EXTENDER BASED RADIN FORCING

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ABSTRACT. We define extender sequences, generalizing measure sequences of Radin forcing.

Using the extender sequences, we show how to combine the Gitik-Magidor forcing for adding many Prikry sequenes with Radin forcing.

We show that this forcing satisfies a Prikry like condition destroys no cardinals, and has a kind of properness.

Depending on the large cardinals we start with this forcing can blow the power of a cardinal together with changing its' cofinality to a prescirbe value. It can even blow the power of a cardinal while keeping it regular or measurable.

1. INTRODUCTION

We give some background on previous work relating directly to the present work. The first forcing which changed the cofinality of a cardinal without changing the cardinal structure was Prikry forcing [6]. In this forcing a measurable cardinal, κ, was 'invested' in order to get $\mathrm{cf}(\kappa) = \omega$ without collapsing any cardinal. Developing that idea, Magidor [2] used a coherent sequence of measures of length $\lambda < \kappa$ in order to get $\mathrm{cf}(\kappa) = \lambda$ without collapsing any cardinals. In [7] Radin, introducing the notion of measure sequence, showed that it is useful to continue the coherent sequence to $\lambda > \kappa$. For example, κ remains regular when $\lambda = \kappa^+$. In general the longer the measure sequence the more resemblance there is between κ in the generic extension and the ground model.

As is well known, and unlike regular cardinals, blowing the power of a singular cardinal is not an easy task. A natural approach to try was to blow the power of a cardinal while it was regular and after that make it singular by one of the above methods. A crucial idea of Gitik and Magidor [3] was to combine the power set blowing and the cofinality change in one forcing. They introduced a forcing notion which added many Prikry sequences at once and still collapsed no cardinals. The 'investment' they needed for this was an extender of length which is the size of power they wanted. Building on the idea of Gitik and Magidor, Segal [8] implemented the idea of adding many sequences to Magidor forcing. So by investing a coherent sequence of extenders of length $\lambda < \kappa$ she was able to get a singular cardinal of cofinality λ together with power as large as the length of the extenders in question. Our work also builds on the idea of Gitik and Magidor. However, we implement the idea of adding many sequences to Radin forcing. So we introduce the notion of

extender sequence and show that it makes sense to deal with quite long extender sequences. As in Radin forcing, for long enough sequences we are left with κ which is regular and even measurable. The power size will be the length of the extenders we start with.

The structure of this work is as follows. In section 2 we define extender sequences. In section 3 we define Radin forcing. The definition is not the usual one and is used to introduce the idea used in section 4. In section 4 we define $P_{\bar{E}}$, the forcing notion which is the purpose of this work. In section 5 we show the chain-conditions satisfied by $P_{\bar{E}}$ and how 'locally' it resembles Radin forcing. We also show here that there are many new subsets in the generic extension. In section 6 we investigate the structure of dense open subsets of $P_{\bar{E}}$. We show that they satisfy a strong homogeneity property. In section 7 we prove Prikry's condition for $P_{\bar{E}}$. The proof is a simple corollary of the strong homogeneity of dense open subsets. In section 8 we show that $P_{\bar{E}}$ satisfies a kind of properness. In section 9 we combine the machinery developed so far in order to show that no cardinals are collapsed. In section 10 we show how the length of the extender sequences affect the properties of κ. Section 11 summarizes what the forcing $P_{\bar{E}}$ does. In section 12 we have a result concerning $P_{\bar{E}}$ when $l(\bar{E}) = 1$. We show that there is, in V, a generic filter over an elementary submodel in an ω-iterate of V. We were not able to prove something equivalent (or weaker) for the general case. Section 13 contains a list of missing or unknown points to check. The last point in this list is in preparation.

Our notation is standard. We assume fluency with forcing and extenders. Some basic properties of Radin forcing are taken for granted.

2. Extender sequences

2.1. **Constructing from elementary embedding.** Suppose we have an elementary embedding $j{:}V \to M \supset V_\lambda$, $\mathrm{crit}(j) = \kappa$. The value of λ is determined later, according to the different applications we have.

Construct from j a nice extender like in [3]:

$$E(0) = \langle\langle E_\alpha(0) \mid \alpha \in \mathcal{A}\rangle, \langle \pi_{\beta,\alpha} \mid \beta \geq \alpha \ \alpha, \beta \in \mathcal{A}\rangle\rangle.$$

We recall the properties of this extender:

1. $\mathcal{A} \subseteq |V_\lambda| \setminus \kappa$,
2. $|\mathcal{A}| = |V_\lambda|$,
3. $\mathcal{A}$ is κ^+-directed,
4. κ is minimal in $\mathcal{A}$ and we write $\pi_{\alpha,0}$ instead of $\pi_{\alpha,\kappa}$,
5. $\forall \alpha, \beta \in \mathcal{A} \ \nu^0 = \pi_{\alpha,0}(\nu) = \pi_{\beta,0}(\nu)$,
6. $\forall \alpha, \beta \in \mathcal{A} \ \pi_{\beta,0}(\nu) = \pi_{\alpha,0}(\pi_{\beta,\alpha}(\nu))$,
7. $\forall \alpha, \beta, \gamma \in \mathcal{A} \ \exists A \in E_\gamma(0) \ \forall \nu \in A \ \pi_{\gamma,\alpha}(\nu) = \pi_{\beta,\alpha}(\pi_{\gamma,\beta}(\nu))$.

If, for example, we need $|E(0)| = \kappa^{+3}$ then, under GCH, we require $\lambda = \kappa + 3$. A typical large set in this extender concentrates on singletons.

If j is not sufficiently closed , then $E(0) \notin M$ and the construction stops. We set

$$\forall \alpha \in \mathcal{A} \ \bar{E}_\alpha = \langle \alpha, E(0) \rangle.$$

We say that $\bar{E}_\alpha$ is an extender sequence of length 1. $(l(\bar{E}_\alpha) = 1)$

If, on the other hand, $E(0) \in M$ we can construct for each $\alpha \in \operatorname{dom} E(0)$ the following ultrafilter

$$A \in E_{\langle \alpha, E(0) \rangle}(1) \iff \langle \alpha, E(0) \rangle \in j(A).$$

Such an A concentrates on elements of the form $\langle \xi, e(0) \rangle$ where $e(0)$ is an extender on ξ^0 and $\xi \in \operatorname{dom} e(0)$. Note that $e(0)$ concentrates on singletons below ξ^0. If, for example, $|E(0)| = \kappa^{+3}$ then on a large set we have $|e(0)| = (\xi^0)^{+3}$.

We define $\pi_{\langle \beta, E(0) \rangle, \langle \alpha, E(0) \rangle}$ as

$$\pi_{\langle \beta, E(0) \rangle, \langle \alpha, E(0) \rangle}(\langle \xi, e(0) \rangle) = \langle \pi_{\beta, \alpha}(\xi), e(0) \rangle.$$

From this definition we get

$$j(\pi_{\langle \beta, E(0) \rangle, \langle \alpha, E(0) \rangle})(\langle \beta, E(0) \rangle) = \langle \alpha, E(0) \rangle.$$

Hence we have here an extender

$$E(1) = \langle \langle E_{\langle \alpha, E(0) \rangle}(1) \mid \alpha \in \mathcal{A} \rangle, \langle \pi_{\langle \beta, E(0) \rangle, \langle \alpha, E(0) \rangle} \mid \beta \geq \alpha \; \alpha, \beta \in \mathcal{A} \rangle \rangle.$$

Note that the difference between $\pi_{\beta, \alpha}$ and $\pi_{\langle \beta, E(0) \rangle, \langle \alpha, E(0) \rangle}$ is quite superficial. We can define $\pi_{\langle \beta, E(0) \rangle, \langle \alpha, E(0) \rangle}$ in a uniform way for both extenders. Just project the first element of the argument using $\pi_{\beta, \alpha}$.

