

# Quantum Generic Attacks on Feistel Schemes

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**Abstract.** The Feistel scheme is an important structure in the block ciphers. The security of the Feistel scheme is related to distinguishability with a random permutation. In this paper, efficient quantum algorithms for distinguishing classical 3,4-round and unbalanced Feistel scheme with contracting functions from random permutation are proposed. Our algorithms realize an exponential speed-up over classical algorithms for these problems. Furthermore, the method presented in this paper can also be used to consider unbalanced Feistel schemes with expanding functions.

## 1. Introduction

Many block ciphers algorithms used in cryptography are Feistel schemes(FS), for example DES, TDES, many AES candidates. Classical (balanced) FS, unbalanced FS with contracting functions, and unbalanced FS with expanding functions have been widely studied. The classical security of the Feistel scheme has been considered in terms of indistinguishability from a random permutation, because the FS is secure against any chosen-plaintext attack if they are indistinguishable [1, 2].

Luby and Rackoff [3] have shown their famous result: all generic attacks on FS that more than 3-round require at least  $O(2^{n/2})$  inputs, even for chosen inputs. Moreover, all the generic attacks on 4-round FS require at least  $O(2^{n/2})$  inputs, even for a stronger attack that combines chosen inputs and chosen outputs [3]. For 5 round or more the question is more complicated. In Patarin's work[1], it was proved that for 5 round or more the number of queries must be at least  $O(2^n)$ .

It is known that quantum algorithms can be used to realize a sub-exponential or even exponential speed-up over known classical algorithms for some problems. Kuwakado[4] showed that quantum algorithms are effective to the 2-round FS and a variant of 3-round FS. A variant of 3-round FS is distinguishable from a random permutation(RP) by making  $O(2^{n/2})$  classical queries. However, Kuwakado[4] showed that  $O(2^{n/3})$  quantum queries are enough for the same task. In this paper, we present a more effective quantum algorithm for this problem by Simon algorithm[5], the query complexity here is only  $O(n)$ . Furthermore, we propose quantum algorithm for 4-round FS and unbalanced FS with contracting functions. Our quantum query complexity achieves an exponential speed-up over known classical query algorithm.

The rest of this paper is organized as follows. In section 2, we give a short overview of the FS and the quantum computation. In section 3, quantum algorithms to distinguish a variant 3-round FS and a 4-round FS from RP are considered. In section 4, we present effective algorithm for unbalanced FS with contracting functions. Conclusions are given in section 5.

## 2. Background

A Feistel scheme(FS) from  $\{0, 1\}^N$  to  $\{0, 1\}^N$  with  $r$ -round is a permutation structured by round functions. When these functions are randomly chosen, we get what is called a "random FS". The attacks on these "random FS" are called "generic attacks" since these attacks are valid for most of the round functions  $f_1 \dots f_r$ . For most of classical FS, we have  $N = 2n$  and the round functions  $f_i$  are from  $\{0, 1\}^n$  to  $\{0, 1\}^n$ . Such schemes are called "balanced" FS.

**Unbalanced Feistel Scheme:** An unbalanced FS  $G_k^d$  with contracting functions is a FS with  $d$  rounds. On some input  $[I^1, I^2, \dots, I^k]$ ,  $G_k^d$  produces an output denoted by  $[S^1, S^2, \dots, S^k]$  by going through  $d$  rounds, where  $I^i, S^i \in \{0, 1\}^N$ , ( $1 \leq i \leq k$ ). The round function  $f_j$  at round  $j$  is a function from  $(k-1)n$  bits to  $n$  bits. At each round,

the last  $(k - 1)n$  bits of the round entry are used as an input to the round function  $f_j$ , which produces  $n$  bits. Those bits are xored to the first  $n$  bits of the round entry. Finally before going to round  $j + 1$ , the  $kn$  bit value is rotated by  $n$  bits.

The first round of  $G_k^d$  is represented in Fig1 below.

