Finite groups with the same character tables, Drinfel'd algebras and Galois algebras

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February 1, 2008

Abstract

We prove that finite groups have the same complex character tables iff the group algebras are twisted forms of each other as Drinfel'd quasi-bialgebras or iff there is non-associative bi-Galois algebra over these groups. The interpretations of class-preserving automorphisms and permutation representations with the same character in terms of Drinfel'd algebras are also given.

1. Introduction. The theory of quasi-Hopf algebras was developed by V.G.Drinfel'd for the description of quantizations of Lie groups and algebras or so-called quantum groups.

Althought the deformational quantization approach which is so useful in the theory of quantum groups can't be applied for the the case of finite groups, the idea of twisting seems to be very suitible for reformulating of various problems from representation theory of finite groups.

The key observations of this article is that any bijection between character tables of finite groups corresponds to the quasi-isomorphism of the group algebras considered as quasi-Hopf algebras and any two homomorphisms of the group algebras define the same map of character tables iff they are twisted forms.

In particular, we can give the definitions in terms of (quasi-)Hopf algebras of such objects as class-preserving automorphisms, permutation representations with the same character, groups with the same character tables. Namely, any class-preserving automorphism is twisted form of identity maps as homomorphisms of Hopf algebras. Two permutation representations have the same complex character iff the corresponding homomorphisms into symmetric group are twisted forms as homomorphisms of Hopf algebras. Two groups have the same character tables iff their group algebras are twisted forms as quasi-Hopf algebras.

This point of view allows to select the subclass of pairs of groups with the same character tables. This subclass consists of pairs of groups whose group algebas are twisted forms as Hopf algebras.

The notion of quasi-homomorphism of group algebras can be formulated in terms of Galois algebras. Using the calculation of automorphisms of associative Galois algebras we can describe quasi-isomorphisms of group algebras as Hopf algebras. These quasi-isomorphisms correspond to normal abelean 2-subgroups equipped with some non-degenerated bimultiplicative forms.

2. Semirings and character tables. A *semiring* is a set S with a collection of non-negative integers $\{m^x_{x_1,x_2}, x, x_1, x_2 \in S\}$ (*structural constants*) which satisfy the (*associativity*) condititon

$$m^x_{x_1,x_2,x_3} = \sum_{t\in S} m^x_{x_1,t} m^t_{x_2,x_3} = \sum_{s\in S} m^s_{x_1,x_2} m^x_{s,x_3}, \quad \forall x, x_1, x_2, x_3 \in S.$$

An element e of the semiring S is an *identity* if $m^s_{t,e} = m^s_{e,t} = \delta_{s,t}$ for all $s, t \in S$.

A *morphism* from the semiring S to the semiring S' is a collection $\{n^s_t, s \in S, t \in S'\}$ of non-negative integers which satisfy the following condition:

$$\sum_{s\in S} m^s_{s_1,s_2} n^t_s = \sum_{t_1,t_2\in S'} {m'}^t_{t_1,t_2} n^{t_1}_{s_1} n^{t_2}_{s_2} \tag{1}$$

for any $s_1, s_2 \in S$ and $t \in S'$. A *degree map* d for the semiring S is a morphism from S to the one-element semiring with identity, e.g. a colection $\{d(s), s \in S\}$ of non-negative integers such that $d(s_1)d(s_2) = \sum_{s\in S} m^s_{s_1,s_2} d(s)$.

The *enveloping ring* $A(S)$ of the semiring S is the free $\mathbf{Z}$-module with the basis $\{[s], s \in S\}$ labeled by the elements of S and with the multiplications $[i][j] = \sum_{s\in S} m^s_{i,j}[s]$. We will denote by $A_{\geq 0}(S)$ the cone of non-negative combinations of basic elements (the cone of *non-negative elements*).

A morphism of semirings defines a homomorphism of their enveloping rings $f : A(S) \to A(S')$ where $f([s]) = \sum_{t\in S'} n^t_s [t]$.

Denote by $(\ ,\ )$ canonical bilinear form on $A(S)$

$$(x, y) = \delta_{x,y}, \quad \forall x, y \in S.$$

The semiring S is *rigid* if there defined an antihomomorphism $(\ )^* : S \to S$ (*conjugation*) such that

$$(xy, z) = (y, x^*z), \quad \forall x, y, z \in S.$$

Note that conjugation is an anti-endomorphism of the enveloping ring $A(S)$:

$$(z, (xy)^*w) = ((xy)z, w) = (x(yz), w) = (yz, x^*w) = (z, y^*x^*w),$$

or $(xy)^* = y^*x^*$.

