest that the inner set of $(k+1)$–positive maps forms a small fraction of the outer set of k–positive maps. These results are related to analogous bounds for the relative volume of the sets of k–entangled states describing a bipartite $d \times d$ system.

PACS: 03.65.Aa, 03.67.Mn, 02.40.Ft

1 Introduction

The structure of the set of entangled quantum states is a subject of vivid scientific interest in view of possible applications in the theory of quantum information processing. However, even in the simplest case of systems composed of two subsystems only, several basic problems related to the phenomenon of quantum entanglement remain unsolved. For instance, sufficient and necessary conditions for separability of an arbitrary quantum state are established only in the case of four and six dimensional Hilbert spaces $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^6 = \mathbb{C}^2 \otimes \mathbb{C}^3$.

In higher dimensions several conditions are known which imply the property of quantum entanglement [1]. Unfortunately these tools are not universal and even in the case of a 3×3 system there exist quantum states, the entanglement of which cannot be diagnosed with the general criteria currently available. The structure of the set of separable states is thus not well understood, and its geometry is still a subject of recent studies [2–4].

The complexity of the set of quantum states for a $d \times d$ system increases quickly with the dimension. For $d \geq 3$ it is useful to distinguish different degrees of quantum entanglement. A state ρ, represented by a Hermitian, positive semi-definite matrix with unit trace (density matrix or density operator), is called *separable*, or 1–*entangled*, if it belongs to the convex hull of the set of product states [5]. More generally, for $1 \leq k \leq d$, one introduces the set of k-entangled states, the states with Schmidt rank not larger than k [6] (see section 2 for a precise definition). Any k–entangled state belongs by definition to the larger set of $(k+1)$–entangled states and, in this convention, the set of d–entangled states coincides with the set of all states of a bi–partite, $d \times d$ system.

For large d, the set of separable mixed states of a bipartite $d \times d$ system is known to cover only a small fraction of the entire body of mixed states of size d^2. Asymptotically sharp estimates for these ratios are known and bounds for the radius of the maximal ball inscribed into the set of separable mixed states were obtained, both in the bipartite and in the multipartite case [7–14]. The ratio between these volumes depends additionally on the measure used [15–18].

The structure of the set of states of a composite, bi–partite system is closely related to properties of the set of linear maps that send the convex body of normalized mixed states of a mono-partite system into itself. A map is called *positive* if any positive (semi-definite) matrix is mapped into a positive matrix. If $k \geq 1$, a map Φ is called k*–positive* if the extended map $\Phi \otimes \mathcal{I}_k$ is positive (here $\mathcal{I}_k$ is the identity map on $\mathcal{M}_k$, the space of $k \times k$ matrices). A map Φ is *completely positive* (CP) if the extended map is positive for all $k \in \mathbb{N}$. Note that if a map $\Phi : \mathcal{M}_d \to \mathcal{M}_d$ is d-positive, it is also completely positive [19], and so only the range $1 \leq k \leq d$ is of interest. In general, the characterization of a set of k–positive maps is not easy [20, 21], and the geometry of the set of maps acting on $\mathcal{M}_d$ is nontrivial even in the simplest case of $d = 2$ [2, 22].

The correspondence between the sets of quantum maps and quantum states can be made precise due to the Choi–Jamiołkowski isomorphism: the set of trace preserving, completely positive maps acting on $\mathcal{M}_d$ is isomorphic with the set of these bi–partite states from $\mathcal{M}_{d^2}$, for which the partial trace over one (say, the second) subsystem is proportional to the identity matrix [23, 24]. Furthermore, the (larger) set of k–positive maps is the dual of the set of k–superpositive maps (see section 3 for details and fine points). These maps, also called k–entanglement breaking channels, correspond by the Choi–Jamiołkowski isomorphism to k–entangled states of a bipartite system [21,25]. Thus investigating relations between sets of maps of different degree of positivity one can establish properties of the subsets of $\mathcal{M}_{d^2}$ characterized by different classes of quantum entanglement [21, 26], and *vice versa*.

One of the main objects of this work is to derive bounds for the volume radius of the convex body of trace preserving, k–positive maps acting on a d–dimensional system. We are working with respect to the Hilbert-Schmidt measure, induced by the Euclidean (Hilbert–Schmidt, or Frobenius) distance in the space of quantum maps. These results allow us to estimate the ratio of the volumes of the different sets. In large dimension, one finds that the property of "additional degrees of positivity" is very rare. While our methods allow only to compare the set of k–positive maps with that of ak–positive maps, where $a > 1$ in a universal constant (independent of k, d, but possibly not-so-small), the results obtained strongly suggest that the set of $(k + 1)$–positive maps occupies only a small part of the larger set of k–positive maps, with the trend particularly pronounced for small k's.

To arrive at these conclusions, we study first the volumes of the nested sets of k–entangled states of a $d \times d$ bi-partite system, with $k = 1, 2, \ldots, d$. Ratio of these volumes can be used to estimate the probabity that a random quantum density matrix of a bipartite system distributed according to the Hilbert-Schmidt measure [27] is k–entangled. Of special importance is the case $k = 2$, as knowledge about the set of 2–entangled states is crucial for problems related to the distillation of quantum entanglement [28]. It would be of substantial interest to rigorously show that, for large dimension d, the set of 2–entangled states is small in comparison to the set of 3-entangled states, but large with respect to the set of separable states.

This work is organized as follows. In the next section we introduce some necessary definitions involving the sets of trace preserving k–positive maps and the set of normalized, k–entangled states of a bi–partite system. The duality relations between convex cones and tools used to estimate volumes of dual sets are discussed in section 3. The main results of this paper are obtained in section 4, where we derive bounds for the volume radius of the set of normalized k–entangled states and related parameters. A summary of the results obtained and their discussion is presented in the final section 5.

2 Trace preserving k-positive maps and normalized k-entangled states: concepts and notation

In this section we recall necessary definitions and introduce notation used throughout the paper.

If $\mathcal{H}$ is a Hilbert space, we will denote by $|\cdot|$ its norm and by $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators on $\mathcal{H}$. Most often we will have $\mathcal{H} = \mathbb{C}^d$ for some $d \in \mathbb{N}$, then operators are just matrices and we will write $\mathcal{M}_d$ for $\mathcal{B}(\mathbb{C}^d)$. We will generally use the Dirac bra-ket notation, but in some contexts we will employ the (more common in the functional analysis literature) symbols $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$ for the scalar product.

Transformations that are discrete in time can be described by linear *quantum maps*, or *super-operators*, $\Phi : \mathcal{M}_d \to \mathcal{M}_d$ (or, more generally, $\Phi : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$). A map is called *positive* (or *positivity-preserving*) if every positive (semi-definite) operator is mapped into a positive operator.

Let $1 \leq k \leq d$. A map $\Phi : \mathcal{M}_d \to \mathcal{M}_d$ is called *k-positive* if the extended map $\Phi \otimes \mathcal{I}_k : \mathcal{M}_d \otimes \mathcal{M}_k \to \mathcal{M}_d \otimes \mathcal{M}_k$ is positive, where $\mathcal{I}_k$ is the identity map on $\mathcal{M}_k$. The set of k-positive maps on $\mathcal{M}_d$ is a convex cone and will be denoted by $\mathcal{P}_k(\mathcal{M}_d)$.

