

Risk Measures in Quantitative Finance

by
Sovan Mitra

Abstract

This paper was presented and written for two seminars: a national UK University Risk Conference and a Risk Management industry workshop. The target audience is therefore a cross section of Academics and industry professionals.

The current ongoing global credit crunch ¹ has highlighted the importance of risk measurement in Finance to companies and regulators alike. Despite risk measurement's central importance to risk management, few papers exist reviewing them or following their evolution from its foremost beginnings up to the present day risk measures.

This paper reviews the most important portfolio risk measures in Financial Mathematics, from Bernoulli (1738) to Markowitz's Portfolio Theory, to the presently preferred risk measures such as CVaR (conditional Value at Risk). We provide a chronological review of the risk measures and survey less commonly known risk measures e.g. Treynor ratio.

Key words: Risk measures, coherent, risk management, portfolios, investment.

1. Introduction and Outline

Investors are constantly faced with a trade-off between adjusting potential returns for higher risk. However the events of the current ongoing global credit crisis and past financial crises (see for instance [Sto99a] and [Mit06]) have demonstrated the necessity for adequate risk measurement. Poor risk measurement can result in bankruptcies and threaten collapses of an entire finance sector [KH05].

Risk measurement is vital to trading in the multi-trillion dollar derivatives industry [Sto99b] and insufficient risk analysis can misprice derivatives [GF99]. Additionally

¹“Risk comes from not knowing what you're doing”, Warren Buffett.

incorrect risk measurement can significantly underestimate particular risk types e.g. market risk, credit risk etc. .

This paper reviews the most significant risk measures with a particular focus on those practised within the Financial Mathematics/Quantitative Finance arena. The evolution of risk measures can be categorised into four main stages:

1. Pre-Markowitz risk measures;
2. Markowitz Portfolio Theory based risk measures;
3. Value at Risk and related risk measures;
4. Risk Measures based on Coherent Risk Measurement Theory.

The outline of the paper is as follows. The paper starts with the first risk measures proposed; unknown to many Financial Mathematicians, these arose *prior* to Markowitz's risk measure. We then introduce Markowitz's Portfolio Theory, which provided the first formal model of risk measurement and diversification. After Markowitz we discuss Markowitz related risk measures, in particular the CAPM model (Capital Asset Pricing Model) and other related risk measures e.g. Treynor ratio.

We then discuss the next most significant introduction to risk measurement: Value at Risk (VaR) and explain its benefits. This is followed by a discussion on the first axioms on risk theory -the coherency axioms by Artzner et al. [ADEH97]. Coherent risk measures and copulas are discussed and finally we mention the future directions of risk measurement.

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2. Pre-Markowitz and Markowitz Risk Measurement Era

2.1. Pre-Markowitz Risk Measurement

A risk measure ρ is a function mapping a distribution of losses \mathcal{G} to \mathbb{R} , that is

$$\rho : \mathcal{G} \longrightarrow \mathbb{R}. \tag{1}$$

We note that some Academics distinguish between risk and uncertainty as first defined by Knight [Kni21]. Knight defines risk as randomness with known probabilities (e.g. probability of throwing a 6 on a die) whereas uncertainty is randomness with unknown probabilities (e.g. probability of rainy weather). However, in Financial risk literature this distinction is rarely made.

A particularly important aspect of risk and risk measurement is portfolio diversification. Diversification is the concept that one can reduce total risk without sacrificing

possible returns by investing in more than one asset. This is possible because not all risks affect all assets, for example, a new Government regulation on phone call charges would affect the telecoms sector and perhaps a few others deriving significant revenues or costs from phone calls, but not every sector or company. Thus by investing in more than 1 asset, one is less exposed to such “asset specific” risks. Note that there also exist risks that cannot be mitigated by diversification, for example a rise in interest rates would affect all businesses as they all save or spend money. We can conceptually categorise all risks into diversifiable and non-diversifiable risk (also known as “systematic risk” or “market risk”).

Contrary to popular opinion, risk measurement and diversification had been investigated prior to Markowitz’s Portfolio Theory (MPT). Bernoulli in 1738 [Ber54] discusses the famous St. Petersburg paradox and that risky decisions can be assessed on the basis of expected utility. Rubinstein [Rub02] states Bernoulli recognised diversification; that investing in a portfolio of assets reduces risk without decreasing return.