It follows from the definition that the kernel of the conjugation $(\ )^*$ lies in the kernel of the bilinear form $(\ ,\ )$

$$(x, y) = (1, x^*y) = 0, \quad \text{for} \quad x \in ker(\ )^*, y \in S.$$

Since the bilinear form (,) is non-degenerated the conjugation is injective. So it is bijective for the finite semiring S. In that case the conjugation has a finite order as an automorphism of the finite set S.

Lemma 1. *Let d be a degree map for the rigid seniring S. Then $\rho = \sum_{s\in S} d(s^*)s \in A(S)$ satisfies to the conditions*

$$x\rho = d(x)\rho, \ \forall x \in A(S).$$

Proof. Since $m^t_{x,s} = (t, xs) = (x^*t, s) = (t^*x, s^*) = m^{s^*}_{t^*,x}$ we have

$$x\rho = \sum_{s\in S} d(s^*)xs = \sum_{s,t\in S} d(s^*)m^t_{x,s}t = \sum_{s,t\in S} d(s^*)m^{s^*}_{t^*,x}t = \sum_{t\in S} d(t^*x)t = \rho(x).$$

□

Proposition 1(Uniqueness of degree map)**.** *Any two degree maps for commutative rigid semiring S coincides.*
Proof. Let d, d' be degree maps for S. Define $\rho = \sum_{s\in S} d(s^*)s, \rho' = \sum_{s\in S} d'(s^*)s$. Then $d(\rho')\rho = \rho'\rho = d'(\rho)\rho'$, which means $d = d'$. □

Since the enveloping ring $A(S)$ of rigid semiring S is equipped with non-degenerated semi-invariant bilinear form, the enveloping algebra $A_{\mathbf{Q}}(S) = A(S) \otimes \mathbf{Q}$ over rational numbers $\mathbf{Q}$ is semisimple.

For commutative rigid semiring S the enveloping algebra $A_{\bar{\mathbf{Q}}}(S) = A(S) \otimes \bar{\mathbf{Q}}$ over algebraic closure $\bar{\mathbf{Q}}$ of $\mathbf{Q}$ is isomorphic to the algebra of functions on some finite set $Cl(S)$ (*"conjugacy classes"* of S). The set $Cl(S)$ can be identified with the set of maximal ideals of $A_{\bar{\mathbf{Q}}}(S)$, so that the value $x(c)$ of $x \in A_{\bar{\mathbf{Q}}}(S)$ on $c \in Cl(S)$ is unique element of $\bar{\mathbf{Q}}$ such that $x \in x(c) + m_c$. Here m_c is maximal ideal of $A_{\bar{\mathbf{Q}}}(S)$ corresponding to c.

For any commutative rigid semiring S we can associate a *character table* $(s(c))_{s\in S, c\in Cl(S)}$, which is $|S| \times |S|$-matrix with entries in $\bar{\mathbf{Q}}$.

Proposition 2. *The map $f : A(S) \to A(S')$ given by the collection $\{n^s_t, s \in S, t \in S'\}$ is a homomorphism of (semi)rings (the collection satisfies to the condition* (1)*) if and only if there is a map $f^* : Cl(S') \to Cl(S)$ such that $f(s)(c) = s(f^*(c))$ for any $s \in S, c \in Cl(S')$.*
Proof. The map $f : A(S) \to A(S')$ is a ring homomorphism iff $f_{\bar{\mathbf{Q}}} : A_{\bar{\mathbf{Q}}}(S) \to A_{\bar{\mathbf{Q}}}(S')$ is a homomorphism of $\bar{\mathbf{Q}}$-algebras. Any homomorphism of algebras of functions corresponds to the map of sets $f^* : Cl(S') \to Cl(S)$. □

Example 1. The set $Irr(G)$ of irreducible characters of the finite group G has a natural semiring structure:

$$\chi\psi = \sum \eta \in Irr(G) m^{\eta}_{\chi,\psi}\eta, \quad \chi, \psi \in Irr(G).$$

The *map* of character tables of the finite groups G_1, G_2 is a pair consisting of

i) the map of the sets of conjugacy classes $cl(G_1) \to cl(G_2), \quad C \mapsto C^*$,

ii) the map from the set of irreducible characters to the semiring of characters $Irr(G_2) \to R_{\geq 0}(G_1), \quad \chi \mapsto \chi^* = sum_{\psi \in Irr(G_1)} n_{\chi,\psi}\psi$, where $n_{\chi,\psi} \geq 0$

such that $\chi^*(C) = \chi(C^*)$ for all $\chi \in Irr(G_2), C \in cl(G_1)$.