If $\langle E(0), E(1) \rangle \notin M$ then the construction stops. In this case we set

$$\forall \alpha \in \mathcal{A} \; \bar{E}_\alpha = \langle \alpha, E(0), E(1) \rangle.$$

We say that $\bar{E}_\alpha$ is an extender sequence of length 2. $(l(\bar{E}_\alpha) = 2)$

If $\langle E(0), E(1) \rangle \in M$ then we construct the extender $E(2)$ in the same way as we constructed $E(1)$ from $E(0)$.

The above private case being worked out we continue with the general case. Assume we have constructed

$$\langle E(\tau') \mid \tau' < \tau \rangle.$$

If $\langle E(\tau') \mid \tau' < \tau \rangle \notin M$ then the construction stops here. We set

$$\forall \alpha \in \mathcal{A} \; \bar{E}_\alpha = \langle \alpha, E(\tau') \mid \tau' < \tau \rangle,$$

and we say that $\bar{E}_\alpha$ is an extender sequence of length τ. $(l(\bar{E}_\alpha) = \tau)$

If, on the other hand, $\langle E(\tau') \mid \tau' < \tau \rangle \in M$ then we construct

$$A \in E_{\langle \alpha, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau \rangle}(\tau) \iff$$
$$\langle \alpha, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau \rangle \in j(A).$$

Defining $\pi_{\langle \beta, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau \rangle, \langle \alpha, E(0), \dots, E(\tau'), \dots, \mid \tau' < \tau \rangle}$ using the first coordinate as before gives the needed projection.

We are quite casual in writing the indices of the projections and ultrafilters. By this we mean that we sometimes write $\pi_{\beta, \alpha}$ when we should have written $\pi_{\langle \beta, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau \rangle, \langle \alpha, E(0), \dots, E(\tau'), \dots, \mid \tau' < \tau \rangle}$ and $E_\alpha(\tau)$ when we should have written $E_{\langle \alpha, E(0), \dots, E(\tau'), \dots, \mid \tau' < \tau \rangle}(\tau)$.

With this abuse of notation the projection we just defined satisfies

$$j(\pi_{\beta, \alpha})(\langle \beta, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau \rangle) =$$
$$\langle \alpha, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau \rangle,$$

and we have the extender

$$E(\tau) = \langle \langle E_\alpha(\tau) \mid \alpha \in \mathcal{A} \rangle, \langle \pi_{\beta, \alpha} \mid \beta \geq \alpha \; \alpha, \beta \in \mathcal{A} \rangle \rangle.$$

We let the construction run until it stops due to the extender sequence not being in M.

Definition 2.1. We call $\bar\mu$ an extender sequence if there is an elementary embedding $j{:}V \to M$ such that $\bar\nu$ is an extender sequence generated as above and $\bar\mu = \bar\nu{\restriction}\tau$ for $\tau \leq \mathrm{l}(\bar\nu)$. $\kappa(\bar\mu)$ is the ordinal at the beginning of the sequence. (i.e. $\kappa(\bar E_\alpha) = \alpha$). $\kappa^0(\bar\mu)$ is $(\kappa(\bar\mu))^0$. (i.e. $\kappa^0(\bar E_\alpha) = \kappa$).

That is, we do not have to construct the extender sequence until it is not in M. We can stop anywhere on the way.

Definition 2.2. A sequence of extender sequences $\langle \bar\mu_1, \ldots, \bar\mu_n \rangle$ is called ^{0}-increasing if $\kappa^0(\bar\mu_1) < \cdots < \kappa^0(\bar\mu_n)$.

Definition 2.3. Let $\langle \bar\mu_1, \ldots, \bar\mu_n \rangle$ be ^{0}-increasing. An extender sequences $\bar\mu$ is called permitted to $\langle \bar\mu_1, \ldots, \bar\mu_n \rangle$ if $\kappa^0(\bar\mu_n) < \kappa^0(\bar\mu)$.

Definition 2.4. We say $A \in \bar E_\alpha$ if $\forall \xi < \mathrm{l}(\bar E_\alpha)\ A \in E_\alpha(\xi)$.

Definition 2.5. $\bar E = \langle \bar E_\alpha \mid \alpha \in \mathcal{A} \rangle$ is an extender sequence system if there is an elementary embedding $j{:}V \to M$ such that all $\bar E_\alpha$ are extender sequences generated from j as prescribed above and $\forall \alpha, \beta \in \mathcal{A}\ \mathrm{l}(\bar E_\alpha) = \mathrm{l}(\bar E_\beta)$. This common length is called the length of the system, $\mathrm{l}(\bar E)$. We write $\bar E(\bar\mu)$ for the extender sequence system to which $\bar\mu$ belongs (i.e. $\bar E(\bar E_\alpha) = \bar E$).

The generalization of the measure on the α coordinate in Gitik-Magidor forcing [3] is $\bar E_\alpha$.

2.2. $\bar E_\alpha$-tree.

Definition 2.6. A tree T is an $\bar E_\alpha$-tree if its' elements are of the form

$$\langle \langle \bar\mu_1, \ldots, \bar\mu_n \rangle, S \rangle$$

where
1. Set $\mathrm{dom}\,T = \{ \langle \bar\mu_1, \ldots, \bar\mu_n \rangle \mid \langle \langle \bar\mu_1, \ldots, \bar\mu_n \rangle, S \rangle \in T \}$. Then the function

$$\langle \bar\mu_1, \ldots, \bar\mu_n \rangle \mapsto \langle \langle \bar\mu_1, \ldots, \bar\mu_n \rangle, S \rangle$$

   from $\mathrm{dom}\,T$ to T is $1-1$ and onto,
2. $t \in \mathrm{Lev}_n(\mathrm{dom}\,T) \implies |t| = n + 1$,
3. $\langle \bar\mu_1, \ldots, \bar\mu_n \rangle$ are ^{0}-increasing extender sequences,
4. $\mathrm{Lev}_0(\mathrm{dom}\,T) \in \bar E_\alpha$ and for each $t \in \mathrm{dom}\,T\ \mathrm{Suc}_{\mathrm{dom}\,T}(t) \in \bar E_\alpha$,
5. S is a $\bar\mu_n$-tree. When $\mathrm{l}(\bar\mu_n) = 0$ we set $S = \emptyset$.
   Note that this clause is recursive.

Note 2.7. Later on, we abuse notation and use T instead of $\mathrm{dom}\,T$. i.e. $\mathrm{Suc}_T(t)$ instead of $\mathrm{Suc}_{\mathrm{dom}\,T}(t)$.

Definition 2.8. Assume T is a $\bar E_\alpha$-tree and $t \in T$, then:
1. $T_t = \{ \langle s, S \rangle \mid \langle t ^\frown s, S \rangle \in T \}$.
2. $T_t(\bar\mu)$ is the tree, S, satisfying $\langle \bar\mu, S \rangle \in \mathrm{Suc}_T(t)$.

Definition 2.9. Let T, S be $\bar E_\alpha$-trees, where $\mathrm{l}(\bar E) = 1$. We say that $T \leq S$ if

1. $\mathrm{Lev}_0(T) \subseteq \mathrm{Lev}_0(S)$,
2. $\forall t \in T \, \mathrm{Suc}_T(t) \subseteq \mathrm{Suc}_S(t)$.

