Integer Sequences associated with Integer Monic Polynomial

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Abstract: To every integer monic polynomial of degree m can be associated m integer sequences having interesting properties to the roots of the polynomial. These sequences can be used to find the real roots of any integer monic polynomial by using recursion relation involving integers only. This method is faster than the conventional methods using floating point arithmetic.
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1. Introduction

A polynomial is said to be integer monic [1] if all its coefficients are integers and the coefficient of highest power is unity. To a given polynomial can be associated an *m*-term recursion relation for generating integer sequences. A set of *m* such sequences, which together exhibit interesting properties related to the roots of the polynomial, can be obtained if the *m* initial terms of each of these *m* sequences is chosen in a special way using the companion matrix [2] of the polynomial. The companion matrix of a polynomial is a matrix such that the characteristic equation of the companion matrix is the given polynomial.

2. Construction of the integer sequences

Let $p(x) = x^m + a_1 x^{m-1} + \ldots.. + a_m$, where a_i are integers, be the given monic polynomial. The companion matrix of the polynomial is given by

$$\mathbf{R} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \ldots & -a_{m-1} & -a_m \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & 0 & . \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix} \quad (1)$$

Let r_i, i = 1, 2, ….., *m* be the roots of p(x). Then each r_i is an eigenvalue of **R** with corresponding eigenvector $[r_i^{m-1}, r_i^{m-2}, \ldots\ldots, r_i, 1]^T$. The matrix **R** satisfies the equation

$$\mathbf{R}^m + a_1\mathbf{R}^{m-1} + \ldots\ldots + a_m = \mathbf{0} \qquad (2)$$

Let

$$\mathbf{S}_0 = (S^{(1)}{}_0, S^{(2)}{}_0, \ldots\ldots, S^{(m)}{}_0)^T \qquad (3)$$

where $S^{(i)}{}_0$ are arbitrary integers. Define

$$\mathbf{S}_j = \mathbf{R}^j\,\mathbf{S}_0 \;, \; j = 1, 2, \ldots\ldots, m-1 \qquad (4)$$

The first *m* terms of the sequence $S^{(i)}$ are taken to be $\{S^{(i)}{}_0, S^{(i)}{}_1, \ldots\ldots, S^{(i)}{}_{m-1}\}$. Beyond this the *m*-term recurrence relation defines the terms of the sequence

$$S^{(i)}{}_j = - a_1 S^{(i)}{}_{j-1} - a_2 S^{(i)}{}_{j-2} - \ldots\ldots - a_m S^{(i)}{}_{j-m} \;, \quad j > m - 1, \; i = 1, 2, \ldots\ldots, m \qquad (5)$$

These sequences satisfy the interesting property that each of the ratios $S^{(i)}{}_j$ to $S^{(i+1)}{}_j$,i = 1, 2, ……., *m* -1 tends to the root of p, having the largest absolute value, if it is unique.

3. Example

Let

$$p(x) = x^2 + 2x - 1 \qquad (6)$$

be the given integer monic polynomial. Here *m* = 2, $a_1 = 2$ and $a_2 = -1$. The companion matrix to the polynomial (6) is given by

$$\mathbf{R} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \qquad (7)$$

We take

$$\mathbf{S}_0 = [1 \;\; 0]^T \qquad (8)$$

Then

$$\mathbf{S}_1 = [-2 \;\; 1]^T \qquad (9)$$

Hence the first two terms of the sequence $S^{(1)}$ are {1, -2} and the first two terms of the sequence $S^{(2)}$ are {0, 1}. Subsequent terms in these sequences are to be obtained by the recursion relation

$$S^{(i)}_j = -2\, S^{(i)}_{j-1} + S^{(i)}_{j-2}, \qquad j > 1,\ i = 1, 2 \tag{10}$$

The application of this recursion leads to the following sequences

j	$S^{(1)}_j$	$S^{(2)}_j$	Ratio
0	1	0	inf
1	-2	1	-2
2	5	-2	-2.5
3	-12	5	-2.4
4	29	-12	-2.4167
5	-70	29	-2.4138
6	169	-70	-2.4143

The ratio $S^{(1)}_j$ to $S^{(2)}_j$ converges to the largest (absolute value) root, namely $(-1 - \sqrt{2})$, of polynomial p(x) in (6).

4. Remark

The eigenvalues of the matrix $\mathbf{R}' = (a\mathbf{I} + b\mathbf{R})$ are $r_i' = (a + br_i)$ with corresponding eigenvectors $[r_i^{m-1}, r_i^{m-2}, \ldots, r_i, 1]^T$, where **I** is the *m x m* identity matrix. A different set of sequences may be obtained by replacing matrix **R** in (2) by **R’** **,** where a, b are suitable integers, chosen so that some other root of p goes to the largest (absolute value) eigenvalue of the **R**’ without change in its corresponding eigenvector. In the above example, the roots of p are $(-1 - \sqrt{2})$ and $(-1 + \sqrt{2})$. The largest (absolute value) root is $(-1 -\sqrt{2})$. If we take a = 2 and b = 1, we get **R**’ given by