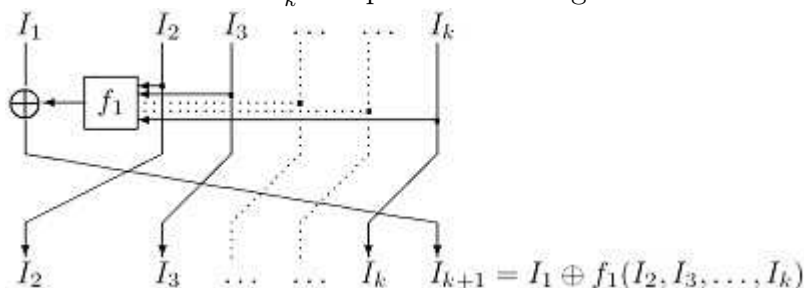


Fig1. First Round of  $G_k^d$

To demonstrate the superior computational ability in quantum computers, many distinguishing problems have been studied. Most of the known results showing that quantum computers outperform their classical counterparts can be phrased as black-box problems. A black-box (oracle) is subroutine that implements some operation or function. It provides no other information other than taking an input and giving the prescribed output. Quantum computers can offer superpolynomial speedups over classical computers, but only for certain "structured" problems. The key question is whether we can find the "structured" for the certain task.

**FS and RP Distinguishing problem:** Giving an oracle  $C$  and promising that  $C$  is either the  $r$ -round FS or RP. The problem is to determine whether  $C$  is FS or RP by making queries to the oracle  $C$  with a complexity as small as possible.

**Query complexity** [6]: When referring to a black-box algorithm, the query complexity is the number of applications of the black-box or oracle used by the algorithm. When referring to a black-box problem, the query complexity is the number of applications of the black-box required by any algorithm to solve the problem.

### Notations

We will use the following notations in this paper.

$I_n = \{0, 1\}^n$  is the set of the binary strings of length  $n$ . Particularly,  $\mathbf{0}$  is  $n$ -bits zero vector.

For  $\mathbf{a}, \mathbf{b} \in I_n$ ,  $\mathbf{a} \oplus \mathbf{b}$  stands for bit by bit exclusive of  $\mathbf{a}$  and  $\mathbf{b}$ .

The set of all function from  $I_n$  to  $I_n$  is  $F_n$ .

Let  $f$  is a function of  $F_n$ . Let  $\mathbf{a}, \mathbf{b}$  be elements of  $I_n$ . Then by definition:

$$\text{FS}(f)[\mathbf{a}, \mathbf{b}] = [\mathbf{b}, \mathbf{a} \oplus f(\mathbf{b})].$$

Let  $f_1, \dots, f_r$  be  $r$  functions of  $F_n$ . Then by definition:

$\text{FS}^r(f_1, \dots, f_r) = \text{FS}(f_r) \circ \dots \circ \text{FS}(f_1)$ , where  $\circ$  is the composition of functions.

$\text{FS}^r(f_1, \dots, f_r)$  is called "a FS with  $r$ -round".

$G_k^d$  is a  $d$  rounds unbalanced FS with contracting functions.

Furthermore, we denote Feistel scheme and random permutation by FS and RP respectively.

### 3. Quantum Attack on Classical Feistel scheme

#### 3.1. Attack on The variant 3-round FS

This section shows that quantum algorithm can make less queries to distinguish a variant of the 3-round Feistel scheme VFS from a RP. The variant considered here is the 3-round FS such that the second internal function  $f_2$  is replaced with a RP on  $\{0,1\}^n$ . Kuwakado[4] showed that a VFS is distinguishable from a RP by making  $O(2^{n/3})$  quantum queries. However, in this section, we will propose a quantum algorithm to distinguish a VFS with less queries. This distinguishing attack is similar to Simon's algorithm[5].

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#### Algorithm 1

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Input: An oracle  $C$  that is promised to be either VFS or RP. A constant  $q \geq \lceil -\log_3 \varepsilon \rceil$ .