We say that two finite groups G_1, G_2 have *the same character tables* if there are one-to-one mappings $\chi \mapsto \chi^*$ and $C \mapsto C^*$ between the sets of irreducible characters and conjugasy classes, respectively, of G_1 and G_2, such that $\chi^*(C^*) = \chi(C)$ for all χ, C. For examples, see [1, 3, 4, 8, 10, 12, 13].

3. Semisimple monoidal categories. The *monoidal category* [6] is a category G with a bifunctor

$$\otimes : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G} \qquad (X, Y) \mapsto X \otimes Y$$

which called *tensor (or monoidal) product*. This functor must be equiped with a functorial collection of isomorphisms (so-called *associativity constraint*)

$$\varphi_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z \qquad \text{for any} \quad X, Y, Z \in \mathcal{G}$$

which satisfies to the following *pentagon axiom*:

$$(X \otimes \varphi_{Y,Z,W})\varphi_{X,Y \otimes Z,W}(\varphi_{X,Y,Z} \otimes W) = \varphi_{X,Y,Z \otimes W}\varphi_{X \otimes Y,Z,W}.$$

A *quasimonoidal functor* between monoidal categories $\mathcal{G}$ and $\mathcal{H}$ is a functor $F : \mathcal{G} \longrightarrow \mathcal{H}$, which is equipped with the functorial collection of isomorphisms (the so-called *quasimonoidal structure*)

$$F_{X,Y} : F(X \otimes Y) \to F(X) \otimes F(Y) \qquad \text{for any} \quad X, Y \in \mathcal{G}.$$

We shal call it *monoidal structure* if

$$F_{X,Y \otimes Z}(I \otimes F_{Y,Z}) = F_{X \otimes Y,Z}(F_{X,Y} \otimes I)$$

for any objects $X, Y, Z \in \mathcal{G}$.

The morphism $\alpha : F \to G$ between two monoidal functors $F, G : \mathcal{G} \to \mathcal{H}$ is *monoidal* if $F_{X,Y}(\alpha_X \otimes \alpha_Y) = \alpha_{x \otimes Y} G_{X,Y}$ for any $X, Y \in \mathcal{G}$.

Monoidal category structures on $\mathcal{G}$ differed by the associativity constraint will be called *twisted forms* of each other.

The structures of monoidal functor for $F : \mathcal{G} \to \mathcal{H}$ will be called *twisted forms* of each other.

The monoidal category $\mathcal{G}$ is *rigid* if it is equipped with the *dualization* functor, which is a contravariant functor $(\ )^* : \mathcal{G} \to \mathcal{G}$ with a collections of morphisms $\iota : 1 \to X \otimes X^*$ and $\epsilon\upsilon : x^* \otimes X \to 1$ for any $X \in \mathcal{G}$ such that the compositions

$$X \xrightarrow{I \otimes \iota} X \otimes (X^* \otimes X) \xrightarrow{\varphi} (X \otimes X^*) \otimes X \xrightarrow{\epsilon\upsilon \otimes I} X,$$

$$X^* \xrightarrow{\iota\otimes I} (X^*\otimes X)\otimes X^* \xrightarrow{\varphi^{-1}} X^*\otimes (X\otimes X^*) \xrightarrow{I\otimes\epsilon\upsilon} X^*$$

are identical.

Let $\mathcal{G}$ be a semisimple monoidal k-linear category over the field algebraically closed k with the set S of isomorphism classes of simple objects. The collection of dimensions $m^x_{y,z} = dim_k Hom_{\mathcal{G}}(X, Y\otimes Z)$ form a semiring structure on the set S. Here X, Y and Z are some representatives of the classes $x, y, z \in S$. Note that the enveloping ring of semiring S coincides with the Grothendieck ring $K_0(\mathcal{G})$ of the category $\mathcal{G}$. The semiring $S(\mathcal{G})$ is rigid for the rigid monoidal category $\mathcal{G}$.