A map Φ is k-positive iff the Choi matrix $C_\Phi = \sum_{i,j=1}^d E_{ij} \otimes \Phi(E_{ij})$ is *k-block positive*, i.e. if

$$\Big\langle C_\Phi \Big(\sum_{i=1}^k u_i \otimes v_i \Big), \sum_{j=1}^k u_j \otimes v_j \Big\rangle \geq 0 \tag{1}$$

for all $u_i, v_j \in \mathbb{C}^d$, $1 \leq i, j \leq d$ (see, e.g., [21]). Thus we can identify the cone $\mathcal{P}_k(\mathcal{M}_d)$ via the Jamiołkowski–Choi isomorphism $\Phi \to C_\Phi$ with the cone of k-block positive operators on $\mathbb{C}^d \otimes \mathbb{C}^d$,

$$\mathcal{P}_k(\mathcal{M}_d) \sim \mathcal{BP}_k(\mathbb{C}^d \otimes \mathbb{C}^d).$$

As is apparent from condition (1), the cone $\mathcal{BP}_k(\mathbb{C}^d \otimes \mathbb{C}^d)$ is dual to the cone of k-entangled operators on $\mathbb{C}^d \otimes \mathbb{C}^d$, i.e., to

$$\mathrm{Ent}_k(\mathbb{C}^d \otimes \mathbb{C}^d) = \mathrm{conv} \left(\Big\{ |\xi\rangle\langle\xi| \,:\, \xi = \sum_{j=1}^k u_j \otimes v_j,\ u_j, v_j \in \mathbb{C}^d \text{ for } j = 1, \ldots, k \Big\} \right) \tag{2}$$

Vectors of the form $\xi = \sum_{j=1}^k u_j \otimes v_j$ will be called k-entangled. Observe that the special case of $k = 1$ coincides with the definition of separable (product) matrices or vectors.

A map $\Phi : \mathcal{M}_d \to \mathcal{M}_d$ is said to be *k-super positive* if its Choi matrix C_Φ is k-entangled. This turns out to be equivalent to the existence of a Kraus representation $\Phi(\rho) = \sum_i A_i^\dagger \rho A_i$, of Φ such that all the operators A_i are of rank smaller than or equal to k [21]. We denote the convex cone of k-superpositive maps on $\mathcal{M}_d$ by $\mathcal{SP}_k(\mathcal{M}_d)$. As before, we have the identification

$$\mathcal{SP}_k(\mathcal{M}_d) \ \sim\ \mathrm{Ent}_k(\mathbb{C}^d \otimes \mathbb{C}^d)$$

and, in the appropriate sense, the cones of maps $\mathcal{P}_k(\mathcal{M}_d)$ and $\mathcal{SP}_k(\mathcal{M}_d)$ are dual to each other. Note that $\mathcal{SP}_d(\mathcal{M}_d) = \mathcal{P}_d(\mathcal{M}_d) = \mathcal{CP}(\mathcal{M}_d)$, the convex cone of completely positive maps, while for an arbitrary $k \in \mathbb{N}$ we have $\mathcal{SP}_k(\mathcal{M}_d) \subset \mathcal{CP}(\mathcal{M}_d) \subset \mathcal{P}_k(\mathcal{M}_d)$.

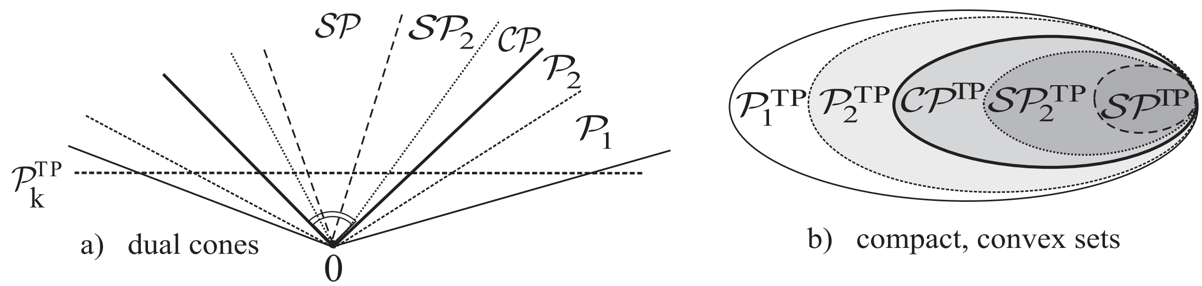

Figure 1: Sketch of sets of maps for the case $d = 3$. a) Convex cone $\mathcal{P} = \mathcal{P}_1$ of positive maps includes the cone $\mathcal{P}_2$ of 2–positive maps and its subcone $\mathcal{P}_3$ containing the 3–positive maps, which in the case of $d = 3$ coincides with the set $\mathcal{CP}$ of completely positive maps. It includes the dual sets of superpositive maps $\mathcal{SP}_2$ and $\mathcal{SP} = \mathcal{SP}_1$. b) Cross-section of the cones with the hyperplane corresponding to the trace preserving condition yields a sequence of nested convex bodies, $\mathcal{P}_k^{TP}$, the volumes of which we aim to estimate.

If we are interested in quantum *states*, or density matrices, we impose the normalization $\operatorname{tr}\rho = 1$. In other words, we are then investigating the base of the corresponding cone, which we will denote by a superscript "^{1}". Thus, in particular,

$$\mathrm{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d) = \mathrm{Ent}_k(\mathbb{C}^d \otimes \mathbb{C}^d) \cap \{M \in \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d) : \operatorname{tr} M = 1\} \tag{3}$$

is the set of *k-entangled states* on $\mathbb{C}^d \otimes \mathbb{C}^d$. These will be the primary objects of our analysis. One similarly defines the dual object $\mathcal{BP}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d)$ of normalized k-block positive operators, which are not necessarily states as they may be non-positive-semi-definite.

On the other hand, when we are interested in quantum *maps*, the *trace preserving* constraint "$\operatorname{tr}\Phi(\rho) = \operatorname{tr}\rho$ for $\rho \in \mathcal{M}_d$" (or, dually, the *unital* constraint) is more appropriate. This will be indicated by a superscript "^{TP}", for example

$$\mathcal{P}_k^{TP}(\mathcal{M}_d) = \mathcal{P}_k \cap \{\Phi : \mathcal{M}_d \to \mathcal{M}_d \ : \ \forall\, \rho \in \mathcal{M}_d \ \operatorname{tr}\Phi(\rho) = \operatorname{tr}\rho\} \tag{4}$$

is the (convex) set of k-positive, trace preserving maps and similarly for $\mathcal{SP}_k^{TP}(\mathcal{M}_d)$. Note that, under the identifications indicated above,

$$\mathcal{SP}_k^{TP}(\mathcal{M}_d) \subset d\,\mathrm{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d) \quad \text{and} \quad \mathcal{P}_k^{TP}(\mathcal{M}_d) \subset d\,\mathcal{BP}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d). \tag{5}$$

The inclusions are proper for $d > 1$ since the unit trace condition involves just one scalar constraint while the trace preserving condition leads to d^2 independent scalar constraints.

We now focus our attention on the set $\mathrm{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d)$ of k-entangled states. From the definitions (2) and (3) one concludes that it is the convex hull of the set of *pure* k-entangled states

$$\mathrm{Ent}_k^P(\mathbb{C}^d \otimes \mathbb{C}^d) =: \big\{|\xi\rangle\langle\xi| \,:\, \xi \ \text{ is } \ k\text{-entangled}, |\xi| = 1\big\}. \tag{6}$$

In other words, this set consists of projections onto 1-dimensional subspaces of $\mathbb{C}^d \otimes \mathbb{C}^d$ spanned by k-entangled vectors. It will be also convenient to assign a symbol to the set of *vectors* appearing in (6). We set

$$\mathrm{Ent}_k^V(\mathbb{C}^d \otimes \mathbb{C}^d) = \left\{\xi = \sum_{j=1}^k u_j \otimes v_j \ : \ u_j, v_j \in \mathbb{C}^d \text{ for } j = 1, \ldots, k, |\xi| = 1\right\}.$$

Note that the sets Ent_k^P and Ent_k^V "live" in different spaces: while the former is a subset of $\mathrm{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d) \subset \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d)$), the latter is a subset of the sphere of $\mathbb{C}^d \otimes \mathbb{C}^d$. We will elaborate on this difference in section 4.