Output: Oracle  $C$  is VFS or RP with success probability  $1 - \varepsilon$ .

Let  $k = 1$ .

1) While  $k < q$  do

2) For  $t$  from 1 to  $n + 5$  do

3) Prepare the  $4n$  qubit state  $\sum_{\mathbf{i}=0}^{2^n-1} |\mathbf{i}\rangle_1 |\mathbf{0}\rangle_2 |\mathbf{0}\rangle_3 |\mathbf{0}\rangle_4$ .

4) Apply  $U_C$  to create the state  $\sum_{\mathbf{i}=0}^{2^n-1} |\mathbf{i}\rangle_1 |\mathbf{0}\rangle_2 |\mathbf{c}_i\rangle_3 |\mathbf{d}_i\rangle_4$ .

Where  $U_C |\mathbf{i}\rangle_1 |\mathbf{0}\rangle_2 |\mathbf{0}\rangle_3 |\mathbf{0}\rangle_4 = |\mathbf{i}\rangle_1 |\mathbf{0}\rangle_2 |\mathbf{c}_i\rangle_3 |\mathbf{d}_i\rangle_4$ , i.e.  $C(\mathbf{i}, \mathbf{0}) = (\mathbf{c}_i, \mathbf{d}_i)$ .

5) Measure the fourth register, and then apply  $U_C$  again to "uncompute" the value of function from the third and the fourth registers; we have a random "coset state"  $\frac{1}{\sqrt{2}}(|\mathbf{i}\rangle + |\mathbf{j}\rangle)$  in the first register. Let  $\mathbf{s} = \mathbf{i} \oplus \mathbf{j}$  and  $K = \{\mathbf{0}, \mathbf{s}\}$ .

6) Apply a Hadamard gate to the first register, the state in the first register is  $\sum_{\mathbf{y} \in K^\perp} (-1)^{\mathbf{y} \cdot \mathbf{i}} |\mathbf{y}\rangle$ , where normalization factor has been omitted.

7) Measure the first register to obtain a string  $\mathbf{y}_t \in \{0,1\}^n$ ;  $t = t + 1$ .

8) Let  $M$  be the  $(n + 5) \times n$  matrix whose  $t$ -th row is the vector  $\mathbf{y}_t$ . Solve the system  $M\mathbf{x}_k = \mathbf{0}$ ;  $k = k + 1$ .

9) For  $1 \leq k \leq q$ , if  $\mathbf{x}_k = \mathbf{0}$  holds, then output "VFS", otherwise, output "RP".

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Correctness: After applying the Hadamard gates in step 6, the "coset state"  $\frac{1}{\sqrt{2}}(|\mathbf{i}\rangle + |\mathbf{j}\rangle)$  gets mapped to  $\sum_{\mathbf{y} \in K^\perp} (-1)^{\mathbf{y} \cdot \mathbf{i}} |\mathbf{y}\rangle$ , where  $K^\perp = \{\mathbf{y} | \mathbf{y} \in \mathbf{Z}_2^n, \mathbf{y} \cdot \mathbf{s} = \mathbf{0}\}$  and normalization factor has been omitted. Thus, with  $n + 5$  random samples from  $K^\perp$ , the samples vectors  $\mathbf{y}_t$  will generated  $K^\perp$  with probability exponentially close to one. Then, we can efficiently compute generators for  $K$  by Gaussian elimination, i.e. we can determine whether  $\mathbf{i} \oplus \mathbf{j} = \mathbf{0}$  holds.

Suppose that  $C$  is VFS. Since the second internal function is a random permutation, the value of the first and the fourth register is one-to-one, so the equality  $\mathbf{i} \oplus \mathbf{j} = \mathbf{0}$  always holds; Suppose that  $C$  is RP. Since the right part of  $\text{RP}(\mathbf{i}, 00\dots 0)$  is a random binary

strings of length  $n$ , then the number of  $|\mathbf{i}\rangle$  that map to the same value  $|\mathbf{b}_i\rangle$  in the fourth register is two on average. So  $\mathbf{i} \oplus \mathbf{j} \neq \mathbf{0}$  holds with high probability.