Definition 2.10. Let T, S be $\bar{E}_\alpha$-trees. We say that $T \leq S$ if

1. $\mathrm{Lev}_0(T) \subseteq \mathrm{Lev}_0(S)$,
2. $\forall t \in T \, \mathrm{Suc}_T(t) \subseteq \mathrm{Suc}_S(t)$,
3. $\forall \langle \bar{\mu} \rangle \in \mathrm{Lev}_0(T) \, T(\bar{\mu}) \leq S(\bar{\mu})$,
4. $\forall t \in T \, \forall \langle \bar{\mu} \rangle \in T_t \, T_t(\bar{\mu}) \leq S_t(\bar{\mu})$.

Note that the last 2 conditions are recursive.

Definition 2.11. Let S be $\bar{E}_\alpha$-tree and $\beta > \alpha$. Define $T = \pi_{\beta,\alpha}^{-1}(S)$ by

1. $\mathrm{dom}\, T = \pi_{\beta,\alpha}^{-1}(\mathrm{dom}\, S)$,
2. $T_{\langle \bar{\mu}_1, \dots, \bar{\mu}_{n-1} \rangle}(\bar{\mu}_n) = \pi_{\bar{\mu}_n, \pi_{\beta,\alpha}(\bar{\mu}_n)}^{-1} S_{\langle \pi_{\beta,\alpha}(\bar{\mu}_1), \dots, \pi_{\beta,\alpha}(\bar{\mu}_{n-1}) \rangle}(\pi_{\beta,\alpha}(\bar{\mu}_n))$.

Definition 2.12. Let T, S be $\bar{E}_\beta$, $\bar{E}_\alpha$-trees respectively, where $\beta \geq \alpha$. We say that $T \leq S$ if

1. $T \leq \pi_{\beta,\alpha}^{-1}(S)$.

Definition 2.13. Assume we have $A_{\langle \bar{\nu}, R \rangle}$ where $\bar{\nu}$ is an extender sequence such that each element in $A_{\langle \bar{\nu}, R \rangle}$ is of the form $\langle \bar{\mu}, S \rangle$ where $\bar{\mu}$ is an extender sequence and S is a tree. (in this work we always have S is $\bar{\mu}$-tree). We define $\triangle^0_{\langle \bar{\nu}, R \rangle} A_{\langle \bar{\nu}, R \rangle}$ as

$$\langle \bar{\mu}, S \rangle \in \underset{\langle \bar{\nu}, R \rangle}{\triangle^0} A_{\langle \bar{\nu}, R \rangle} \iff \forall \langle \bar{\nu}, R \rangle \, \kappa(\bar{\nu}) < \kappa^0(\bar{\mu}) \to \langle \bar{\mu}, S_{\langle \bar{\nu}, R \rangle} \rangle \in A_{\langle \bar{\nu}, R \rangle}.$$

3. RADIN FORCING

The main aims of this section are 3.8, 3.10. As the simplest way we found to formulate them was with Radin forcing [7, 5, 10] we took the opportunity to depart from usual formulation in order to introduce ideas that we use in the extender based forcing later.

The main point is that possible extensions of a condition are stored in $\bar{E}_\alpha$-tree and not in a set. The α is be fixed, so practically we deal here with a measure sequence and not an extender sequence.

Definition 3.1. A condition in R_α is of the form

$$\langle \langle \bar{\mu}_n, s^n \rangle, S^n, \dots, \langle \bar{\mu}_1, s^1 \rangle, S^1, \langle \bar{\mu}_0, s^0 \rangle, S^0 \rangle$$

where

1. $\bar{\mu}_0 = \bar{E}_\alpha$,
2. $\forall i \leq n \, S^i$ is $\bar{\mu}_i$-tree,
3. $\forall i \leq n \, s^i \in V_{\kappa^0(\bar{\mu}_i)}$ is an extender sequence.

Definition 3.2. Let $p, q \in R_\alpha$. We say that p is Prikry extension of q ($p \leq^* q$ or $p \leq^0 q$) if p, q are of the form

$$p = \langle \langle \bar{\mu}_n, s^n \rangle, S^n, \dots, \langle \bar{\mu}_1, s^1 \rangle, S^1, \langle \bar{E}_\alpha, s^0 \rangle, S^0 \rangle,$$
$$q = \langle \langle \bar{\mu}_n, t^n \rangle, T^n, \dots, \langle \bar{\mu}_1, t^1 \rangle, T^1, \langle \bar{E}_\alpha, t^0 \rangle, T^0 \rangle,$$

and

- $\forall i \leq n \; S^i \leq T^i$,
- $\forall i \leq n \; s^i = t^i$.

Definition 3.3. Let $p = \langle \langle \bar{\mu}_n, s^n \rangle, S^n, \ldots, \langle \bar{\mu}_1, s^1 \rangle, S^1, \langle \bar{\mu}_0, s^0 \rangle, S^0 \rangle$ where $\bar{\mu}_0 = \bar{E}_\alpha$. Let $\langle \bar{\nu} \rangle \in S^i$. We define $(p)_{\langle \bar{\nu} \rangle}$ to be

$$(p)_{\langle \bar{\nu} \rangle} = \langle \langle \bar{\mu}_n, s^n \rangle, S^n, \ldots, \langle \bar{\mu}_{i+1}, s^{i+1} \rangle, S^{i+1},$$
$$\langle \bar{\nu}, s^i \rangle, S^i(\bar{\nu}), \langle \bar{\mu}_i, \bar{\nu} \rangle, S^i_{\langle \bar{\nu} \rangle},$$
$$\langle \bar{\mu}_{i-1}, s^{i-1} \rangle, S^{i-1}, \ldots, \langle \bar{\mu}_0, s^0 \rangle, S^0 \rangle.$$

Note the degenerate case in this definition when $\mathrm{l}(\bar{\nu}) = 0$. In this case $S^i(\bar{\nu}) = \emptyset$.

Definition 3.4. Let $p, q \in R_\alpha$ where

$$q = \langle \langle \bar{\mu}_n, s^n \rangle, S^n, \ldots, \langle \bar{\mu}_1, s^1 \rangle, S^1, \langle \bar{\mu}_0, s^0 \rangle, S^0 \rangle.$$

We say that p is 1-point extension of q ($p \leq^1 q$) if there is $\langle \bar{\nu} \rangle \in S^i$ such that $p \leq^* (q)_{\langle \bar{\nu} \rangle}$.

Definition 3.5. Let $p, q \in R_\alpha$. We say that p is n-point extension of q ($p \leq^n q$) if there are $p^n, \ldots, p^0$ such that

$$p = p^n \leq^1 \ldots \leq^1 p^0 = q.$$

Definition 3.6. Let $p, q \in R_\alpha$. We say that p is an extension of q ($p \leq q$) if there is n such that $p \leq^n q$.

3.8 is needed in the proof of 7.1. Very loosly speaking 3.8 means that if "something" happens on a measure one set for one of the measures, that "something" is happening on a measure one set for all the measures.

3.8 is proved by induction and 3.7 is the first case of the induction.