Proposition 3. *Semisimple monoidal categories are twisted forms of each other iff their semirings of simple objects are isomorphic. Isomorphism classes of quasimonoidal functors $F : \mathcal{G} \to \mathcal{H}$ between semisimple monoidal categories are in one-to-one correspondence with the homomorphisms $S(\mathcal{G}) \to S(\mathcal{H})$ of the semirings of simple objects. In particular, monoidal functors $F, G : \mathcal{G} \to \mathcal{H}$ between semisimple monoidal categories are twisted forms of each other iff they induce the same map $K_0(\mathcal{G}) \to K_0(\mathcal{H})$ of the Grothendieck rings.*

Proof. The proposition follows from the fact that two fuctors $F, G : \mathcal{G} \to \mathcal{H}$ between semisimple categories are isomorphic iff they induce the same map $S(\mathcal{G}) \to S(\mathcal{H})$ of the semirings of simple objects. □

4. Drinfel'd algebras. A *Drinfel'd algebra* or *quasi-bialgebra* [7] is an algebra H together with a homomoprhisms of algebras

$$\Delta : H \to H\otimes H \quad \text{(coproduct)}, \quad \varepsilon : H \to k \quad \text{(counit)}$$

and an invertible element $\Phi \in H^{\otimes 3}$ (*associator*) for which

$$(\Delta\otimes I)(\Delta(h)) = \Phi(I\otimes\Delta)(\Delta(h))\Phi^{-1} \quad \forall h\in H \quad \text{(coassociativity)},$$

$$(I\otimes I\otimes\Delta)(\Phi)(\Delta\otimes I\otimes I)(\Phi) = (1\otimes\Phi)(I\otimes\Delta\otimes I)(\Phi)(\Phi\otimes 1),$$

$$(\varepsilon\otimes I)\Delta = (I\otimes\varepsilon)\Delta = I.$$

Drinfel'd algebra is a generalization of the well-known notion of *bialgebra* which corresponds to the case of trivial associator $\Phi = 1$.

Drinfel'd algebras structures on the algebra H which is differed only by associator will be called *twisted forms* of each other.

A *quasi-homomorphism* of quasi-bialgebras H_1 and H_2 is pair (f, F) consisting of a homomorphism of algebras $f : H_1 \to H_2$ and an invertible element $F \in H_2^{\otimes 2}$ such that

$$\Delta(f(h)) = F(f\otimes f)(\Delta(h))F^{-1}.$$

It is a *homomorphism* of quasi-bialgebras if, additionly,

$$(\Delta \otimes I)(F)(F \otimes 1)(f \otimes f \otimes f)(\Phi_1) = \Phi_2(I \otimes \Delta)(F)(1 \otimes F).$$

Two homomorphisms of quasi-bialgebras are *twisted forms* of each other if they differ only by the invertible element F. We can define the *morphism* between two homomorphisms $(f, H), (g, G) : H_1 \to H_2$ as an element $c \in H_2$ such that $cf(h) = g(h)c$ for any $h \in H_1$ and $\Delta(c)G = F(c \otimes c)$.

A *homomorphism of bialgebras* H_1, H_2 is a homomorphism of algebras $f : H_1 \to H_2$ such that $\Delta f = (f \otimes f)\Delta$.

Now we discuss the connection between monoidal categories and quasi-bialgebras. Coproduct allows to define the structure of H-module on the tensor product $M \otimes_k N$ of two H-modules:

$$h * (m \otimes n) = \Delta(h)(m \otimes n), \qquad h \in H, m \in M, n \in N.$$

The associator Φ defines the associativity constraint

$$\varphi : L \otimes M \otimes N \to L \otimes M \otimes N, \qquad \varphi(l \otimes m \otimes n) = \Phi(l \otimes m \otimes n).$$

Thus the category $\mathcal{M}(H)$ of H-modules is a monoidal category.
The homomorphism of quasi-bialgebras $f : H_1 \to H_2$ defines the monoidal functor

$$f^* : \mathcal{M}(H_2) \to \mathcal{M}(H_1)$$

with the monoidal structure defined by the invertible element $F \in H_2^{\otimes 2}$

$$f^*{}_{M,N} : f^*(M \otimes N) \to f^*(M) \otimes f^*(M) \quad f^*{}_{M,N}(m \otimes n) = F(m \otimes n).$$

The morphisms between homomorphisms $f, g : H_1 \to H_2$ of quasi-bialgebras correspond to the monoidal morphisms between monoidal functors $f^*, g^* : \mathcal{M}(H_2) \to \mathcal{M}(H_1)$.