Sometimes, for a RP, we will obtain only one or three(or more) different  $\mathbf{i}$  in the first register that map to the same value  $\mathbf{d}$  in the fourth register. In this case, we will also get the zero solution to the system  $M\mathbf{x}_k = \mathbf{0}$ , the probability will be considered later.

Query complexity: If  $C$  is RP, then the value  $|\mathbf{i}\rangle$  in the first register and  $|\mathbf{d}_i\rangle$  in the fourth register is two-to-one in general, particularly, we take the probability is  $2/3$ , i.e. the error probability of algorithm 1 can be evaluated as follows:

$$\begin{aligned} P_{err} &= \Pr[\text{VFS}|\text{RP}] \Pr[\text{RP}] + \Pr[\text{RP}|\text{VFS}] \Pr[\text{VFS}] \\ &\leq \left(\frac{1}{3}\right)^q \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} \approx 3^{-q} \end{aligned}$$

$P_{err} \leq \varepsilon$ , so we have  $q \geq -\log_3 \varepsilon$ .

Furthermore, to solve the system  $M\mathbf{x}_k = \mathbf{0}$ , we need only  $O(n^3)$  elementary classical operations. So the above algorithm makes  $O(2q * (n + 5)) \approx O(n)$  quantum queries and  $O(n)$  elementary quantum operations and  $O(n^3)$  classical operations.

**Theorem 1** Given an oracle  $C$  that is promised to be either VFS or RP, there is a quantum algorithm to distinguish VFS from RP with  $O(n)$  quantum queries.

Proof. The result is obviously from the above analysis.

### 3.2. Attack on The 4-round Feistel scheme

Let  $C = \text{FS}^4(f_1, f_2, f_3, f_4)$  is a 4-round FS, where  $C(\mathbf{a}, \mathbf{b}) = (\mathbf{c}, \mathbf{d})$ . In Patarin's work[1], for a RP, the number  $N$  of  $(i, j)$ ,  $1 \leq i < j \leq m$  such that  $\mathbf{b}_i = \mathbf{b}_j$  and  $\mathbf{c}_i \oplus \mathbf{a}_i = \mathbf{c}_j \oplus \mathbf{a}_j$  is  $\frac{m^2}{2 \cdot 2^{2n}}$ . However, for a  $\text{FS}^4(f_1, f_2, f_3, f_4)$ , the number  $N$  is  $\frac{m^2}{2^{2n}}$ . So we take all  $\mathbf{a}_i \in I_n$ ,  $\mathbf{b}_i = \mathbf{0}$  and count the number  $N$  of equalities of the form  $\mathbf{c}_i \oplus \mathbf{a}_i = \mathbf{c}_j \oplus \mathbf{a}_j$ ,  $i < j$ . So for a RP  $N \approx 2^{n-1}$ , and for a  $\text{FS}^4(f_1, f_2, f_3, f_4)$  we have  $N \approx 2^n$ . Then, given  $\mathbf{c} \oplus \mathbf{a} \in I_n$ , for a RP the number of  $\mathbf{a}_i \in I_n$  such that  $C(\mathbf{a}_i, \mathbf{0}) \oplus (\mathbf{a}_i, \mathbf{0}) = (\mathbf{c}_i \oplus \mathbf{a}_i, \mathbf{d}_i \oplus \mathbf{0}) = (\mathbf{c} \oplus \mathbf{a}, \mathbf{d}_i)$  is 2 on average. However, for a 4-round FS, the number is 3 on average.

Now a quantum algorithm to distinguish 4-round FS from RP will be proposed.

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#### Algorithm 2

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Input: An oracle  $C$  that is promised to be either 4-round FS or RP. A constant  $q \geq \lceil -20 \log_3 \varepsilon \rceil$ .