Lemma 3.7. *Suppose* $\mathrm{l}(\bar{E}_\alpha) = 2$ *and* T *is a tree such that* $\mathrm{Lev}_0(T) \in E_\alpha(i)$ *for* $i < 2$, *and* $\forall \langle \bar{\nu} \rangle \in T \; T_{\langle \bar{\nu} \rangle}$ *is an* $\bar{E}_\alpha$-*tree. Then there is an* $\bar{E}_\alpha$-*tree,* T^*, *satisfying*

1. $\forall \langle \bar{\nu} \rangle \in T \cap T^* \; T^*_{\langle \bar{\nu} \rangle} \leq T_{\langle \bar{\nu} \rangle}, \; T^*(\bar{\nu}) \leq T(\bar{\nu})$,
2. *If* $p \leq \langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^* \rangle$ *then there is* $\langle \bar{\mu} \rangle \in T^* \cap T$ *such that*

$$p \parallel \left(\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^* \rangle \right)_{\langle \bar{\mu} \rangle}.$$

Proof. There are 2 cases which to deal with:

- $A_0 = \mathrm{Lev}_0(T) \in E_\alpha(0)$: If $A_0 \in E_\alpha(1)$ we set $T^* = T$ and the proof is finished. So suppose $A_0 \notin E_\alpha(1)$. We would like to build $A_1 \in E_\alpha(1)$. Set

$$\forall \langle \bar{\mu}_0 \rangle \in T \; A_{\langle \bar{\mu}_0 \rangle, 1} = \left\{ \langle \bar{\mu}_1, S \rangle \in T_{\langle \bar{\mu}_0 \rangle} \mid A_0 \cap \kappa^0(\bar{\mu}_1) \in \bar{\mu}_1(0)_{\kappa(\bar{\mu}_1)} \right\}.$$

As $A_0 \in E_\alpha(0)$ and $\mathrm{Suc}_T(\langle \bar{\mu}_0 \rangle) \in E_\alpha(1)$ we get that $A_{\langle \bar{\mu}_0 \rangle, 1} \in E_\alpha(1)$. Let

$$A_1 = \underset{\langle \bar{\mu}_0 \rangle \in T}{\triangle^0} A_{\langle \bar{\mu}_0 \rangle, 1}.$$

We can construct now T^*:

$$\mathrm{Lev}_0(T^*) = A_0 \cup A_1,$$
$$\forall \bar{\mu}_0 \in A_0 \; T^*_{\langle \bar{\mu}_0 \rangle} = T_{\langle \bar{\mu}_0 \rangle},$$
$$\forall \bar{\mu}_1 \in A_1 \; T^*_{\langle \bar{\mu}_1 \rangle} = \bigcap_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle \in T} T_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle}.$$

- $A_1 = \text{Lev}_0(T) \in E_\alpha(1)$: If $A_1 \in E_\alpha(0)$ we set $T^* = T$ and finish the proof. So assume $A_1 \notin E_\alpha(0)$. We would like to build $A_0 \in E_\alpha(0)$. Set

$$S = j(T)\big(\langle \alpha, E(0) \rangle\big),$$
$$A_0 = \text{Lev}_0(S) \setminus A_1.$$

We construct T^*:

$$\text{Lev}_0(T^*) = A_0 \cup A_1,$$
$$\forall \bar{\mu}_1 \in A_1 \ T^*_{\langle \bar{\mu}_1 \rangle} = T_{\langle \bar{\mu}_1 \rangle}.$$

We are left with the construction of $T^*_{\langle \bar{\mu}_0 \rangle}$ for $\bar{\mu}_0 \in A_0$. For all $\bar{\mu} \in A_0$ set

$$A_{\langle \bar{\mu}_0 \rangle, 0} = \text{Suc}_S\big(\langle \bar{\mu}_0 \rangle\big),$$
$$A_{\langle \bar{\mu}_0 \rangle, 1} = \big\{ \langle \bar{\mu}_1, T(\bar{\mu}_1)_{\langle \bar{\mu}_0 \rangle} \rangle \mid \langle \bar{\mu}_1 \rangle \in T, \ \langle \bar{\mu}_0 \rangle \in T(\bar{\mu}_1) \big\},$$
$$\text{Suc}_{T^*}\big(\langle \bar{\mu}_0 \rangle\big) = A_{\langle \bar{\mu}_0 \rangle, 0} \cup A_{\langle \bar{\mu}_0 \rangle, 1},$$
$$\forall \bar{\mu}_1 \in A_{\langle \bar{\mu}_0 \rangle, 1} \ T^*_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle} = T_{\langle \bar{\mu}_1 \rangle}.$$

We continue one more level and hope this will convince the reader we indeed can complete T^*. We are left with the construction of $T^*_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle}$ for $\langle \bar{\mu}_0, \bar{\mu}_1 \rangle \in A_0 \times A_{\langle \bar{\mu}_0 \rangle, 0}$. For all $\langle \bar{\mu}_0, \bar{\mu}_1 \rangle \in A_0 \times A_{\langle \bar{\mu}_0 \rangle, 0}$ set

$$A_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle, 0} = \text{Suc}_S\big(\langle \bar{\mu}_0, \bar{\mu}_1 \rangle\big),$$
$$A_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle, 1} = \big\{ \langle \bar{\mu}_2, T(\bar{\mu}_2)_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle} \rangle \mid \langle \bar{\mu}_2 \rangle \in T, \langle \bar{\mu}_0, \bar{\mu}_1 \rangle \in T(\bar{\mu}_2) \big\},$$
$$\text{Suc}_{T^*}\big(\langle \bar{\mu}_0, \bar{\mu}_1 \rangle\big) = A_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle, 0} \cup A_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle, 1},$$
$$\forall \bar{\mu}_2 \in A_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle, 1} \ T^*_{\langle \bar{\mu}_0, \bar{\mu}_1, \bar{\mu}_2 \rangle} = T_{\langle \bar{\mu}_2 \rangle}.$$

We are left with the construction of $T^*_{\langle \bar{\mu}_0, \bar{\mu}_1, \bar{\mu}_2 \rangle}$ for $\langle \bar{\mu}_0, \bar{\mu}_1, \bar{\mu}_2 \rangle \in A_0 \times A_{\langle \bar{\mu}_0 \rangle, 0} \times A_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle, 0}$ and we hope that by now the continuation is clear. $\square$

Lemma 3.8. *Let $\xi_0 < \text{l}(\bar{E}_\alpha)$, T a tree such that $\text{Lev}_0(T) \in E_\alpha(\xi_0)$ and $\forall \langle \bar{\mu} \rangle \in T$ $T_{\langle \bar{\mu} \rangle}$ is an $\bar{E}_\alpha$-tree. Then there is an $\bar{E}_\alpha$-tree, T^*, satisfying*

1. $\forall \langle \bar{\nu} \rangle \in T \cap T^* \ T^*_{\langle \bar{\nu} \rangle} \leq T_{\langle \bar{\nu} \rangle}, \ T^*(\bar{\nu}) \leq T(\bar{\nu}),$
2. *If $p \leq \big\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^* \big\rangle$ then there is $\langle \bar{\mu} \rangle \in T^* \cap T$ such that*

$$p \parallel \big(\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^* \rangle\big)_{\langle \bar{\mu} \rangle}.$$

Proof. Our induction hypothesis is that this lemma is true for $\bar{\mu}$'s with $\text{l}(\bar{\mu}) < \text{l}(\bar{E}_\alpha)$. The previous lemma is the case for $\text{l}(\bar{\mu}) = 2$.