Prior to Markowitz a number of Economists had used variance as a measure of portfolio risk. For example Irving Fisher [Fis06] in 1906 suggested measuring Economic risk by variance, Tobin [Tob58] related investment portfolio risks to variance of returns.

Before significant contributions were made in Financial Mathematics to risk measurement theory, risk measurement was primarily a securities analysis based topic. Furthermore, securities analysis in itself was still in its infancy in the first half of the twentieth century. Benjamin Graham, widely considered the father of modern securities analysis, proposed the idea of margin of safety as a measure of risk in [Gra03]. Graham also recommended portfolio diversification to reduce risks. Graham’s methodology of investing (widely known as value investing) has not been pursued by the Financial Mathematics community, partly because of its reliance of securities rather than mathematical analysis. Exponents of the value based investment methodology include renowned investors Jeremy Grantham, Warren Buffett [Hag05] and Walter Schloss.

2.2. Markowitz’s Portfolio Theory (MPT)

Although Financial Analysts and Economists were aware of risk, prior to Markowitz’s risk measure it was more concerned with standard financial statement analysis, following a similar line of enquiry to Graham [Gra03]. However Markowitz ([Mar52], [Mar91b]) was the first to *formalise* portfolio risk, diversification and asset selection in a mathematically consistent framework. All that was needed were asset return means, variances and covariances. In this respect MPT was a significant innovation in risk measurement, for which Markowitz won the Nobel prize.

Markowitz proposed a portfolio's risk is equal to the variance of the portfolio's returns. If we define the weighted expected return of a portfolio μ_p as

$$\mu_p = \sum_{i=1}^N w_i \mu_i, \quad (2)$$

then the portfolio's variance σ_p^2 is

$$\sigma_p^2 = \sum_{i=1}^N \sum_{j=1}^N \sigma_{ij} w_i w_j, \quad (3)$$

where

- N is the number of assets in a portfolio;
- i, j are the asset indices and $i, j \in \{1, \dots, N\}$;
- w_i is the asset weight, subject to the constraints:

$$0 \leq w_i \leq 1,$$

$$\sum_{i=1}^N w_i = 1;$$

- σ_{ij} is the covariance of asset i with asset j ;
- μ_i is the expected return for asset i .

Markowitz's portfolio theory was the first to explicitly account for portfolio diversification as the correlation (or covariance) between assets. From equation 3 one observes that σ_p^2 decreases as σ_{ij} without necessarily reducing μ_p . The MPT also introduced the idea of optimising portfolio selection by selecting assets lying on an efficient frontier. The efficient frontier is found by minimising risk (σ_p^2) by adjusting w_i subject to the constraint μ_p is fixed; hence such portfolios provide the best μ_p for minimal risk [Mar91a]. Additionally, it can be shown that the efficient frontier follows a concave relation between μ_p and σ_p^2 . This reflects the idea of expected utility concavely increasing with risk. Most portfolio managers apply a MPT framework to optimise portfolio selection [Rub02].

Based on MPT portfolio risk measurement, Sharpe [Sha66] invented the Sharpe Ratio \mathcal{S} :

$$\mathcal{S} = \frac{\mu_p - R_f}{\sigma_p}, \quad (4)$$

where R_f is the risk-free rate of return. Sharpe's ratio can be interpreted as the excess return above the risk free rate per unit of risk, where risk is measured by MPT. The Sharpe ratio provides a portfolio risk measure in terms of determining the *quality* of the portfolio's return at a given level of risk. It is worth noting the Sharpe ratio's similarity to the t-statistic. A discussion on the Sharpe ratio can be found at Sharpe's website: www.stanford.edu/~wfs Sharpe/.

A variant on the Sharpe ratio is the Sortino Ratio [SP94], where we replace the denominator by the standard deviation of the portfolio returns below μ_p . This ratio essentially performs the same measurement as the Sharpe ratio but does not penalise portfolio performance for returns above μ_p .