The quasi-Hopf algebra H will be called *rigid* if the monoidal category $\mathcal{M}(H)$ is rigid. The dualization functor for $\mathcal{M}(H)$ corresponds to the antihomomorphism $S : H \to H$ (*antipode*) with some additional properties (see [7]). For bialgebra these properties takes a form of the relation

$$\mu(S \otimes I)\Delta = \mu(I \otimes S)\Delta = \varepsilon,$$

where $\mu : H \otimes H \to H$ is the multiplication in H. Bialgebra with an antipode is called *Hopf algebra*.

Example 2. Group algebra $k[G]$ of the group G. As k-vector space $k[G]$ is spanned by the elements of the group G. The structure maps have the following forms on the basis:

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}.$$

The homomorphism of the groups $f : G_1 \to G_2$ defines the homomorphism of their group algebras and any homomorphism of bialgebras $k[G_1] \to k[G_2]$ is of this kind. The group algebra provides an example of so-called *cocommutative* Hopf algebra $t\Delta = \Delta$. Over the algebraically closed field of characteristic zero group algebras are characterized by this property (Kostant theorem): any cocommutative finite dimensional Hopf algebra is isomorphic to the group algebra.

For semisimple quasi-bialgebra H denote by $S(H) = S(\mathcal{M}(H))$ the semiring of simple modules. The semiring $S(H)$ is rigid for quasi-Hopf algebra H.

The next proposition is the direct consequence of the definitions and proposition Proposition 3.

Proposition 4. *The homomorphisms of quasi-bialgebras $f_1, f_2 : H_1 \to H_2$ are twisted forms if and only if the monoidal functors $(f_1)^*, (f_2)^*$ are twisted forms. In particular, the homomorphisms of semisimple quasi-bialgebras $f_1, f_2 : H_1 \to H_2$ induce the same homomorphism $K_0(f_1), K_0(f_2) : K_0(H_2) \to K_0(H_1)$ of Grothendieck rings if and only if one is isomorphic to the twisted form of the other.*

The generalization of the so-called Tannaka-Krein theory [5, 6] states that we can reconstruct a quasi-bialgebra from the monoidal category $\mathcal{G}$ and a quasi-monoidal functor $F : \mathcal{G} \to \mathcal{M}(k)$ to the category of vector spaces as endomorphisms $End(F)$ of the functor F.

Theorem 1. *Finite dimensional semisimple quasi-Hopf algebras H_1, H_2 are quasi-isomorphic if and only if their semirings of simple objects $S(H_2), S(H_1)$ are isomorphic.*

Proof. Since twisting does not change the semiring of simple objects we need to prove the if statement. Let $f^* : S(H_2) \to S(H_1)$ be an isomorphism of semirimgs of simple objects. By Proposition 3 we can costruct a quasi-monoidal functor (equivalence) $F : \mathcal{M}(H_2) \to \mathcal{M}(H_1)$ which induces the given homomorphisms f^*. By Proposition 1 the composition $d_1 f^*$ coincides with d_2, where d_i is a degree map for $S(H_i)$. Hence the composition $F_1 F$ of functor F with the forgetful funtor $F_1 : \mathcal{M}(H_1) \to \mathcal{M}(k)$ is isomorphic to the forgetful funtor $F_2 : \mathcal{M}(H_2) \to \mathcal{M}(k)$ as quasi-monoidal functor. Using Tannaka-Krein theory we can construct the isomomorphism of quasi-bialgebras $f : H_1 \to H_2$ as

$$H_1 = End(F_1) \to End(F_1 F) \to End(F_2) = H_2.$$

□

Remark 1. The weak version of the theorem Theorem 1 for Hopf algebras was proved in [11] where it was assumed that the isomorphism of (enveloping algebras of) semiring preserves the class of regular representation.