Let $S = j(T)(\bar{E}_\alpha \restriction \xi_0)$. The tree S is an $\bar{E}_\alpha \restriction \xi_0$-tree. We extend it step by step to a full $\bar{E}_\alpha$-tree as requested.
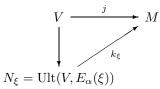
Let

$$A_{\xi_0} = \text{Lev}_0(T),$$
$$A_{<\xi_0} = \text{Lev}_0(S) \setminus A_{\xi_0}.$$

For $\xi_0 < \xi < \text{l}(\bar{E}_\alpha)$ do the following:

$$N_\xi = \text{Ult}(V, E_\alpha(\xi)),$$
$$k_\xi([h]_{E_\alpha(\xi)}) = j(h)(\bar{E}_\alpha \restriction \xi).$$

$$V \xrightarrow{\ j\ } M$$

$$k_\xi$$

$$N_\xi = \mathrm{Ult}(V, E_\alpha(\xi))$$

and recall that $\mathrm{crit}(k_\xi) = (\kappa^{++})^{N_\xi}$. As

$$\langle \alpha, E(0), \ldots, E(\tau), \ldots \mid \tau < \xi \rangle = k_\xi([id]_{E_\alpha(\xi)}) \in \mathrm{ran}(k_\xi)$$

there is in N_ξ a preimage for it:

$$\langle \alpha', E'(0), \ldots, E'(\tau') \ldots \mid \tau' < \xi' \rangle.$$

As $A_{\xi_0} \in E_\alpha(\xi_0)$, $\xi_0 < \xi$ we have $\tau' < \xi'$ such that $A_{\xi_0} \in E'_{\alpha'}(\tau')$, (where $\alpha' = [\kappa(id)]_{E_\alpha(\xi)}$. Taking a function h_ξ such that $[h_\xi]_{E_\alpha(\xi)} = \tau'$ we get

$$\left\{ \bar{\mu} \mid A_{\xi_0} \cap \kappa^0(\bar{\mu}) \in \bar{\mu}(h_\xi(\bar{\mu}))_{\kappa(\bar{\mu})} \right\} \in E_\alpha(\xi).$$

For each $\bar{\mu}_0 \in A_{\xi_0}$ we set

$$A_{\xi, \langle \bar{\mu}_0, T(\bar{\mu}_0) \rangle} = \left\{ \langle \bar{\mu}_1, R \rangle \in T_{\langle \bar{\mu}_0 \rangle} \mid A_{\xi_0} \cap \kappa^0(\bar{\mu}_1) \in \bar{\mu}_1(h_\xi(\bar{\mu}_1))_{\kappa(\bar{\mu})} \right\},$$

$$A'_\xi = \underset{\langle \bar{\mu}_0, T(\bar{\mu}_0) \rangle}{\triangle^0} A_{\xi, \langle \bar{\mu}_0, T(\bar{\mu}_0) \rangle}.$$

For any tree R which appears in a pair $\langle \bar{\mu}_1, R \rangle \in A'_\xi$ we can invoke by induction our lemma and generate R^* which is a $\bar{\mu}_1$-tree. Define now A_ξ as:

$$\langle \bar{\mu}_1, R^* \rangle \in A_\xi \iff \langle \bar{\mu}_1, R \rangle \in A'_\xi.$$

When we have $\left\{ A_\xi \mid \xi_0 < \xi < \mathrm{l}(\bar{E}_\alpha) \right\}$ we set

$$A_{>\xi_0} = \bigcup_{\xi_0 < \xi < \mathrm{l}(\bar{E}_\alpha)} A_\xi \setminus (A_{<\xi_0} \cup A_{\xi_0}),$$

$$\mathrm{Lev}_0(T^*) = A_{<\xi_0} \cup A_{\xi_0} \cup A_{>\xi_0},$$

$$\forall \langle \bar{\mu}_0 \rangle \in A_{\xi_0} \ T^*_{\langle \bar{\mu}_0 \rangle} = T_{\langle \bar{\mu}_0 \rangle},$$

$$\forall \langle \bar{\mu}_1 \rangle \in A_{>\xi_0} \ T^*_{\langle \bar{\mu}_1 \rangle} = \bigcap T_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle}.$$

We are left to define $T_{\langle \bar{\mu}_0 \rangle}$ for $\langle \bar{\mu}_0 \rangle \in A_{<\xi_0}$. For each $\bar{\mu}_0 \in A_{<\xi_0}$ set:

$$A_{\langle \bar{\mu}_0 \rangle, \xi_0} = \left\{ \langle \bar{\mu}_1, R_{\langle \bar{\mu}_0 \rangle} \rangle \mid \langle \bar{\mu}_1, R \rangle \in A_{\xi_0}, \ \langle \bar{\mu}_0 \rangle \in R \right\},$$

$$A_{\langle \bar{\mu}_0 \rangle, <\xi_0} = \mathrm{Suc}_S(\langle \bar{\mu}_0 \rangle) \setminus A_{\langle \bar{\mu}_0 \rangle, \xi_0}.$$

For each $\bar{\mu}_1 \in A_{\xi_0}$ we set

$$A_{\langle \bar{\mu}_0 \rangle, \xi, \langle \bar{\mu}_1, T(\bar{\mu}_1)_{\langle \bar{\mu}_0 \rangle} \rangle} = \left\{ \langle \bar{\mu}_2, R \rangle \in T_{\langle \bar{\mu}_1 \rangle} \mid A_{\xi_0} \cap \kappa^0(\bar{\mu}_2) \in \bar{\mu}_2(h_\xi(\bar{\mu}_2))_{\kappa(\bar{\mu}_2)} \right\},$$

$$A'_{\langle \bar{\mu}_0 \rangle, \xi} = \underset{\langle \bar{\mu}_1, T(\bar{\mu}_1)_{\langle \bar{\mu}_0 \rangle} \rangle}{\triangle^0} A_{\langle \bar{\mu}_0 \rangle, \xi, \langle \bar{\mu}_1, T(\bar{\mu}_1)_{\langle \bar{\mu}_0 \rangle} \rangle}.$$

We define

$$\langle \bar{\mu}_0, R^* \rangle \in A_{\langle \bar{\mu}_0 \rangle, \xi} \iff \langle \bar{\mu}_0, R \rangle \in A'_{\langle \bar{\mu}_0 \rangle, \xi}$$

where R^* is generated from R using the current lemma by induction. Now we set

$$A_{\langle \bar{\mu}_0 \rangle, > \xi_0} = \bigcup_{\xi_0 < \xi < \mathrm{l}(\bar{E}_\alpha)} A_{\langle \bar{\mu}_0 \rangle, \xi} \setminus (A_{\langle \bar{\mu}_0 \rangle, < \xi_0} \cup A_{\langle \bar{\mu}_0 \rangle, \xi_0}),$$

$$\mathrm{Suc}_{T^*}(\langle \bar{\mu}_0 \rangle) = A_{\langle \bar{\mu}_0 \rangle, < \xi_0} \cup A_{\langle \bar{\mu}_0 \rangle, \xi_0} \cup A_{\langle \bar{\mu}_0 \rangle, > \xi_0},$$

$$\forall \langle \bar{\mu}_1 \rangle \in A_{\bar{\mu}_0, \xi_0} \ T^*_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle} = T_{\langle \bar{\mu}_1 \rangle},$$

$$\forall \langle \bar{\mu}_2 \rangle \in A_{\bar{\mu}_0, > \xi_0} \ T^*_{\langle \bar{\mu}_0, \bar{\mu}_2 \rangle} = \bigcap T_{\langle \bar{\mu}_1, \bar{\mu}_2 \rangle}.$$

This leaves us with the definition of $T_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle}$ for $\langle \bar{\mu}_0, \bar{\mu}_1 \rangle \in A_{< \xi_0} \times A_{\langle \bar{\mu}_0 \rangle, < \xi_0}$ which is done exactly as in this step. $\qquad \square$

3.10 is needed in the proof of 6.1. Loosly speaking it says that if "something" happens on all extensions which are taken from $\mathrm{dom}\, T$, then that "something" happens on all extensions from T.

3.10 is proved by induction where 3.9 is the first case.