It is worth mentioning that Roy [Roy52] formulated Markowitz's Portfolio Theory at the same time as Markowitz. As Markowitz says [Rub02]: "*On the basis of Markowitz (1952), I am often called the father of modern portfolio theory (MPT), but Roy (1952) can claim an equal share of this honor.*"

2.3. Capital Asset Pricing Model (CAPM)

The MPT in the 1960s' was computationally infeasible; it required covariance calculations for all the assets where $N \geq 100$. This motivated another risk measurement technique by Sharpe [Sha64] called CAPM, which was based on the MPT risk model:

$$\mu_i = R_f + \beta_i(\mu_m - R_f), \quad (5)$$

$$\beta_i = \frac{\sigma_{im}}{\sigma_m}, \quad (6)$$

where

- R_f is the risk-free rate of return;
- μ_m is the expected market return;
- β_i is known as the beta coefficient for asset i;
- σ_{im} is the covariance of asset i and the market;
- σ_m is the standard deviation of the market.

The β_i measures the sensitivity of the asset i's returns to the market; a high β_i implies asset i's returns increase with the market. In the CAPM model the term $(\mu_m - R_f)$ is the market risk premium, which is the return awarded above the risk-free rate for investing in a risky asset.

The CAPM theory postulates that all investors of different risk aversion would all hold the same portfolio. This portfolio would be a mixture of riskless and risky assets,

weighted according to the asset’s market capital (number of shares outstanding \times share price). Thus CAPM theory essentially suggests investors would hold an index tracker fund and encouraged the development of index funds. Index funds have been pioneered by investment managers such as John Bogle [Bog93] (e.g. Vanguard 500 Index Fund).

The CAPM theory gave portfolio fund managers the first “standard” portfolio performance benchmark by measuring against an index’s performance. Benchmark examples are given from [Jia03]:

Index	Portfolio Benchmark
S & P 500	Large Market Capital Equity Funds
S & P 400	Mid-Capital Equity Funds
Russell 2000	Small Capital Equity Funds
NASDAQ Composite	Technology Sector Equity Funds

Variations on the CAPM model include Jensen’s risk measure. Jensen quantifies portfolio returns above that predicted by CAPM with α :

$$\alpha = \mu_p - [R_f + \beta_p(\mu_m - R_f)], \quad (7)$$

where β_p is the portfolio’s beta. The term α can be interpreted as a measure of a portfolio manager’s investment ability or “beating the market”.

Finally, another lesser well known portfolio ratio measure is the Treynor ratio [Hüb05] \mathcal{T} :

$$\mathcal{T} = \frac{\mu_p - R_f}{\beta_p}. \quad (8)$$

Similar to the Sharpe ratio, the Treynor ratio can be interpreted as the “quality” of portfolio return above R_f per unit of risk but with risk measured on a CAPM basis.

3. Value at Risk (VaR)

3.1. VaR Risk Measure

The next era of risk measurement after MPT can be traced to the introduction of Value at Risk (VaR). This represented a significant change in direction of risk measurement for the following reasons:

- firstly, VaR initiated a shift in focus of using risk measures for the *management* of risk in an industry context. In 1994 JP Morgan created the VaR risk measure, apparently to measure risk across the whole institution under one holistic risk measure [Dow02]. Previous risk measures did not focus on such a holistic approach to risk measurement or management.
- secondly, the Basel Committee on Banking Supervision, which standardises international banking regulations and practises, stipulated a market risk capital requirement based upon VaR in 1995. This factor has subsequently fuelled interest in VaR and VaR related measures as well as becoming a popular risk measure [DSZ04].

- finally, previous measures focussed on explaining the return on an asset based on some theoretical model of the risk and return relation e.g. CAPM. VaR on the other hand shifted the focus to measuring and quantifying the risk itself and in terms of losses (rather than expected return).

VaR's purpose is to simply address the question "How much can one expect to lose, with a given cumulative probability ζ , for a given time horizon T?". VaR is therefore defined as [Sze05]:

$$F(Z(T) \leq VaR) = \zeta, \quad (9)$$

where

- $F(\cdot)$ is the cumulative probability distribution function;
- $Z(T)$ is the loss. The loss $Z(t)$ is defined by

$$Z(t) = S(0) - S(t), \quad (10)$$

where $S(t)$ is the stock price at time t ;

- ζ is a cumulative probability associated with threshold value VaR, on the loss distribution of $Z(t)$.

To give an example of VaR, a portfolio may have a VaR of \$10,000,000, for one day, with a cumulative probability of 90%. This means that the portfolio can expect a maximum loss of \$10,000,000, over one day, with a 90% cumulative probability. An alternative interpretation would be that the portfolio's loss could exceed \$10,000,000, in one day, with a cumulative probability of 100-90=10%. Typically ζ is chosen to be 0.90, 0.95 and 0.99.