Output: Oracle  $C$  is 4-round FS or RP with success probability  $1 - \varepsilon$ .

Let  $k = 1$ .

1) While  $k < q$  do

2) For  $t$  from 1 to  $n + 5$  do

3) Prepare the  $4n$  qubit state  $\sum_{\mathbf{i}=0}^{2^n-1} |\mathbf{i}\rangle_1 |\mathbf{0}\rangle_2 |\mathbf{0}\rangle_3 |\mathbf{0}\rangle_4$ .

4) Apply  $U_C$  to create the state  $\sum_{\mathbf{i}=0}^{2^n-1} |\mathbf{i}\rangle_1 |\mathbf{0}\rangle_2 |\mathbf{c}_i\rangle_3 |\mathbf{d}_i\rangle_4$ .

Where  $U_C |\mathbf{i}\rangle_1 |\mathbf{0}\rangle_2 |\mathbf{0}\rangle_3 |\mathbf{0}\rangle_4 = |\mathbf{i}\rangle_1 |\mathbf{0}\rangle_2 |\mathbf{c}_i\rangle_3 |\mathbf{d}_i\rangle_4$ .

- 5) Apply  $U_D$  to create the state  $\sum_{\mathbf{i}=0}^{2^n-1} |\mathbf{i}\rangle_1 |\mathbf{0}\rangle_2 |\mathbf{i} \oplus \mathbf{c}_i\rangle_3 |\mathbf{d}_i\rangle_4$ ,  
Where  $U_D |\mathbf{i}\rangle_1 |\mathbf{0}\rangle_2 |\mathbf{c}_i\rangle_3 |\mathbf{d}_i\rangle_4 = |\mathbf{i}\rangle_1 |\mathbf{0}\rangle_2 |\mathbf{i} \oplus \mathbf{c}_i\rangle_3 |\mathbf{d}_i\rangle_4$
- 6) Measure the third register, apply  $U_D$  and  $U_C$  again, then we have a random “coset state”  $\frac{1}{\sqrt{2}}(|\mathbf{i}\rangle + |\mathbf{j}\rangle)$  or  $\frac{1}{\sqrt{3}}(|\mathbf{i}\rangle + |\mathbf{j}\rangle + |\mathbf{k}\rangle)$  in the first register.
- 7) Call step 6 to step 8 of Algorithm 1.
- 8) Denote  $x_k = 1$  if  $\mathbf{x}_k \neq \mathbf{0}$ , else  $x_k = 0$ . Determine the oracle  $C$  is 4-round FS or RP through binary sequence  $\mathbf{x} = (x_1, \dots, x_q)$ .

Correctness: The distribution of a secure FS should be smooth, i.e., for given  $\mathbf{c} \oplus \mathbf{a} \in I_n$ , the number of  $\mathbf{a}_i \in I_n$  such that  $C(\mathbf{a}_i, \mathbf{0}) \oplus (\mathbf{a}_i, \mathbf{0}) = (\mathbf{c} \oplus \mathbf{a}, \mathbf{d}_i)$  is 3 in general. So, for simplicity, we take the probability for 2/3. Note that  $x_i = 1$  if the number is 2, and  $x_i = 0$  if the number is 3. Suppose that  $C$  is RP then the equality  $x_i = 1$  holds with probability at least 2/3, however, the probability is at most 1/3 for 4-round FS. Denote the number of  $x_i = 0$  and  $x_i = 1$  ( $1 \leq i \leq q$ ) in the sequence  $\mathbf{x}$  is  $N_0, N_1$ , respectively. So if  $N_1 > N_0$ , we consider that  $C$  is RP, otherwise,  $C$  is FS.