Corollary 1. *The finite groups G_1, G_2 have the same character table if and only if their group algebras are isomorphic as quasi-Hopf algebras, e.g. there is an isomorphism of algebras $f : k[G_1] \to k[G_2]$ and an invertible element $F \in k[G_2]^{\otimes 2}$ such that $F\Delta_2(f(x)) = (f \otimes f)(\Delta_1(x))$ for any $x \in k[G_1]$.*

If we denote by Δ_F the twisted by F comultiplication on $k[G_2]$ $\Delta_F(x) = F\Delta_2(x)F^{-1}$ then the map f would be an isomorphism of Hopf algebras $(k[G_1], \Delta_1)$ and $(k[G_2], \Delta_F)$. The existence of such isomorphism is equivalent to the existence of an isomorphism of groups

$$G_1 \to G(F) = G(k[G_2], \Delta_F) = \{x \in k[G_2], F\Delta_2(x) = (x \otimes x)F\}.$$

The cocommutativity of the coproduct Δ_F is equivalent to the condition

$$t(F) = F \tag{2}$$

The coassociativity of the twisted coproduct Δ_F is equivalent to the equation on F

$$(1 \otimes F)(I \otimes \Delta)(F) = (F \otimes 1)(\Delta \otimes I)(F)\Phi, \tag{3}$$

where Φ is some invertible G_2-invariant element of $k[G_2]^{\otimes 3}$ (*associator*). In particular, such Φ satisfy to the equation

$$(\Phi \otimes 1)(I \otimes \Delta \otimes I)(\Phi)(1 \otimes \Phi) = (\Delta \otimes I \otimes I)(\Phi)(I \otimes I \otimes \Delta)(\Phi). \tag{4}$$

We can replace F by the product FC for any G_2-invariant element $C \in k[G_2]^{\otimes 2}$ without changing the twisted coproduct $\Delta_{FC} = \Delta_F$. The G_2-invariance of C allows to write the associator Φ^C for the product FC as

$$\Phi^C = (\Delta \otimes I)(C)^{-1}(C \otimes 1)^{-1}\Phi(1 \otimes C)(I \otimes \Delta)(C). \tag{5}$$

Thus the element Φ is defined up to the transformations (5).

We can also replace F by $F^c = (c \otimes c)F\Delta(c)^{-1}$ for invertible $c \in k[G_2]$. Then the corresponding twisted coproducts will be connected by conjugation by c

$$\Delta_{F^c}(cxc^{-1}) = (c \otimes c)\Delta_F(x)(c \otimes c)^{-1}.$$

The preveous theorem reduces the problem of finding finite groups whose character tables coincide with the character table of G to the problem of finding the solutions (Φ, F) to the equations (2), (3), (4) such that the order of the group $G(F)$ equals $|G|$. If the ground field k is algebraically closed of characteristics zero, then we can ommite the condition $|G(F)| = |G|$ using Kostant theorem.

In [7] V.Drinfeld suggested the following way of solving the equation (3) for general Hopf algebra. Introduce the new multiplication μ_F on the dual Hopf algebra $k(G) = k[G]^*$

$$\mu_F(l \otimes m)(x) = (l \otimes m)(F\Delta(x)), \quad l, m \in k(G), x \in k[G].$$

This multiplication will be invariant under the action of the group G on $k(G)$

$$(gl)(x) = l(xg).$$

By another words, elements of the group G act as automorphisms of the algebra $R_F = (k(G), \mu_F)$. Moreover, the algebra R_F is a so-called *Galois* G-algebra. It means, that the natural map of vector spaces

$$R_F \otimes R_F \to Hom(k[G], R_F), \quad l \otimes m \mapsto (g \mapsto g(l)m)$$

is an isomorphism. The group $G(F)$ appears as automorphisms group $Aut_G(R_F)$ of G-algebra R_F. It is not hard to see that if $|G(F)| = |G|$, then R_F is also Galois $G(F)$-algebra, or bi-Galois $G - G(F)$-algebra.

The algebras R_{F_1}, R_{F_2} are isomorphic as G-algebras iff there is an invertible $c \in k[G]$ such that $F_1 = F_2^c$. This method is mostly applicable for the case of $\Phi = 1$, because of the following fact:

the algebra R_F is associative iff $\Phi = 1$.