3.2. VaR Implementation

The measurement of VaR on a portfolio presents a theoretical and computational challenge as it is difficult to model the evolution of a portfolio over time containing hundreds of assets [MR05],[Hul00]. Hence the implementation of VaR is of particular interest to industry and academic Researchers; the four main methods are [Dow02]:

VaR Historical Simulation

Using a set of historical data we obtain an empirical probability distribution of losses for the portfolio. One can then determine the appropriate VaR by extracting it from the associated quantile. Such an approach is convenient to implement and can be improved by a range of statistical methods. For example one can apply bootstrapping [Efr79] when one has a small data set or implement a weighted historical simulation approach.

VaR Parametric Approach

The parametric approach requires an analytic solution to determining the VaR for any cumulative probability. Unfortunately not all distributions have solutions, however one can apply Extreme Value Theory (Peak over Threshold and Generalised Extreme Value distributions) to estimate VaR; the reader is referred to [CT04] for more information.

VaR Monte Carlo Simulation

Monte Carlo simulation is a generic method of simulating some random process (e.g. stochastic differential equation) representing the assets or the portfolio itself. Consequently after sufficient simulations we can obtain a loss distribution and therefore extract VaR for different probabilities as was done for historical simulation. One can improve Monte Carlo simulation through various computational techniques [Gla04] such as importance sampling, stratified sampling and antithetic sampling.

VaR Variance-Covariance Method (also known as the Delta-Normal Method)

Under the variance-covariance method, we model the portfolio's loss distribution by making two assumptions:

- the portfolio is linear: the change in the portfolio's price $V(t)$ is linearly dependent on its constituent asset prices $S_i(t)$. In other words:

$$\Delta V(t) = \sum_{i=1}^N \Delta S_i(t). \quad (11)$$

A portfolio will be linear if it contains no derivatives. Furthermore, in practise some modellers assume non-linear portfolios are linear for analytical tractability. This assumption is made in [RU00].

- the constituent assets have a joint Normal return distribution, which implies the portfolio's returns are Normally distributed. It is worth noting that the sum of Normally distributed functions is not strictly always Normal; the specific property of joint Normality however guarantees the portfolio's return is Normal. Hence the linear portfolio assumption alone cannot guarantee the portfolio's return is Normal.

Given the two assumptions enables us to describe the portfolio's loss using a Normal distribution, for which numerous analytical equations and distribution fitting methods exist. Therefore VaR calculation and implementation becomes significantly simpler.

4. Coherent Risk Measures

4.1. Axioms of Risk Measurement

A significant milestone in risk measurement was achieved when Artzner et al. [ADEH97] proposed the first axioms of risk measurement; risk measures that obeyed such axioms were called coherent risk measures. The coherency axioms had far reaching implications as it was no longer possible to arbitrarily assign a function for risk measurement unless it obeyed these axioms, consequently VaR was no longer considered an adequate risk measure.

We now define a coherent risk measure $\rho(\cdot)$. Let X and Y denote the future loss of two portfolios, then a risk measure ρ is coherent if it adheres to the four axioms:

1. risk is monotonic: if $X \leq Y$ then $\rho(X) \leq \rho(Y)$;
2. risk is homogeneous: $\rho(\lambda X) = \lambda \rho(X)$ for $\lambda > 0$;
3. riskless translation invariance: $\rho(X + \chi) = \rho(X) - \chi$, where χ is a riskless bond;

4. risk is sub-additive: $\rho(X+Y) \leq \rho(X)+\rho(Y)$.

We will now explain each axiom in turn. The monotonicity axiom tells us that we associate higher risk with higher loss. The homogeneity axiom ensures that we cannot increase or decrease risk by investing differing amounts in the same stock; in other words the risk arises from the stock itself and is not a function of the quantity purchased².

The translation invariance axiom can be explained by the fact that the investment in a riskless bond bears no loss with probability 1. Hence we must always receive the initial amount invested. The initial investment is subtracted because risk measures measure loss as a positive amount, a hence gain is negative.

The subadditivity is the most important axiom because it ensures that a coherent risk measure takes into portfolio diversification. The axioms shows that investing in both portfolio X *and* Y results in a lower risk overall than the sum of the risks in investing in portfolio X plus the risk in portfolio Y separately. VaR is not a coherent risk measure because it does not obey the subadditivity axiom, consequently it can result in higher risk arising from diversification.