The tensor product $\mathbb{C}^d \otimes \mathbb{C}^d$ can be canonically identified with the space of $d \times d$ matrices $\mathcal{M}_d$, or with the space of operators on $\mathbb{C}^d$, via the map induced by

$$u \otimes v \to |u\rangle\langle v| \tag{7}$$

(To be precise, operators on $\mathbb{C}^d$ correspond *canonically* to the tensor product $\mathbb{C}^d \otimes \overline{\mathbb{C}^d}$, but we will not dwell on this distinction.) Under this identification, the set Ent_k^V corresponds to the set of operators on $\mathbb{C}^d$ whose rank is at most k, normalized in the Hilbert-Schmidt norm. In the sequel we will tend to not distinguish carefully between these sets, and between tensors and operators.

Since every $\xi \in \mathrm{Ent}_k^V$ admits a Schmidt decomposition, we also have

$$\mathrm{Ent}_k^V(\mathbb{C}^d \otimes \mathbb{C}^d) = \Big\{\xi \ : \ \xi = \sum_{j=1}^k s_j \, u_j \otimes v_j\Big\},$$

where (u_j) and (v_j) are orthonormal sequences in $\mathbb{C}^d$ and $s_j \geq 0$ with $\sum_{j=1}^k s_j^2 = 1$.

Our purpose is to give two-sided estimates for various geometric parameters (such as the volume radius or the mean width) of the convex sets $\mathrm{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d)$, $\mathcal{P}_k^{TP}(\mathcal{M}_d)$ and $\mathcal{SP}_k^{TP}(\mathcal{M}_d)$. Our approach will be to study first the set $\mathrm{Ent}_k^V(\mathbb{C}^d \otimes \mathbb{C}^d)$, and then to deduce the needed estimates for the remaining sets.

3 Duality relations and generalities on volume radii

In this section we recall the duality relations between the sets of k–positive maps and k–superpositive maps. Due to the classical Urysohn and Santaló inequalities, and to the relatively more recent inverse Santaló inequality, duality allows us to relate estimates for the volumes of the convex bodies of appropriately normalized trace preserving k–positive maps, k–superpositive maps and the corresponding sets of k–entangled states.

3.1 Dual cones and dual sets

Let $\mathcal{C}$ be a (closed convex) cone in a real inner product space $\mathcal{K}$ and $\mathcal{C}^*$ the dual cone, i.e.,

$$\mathcal{C}^* = \{y \in \mathcal{K} \,:\, \langle y, x\rangle \geq 0 \;\text{ for all }\; x \in \mathcal{C}\}.$$

As was already indicated above, we have the following duality relation for our cones of interest

$$\mathcal{BP}_k(\mathbb{C}^d \otimes \mathbb{C}^d) = \Big(\mathrm{Ent}_k(\mathbb{C}^d \otimes \mathbb{C}^d)\Big)^*, \tag{8}$$

where the ambient inner product space is the Hermitian part of $\mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d)$ endowed with the Hilbert-Schmidt scalar product $\langle A, B\rangle_{HS} := \mathrm{tr}AB^\dagger$ (which in the present context is just $\mathrm{tr}AB$ since B is Hermitian), and similarly

$$\mathcal{SP}_k(\mathcal{M}_d) = \Big(\mathcal{P}_k(\mathcal{M}_d)\Big)^*, \tag{9}$$

where the duality for maps is defined via their Choi matrices [2, 19] by

$$(\Phi, \Psi) := \langle C_\Phi, C_\Psi\rangle_{HS}.$$

Since for closed convex cones we have the bipolar theorem $\big(\mathcal{C}^*\big)^* = \mathcal{C}$, the roles of the cones in (8), (9) can be exchanged.

It is elementary, but not very well known that the duality of cones passes to duality $^\circ$ (polarity) of *bases* of cones. Here $^\circ$ is the standard polar defined by $K^\circ = \{x : \langle x, y\rangle \leq 1 \text{ for all } y \in K\}$. (In particular, if K is the unit ball in some norm, K° is the unit ball in the dual norm.) Let $\mathcal{C} \subset \mathcal{K}$ be a closed convex cone and let $e \in \mathcal{C} \cap \mathcal{C}^*$ be a unit vector. Put $V^{\mathrm{b}} := \{x \in \mathcal{K} \,:\, \langle x, e\rangle = 1\}$ and let $\mathcal{C}^{\mathrm{b}} = \mathcal{C} \cap V^{\mathrm{b}}$ be the base of the cone $\mathcal{C}$. Making use of Lemma 1 from [26] one obtains a relation

$$(\mathcal{C}^*)^{\mathrm{b}} := \mathcal{C}^* \cap V^{\mathrm{b}} = \{y \in V^{\mathrm{b}} \,:\, \forall x \in \mathcal{C}^{\mathrm{b}} \;\; \langle -(y-e), x-e\rangle \leq 1\}. \tag{10}$$

If we think of V^{b} as a vector space with the origin at e, and of $\mathcal{C}^{\mathrm{b}}$ and $(\mathcal{C}^*)^{\mathrm{b}}$ as subsets of that vector space, then $(\mathcal{C}^*)^{\mathrm{b}} = -(\mathcal{C}^{\mathrm{b}})^\circ$.

In our case the two dual objects (modulo the reflection with respect to e) are the appropriately *rescaled* sets $d\,\mathrm{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d)$ of k–entangled states and of k–positive operators $d\,\mathcal{BP}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d)$. The rescaling by the factor d is needed since the maximally mixed state $\rho_* = d^{-2} I_{\mathbb{C}^d \otimes \mathbb{C}^d}$ is not of unit length in the Hilbert-Schmidt norm, and the correct normalization is $e = d^{-1} I_{\mathbb{C}^d \otimes \mathbb{C}^d} = d\,\rho_*$. In the case of maps it is not necessary to renormalize; however, since we normally insist on the trace preserving condition ^{TP} (which also "destroys" precise duality), estimating the size of the sets $\mathcal{SP}_k^{TP}(\mathcal{M}_d)$ and $\mathcal{P}_k^{TP}(\mathcal{M}_d)$ requires an additional elementary technical tool (Proposition 6 of [26]).

3.2 On volume radii, mean widths, and volumes of dual sets

Let K be a compact subset of $\mathbb{R}^n$. The volume radius of K is defined as

$$\mathrm{vrad}(K) = \left(\frac{\mathrm{vol}(K)}{\mathrm{vol}(B_2^n)} \right)^{1/n},$$

where B_2^n is the unit Euclidean ball. In other words, $\mathrm{vrad}(K)$ is the radius of a Euclidean ball, whose volume is equal to that of K.

Another measure of the size of K is the mean width defined by

$$w(K) = 2 \int_{S^{n-1}} h_K(u)\, du,$$

where du is the normalized measure on S^{n-1} and $h_K(u) = \max_{x \in K} \langle x, u \rangle$ is the support function of K. Urysohn's inequality (see, e.g., [29] or [30]) asserts then that

$$\mathrm{vrad}(K) \;\leq\; \frac{1}{2}\, w(K)\,. \tag{11}$$

Urysohn's inequality is usually stated for convex bodies (i.e., convex compact subsets of $\mathbb{R}^n$ with nonempty interior), but since clearly the width of a set and of its convex hull coincide, it is a posteriori true also for non-convex sets.