Query complexity: By the analysis above, the error probability is evaluated as follows:

$$\begin{aligned} P_{err} &= \Pr[\text{VFS}|\text{RP}] \Pr[\text{RP}] + \Pr[\text{RP}|\text{VFS}] \Pr[\text{VFS}] \\ &= 3^{-q} [C_q^0 + C_q^1 \cdot 2 + C_q^2 \cdot 2^2 + \dots C_q^r \cdot 2^r] \\ &\leq \frac{2^r}{3^q} [C_q^0 + C_q^1 + C_q^2 + \dots C_q^r] \leq \frac{2^{3r}}{3^{2r+1}} \end{aligned}$$

$$P_{err} \leq \varepsilon, \text{ so we have } r \geq \frac{\log_3 \varepsilon}{3 \log_3 2 - 2} \approx -10 \log_3 \varepsilon, \text{ i.e. } q \geq -20 \log_3 \varepsilon.$$

Furthermore, to solve the system  $M\mathbf{x}_k = \mathbf{0}$ , we need only  $O(n^3)$  elementary classical operations.

The above algorithm makes  $O(q * (n + 5)) \approx O(rn)$  quantum queries and  $O(rn)$  elementary quantum operations and  $O(rn^3)$  classical operations.

**Theorem 2** Given an oracle  $C$  that is promised to be either 4-round FS or RP, there is a quantum algorithm to distinguish FS from RP with  $O(n)$  quantum queries.

The result is obviously from the above analysis. Patarin[1][2] showed that to distinguish 4-round FS from RP, classical generic attack required  $O(2^{n/2})$  random queries and  $O(2^{n/2})$  computations. So quantum algorithm here realizes an exponential speed-up.

## 4. Unbalanced FS with contracting functions

### 4.1. Attacks on 4-round: $G_3^4$

On some input  $[I^1, I^2, I^3]$ ,  $G_3^4$  produces an output denoted by  $[S^1, S^2, S^3]$ . We choose  $m$  messages such that  $\forall i, I_i^3 = \mathbf{0}$  and  $I_i^2 \neq I_j^2$  for all  $i \neq j$ , where  $I_i^1, I_i^2, I_i^3 \in I_n$ . Then count the number  $N$  of pairs  $(i, j)$  with  $i < j$  such that  $I_i^2 \oplus I_j^2 = S_i^1 \oplus S_j^1$ . For a RP, this condition appears only by chance. Thus, from Patarin’s work[7], we get  $N \approx \frac{m^2}{2^{n+1}} + \sigma \left( \frac{m}{2^{n/2}} \right)$ , where  $\sigma \left( \frac{m}{2^{n/2}} \right)$  denotes the standard deviation. For  $G_3^4$  we have  $N \approx \frac{m^2}{2^n} + \sigma \left( \frac{m}{2^{n/2}} \right)$ . So take  $m = 2^n$ , for a RP  $N \approx 2^{n-1}$ , and for  $G_3^4$  we have  $N \approx 2^n$ .

So we can obtain an effective algorithm to distinguish  $G_3^4$  from RP.

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Algorithm 3

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Input: An oracle  $C$  that is promised to be either  $G_3^4$  or RP. A constant  $q \geq \lceil -20 \log_3 \varepsilon \rceil$ .

Output: Oracle  $C$  is  $G_3^4$  or RP with success probability  $1 - \varepsilon$ .

Let  $k = 1$ .

1) While  $k < q$  do

2) For  $t$  from 1 to  $n + 5$  do

3) Prepare the  $4n$  qubit state  $\sum_{\mathbf{i}=0}^{2^n-1} |\mathbf{0}\rangle_1 |\mathbf{i}\rangle_2 |\mathbf{0}\rangle_3 |\mathbf{0}\rangle_4 |\mathbf{0}\rangle_5 |\mathbf{0}\rangle_6$ .

4) Apply  $U_C$  to create the state  $\sum_{\mathbf{i}=0}^{2^n-1} |\mathbf{0}\rangle_1 |\mathbf{i}\rangle_2 |\mathbf{0}\rangle_3 |\mathbf{S}_i^1\rangle_4 |\mathbf{S}_i^2\rangle_5 |\mathbf{S}_i^3\rangle_6$ .