In general, it would be only Φ-*associative* in the following sense:

$$x(yz) = \Phi(xy)z, \quad \forall x, y, z \in R,$$

where the product $\Phi(xy)z = \mu(\mu \otimes I)(\Phi(x \otimes y \otimes z))$ is defined by the action of $k[G]^{\otimes 3}$ on $R^{\otimes 3}$.

5. Galois algebras. Here we give brief description of bi-Galois associative algebras. As was explained above they correspond to the isomorphisms of character tables with trivial associators.

Let R be an algebra with the action of the group G (G-*algebra*). The *cross product* $R * G$ is a vector space spanned by the elements $a * g \quad a \in R, g \in G$ satisfying $(a + b) * g = a * g + b * g$. The multiplication is given by the formula

$$(a * g)(b * f) = ag(b) * gf \qquad \forall a, b \in R, g \in G.$$

A G-algebra R is *Galois* if the map

$$\theta : R * G \to End(R) \qquad \theta(a * g)(b) = ag(b)$$

is an isomorphism.

A Galois G-algebra R has the following properties:

R has no non-trivial G-invariant twosided ideals,

R is semisimple,

G acts transitively on the set of maximal twosided ideals of R.

Let S be a stabilizer of some maximal ideal M of R. Then the quotient algebra $B = R/M$ is simple Galois S-algebra and R can be identified with the algebra of S-invariant functions

$$ind_S^G(B) = \{a : G \to B, \quad a(sg) = s(a(g)) \qquad \forall s \in S, g \in G\}$$

with the G-action $(fa)(g) = a(gf)$.

The S-algebra $B = End(V)$ is Galois iff the multiplier of the projective representation $S \to PGL(V) = Aut(B)$ is a non-degenerated 2-cocycle. We call a 2-cocycle $\alpha \in Z^2(G, k^*)$ *non-degenerated* if for any $s \in S$ the map from the centralizer

$$C_S(s) \to k^* \qquad t \mapsto \alpha(s,t)\alpha(t,s)^{-1}$$

is non-trivial.

Example 3. Let A be a finite abelian group. Denote by $\hat{A}$ the dual group $Hom(A, k^*)$. The 2-cocycle α on $S = A \oplus \hat{A}$

$$\alpha((a,\chi),(b,\psi)) = \chi(b), \quad a, b \in A, \chi, \psi \in \hat{A}$$

is non-degenerated.

Describe the automorphisms of Galois algebras.

The set of maximal ideals of G-Galois algebra R can be identified as G-set with G/S where S is a stabilizer of some maximal ideal.

The action of automorphisms on maximal ideals defines the homomorphism $Aut_G(R) \to N_G(S)/S$.

The kernel of this homomorphism coincides with $Aut_S(B)$. The group of automorphisms of the simple Galois S-algebra is isomorphic to the character group $\hat{S} = Hom_{group}(S, k^*)$.

The image of this homomorphism coincides with the stabilizer $St_{N_G(S)/S}(\alpha)$ of the cohomological class $\alpha \in H^2(S, k^*)$.

Thus we have a short exact sequence

$$\hat{S} \to Aut_G(R) \to St_{N_G(S)/S}(\alpha).$$

The class of this extension in $H^2(St_{N_G(S)/S}(\alpha), \hat{S})$ is the image of the class $\alpha \in H^2(S, k^*)$ by

$$d_2^{0,2} : H^0(St_{N_G(S)/S}(\alpha), H^2(S, k^*)) \to H^2(St_{N_G(S)/S}(\alpha), H^1(S, k^*))$$

the differential of Hochschild-Serre spectral sequence corresponding to the extension $S \to N_G(S) \to N_G(S)/S$.

We can apply now the outlined description of automorphisms of Galois algebras to the investigation of bi-Galois algebras.

A *biGalois* $G_1 - G_2$-algebra is an algebra R with the commuting actions of the groups G_1, G_2 such that R is both Galois G_1-algebra and G_2-algebra.

The $G_1 - G_2$-biGalois algebra corresponds to the following data:

the normal inclusions $S \to G_i$ of the abelian group S with the same quotient group F, the non-degenerated class $\alpha \in H^2(S, k^*)$ such that

$$d(\alpha) = \gamma_1 - \gamma_2,$$

where γ_i is the class of the extension $S \to G_i \to F$ in $H^2(F, S)$ and d is the differential of Hochschild-Serre spectral sequence corresponding to the splitting extension of F by S.