It is convenient to note that the spherical integral implicit in the definition of the width can be expressed as an integral with respect to μ_n, the standard Gaussian measure on $\mathbb{R}^n$,

$$\frac{1}{2}\, w(K) = \int_{S^{n-1}} \max_{x \in K} \langle x, u \rangle du = \gamma_n \int_{\mathbb{R}^n} \max_{x \in K} \langle x, y \rangle\, d\mu_n(y), \tag{12}$$

where $\gamma_n = \frac{\Gamma(n/2)}{\sqrt{2}\Gamma(n/2+1/2)} \sim \frac{1}{\sqrt{n}}$. In turn, the Gaussian integral can be interpreted as the expected value of the maximum of a Gaussian process. Such quantities have been extensively studied in probability theory. In particular, the Dudley's inequality ([31], or see [29], Theorem 5.6) asserts that

$$\int_{\mathbb{R}^n} \max_{x \in K} \langle x, y \rangle\, d\mu_n(y) \;\leq\; C \int_0^\infty \sqrt{\log N(K, \varepsilon)} d\varepsilon, \tag{13}$$

where $C > 0$ is a universal numerical constant and $N(K, \varepsilon)$ (the covering number) is the smallest number N such that there are points $x_1 \ldots, x_N$ such $K \subset \cup_{i=1}^N x_i + \varepsilon B_2^n$ (or, more generally, in an arbitrary metric space, the smallest number of balls of radius ε whose union covers K). The expression on the right hand side of (13) is sometimes called *the entropy integral.*

3.3 Santaló and inverse Santaló inequalities

The classical Santaló inequality [32] asserts that if $K \subset \mathbb{R}^m$ is a 0-symmetric convex body and K° its polar body, then $\mathrm{vol}(K)\ \mathrm{vol}(K^\circ) \ \le\ \big(\mathrm{vol}\big(B_2^m\big)\big)^2$ or, in other words,

$$\mathrm{vrad(K)\ vrad(K^\circ)} \le 1. \tag{14}$$

Moreover, the inequality holds also for not-necessarily-symmetric convex sets after an appropriate translation. In particular, if the origin is the centroid of K or of K°, a condition that will be satisfied for all sets we will consider in what follows. Even more interestingly, there is a converse inequality [33], usually called "the inverse Santaló inequality,"

$$\mathrm{vrad(K)\ vrad(K^\circ)} \ge c \tag{15}$$

for some universal numerical constant $c > 0$, independent of the convex body K (symmetric or not) and, most notably, of its dimension m. An argument yielding reasonable value of c, particularly for *symmetric* bodies, was given by Kuperberg [34].

The inequalities (14) and (15) together imply that, under some natural hypotheses (which are verified in most of cases of interest), the volume radii of a convex body and of its polar are approximately (i.e., up to a multiplicative universal numerical constant) reciprocal.

Whenever an upper bound on volume radius is obtained via an estimate for the mean width (and then applying (11)), a lower bound on the size of the polar body can be derived without resorting to inverse Santaló inequality. Instead, one may use the following elementary fact

$$\mathrm{vrad}\big(K^\circ\big) \ge \frac{1}{2}\Big(w(K)\Big)^{-1}, \tag{16}$$

which is just a consequence of Hölder inequality – see e.g. Appendix A in [11].

4 Volume radii for the set of k–entangled states

We are now in a position to derive the key results of this work: bounds for the volume radius of $\mathrm{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d)$, the set of k–entangled states of a $d \times d$ system. We start by estimating the covering numbers of $\mathrm{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d)$. Then we will use the inequality (13) and the identity (12) to estimate the mean width $w\big(\mathrm{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d)\big)$, and subsequently the inequality (11) to obtain an upper bound on its volume radius. A lower bound on the volume radius will follow from inequalities for some norms associated with the sets $\mathrm{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d)$. Due to the Choi-Jamiołkowski isomorphism one deduces then analogous estimates for the volumes of the set $\mathcal{SP}_k^{TP}(\mathcal{M}_d)$ of k–superpositive maps. Eventually, by duality relations, these results lead to bounds for the volume radius of the set $\mathcal{P}_k^{TP}(\mathcal{M}_d)$ of trace preserving k–positive maps acting on the d-dimensional system.

Before proceeding, we offer a few comments on the relation between the unit vectors $\xi \in S_{\mathbb{C}^n} = \{\xi \in \mathbb{C}^n : |\xi| = 1\}$, the cosets $[\xi] \in S_{\mathbb{C}^n}/S_{\mathbb{C}^1} = \mathbb{C}P^{n-1}$ (the complex projective space), and the rank one projections $|\xi\rangle\langle\xi| \in \mathcal{M}_n$. These distinct objects usually are not carefully distinguished in the physics literature since, most of the time, no confusion arises. However, here we need to be careful: even though the correspondence $[\xi] \to |\xi\rangle\langle\xi|$ is a bijection between $S_{\mathbb{C}^n}/S_{\mathbb{C}^1}$ and the set of pure states on $\mathbb{C}^n$, these two distinct "identities" lead to different metric structures. For the projective space $\mathbb{C}P^{n-1} = S_{\mathbb{C}^n}/S_{\mathbb{C}^1}$, the canonical metric is induced by the Euclidean distance on $S_{\mathbb{C}^n} \subset \mathbb{C}^n$ and the quotient map $\xi \to [\xi]$, i.e., the distance between $[\xi]$ and $[\eta]$ is $\min_{z\in\mathbb{C},|z|=1} |\xi - z\eta| = 2^{1/2}(1 - |\langle\xi|\eta\rangle|)^{1/2}$. This distance, based on the Bures metric (see [2]), differs from the Hilbert-Schmidt metric, natural to measure the distance between two density operators, $\||\xi\rangle\langle\xi| - |\eta\rangle\langle\eta|\|_{HS} = 2^{1/2}(1 - |\langle\xi|\eta\rangle|^2)^{1/2}$. The ratio between the Hilbert-Schmidt and the Bures distance *for pure states* is $(1 + |\langle\xi|\eta\rangle|)^{1/2}$, which can take any value between 1 and $\sqrt{2}$. This is actually good news since it tells us that when passing from one framework to the other we *at worst* need to pay the price of a dimension independent factor of $\sqrt{2}$.

4.1 Upper bound for $\mathrm{vrad}\left(\mathrm{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d)\right)$

Since, as mentioned earlier, the width of a set is the same as that of its convex hull, we will be estimating via the Dudley's inequality (13) the mean width of $\mathrm{Ent}_k^P(\mathbb{C}^d \otimes \mathbb{C}^d)$, the set of the extreme points of the set $\mathrm{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d)$ of k-entangled states. In turn, by the remark above, the latter problem reduces – at the cost of a multiplicative factor not exceeding $\sqrt{2}$ – to estimating the entropy integral of the sets of k-entangled *vectors* $\mathrm{Ent}_k^V(\mathbb{C}^d \otimes \mathbb{C}^d) \subset S_{\mathbb{C}^d \otimes \mathbb{C}^d}$. We will employ the identification indicated in (7), i.e.,

$$\mathrm{Ent}_k^V(\mathbb{C}^d \otimes \mathbb{C}^d) \sim \Big\{ \sum_{j=1}^k s_j |u_j\rangle\langle v_j| \ : \ s_j \geq 0, \ \sum_{j=1}^k s_j^2 = 1 \Big\} \subset \mathcal{M}_d,$$

where (u_j) and (v_j) vary over orthonormal sequences in $\mathbb{C}^d$, and the problem reduces to finding, for $\varepsilon \in (0,1)$, ε-nets of this set of $d \times d$ matrices (with respect to the Euclidean, or Hilbert-Schmidt metric) with good bounds on their cardinalities.