Where  $U_C |\mathbf{0}\rangle_1 |\mathbf{i}\rangle_2 |\mathbf{0}\rangle_3 |\mathbf{0}\rangle_4 |\mathbf{0}\rangle_5 |\mathbf{0}\rangle_6 = |\mathbf{0}\rangle_1 |\mathbf{i}\rangle_2 |\mathbf{0}\rangle_3 |\mathbf{S}_i^1\rangle_4 |\mathbf{S}_i^2\rangle_5 |\mathbf{S}_i^3\rangle_6$ .

5) Apply  $U_D$  to create the state,  $\sum_{\mathbf{i}=0}^{2^n-1} |\mathbf{0}\rangle_1 |\mathbf{i}\rangle_2 |\mathbf{0}\rangle_3 |\mathbf{i} \oplus \mathbf{S}_i^1\rangle_4 |\mathbf{S}_i^2\rangle_5 |\mathbf{S}_i^3\rangle_6$

Where  $U_D |\mathbf{0}\rangle_1 |\mathbf{i}\rangle_2 |\mathbf{0}\rangle_3 |\mathbf{S}_i^1\rangle_4 |\mathbf{S}_i^2\rangle_5 |\mathbf{S}_i^3\rangle_6 = |\mathbf{0}\rangle_1 |\mathbf{i}\rangle_2 |\mathbf{0}\rangle_3 |\mathbf{i} \oplus \mathbf{S}_i^1\rangle_4 |\mathbf{S}_i^2\rangle_5 |\mathbf{S}_i^3\rangle_6$

6) Measure the fourth register, apply  $U_D$  and  $U_C$  again, then we have a random “coset state”  $\frac{1}{\sqrt{2}}(|\mathbf{i}\rangle + |\mathbf{j}\rangle)$  or  $\frac{1}{\sqrt{3}}(|\mathbf{i}\rangle + |\mathbf{j}\rangle + |\mathbf{k}\rangle)$  in the second register.

7) Call step 6 to step 8 of Algorithm 1.

8) Denote  $x_k = 1$  if  $\mathbf{x}_k \neq \mathbf{0}$ , else  $x_k = 0$ . Determine the oracle  $C$  is  $G_3^4$  or RP through binary sequence  $\mathbf{x} = (x_1, \dots, x_q)$ .

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Correctness: The equality  $I_i^2 \oplus I_j^2 = S_i^1 \oplus S_j^1$  holds if and only if  $I_i^2 \oplus S_i^1 = I_j^2 \oplus S_j^1$  holds. So by measuring the fourth register, applying  $U_D$  and  $U_C$  again, we have a random “coset state”

$$\frac{1}{\sqrt{2}}(|\mathbf{i}\rangle + |\mathbf{j}\rangle) \tag{1}$$

or

$$\frac{1}{\sqrt{2}}(|\mathbf{i}\rangle + |\mathbf{j}\rangle + |\mathbf{k}\rangle) \tag{2}$$

in the second register.

We know that  $x_k = 1$  if equality (1) holds, and  $x_k = 0$  if equality (2) holds. Similarly algorithm 2, if the oracle  $C$  is  $G_3^4$ , then  $x_k = 1$  with probability at most  $1/3$ , however, the probability at least  $2/3$  for a RP. So it will be enough to distinguish  $G_3^4$  from RP by the observed sequence  $\mathbf{x}$ .

Query complexity: The complexity is similar to algorithm 2. By the analysis of algorithm 2, the error probability is evaluated as follows:

$$P_{err} = \Pr[\text{VFS}|\text{RP}] \Pr[\text{RP}] + \Pr[\text{RP}|\text{VFS}] \Pr[\text{VFS}] \leq \frac{2^{3r}}{3^{2r+1}}$$

$$P_{err} \leq \varepsilon, \text{ so we have } r \geq \frac{\log_3 \varepsilon}{3 \log_3 2 - 2} \approx -10 \log_3 \varepsilon, \text{ i.e. } q \geq -20 \log_3 \varepsilon.$$

Furthermore, to solve the system  $M\mathbf{x}_k = \mathbf{0}$ , we need only  $O(n^3)$  elementary classical operations.