Lemma 3.9. *Assume* $\mathrm{l}(\bar{E}_\alpha) = 2$ *and let T be $\bar{E}_\alpha$-tree. Then there is $T^* \leq T$ such that if*

$$p \leq \left\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^* \right\rangle$$

then there is $\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in T$ such that

$$p \leq^* \left(\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T \rangle \right)_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle}.$$

Proof. As is usual in this section the proof is done level by level. Let us set $T^1 = T$ and it is trivially true that if

$$p \leq^1 \left\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^1 \right\rangle$$

then there is $\langle \bar{\nu}_1 \rangle \in T^1 = T$ such that

$$p \leq^* \left(\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^1 \rangle \right)_{\langle \bar{\nu}_1 \rangle}.$$

We continue to the second level. Let us set

$$A_{\langle \bar{\nu}_1, S^1 \rangle} = \mathrm{Suc}_{T^1}(\langle \bar{\nu}_1 \rangle),$$

$$A_{\langle \rangle} = \mathop{\triangle}\limits_{\langle \bar{\nu}_1, S^1 \rangle}^0 A_{\langle \bar{\nu}_1, S^1 \rangle}$$

$$B_{\langle \rangle} = \left\{ \langle \bar{\nu}_2 \rangle \mid \mathrm{l}(\bar{\nu}_2) = 0 \text{ or } \mathrm{Lev}_0(T^1) \cap \kappa^0(\bar{\nu}_2) \in \bar{\nu}_2 \right\},$$

$$\mathrm{Lev}_0(T^{(0)}) = A_{\langle \rangle} \cap B_{\langle \rangle},$$

$$T^{(0)}_{\langle \bar{\nu}_2 \rangle} = \bigcap_{\langle \bar{\nu}_1 \rangle \in T^{(0)}(\bar{\nu}_2)} T^1_{\langle \bar{\nu}_1, \bar{\nu}_2 \rangle},$$

$$T^2 = T^1 \cap T^{(0)}.$$

Let us assume that

$$p \leq^2 \left\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^2 \right\rangle.$$

There are 2 cases to consider here:

1. $p \leq^2 \left(\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^2 \rangle \right)_{\langle \bar{\nu}_1, \bar{\nu}_2 \rangle}$ where $\langle \bar{\nu}_1, \bar{\nu}_2 \rangle \in T^2$: At once we have $\langle \bar{\nu}_1, \bar{\nu}_2 \rangle \in T^1 \leq T$.

2. $p \leq^2 \left(\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^2 \rangle \right)_{\langle \bar{\nu}_2, \bar{\nu}_1 \rangle}$ where $\langle \bar{\nu}_2 \rangle \in T^2$, $\langle \bar{\nu}_1 \rangle \in T^2(\bar{\nu}_2)$: By construction $\forall \langle \bar{\mu} \rangle \in T^2(\bar{\nu}_2) \ \langle \bar{\nu}_2 \rangle \in T^1_{\langle \bar{\mu} \rangle}$. As $\langle \bar{\nu}_1 \rangle \in T^2(\bar{\nu}_2)$ we get $\langle \bar{\nu}_1, \bar{\nu}_2 \rangle \in T^1 \leq T$.

We show how to continue to the third level. Let us set

$$A_{\langle \bar{\nu}_1, S^1, \bar{\nu}_2, S^2 \rangle} = \mathrm{Suc}_{T^2}(\langle \bar{\nu}_1, \bar{\nu}_2 \rangle),$$

$$A_{\langle \bar{\nu}_1, S^1 \rangle} = \underset{\langle \bar{\nu}_2, S^2 \rangle}{\triangle^0} A_{\langle \bar{\nu}_1, S^1, \bar{\nu}_2, S^2 \rangle},$$

$$B_{\langle \bar{\nu}_1, S^1 \rangle} = \left\{ \langle \bar{\nu}_3 \rangle \mid \mathrm{l}(\bar{\nu}_3) = 0 \text{ or } \mathrm{Suc}_{T^2}(\langle \bar{\nu}_1, S^1 \rangle) \cap \kappa^0(\bar{\nu}_3) \in \bar{\nu}_3 \right\},$$

$$\mathrm{Lev}_0(T^{(0)}) = \mathrm{Lev}_0(T^2),$$

$$\mathrm{Suc}_{T^{(0)}}(\langle \bar{\nu}_1 \rangle) = A_{\langle \bar{\nu}_1, S^1 \rangle} \cap B_{\langle \bar{\nu}_1, S^1 \rangle},$$

$$T^{(0)}_{\langle \bar{\nu}_1, \bar{\nu}_3 \rangle} = \bigcap_{\langle \bar{\nu}_2 \rangle \in T^{(0)}_{\langle \bar{\nu}_1 \rangle}(\bar{\nu}_3)} T^2_{\langle \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3 \rangle},$$

$$A_{\langle \rangle} = \underset{\langle \bar{\nu}_1, S^1 \rangle}{\triangle^0} A_{\langle \bar{\nu}_1, S^1 \rangle},$$

$$B_{\langle \rangle} = \left\{ \langle \bar{\nu}_3 \rangle \mid \mathrm{l}(\bar{\nu}_3) = 0 \text{ or } \mathrm{Lev}_0(T^2) \cap \kappa^0(\bar{\nu}_3) \in \bar{\nu}_3 \right\},$$

$$\mathrm{Lev}_0(T^{(1)}) = A_{\langle \rangle} \cap B_{\langle \rangle},$$

$$\mathrm{Suc}_{T^{(1)}}(\langle \bar{\nu}_3 \rangle) = \bigcap_{\langle \bar{\nu}_1, \bar{\nu}_2 \rangle \in T^{(0)}(\bar{\nu}_3)} T^2_{\langle \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3 \rangle},$$

$$T^3 = T^2 \cap T^{(0)} \cap T^{(1)}.$$

Let us assume that

$$p \leq^3 \left\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^3 \right\rangle.$$

There are 3 cases to consider here:

1. $p \leq^* \left(\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^3 \rangle \right)_{\langle \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3 \rangle}$ where $\langle \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3 \rangle \in T^3$: At once we get $\langle \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3 \rangle \in T$.

2. $p \leq^* \left(\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^3 \rangle \right)_{\langle \bar{\nu}_1, \bar{\nu}_3, \bar{\nu}_2 \rangle}$ where $\langle \bar{\nu}_1, \bar{\nu}_3 \rangle \in T^3$, $\langle \bar{\nu}_2 \rangle \in T^3_{\langle \bar{\nu}_1 \rangle}(\bar{\nu}_3)$: In this case $\forall \langle \bar{\mu} \rangle \in T^3_{\langle \bar{\nu}_1 \rangle}(\bar{\nu}_3) \ \langle \bar{\nu}_3 \rangle \in T^3_{\langle \bar{\nu}_1, \bar{\mu} \rangle}$. As $\langle \bar{\nu}_2 \rangle \in T^3_{\langle \bar{\nu}_1 \rangle}(\bar{\nu}_3)$ we get $\langle \bar{\nu}_3 \rangle \in T_{\langle \bar{\nu}_1, \bar{\nu}_2 \rangle}$.

3. $p \leq^* \left(\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^3 \rangle \right)_{\langle \bar{\nu}_3, \bar{\nu}_1, \bar{\nu}_2 \rangle}$ where $\langle \bar{\nu}_3 \rangle \in T^3$, $\langle \bar{\nu}_1, \bar{\nu}_2 \rangle \in T^3(\bar{\nu}_3)$: $\forall \langle \bar{\mu}_1, \bar{\mu}_2 \rangle \in T^3(\bar{\nu}_3) \ \langle \bar{\nu}_3 \rangle \in T^3_{\langle \bar{\mu}_1, \bar{\mu}_2 \rangle}$ hence $\langle \bar{\nu}_3 \rangle \in T_{\langle \bar{\nu}_1, \bar{\nu}_2 \rangle}$.