Given $\tau = \sum_{j=1}^k t_j |u_j\rangle\langle v_j| \in \mathrm{Ent}_k^V(\mathbb{C}^d \otimes \mathbb{C}^d)$, set

$$E = E_\tau = \mathrm{span}\{u_i : 1 \leq i \leq k\}, \quad F = F_\tau = \mathrm{span}\{v_i : 1 \leq i \leq k\} \tag{17}$$

and let T be the matrix τ considered as an operator acting from F to E. Thus, to each $\tau \in \mathrm{Ent}_k^V(\mathbb{C}^d \otimes \mathbb{C}^d)$ there correspond $(E,F) \in G_{d,k} \times G_{d,k}$ and $T \in S_{HS(F,E)}$ such that

$\tau = TP_F$, where P_H stands for the orthogonal projection of $\mathbb{C}^d$ onto H. Accordingly, the problem reduces to estimating the appropriate covering numbers of the Grassmann manifold $G_{d,k}$ and of $S_{HS(F,E)}$, which – geometrically – is just the unit sphere in a k^2-dimensional (complex) Euclidean space.

Before proceeding, we shall make more precise the metric structure of $G_{d,k}$ implicit in the concept of a net needed for our construction. It will be based on the operator norm $\|\cdot\|_{op}$ on $\mathcal{M}_d$; if E, E' are k-dimensional subspaces of $\mathbb{C}^d$, we set

$$d_{op}(E, E') := \min\{\|U - I\|_{op} \ : \ U \in \mathcal{U}(d), UE = E'\},$$

where $\mathcal{U}(d)$ is the unitary group. We note that d_{op} is the quotient distance induced by the *extrinsic* operator norm on $\mathcal{M}_d \supset \mathcal{U}(d)$ and *not* an intrinsic (geodesic) distance. The corresponding intrinsic distance is the largest principal angle between E and E' and is induced in the same way by the geodesic distance on $\mathcal{U}(d)$. For clarity, we note that if α is the largest principal angle between E and E', then $d_{op}(E, E') = |e^{i\alpha}-1| = 2\sin(\alpha/2) \leq \alpha$, and it is elementary to check that then $\|P_E - P_{E'}\|_{op} = \sin\alpha \leq d_{op}(E, E')$ (this is not going to be used). We now claim that an ε-net $\mathcal{N}_1$ on $G_{d,k}$ (with respect to d_{op}) and ε-nets (with respect to the Hilbert-Schmidt norm $\|\cdot\|_{HS}$) $\mathcal{N}_{E,F}$ on $S_{HS(F,E)} \sim S_{\mathbb{C}^{k^2}}$ for $E, F \in \mathcal{N}_1$ lead to a 3ε-net

$$\mathcal{N} = \{TP_F \ : \ E, F \in \mathcal{N}_1, T \in \mathcal{N}_{E,F}\}$$

in $\mathrm{Ent}_k^V(\mathbb{C}^d \otimes \mathbb{C}^d)$. Indeed, consider an arbitrary element of $\mathrm{Ent}_k^V(\mathbb{C}^d \otimes \mathbb{C}^d)$ induced by $E', F' \in G_{d,k}$ and $T' : F' \to E'$, i.e., $T'P_{F'}$. Let E and F in $\mathcal{N}_1$ be such that

$$d_{op}(E, E') \leq \varepsilon, \quad d_{op}(F, F') \leq \varepsilon.$$

Next, let $U, V \in \mathcal{U}(d)$ be such that $VF = F'$, $UE' = E$ and

$$\|V - I\|_{op} \leq d_{op}(F, F') \leq \varepsilon \ \text{ and } \ \|U - I\|_{op} \leq d_{op}(E, E') \leq \varepsilon$$

Set $S = UT'V$. Then $S \in S_{HS(F,E)}$ and consequently SP_F can be approximated within ε (in the Hilbert-Schmidt norm) by an element of $\mathcal{N}$. Thus it will follow that $\mathcal{N}$ is an 3ε-net of $\mathrm{Ent}_k^V(\mathbb{C}^d \otimes \mathbb{C}^d)$ if we show that SP_F is within 2ε of $T'P_{F'}$. To that end, we note first that $SP_F = UT'VP_F = UT'P_{F'}V$ (the second equality because $P_{F'} = VP_FV^\dagger$) and so

$$\begin{aligned}
\|SP_F - T'P_{F'}\|_{HS} &= \|UT'P_{F'}V - T'P_{F'}\|_{HS} \\
&\leq \|UT'P_{F'}V - T'P_{F'}V\|_{HS} + \|T'P_{F'}V - T'P_{F'}\|_{HS} \\
&\leq \|U - I\|_{op}\|T'P_{F'}V\|_{HS} + \|T'P_{F'}\|_{HS}\|V - I\|_{op} \\
&\leq \varepsilon\|T'\|_{HS} + \varepsilon\|T'\|_{HS} = 2\varepsilon
\end{aligned}$$

as required. It now remains to collect known estimates on covering numbers (with respect to the appropriate metrics) of $G_{d,k}$ and $S_{\mathbb{C}^{k^2}}$. In both cases these estimates are of the form $N(K,\delta) \le \big(C/\delta\big)^{\dim_{\mathbb{R}} K}$, where $C>0$ is a universal constant (for $G_{d,k}$, see Remark 8.4 in [35] for the statement and [36, 37] for proofs and more general setting; the case of $S_{\mathbb{C}^m}$ is classical, see [29], Lemma 4.10). This yields an estimate

$$N(\mathrm{Ent}_k^V(\mathbb{C}^d\otimes\mathbb{C}^d),\varepsilon) \le \#\mathcal{N} \le \big(3C/\varepsilon\big)^{2k^2}\Big(\big(3C/\varepsilon\big)^{4k(d-k)}\Big)^2 = \big(C'/\varepsilon\big)^{2k(4d-3k)} \le \big(C'/\varepsilon\big)^{8kd}$$

as we assumed that $k\le d$. By prior remarks, passing from $\mathrm{Ent}_k^V(\mathbb{C}^d\otimes\mathbb{C}^d)$ to $\mathrm{Ent}_k^P(\mathbb{C}^d\otimes\mathbb{C}^d)$ introduces at worst an extra $\sqrt{2}$ factor, or replacing the constant C' by $C_1=\sqrt{2}C'$. Accordingly, the integrand $\sqrt{\log N(K,\varepsilon)}$ in (13) for $K=\mathrm{Ent}_k^P(\mathbb{C}^d\otimes\mathbb{C}^d)$ is at most $\sqrt{8kd}\,\sqrt{\log(C_1/\varepsilon)}$, while the upper limit of integration is 1. This means that the singularity at $\varepsilon=0$ is integrable, and the final estimate is

$$\begin{aligned}\mathrm{vrad}\,\big(\mathrm{Ent}_k^1(\mathbb{C}^d\otimes\mathbb{C}^d)\big) &\le \frac{1}{2}w(\mathrm{Ent}_k^1(\mathbb{C}^d\otimes\mathbb{C}^d)) = \frac{1}{2}w(\mathrm{Ent}_k^P(\mathbb{C}^d\otimes\mathbb{C}^d)) \le C\gamma_{d^4}C'\sqrt{kd}\\ &\le C_0\frac{k^{1/2}}{d^{3/2}} \qquad (18)\end{aligned}$$

For $k=1$ (separable states) and for $k=d$ (all states) the above bound is known to give the correct order (see, e.g., [13]). In the next subsection we are going to show that the obtained bound is a tight estimate for the volume also for the intermediate cases $k=2,\dots,d-1$.