Functoriality of the differential d allows to reduce consideration to the case of p-group S. The diffirential d is trivial for abelian p-groups if $p \neq 2$.

Example (see, also [8]). Let S be $2n$-dimensional vector space over the field $\mathbf{F}_2$ of two elements. The standard symplectic form $< \, , \, >$ on S defines a non-degenerated 2-cocycle

$$\alpha \in Z^2(S, k^*), \quad \alpha(s,t) = (-1)^{\beta(s,t)},$$

where β is bilinear form on S such that $< s, t >= \beta(s,t) - \beta(t,s)$.
Let $F = Sp_{2n}(2)$ be the group of automorphisms of the form $< \, , \, >$. For $n > 1$ the cohomology group $H^2(F, \hat{S}) = H^2(Sp_{2n}, \mathbf{F}_2^{2n})$ is one dimensional $\mathbf{F}_2$-vector space generated by the class $d(\alpha)$. Thus the affine symplectic group $AffSp_{2n}(2)$ and the (unique) non-split extension of $Sp_{2n}(2)$ by $\mathbf{F}_2^{2n}$ have the same character tables.
For $n = 1$ the cohomology group $H^2(Sp_{2n}, \mathbf{F}_2^{2n})$ is trivial and the pair (S, α) defines the automorphism of character table of $AffSp_2(2) = S_4$ which doesn't correspond to any group automorphism. This isomorphism intertwings the characters χ_4 and χ_5 and the classes $2A$ and $4A$.

S_4	1	2A	2B	3A	4A
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	2	-1	0
χ_4	3	1	-1	0	-1
χ_5	3	-1	-1	0	1

6. Class-preserving automorphisms and permutation representations with the same character. We will call an automorphism $\phi \in Aut(G)$ of the finite group G by *class-preserving* if ϕ preserves all conjugacy classes of G $\phi(g) \in g^G$ for all $g \in G$ (see [2, 9, 14]).

Proposition 5. *For any class-preserving automorphism ϕ of the finite group G there is an invertible element $c \in k[G]$ such that $\phi(g) = cgc^{-1}$ for any $g \in G$ and $F = \Delta(c)^{-1}(c \otimes c)$ is G-invariant element of $k[G]^{\otimes 2}$.*
Proof. It follows directly from the Proposition 4 and the fact that class-preserving automorphism induces trivial automorphism of the character ring $R(G)$. □

Permutation representation of the group G is a homomorphism $\phi : G \to S_n$ to the group of automorphisms $S_n = Aut(X)$ of the finite set of order $|X| = n$. Define the *character* χ_ϕ of the permutation representation $\phi : G \to S_n$ as $\chi_\phi(g) = |\{x \in X, \phi(g)(x) = x\}|$. So χ_ϕ is an image of the natural n-dimensional character of S_n under the homomorphism $\phi^* : R(S_n) \to R(G)$.

Proposition 6. *For any two permutation representations $\phi, \psi : G \to S_n$ of the finite group G there is an invertible element $c \in k[S_n]$ such that $\phi(g) = c\psi(g)c^{-1}$ for any $g \in G$ and $F = \Delta(c)^{-1}(c \otimes c)$ is $\psi(G)$-invariant element of $k[G]^{\otimes 2}$.*
Proof. It can be proved that the homomorphisms $\phi^*, \psi^* : R(S_n) \to R(G)$ of character tables corresponded to permutation representations $\phi, \psi : G \to S_n$ coincides if coincides the characters χ_ϕ, χ_ψ. Hence we can apply the Proposition 4. □

7. Concluding remarks. The cohomological nature of the sets of possible twistings was actively explored in the theory of quantum groups. Nonabelianity of those cohomology is probably a major difficulty of the theory. In quantum group theory this difficulty was overcome by methods of tangent cohomology which are unapplicable for finite groups. In this case non-abelian cohomology sets of twistings can be abelianized by means of algebraic K-theory. Namely, the maps from the sets of twistings to some Hochschild cohomology of representation ring can be costructed [5]. The detailed description of those maps would be the subject of subsequent paper.

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