In this way we continue to all levels. $\square$

Lemma 3.10. *Let T be $\bar{E}_\alpha$ tree. Then there is $T^* \leq T$ such that if*

$$p \leq \left\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^* \right\rangle$$

then there is $\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in T$ such that

$$p \leq^* \left(\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T \rangle \right)_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle}.$$

Proof. The proof is by induction on $\mathrm{l}(\bar{E}_\alpha)$. The first case was done in 3.9. The proof is almost the same. We just make sure to invoke the induction hypothesis while repeating the construction.

Construction of T^1 and T^2 is exactly like in 3.9. We show the construction at the 3rd level.

Let us set

$$A_{\langle \bar{\nu}_1, S^1, \bar{\nu}_2, S^2 \rangle} = \mathrm{Suc}_{T^2}(\langle \bar{\nu}_1, \bar{\nu}_2 \rangle),$$

$$A_{\langle \bar{\nu}_1, S^1 \rangle} = \underset{\langle \bar{\nu}_2, S^2 \rangle}{\triangle^0} A_{\langle \bar{\nu}_1, S^1, \bar{\nu}_2, S^2 \rangle},$$

$$B_{\langle \bar{\nu}_1, S^1 \rangle} = \left\{ \langle \bar{\nu}_3 \rangle \mid \mathrm{l}(\bar{\nu}_3) = 0 \text{ or } \mathrm{Suc}_{T^2}(\langle \bar{\nu}_1, S^1 \rangle) \cap \kappa^0(\bar{\nu}_3) \in \bar{\nu}_3 \right\},$$

$$\mathrm{Lev}_0(T^{(0)}) = \mathrm{Lev}_0(T^2),$$

$$\mathrm{Suc}_{T^{(0)}}(\langle \bar{\nu}_1 \rangle) = A_{\langle \bar{\nu}_1, S^1 \rangle} \cap B_{\langle \bar{\nu}_1, S^1 \rangle},$$

$$T^{(0)}_{\langle \bar{\nu}_1, \bar{\nu}_3 \rangle} = \bigcap_{\langle \bar{\nu}_2 \rangle \in T^{(0)}_{\langle \bar{\nu}_1 \rangle}(\bar{\nu}_3)} T^2_{\langle \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3 \rangle},$$

$$A'_{\langle \rangle} = \underset{\langle \bar{\nu}_1, S^1 \rangle}{\triangle^0} A_{\langle \bar{\nu}_1, S^1 \rangle},$$

$$A_{\langle \rangle} = \left\{ \langle \bar{\nu}_3, S^3 \rangle \mid \langle \bar{\nu}_3, S^{3\prime} \rangle \in A'_{\langle \rangle} \ S \text{ is generated from } S^{3\prime} \text{ by induction} \right\},$$

$$B_{\langle \rangle} = \left\{ \langle \bar{\nu}_3 \rangle \mid l(\bar{\nu}_3) = 0 \text{ or } \mathrm{Lev}_0(T^2) \cap \kappa^0(\bar{\nu}_3) \in \bar{\nu}_3 \right\},$$

$$\mathrm{Lev}_0(T^{(1)}) = A_{\langle \rangle} \cap B_{\langle \rangle},$$

$$\mathrm{Suc}_{T^{(1)}}(\langle \bar{\nu}_3 \rangle) = \bigcap_{\langle \bar{\nu}_1, \bar{\nu}_2 \rangle \in T^{(0)}(\bar{\nu}_3)} T^2_{\langle \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3 \rangle},$$

$$T^3 = T^2 \cap T^{(0)} \cap T^{(1)}.$$

Let us assume that

$$p_2 \frown p_1 \frown p_0 = p \leq^3 \left\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^3 \right\rangle.$$

There are 3 cases to consider here:

1. $p \leq^* \left(\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^3 \rangle \right)_{\langle \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3 \rangle}$ where $\langle \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3 \rangle \in T^3$: At once we get $\langle \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3 \rangle \in T$.

2. $p \leq^* \left(\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^3 \rangle \right)_{\langle \bar{\nu}_1, \bar{\nu}_3, \bar{\nu}_2 \rangle}$ where $\langle \bar{\nu}_1, \bar{\nu}_3 \rangle \in T^3$, $\langle \bar{\nu}_2 \rangle \in T^3_{\langle \bar{\nu}_1 \rangle}(\bar{\nu}_3)$: In this case $\forall \langle \bar{\mu} \rangle \in T^3_{\langle \bar{\nu}_1 \rangle}(\bar{\nu}_3) \ \langle \bar{\nu}_3 \rangle \in T^3_{\langle \bar{\nu}_1, \bar{\mu} \rangle}$. As $\langle \bar{\nu}_2 \rangle \in T^3_{\langle \bar{\nu}_1 \rangle}(\bar{\nu}_3)$ we get $\langle \bar{\nu}_3 \rangle \in T_{\langle \bar{\nu}_1, \bar{\nu}_2 \rangle}$.

3. $p \leq^2 \left(\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^3 \rangle \right)_{\langle \bar{\nu}_3 \rangle}$ where $\langle \bar{\nu}_3 \rangle \in T^3$ and $p_2 \frown p_1 \leq^2 \left(\langle \langle \bar{\nu}_3, \langle \rangle \rangle, T^3(\bar{\nu}_3) \rangle \right)$ By induction there is $\langle \bar{\nu}_1, \bar{\nu}_2 \rangle \in T(\bar{\nu}_3)$ such that

$$p_2 \frown p_1 \leq^* \left(\langle \langle \bar{\nu}_3, \langle \rangle \rangle, T^2(\bar{\nu}_3) \rangle \right)_{\langle \bar{\nu}_1, \bar{\nu}_2 \rangle}.$$

By construction $\forall \langle \bar{\mu}_1, \bar{\mu}_2 \rangle \in T^2(\bar{\nu}_3) \ \langle \bar{\nu}_3 \rangle \in T^2_{\langle \bar{\mu}_1, \bar{\mu}_2 \rangle}$ hence $\langle \bar{\nu}_3 \rangle \in T_{\langle \bar{\nu}_1, \bar{\nu}_2 \rangle}$.

In this way we continue to all levels. $\square$

4. $P_{\bar{E}}$-FORCING

Definition 4.1. A condition in $P^*_{\bar{E}}$ is of the form

$$\left\{ \langle \bar{\gamma}, p^{\bar{\gamma}} \rangle \mid \bar{\gamma} \in g \right\} \cup \{T\}$$

where

1. $g \subseteq \bar{E}$, $|g| \leq \kappa$,
2. $\min \bar{E} \in g$ and g has a maximal element,
3. $p^{\bar{\gamma}} \in V_\kappa$ is an extender sequence. We allow $p^{\bar{\gamma}} = \emptyset$,

4. $p^0 = (p^{\max g})^0$.

   This condition is not really needed here. It is needed in a later forcing based on this one,

5. T is a $\max g$-tree such that for all $t \in T$ $p^{\max g} \frown t$ is ^{0}-increasing,
6. For all $\bar{\gamma} \in g$, $p^{\max g}$ is not permitted to $p^{\bar{\gamma}}$,
7. $\forall \langle \bar{\nu} \rangle \in T$ $|\{\bar{\gamma} \in g \mid \bar{\nu}$ is permitted to $p^{\bar{\gamma}}\}| \le \kappa^0(\bar{\nu})$,
8. $\forall \langle \bar{\nu} \rangle \in T$ if $\bar{\nu}$ is permitted to $p^{\bar{\beta}}, p^{\bar{\gamma}}$ then $\pi_{\max g, \bar{\beta}}(\bar{\nu}) \ne \pi_{\max g, \bar{\gamma}}(\bar{\nu})$.