4.2 The lower bound

We start with an inequality concerning norms on $\mathcal{H}=\mathbb{C}^d\otimes\mathbb{C}^d$. Given $k\in\{1,2,\dots,d\}$, we set, for $\xi\in\mathbb{C}^d\otimes\mathbb{C}^d$

$$\|\xi\|^{(k)} := \max_{\zeta\in\mathrm{Ent}_k^V(\mathbb{C}^d\otimes\mathbb{C}^d)} |\langle\xi,\zeta\rangle| \qquad (19)$$

(Under the identification (7), $\|\cdot\|^{(k)}$ corresponds to the operator norm and $\|\cdot\|^{(d)}$ to the Hilbert-Schmidt/Frobenius norm.) We will need the following elementary inequality

$$\|\xi\|^{(k)} \ge \sqrt{k/d}\,\|\xi\|^{(d)} = \sqrt{k/d}\,|\xi| \quad \text{for all } \ \xi\in\mathbb{C}^d\otimes\mathbb{C}^d. \qquad (20)$$

The norms defined by (19) were studied – independently of this paper – in [38], where they were denoted $\|\cdot\|_{s(k)}$. (Note that the arxiv version of [38] corrects some minor errors present in the published version that are relevant to our argument.) In particular, (20) is a special case of Corollary 3.4 in [38]. We will sketch the argument for completeness. To this end, let $\xi=\sum_{j=1}^d s_j\,u_j\otimes v_j$ be the Schmidt decomposition and, for $\Lambda\subset\{1,2,\dots,d\}$, set $\xi_\Lambda=\sum_{j\in\Lambda} s_j\,u_j\otimes v_j$. If Λ varies over all subsets of $\{1,2,\dots,d\}$ of size k, then $\mathbb{E}\xi_\Lambda=\frac{k}{d}\xi$,

$\mathbb{E}|\xi_\Lambda|^2 = \frac{k}{d}|\xi|^2$ (where $\mathbb{E}$ is the average over all choices of Λ) and $\xi_\Lambda/|\xi_\Lambda| \in \mathrm{Ent}_k^V(\mathbb{C}^d \otimes \mathbb{C}^d)$ for all Λ such that $\xi_\Lambda \neq 0$. We have, on the one hand,

$$\mathbb{E}\langle \xi, \xi_\Lambda\rangle = \frac{k}{d}\langle \xi, \xi\rangle = \frac{k}{d}|\xi|^2$$

while, on the other hand, ignoring Λ's for which $\xi_\Lambda = 0$,

$$\begin{aligned} \mathbb{E}\langle \xi, \xi_\Lambda\rangle &= \mathbb{E}\Big(\Big\langle \xi, \frac{\xi_\Lambda}{|\xi_\Lambda|}\Big\rangle |\xi_\Lambda|\Big) \leq \|\xi\|^{(k)}\mathbb{E}|\xi_\Lambda| \\ &\leq \|\xi\|^{(k)}\left(\mathbb{E}|\xi_\Lambda|^2\right)^{1/2} = \|\xi\|^{(k)}\sqrt{\frac{k}{d}}|\xi| \end{aligned}$$

and it remains to compare the two expressions.

We now want to show that the symmetrized set of (mixed) $2k$-separable states, i.e., $\mathrm{conv}\big(-\mathrm{Ent}_{2k}^1(\mathbb{C}^d \otimes \mathbb{C}^d) \cup \mathrm{Ent}_{2k}^1(\mathbb{C}^d \otimes \mathbb{C}^d)\big)$, considered as a subset of the $\mathbb{R}$-linear subspace of $\mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d)$ consisting of Hermitian matrices, contains a Hilbert-Schmidt ball of radius $\frac{k^{1/2}}{d^{3/2}}$. A standard argument based on Rogers-Shephard inequality – as in [11], Appendix C – will then imply that $\mathrm{vrad}\big(\mathrm{Ent}_{2k}^1(\mathbb{C}^d \otimes \mathbb{C}^d)\big) \geq \frac{1}{2}\frac{k^{1/2}}{d^{3/2}}$, hence

$$\mathrm{vrad}\big(\mathrm{Ent}_{k}^1(\mathbb{C}^d \otimes \mathbb{C}^d)\big) \geq \frac{1}{2}\frac{\lfloor k/2 \rfloor^{1/2}}{d^{3/2}}, \tag{21}$$

which gives the needed lower bound. (Note that we can assume that $k \geq 2$ since the case $k = 1$ was handled already in [13].)

To show that $\mathrm{conv}\big(-\mathrm{Ent}_{2k}^1(\mathbb{C}^d \otimes \mathbb{C}^d) \cup \mathrm{Ent}_{2k}^1(\mathbb{C}^d \otimes \mathbb{C}^d)\big)$ contains the appropriate Hilbert-Schmidt ball we will argue by duality: we will prove that the support function of $-\mathrm{Ent}_{2k}^1(\mathbb{C}^d \otimes \mathbb{C}^d) \cup \mathrm{Ent}_{2k}^1(\mathbb{C}^d \otimes \mathbb{C}^d)$ is bounded from below by $\frac{k^{1/2}}{d^{3/2}}$. To that end, consider an arbitrary Hermitian operator A on $\mathbb{C}^d \otimes \mathbb{C}^d$ with $\|A\|_{HS} = 1$. We need to show that

$$\begin{aligned} \max_{M \in \mathrm{Ent}_{2k}^1(\mathbb{C}^d \otimes \mathbb{C}^d)} |\operatorname{tr} AM| &= \max_{|\eta\rangle \in \mathrm{Ent}_{2k}^V(\mathbb{C}^d \otimes \mathbb{C}^d)} \big|\operatorname{tr} A\,|\eta\rangle\langle\eta|\,\big| \\ &= \max_{|\eta\rangle \in \mathrm{Ent}_{2k}^V(\mathbb{C}^d \otimes \mathbb{C}^d)} \big|\langle\eta| A\,|\eta\rangle\big| \\ &\geq \frac{k^{1/2}}{d^{3/2}}\,. \end{aligned}$$

The equalities are immediate; the inequality will be shown by establishing a chain of identities and inequalities

$$\begin{aligned} \max_{|\eta\rangle \in \mathrm{Ent}_{2k}^V(\mathbb{C}^d \otimes \mathbb{C}^d)} \big|\langle\eta| A\,|\eta\rangle\big| &\geq \max_{|\phi\rangle, |\psi\rangle \in \mathrm{Ent}_{k}^V(\mathbb{C}^d \otimes \mathbb{C}^d)} \mathrm{Re}\big(\langle\phi| A\,|\psi\rangle\big) \\ &= \max_{|\phi\rangle, |\psi\rangle \in \mathrm{Ent}_{k}^V(\mathbb{C}^d \otimes \mathbb{C}^d)} \big|\langle\phi| A\,|\psi\rangle\big| \\ &\geq \frac{k^{1/2}}{d^{3/2}}\,. \end{aligned} \tag{22}$$

Again, the equality is easy: it follows from the fact that the quantity under the third maximum does not change when we multiply $|\phi\rangle$ or $|\psi\rangle$ by $z \in \mathbb{C}$ with $|z| = 1$.