**Theorem 3** Given an oracle  $C$  that is promised to be either  $G_3^4$  or RP, there is a quantum algorithm to distinguish  $G_3^4$  from RP with  $O(-10n \log_3 \varepsilon)$  quantum queries.

#### 4.2. Attacks for $k+1$ round with $k \geq 4$

The input and output of  $G_k^d$  is denoted by  $[I^1, I^2, \dots, I^k]$  and  $[S^1, S^2, \dots, S^k]$ , respectively. For  $G_k^{k+1}$ , we choose  $\forall i, I_i^3 = \dots = I_i^k = 0$  and pairwise distinct  $I_i^1$ . Then from Patarin's work[7], if we take all  $I_i^1 \in I_n$ , the number  $N$  of pairs  $(i, j)$ ,  $i < j$ , such that  $I_i^2 \oplus I_j^2 = S_i^1 \oplus S_j^1$  holds is  $2^n$ . However, for a RP, we have  $N \approx 2^{n-1}$ .

So we can obtain an algorithm similar algorithm 3.

It is enough to substitute step 3 to step 6 in algorithm 3 as follows:

3\*) Prepare the  $2kn$  qubit state  $\sum_{\mathbf{i}=0}^{2^n-1} |\mathbf{i}\rangle_1 |\mathbf{0}\rangle_2 \dots |\mathbf{0}\rangle_k \dots |\mathbf{0}\rangle_{2k}$ .

4\*) Apply  $U_C$  to create the state  $\sum_{\mathbf{i}=0}^{2^n-1} |\mathbf{i}\rangle_1 |\mathbf{0}\rangle_2 \dots |\mathbf{0}\rangle_k |\mathbf{S}_i^1\rangle_{k+1} \dots |\mathbf{S}_i^k\rangle_{2k}$ ,

where  $U_C |\mathbf{i}\rangle_1 |\mathbf{0}\rangle_2 \dots |\mathbf{0}\rangle_k \dots |\mathbf{0}\rangle_{2k} = |\mathbf{i}\rangle_1 |\mathbf{0}\rangle_2 \dots |\mathbf{0}\rangle_k |\mathbf{S}_i^1\rangle_{k+1} \dots |\mathbf{S}_i^k\rangle_{2k}$

5\*) Measure the  $(k+1)$ -th register, apply  $U_C$  again, then we have a random "coset state"  $\frac{1}{\sqrt{2}}(|\mathbf{i}\rangle + |\mathbf{j}\rangle)$  or  $\frac{1}{\sqrt{3}}(|\mathbf{i}\rangle + |\mathbf{j}\rangle + |\mathbf{k}\rangle)$  in the first register.

The correctness and query complexity is obviously from algorithm 3, we have the following theorem.

**Theorem 4** Given an oracle  $C$  that is promised to be either  $G_k^{k+1}$  or RP, there is a quantum algorithm to distinguish  $G_k^{k+1}$  from RP with  $O(n)$  quantum queries.

## 5. Conclusion

In this paper, we presented quantum generic attacks against classical FS. We shows that the 3,4-round FS and unbalanced FS with contracting functions  $G_k^d$  are distinguishable from a RP by making less queries than classical queries. Moreover, the method in this paper can also be used to distinguish unbalanced FS with expanding functions.

Here we will discuss a few more open problems. The main problem is that we haven't considered the more round FS. For 6 round or more, it is still an open problem whether or not the number of quantum queries can achieve exponential speed-up over classical queries. Furthermore, finding another scheme used in classical cryptography which realizes speed-up by quantum algorithm is more challengingly.

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