We say a risk measure is weakly coherent if it is convex, translationally invariant and homogeneous. It is also worth mentioning that coherency axioms ensure the risk measure is convex and so is amenable to computational optimisation; for more information the reader is referred to [RU00]. VaR on the other hand is non-convex and so possesses many local minima.

4.2. Coherent Risk Measures

Given the introduction of coherency axioms and conclusion that VaR was not coherent, new risk coherent measures were proposed to capture the advantages of VaR. In particular, there was a need for a “holistic” risk measure and one that was simple to grasp in that it could capture all the key risk information with three pieces of information: probability, loss and time horizon.

In response to a coherent equivalent to VaR, a variety of VaR related risk measures were proposed. Examples include TVaR (tail value at risk) [ADE⁺03], WCE (Worst conditional expectation) [Ino03] and CVaR (conditional value at risk).

CVaR has become a particularly popular risk measure due to its similarity to VaR but also it assesses “how bad things can get” if the VaR loss is exceeded. CVaR is the expected loss given that the VaR loss is exceeded; it is defined by:

$$CVaR = E[Z(T) \mid Z(T) > VaR]. \quad (12)$$

An alternative definition of CVaR is the mean of the tail distribution of the VaR losses. An additional advantage of CVaR is that the portfolio weights can be easily optimised by linear programming [RU00] to minimise CVaR.

Spectral risk measures are a group of coherent risk measures, whereby the risk is given as the sum of a weighted average of outcomes. The weights can be chosen to reflect risk preferences towards particular outcomes. For more information the reader is referred to [Ace02].

²Note: this assumes we have no liquidity risks. In reality this is not true, particularly during the current global credit crisis.

4.3. Copulas

The subadditivity axiom demonstrates the importance to capture dependencies between stocks when measuring the risk of a portfolio. Consequently this gave rise to the interest in copulas [Nel06]; these are functions mapping a set of marginal distributions into a multivariate distribution and vice versa. Sklar's Theorem underpins copula theory, which states that for a given multivariate distribution there exists a copula that can combine all the marginal distributions to give the joint distribution. For example, in the bivariate case if we have two marginal distributions $F(x)$ and $G(y)$ then there exists a copula function \mathcal{C} to give the multivariate distribution $H(x,y)$:

$$H(x, y) = \mathcal{C}(F(x), G(y)). \quad (13)$$

There exist a variety of copulas and examples include the Gaussian copula [FMN01] and Clayton copula [CNF05]. Prior to their application in Financial Mathematics copulas have been used for many years in Actuarial Sciences, Reliability Engineering, Civil and Mechanical Engineering.

In Extreme Value Theory copulas become extremely important because it is not possible to capture dependencies between random variables using standard correlation. To use multivariate Extreme Value Theory, instead we must apply copulas to capture dependencies [Dow02].

Despite the number of copulas in existence this continues to be an area of active research as it is important to have copulas that capture the correct type of dependencies between stocks. For instance copulas are used in the pricing of collateralized debt obligations (CDOs) [MV04]. However the usage of credit derivatives have been widely cited as an important cause of the current global credit crisis.

5. Future Directions of Risk Measurement

Risk measurement is a thriving area of research. A current area of interest is to find satisfactory methods of modelling dependencies between stocks other than by copulas and correlations. Alternatively, there is much interest in finding copulas that can meaningfully capture dependency behaviours.

Another area of risk measurement research is dynamic risk measurement. This involves measuring risk in continuous time, rather than applied to a static distribution. Examples of dynamic risk measurement include [Rie04],[GO08], [DS05].

Finally, another future direction of risk measurement is devising risk specific risk measures, such credit risk measures, liquidity risk measure etc. . However it should be noted that such risk measures already exist in these areas (for example Merton's structural credit default model [Mer74] and the KMV model [Kea03]).

6. Conclusions

This paper has surveyed the key risk measures in Financial Mathematics as well as its progressive development since the beginning. We have also mentioned, contrary to popular knowledge, that risk measures existed prior to Markowitz.

We have examined the key contributions of the major risk measures, such as Markowitz's Portfolio Theory, whilst also highlighting their influence within the financial industry. We have also discussed newer risk measures such as spectral risk measures, VaR and its variants (e.g. CVaR) and mentioned future areas of research.

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