For the second inequality in (22), we notice that for a fixed $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$,

$$\max_{|\phi\rangle \in \mathrm{Ent}_k^V(\mathbb{C}^d \otimes \mathbb{C}^d)} \big|\langle\phi|A\,|\psi\rangle\big| = \|A\,|\psi\rangle\|^{(k)} \geq \sqrt{\frac{k}{d}}\,\big|A\,|\psi\rangle\big|$$

by (20). Next,

$$\max_{|\psi\rangle \in \mathrm{Ent}_k^V(\mathbb{C}^d \otimes \mathbb{C}^d)} \big|A\,|\psi\rangle\big| \geq \max_{|\psi\rangle \in \mathrm{Ent}_1^V(\mathbb{C}^d \otimes \mathbb{C}^d)} \big|A\,|\psi\rangle\big| \geq \sqrt{\big\langle \big|A|\psi\rangle\big|^2 \big\rangle_\psi},$$

where the symbol $\big\langle \cdot \big\rangle_\psi$ under the square root stands for the average, taken with respect to the natural product measure, over $\psi \in \mathrm{Ent}_1^V(\mathbb{C}^d \otimes \mathbb{C}^d) = \{u \otimes v \,:\, u, v \in S_{\mathbb{C}^d}\} \sim S_{\mathbb{C}^d} \times S_{\mathbb{C}^d}$, the set of normalized product pure states. It remains to check that this average equals

$$\int_{S_{\mathbb{C}^d}} \int_{S_{\mathbb{C}^d}} \big|A\,|u \otimes v\rangle\big|^2\, du\, dv = \frac{1}{d^2}\|A\|_{HS}^2 = \frac{1}{d^2}$$

and combine the estimates.

The first inequality in (22) follows via a standard polarization argument, with a small additional twist to obtain an estimate without any additional multiplicative constants. First, since A is Hermitian one has

$$\mathrm{Re}\big(\langle\phi|A\,|\psi\rangle\big) = \mathrm{Re}\big(\mathrm{tr}(A\,|\psi\rangle\langle\phi|)\big) = \mathrm{tr}\big(A\,\mathrm{Re}(|\psi\rangle\langle\phi|)\big)\ .$$

Next, we have an elementary identity,

$$\Big|\frac{\phi+\psi}{2}\Big\rangle\Big\langle\frac{\phi+\psi}{2}\Big| - \Big|\frac{\phi-\psi}{2}\Big\rangle\Big\langle\frac{\phi-\psi}{2}\Big| = \mathrm{Re}\big(\,|\psi\rangle\langle\phi|\big).$$

Combining the two we obtain

$$\mathrm{Re}\big(\langle\phi|A\,|\psi\rangle\big) = \langle\eta_1|A\,|\eta_1\rangle - \langle\eta_2|A\,|\eta_2\rangle,$$

where $\eta_1 = |\frac{\phi+\psi}{2}\rangle$ and $\eta_2 = |\frac{\phi-\psi}{2}\rangle$. Since the Schmidt rank of η_1 and η_2 is at most $2k$, it follows that $\mathrm{Re}\big(\langle\phi|A\,|\psi\rangle\big) \leq \big(|\eta_1|^2 + |\eta_2|^2\big) \max_{|\eta\rangle \in \mathrm{Ent}_{2k}^V(\mathbb{C}^d \otimes \mathbb{C}^d)} \big|\langle\eta|A\,|\eta\rangle\big|$. On the other hand, the parallelogram identity yields $|\eta_1|^2 + |\eta_2|^2 = |\frac{\phi+\psi}{2}|^2 + |\frac{\phi-\psi}{2}|^2 = \frac{|\phi|^2+|\psi|^2}{2} = 1$, which gives the needed first inequality in (22).

The inequalities (22) may be of interest in themselves. Let us just mention that, in the notation of [38], they read

$$\max_{|\eta\rangle \in \mathrm{Ent}_{2k}^V(\mathbb{C}^d \otimes \mathbb{C}^d)} \big|\langle\eta|A\,|\eta\rangle\big| \geq \|A\|_{S(k)} \geq \frac{k^{1/2}}{d^{3/2}}\|A\|_{HS} \tag{23}$$

for $A \in \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d)$, with the first inequality requiring additionally $A = A^\dagger$.

4.3 Consequences for the remaining sets

Since we determined, in (18) and (21), the precise asymptotic order of the volume radius of the set $\mathrm{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d)$ as $\Theta\big((k/d^3)^{1/2}\big)$, it follows from (8), (10) and the discussion in section 3.3 that, for the dual set, $\mathrm{vrad}\big(\mathcal{BP}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d)\big) = \Theta\big((kd)^{-1/2}\big)$ (remember the "rescaling" issue pointed out in the last paragraph of section 3.1).

To deduce sharp estimates on the volume radius of $\mathcal{SP}_k^{TP}(\mathcal{M}_d)$ and $\mathcal{P}_k^{TP}(\mathcal{M}_d)$ we appeal to (5) and then to Proposition 6 of [26]. Heuristically, Proposition 6 of [26] says that if an m-dimensional convex body K is "reasonably balanced," then all its central sections whose codimension is "substantially smaller" than m have volume radii close to that of K. In the present setting the dimensions and codimensions are exactly the same as in the applications discussed in [26], and the bodies we consider are intermediate with respect to those in [26]. Accordingly, all the arguments carry over and the heuristic principle described above can be rigorously shown to hold, and we can conclude that the volume radii of the bodies on the left hand side of each of the inclusions in (5) are essentially the same as those of the respective bodies on the right hand side. In other words, $\mathrm{vrad}\big(\mathcal{SP}_k^{TP}(\mathcal{M}_d)\big) = \Theta\big((k/d)^{1/2}\big)$ and $\mathrm{vrad}\big(\mathcal{P}_k^{TP}(\mathcal{M}_d)\big) = \Theta\big((d/k)^{1/2}\big)$.

5 Concluding Remarks

In this work we obtained explicit and asymptotically sharp estimates for the volume radius (in the sense of Hilbert-Schmidt measure) of the convex body $\mathrm{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d)$ of normalized k–entangled states. For $k = 1, \ldots, d-1$ these bodies form a nested family of subsets of the set of all states on a bipartite $d \times d$ system, which in the present notation coincides with $\mathrm{Ent}_d^1(\mathbb{C}^d \otimes \mathbb{C}^d) \subset \mathcal{M}_{d^2}$.

Making use of the Choi-Jamiołkowski isomorphism we deduce then bounds for the volumes of the sets $\mathcal{SP}_k^{TP}$ of trace-preserving k–superpositive maps, also called k–entanglement breaking channels. Finally, appealing to the Santaló and inverse Santaló inequalities, which relate the volumes of two dual sets, we obtain tight estimates for the volumes of sets $\mathcal{P}_k^{TP}$ of trace preserving k–positive maps, acting on density matrices of a given size d. (This is again a nested family, with extreme cases $k = 1$ and $k = d$ corresponding respectively to positivity preserving and completely positive quantum maps.)

Our findings show that, in large dimension, the property of "additional degrees of positivity" is uncommon. On the other hand, allowing "additional degrees of entanglement" is a major relaxation. While our methods allow only to compare the set of k–entangled states with that of ak–entangled states, where $a > 1$ in a universal constant (and similarly for maps), the estimates obtained strongly suggest that, for large dimension d, the set of k–entangled states covers only a small fraction of the larger set of $(k+1)$–entangled states, with the trend particularly pronounced for small k's. Indeed, we did show that, for $\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^d$, the ratio $R_{k,d} := \frac{\mathrm{vrad}\left(\mathrm{Ent}_k^1(\mathcal{H})\right)}{(k/d^3)^{1/2}}$ verifies $c_0 \leq R_{k,d} \leq C_0$ for some positive constants c_0, C_0 independent of k, d. If, instead, we had $R_{k,d} = c(d)$, with $c(d) \in [c_0, C_0]$ independent of k, it would follow that $\frac{\mathrm{vol}\left(\mathrm{Ent}_{k+1}^1(\mathcal{H})\right)}{\mathrm{vol}\left(\mathrm{Ent}_k^1(\mathcal{H})\right)} = \left(1 + \frac{1}{k}\right)^{(d^2-1)/2}$. While it is hard to expect that the equality $R_{k,d} = c(d)$ holds *precisely*, it would hold *approximately* if the dependence of $R_{k,d}$ on the parameters k, d was regular enough (which we can not prove rigorously). However, it *does* follow form our estimates that "in the mean" the ratios of the volumes of successive sets do behave as indicated above, i.e., are exponential in d^2 for small k, and then taper off to exponential in d when k is of order d. Further, it follows that if a state is randomly chosen (with respect to the Hilbert-Schmidt volume), then the probability that it is k entangled can be upper-bounded by $\left(\frac{Ck}{d}\right)^{(d^2-1)/2}$ and lower-bounded by $\left(\frac{ck}{d}\right)^{(d^2-1)/2}$, where $C \geq c > 0$ are universal constants. Similar comments can be made about volumes of the sets $\mathcal{P}_k^{TP}$ of k-positive maps, except that in that setting the volumes *decrease* with k.