$$\mathbf{R}' = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \tag{11}$$

The eigenvalues of **R**’ are $(1 - \sqrt{2})$ and $(1 + \sqrt{2})$. The second eigenvalue now becomes the largest (absolute value) but the eigenvector remains unchanged. Since $(\mathbf{R}' - 2\,\mathbf{I})$ satisfies the polynomial in (6), therefore, $(\mathbf{R}' - 2\,\mathbf{I})^2 + 2(\mathbf{R}' - 2\,\mathbf{I}) - \mathbf{I} = \mathbf{0}$, leading to

$$R'^2 - 2\,\mathbf{R}' - \mathbf{I} = \mathbf{0} \tag{12}$$

We take

$$\mathbf{S}_0 = [\,1\ \ 0]^T \tag{13}$$

Then

$$\mathbf{S}_1 = [\,-2\ \ 1]^T \tag{14}$$

Hence the first two terms of the sequence $S^{(1)}$ are {1, -2} and the first two terms of the sequence $S^{(2)}$ are {0, 1}. Subsequent terms in these sequences are to be obtained by the recursion relation

$$S^{(i)}_j = 2\,S^{(i)}_{j-1} + S^{(i)}_{j-2}, \qquad j > 1,\ i = 1, 2 \tag{15}$$

The application of this recursion leads to the following sequences:

j	$S^{(1)}_j$	$S^{(2)}_j$	Ratio
0	1	0	inf
1	0	1	0
2	1	2	0.5
3	2	5	0.4
4	5	12	0.41667
5	12	29	0.41379
6	29	70	0.41423
7	70	169	0.41420

The ratio $S^{(1)}_j$ to $S^{(2)}_j$ converges to the other root, namely $(-1 + \sqrt{2})$, of polynomial p(x) in (6). This suggests that the method can be extended to obtain different sets of integer sequences related to the different roots of a polynomial, except in the case of degenerate roots.

5. The case when all roots have the same absolute value

Consider the polynomial $p(x) = x^m - N$. In this case all the *m* roots have the same absolute value. The method as described in sec 2 above will therefore not be applicable. However, one can use the modification described in sec 4 to make the method applicable. The companion matrix, as modified by taking a = 1 and b = 1 is given by

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & N \\ 1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 1 & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & 1 & . \\ 0 & 0 & 0 & \ldots & 1 & 1 \end{bmatrix} \tag{16}$$

As an example take N = 2 and *m* = 3 case. Then $(\mathbf{R} - \mathbf{I})^3 = 2$, so that

$$\mathbf{R}^3 - 3\,\mathbf{R}^2 + 3\,\mathbf{R} - \mathbf{I} = \mathbf{0} \qquad (17)$$

We take arbitrarily

$$\mathbf{S}_0 = [\,1 \;\; 1 \;\; 0]^T \qquad (18)$$

Then

$$\mathbf{S}_1 = \mathbf{R}\,\mathbf{S}_0 = [\,1 \;\; 2 \;\; 1]^T \qquad (19)$$

and

$$\mathbf{S}_2 = \mathbf{R}\,\mathbf{S}_1 = [\,3 \;\; 3 \;\; 3]^T \qquad (20)$$

Hence the first three terms of the sequence $S^{(1)}$ are {1, 1, 3}, the first three terms of the sequences $S^{(2)}$ are {1, 2, 3}, and the first three terms of the sequence $S^{(3)}$ are {0, 1, 3}. Subsequent terms in these sequences are obtained by the recursion relation

$$S^{(i)}_j = 3\,S^{(i)}_{j-1} - 3\,S^{(i)}_{j-2} + 3\,S^{(i)}_{j-3}\,, \quad j > 1,\; i = 1, 2 \qquad (21)$$

The application of this recursion leads to the following sequences:

j	$S^{(1)}_j$	$S^{(2)}_j$	$S^{(3)}_j$	Ratio $S^{(1)}_j / S^{(2)}_j$	Ratio $S^{(2)}_j / S^{(3)}_j$
0	1	1	0	1	infinity
1	1	2	1	0.5	2
2	3	3	3	1	1
3	9	6	6	1.5	1
4	21	15	12	1.400000	1.250000
5	45	36	27	1.250000	1.333300
6	99	81	63	1.222222	1.285714
7	225	180	144	1.250000	1.250000
8	513	405	324	1.266667	1.250000
9	1161	918	729	1.264706	1.259259
10	2619	2079	1647	1.259740	1.262295
11	5913	4698	3726	1.258621	1.260870
12	13365	10611	8424	1.259542	1.259615
13	30213	23976	19035	1.260135	1.259574
14	68283	54189	43011	1.260090	1.259887
15	154305	122472	97200	1.259921	1.260000
16	348705	276777	219672	1.259877	1.259956
17	788049	625482	496449	1.259907	1.259912
18	1780947	1413531	1121931	1.259928	1.259910
19	4024809	3194478	2535462	1.259927	1.259919
20	9095733	7219287	5729940	1.259921	1.259924
21	20555613	16315020	12949227	1.259920	1.259922
22	46454067	36870633	29264247	1.259921	1.259921
23	104982561	83324700	66134880	1.259921	1.259921
24	237252321	188307261	149459580	1.259921	1.259921
25	536171481	425559582	337766841	1.259921	1.259921

It may be observed that both of the ratios converge to $(2)^{1/3} = 1.259921$.

6. **Conclusions**

We have obtained a method for finding the real roots of any integer monic polynomial by using recursion relation involving integers only. This method is faster than the conventional methods using floating point arithmetic. The details will be discussed elsewhere.

7. Bibiliography

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