Of special importance are the cases involving $k = 2$. This is because understanding the set of 2–entangled states is crucial (for example) for problems related to the distillation of quantum entanglement. It thus would be of substantial interest to rigorously show that, for large dimension d, the ratios $\frac{\mathrm{vol}\left(\mathrm{Ent}_2^1(\mathcal{H})\right)}{\mathrm{vol}\left(\mathrm{Ent}_1^1(\mathcal{H})\right)}$ and $\frac{\mathrm{vol}\left(\mathrm{Ent}_3^1(\mathcal{H})\right)}{\mathrm{vol}\left(\mathrm{Ent}_2^1(\mathcal{H})\right)}$ do indeed behave as predicted by our "global" estimates, i.e., that the set of 2–entangled states is exponentially (in d^2) small in comparison to the set of 3-entangled states, but exponentially large with respect to the set of separable states.

Acknowledgements: It is a pleasure to thank to the Oberwolfach Institute for Mathematics, where this project was initiated during the December 2009 workshop "Geometry of Quantum Entanglement." The first-named and the second-named authors were partially supported by grants from the National Science Foundation (U.S.A.). The third-named author acknowledges support under the grant N N202 090239 of Polish Ministry of Science and Higher Education, Foundation for Polish Science and European Regional Development Fund (agreement no MPD/2009/6), and under the SFB Transregio-12 project financed by the DFG.

References

[1] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 Quantum entanglement *Rev. Mod. Phys.* **81** 865–942

[2] Bengtsson I and Życzkowski K 2006 *Geometry of Quantum States* (Cambridge: Cambridge University Press)

[3] Szarek S J, Bengtsson I and Życzkowski K 2006 On the structure of the body of states with positive partial transpose *J. Phys.* **A 39** L119-L126

[4] Beigi S and Shor P W 2010 Approximating the set of separable states using the positive partial transpose test *J. Math. Phys.* **51** 042202

[5] Werner R F 1989 Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model *Phys. Rev.* **A 40** 4277–4281

[6] Terhal B and Horodecki P 2000 A Schmidt number for density matrices *Phys. Rev.* **A 61** 040301

[7] Życzkowski K, Horodecki P, Sanpera A and Lewenstein M 1998 Volume of the set of separable states *Phys. Rev.* **A 58** 883–892

[8] Braunstein S L, Caves C M, Jozsa R, Linden N, Popescu S and Schack R 1999 Separability of very noisy mixed states and implications for NMR quantum computing *Phys. Rev. Lett.* **83** 1054–1057

[9] Gurvits L and Barnum H 2002 Largest separable balls around the maximally mixed bipartite quantum state *Phys. Rev.* **A 66** 062311

[10] Gurvits L and Barnum H 2003 Separable balls around the maximally mixed multipartite quantum states *Phys. Rev.* **A 68** 042312

[11] Szarek S J 2005 The volume of separable states is super-doubly-exponentially small in the number of qubits *Phys. Rev.* **A 72** 032304

[12] Gurvits L and Barnum H 2005 Better bound on the exponent of the radius of the multipartite separable ball *Phys. Rev.* **A 72** 032322

[13] Aubrun G and Szarek S J 2006 Tensor products of convex sets and the volume of separable states on N qudits *Phys. Rev.* **A 73** 022109

[14] Hildebrand R 2007 Entangled states close to the maximally mixed state *Phys. Rev.* **A 75** 062330

[15] Życzkowski K 1999 Volume of the set of separable states II *Phys. Rev.* **A 60** 3496–3507

[16] Slater P B 2005 Silver mean conjectures for 15–d volumes and 14–d hyperareas of the separable two–qubit systems *J. Geom. Phys.* **53** 74–97

[17] Ye D 2009 On the Bures volume of separable quantum states *J. Math. Phys.* **50** 083502

[18] Ye D 2010 On the comparison of volumes of quantum states *J. Phys. A: Math. Theor.* **43** 315301

[19] Choi M D 1975 Completely positive linear maps on complex matrices *Linear Alg. Appl.* **10** 285–290

[20] Chruściński D and Kossakowski A 2009 Spectral conditions for positive maps *Commun. Math. Phys.* **290** 1051–1064

[21] Skowronek L, Størmer E and Życzkowski K 2009 Cones of positive maps and their duality relations *J. Math. Phys.* **50** 870–892

[22] Ruskai M B, Szarek S and Werner E 2002 An Analysis of completely–positive trace–preserving maps on 2×2 Matrices *Linear Algebra Appl.* **347** 159–187

[23] Życzkowski K and Bengtsson I 2004 On duality between quantum states and quantum maps *Open Syst. Inf. Dyn.* **11** 3–42

[24] Asorey M, Kossakowski A, Marmo G, Sudarshan E C G 2006 Relations Between Quantum Maps and Quantum States *Open Syst. Inf. Dyn.* **12** 319–329

[25] Ranade R S and Ali M 2007 The Jamiołkowski isomorphism and a conceptionally simple proof for the correspondence between vectors having Schmidt number k and k-positive maps *Open Sys. Inf. Dyn.* **14** 371–378

[26] Szarek S J, Werner E and Życzkowski K 2008 Geometry of sets of quantum maps: a generic positive map acting on a high-dimensional system is not completely positive *J. Math. Phys.* **49** 032113

[27] Sommers H–J and Życzkowski K 2004 Statistical properties of random density matrices *J. Phys.* **A 37** 8457–8466

[28] Clarisse L 2005 Characterization of distillability of entanglement in terms of positive maps *Phys. Rev.* **A 71** 032332

[29] Pisier G 1989 *The volume of convex bodies and Banach space geometry* Cambridge Tracts in Mathematics **94** (Cambridge: Cambridge University Press)

[30] Schneider R 1993 *Convex Bodies: The Brunn-Minkowski theory* (Cambridge: Cambridge University Press)

[31] Dudley R M 1967 The sizes of compact subsets of Hilbert space and continuity of Gaussian processes *J. Funct. Anal.* **1** 290–330

[32] Santaló L A 1949 An affine invariant for convex bodies of n-dimensional space (Spanish) *Portugaliae Math.* **8** 155–161

[33] Bourgain J and Milman V 1987 New volume ratio properties for convex symmetric bodies in $\mathbb{R}^n$ *Invent. Math.* **88** 319–340

[34] Kuperberg G 2008 From the Mahler conjecture to Gauss linking integrals *Geom. Funct. Anal.* **18** 870–892

[35] Szarek S J 1983 The finite-dimensional basis problem with an appendix on nets of Grassmann manifolds *Acta Math.***151** 153–179

[36] Szarek S J 1982 Nets of Grassmann manifold and orthogonal groups *Proceedings of Banach Space Workshop* 169–185 (University of Iowa Press)

[37] Szarek S J 1998 Metric entropy of homogeneous spaces, in *Quantum probability* Banach Center Publ. **43** 395–410 (Warsaw: Polish Acad. Sci.)

[38] Johnston N and Kribs D W 2010 Schmidt Norms for Quantum States *J. Math. Phys.* **51** 082202; arXiv:0909.3907v4