We write $\mathrm{mc}(p)$, p^{mc}, T^p, $\bar{E}(p)$, $\mathrm{supp}\, p$ for $\max g$, $p^{\max g}$, T, $\bar{E}$, g respectively.

	$\bar{\mu}_0$	$\bar{\mu}_1$	$\bar{\mu}_2$	$\bar{\mu}_3$	$\bar{\mu}_4$	T
Support	$\bar{E}_\kappa$	$\bar{E}_{\alpha_1}$	$\bar{E}_{\alpha_2}$	$\bar{E}_{\alpha_3}$	$\bar{E}_{\alpha_4} = \mathrm{mc}$	

FIGURE 1. An Example of Condition in $P^*_{\bar{E}}$

Definition 4.2. Let $p, q \in P^*_{\bar{E}}$. We say that p is a Prikry extension of q ($p \le^* q$ or $p \le^0 q$) if

1. $\mathrm{supp}\, p \supseteq \mathrm{supp}\, q$,
2. $\forall \gamma \in \mathrm{supp}\, q$ $p^\gamma = q^\gamma$,
3. $T^p \le T^q$.

We include in this definition the degenerate case $\mathrm{l}(\bar{E}) = 0$. There is neither extender nor tree in this case. By $p \le^* p$ we mean $p = q$.

	$\bar{\mu}_0$	$\bar{\nu}_0$	$\bar{\mu}_1$	$\bar{\nu}_1$	$\bar{\mu}_2$	$\bar{\mu}_3$	$\bar{\mu}_4$	$\bar{\nu}_2$	$\pi^{-1}_{\beta_2,\alpha_4} T$
Support	$\bar{E}_\kappa$	$\bar{E}_{\beta_0}$	$\bar{E}_{\alpha_1}$	$\bar{E}_{\beta_1}$	$\bar{E}_{\alpha_2}$	$\bar{E}_{\alpha_3}$	$\bar{E}_{\alpha_4}$	$\bar{E}_{\beta_2} = \mathrm{mc}$	

FIGURE 2. An Example of Direct Extension

Definition 4.3. A condition in $P_{\bar{E}}$ is of the form

$$p_n \frown \cdots \frown p_0$$

where

- $p_0 \in P^*_{\bar{E}}$,
- $p_1 \in P^*_{\bar{\mu}_1}$,

- $\vdots$,
- $p_n \in P^*_{\bar{\mu}_n}$,

where $\bar{E}, \bar{\mu}_1, \ldots, \bar{\mu}_n$ are extender sequence systems satisfying

$$\kappa^0(\bar{\mu}_n) < \cdots < \kappa^0(\bar{\mu}_1) < \kappa^0(\bar{E}).$$

$$\begin{array}{ccccc|c|ccccc|c|ccccccc|c}
\tau_0 & \tau_1 & \tau_2 & \tau_3 & & \tau_4 & R & \bar{\mu}_5 & \bar{\mu}_6 & \bar{\mu}_7 & \bar{\mu}_8 & & \bar{\mu}_9 & S & \bar{\nu}_0 & \bar{\nu}_1 & \bar{\nu}_5 & \bar{\nu}_6 & & \bar{\nu}_4 & T \\
\hline
\bar{\mu}_0 & \bar{\mu}_1 & \bar{\mu}_2 & \bar{\mu}_3 & \bar{\mu}_4 = \mathrm{mc} & & & \bar{\nu}_0 & \bar{\nu}_1 & \bar{\nu}_2 & \bar{\nu}_3 & \bar{\nu}_4 = \mathrm{mc} & & & \bar{E}_\kappa & \bar{E}_{\alpha_1} & \bar{E}_{\alpha_2} & \bar{E}_{\alpha_3} & \bar{E}_{\alpha_4} = \mathrm{mc}
\end{array}$$

FIGURE 3. An Example of a Condition in $P_{\bar{E}}$

Definition 4.4. Let $p, q \in P_{\bar{E}}$. We say that p is a Prikry extension of q ($p \leq^* q$ or $p \leq^0 q$) if p, q are of the form

$$p = p_n \frown \cdots \frown p_0,$$
$$q = q_n \frown \cdots \frown q_0,$$

and

- $p_0, q_0 \in P_{\bar{E}}^*$, $p_0 \leq^* q_0$,
- $p_1, q_1 \in P_{\bar{\mu}_1}^*$, $p_1 \leq^* q_1$,
- $\vdots$,
- $p_n, q_n \in P_{\bar{\mu}_n}^*$, $p_n \leq^* q_n$.

Definition 4.5. Let $p \in P_{\bar{E}}^*$ and $\langle \bar{\nu} \rangle \in T^p$. We =define $(p)_{\langle \bar{\nu} \rangle}$ to be $p_1' \frown p_0'$ where

1. $\operatorname{supp} p_1' = \left\{ \pi_{\mathrm{mc}(p), \bar{\gamma}}(\bar{\nu}) \mid \bar{\gamma} \in \operatorname{supp} p, \ \kappa^0(p^{\bar{\gamma}}) < \kappa^0(\bar{\nu}) \right\}$,
2. $p_1'^{\pi_{\mathrm{mc}(p), \bar{\gamma}}(\bar{\nu})} = p^{\bar{\gamma}}$,
3. $T^{p_1'} = T^p(\bar{\nu})$,
4. $\operatorname{supp} p_0' = \operatorname{supp} p$,
5. $\forall \bar{\gamma} \in \operatorname{supp} p_0' \ \ p_0'^{\bar{\gamma}} = \begin{cases} \pi_{\mathrm{mc}(p), \bar{\gamma}}(\bar{\nu}) & \kappa^0(p^{\bar{\gamma}}) < \kappa^0(\bar{\nu}) \\ p^{\bar{\gamma}} & \text{otherwise} \end{cases}$,
6. $T^{p_0'} = T^p_{\langle \bar{\nu} \rangle}$.

Definition 4.6. Let $p, q \in P_{\bar{E}}$. We say that p is a 1-point extension of q ($p \leq^1 q$) if p, q are of the form

$$p = p_{n+1} \frown p_n \frown \cdots \frown p_0,$$
$$q = q_n \frown \cdots \frown q_0,$$

and there is $0 \leq k \leq n$ such that

- $p_i, q_i \in P_{\bar{\mu}_i}^*$, $p_i \leq^* q_i$ for $i = 0, \ldots, k-1$,
- $p_{i+1}, q_i \in P_{\bar{\mu}_i}^*$, $p_{i+1} \leq^* q_i$ for $i = k+1, \ldots, n$,
- There is $\langle \bar{\nu} \rangle \in T^{q_k}$ such that $p_{k+1} \frown p_k \leq^* (q_k)_{\langle \bar{\nu} \rangle}$.

$$\begin{array}{ccccc|ccccc}
\bar{\mu}_0 & \bar{\mu}_1 & & \bar{\mu}_3 & \bar{\mu}_4 \ T(\bar{\nu}) & \bar{\nu}^0 & \pi_{\alpha_4, \alpha_1}(\bar{\nu}) & \bar{\mu}_2 & \pi_{\alpha_4, \alpha_3}(\bar{\nu}) & \bar{\nu} \ T_{\langle \bar{\nu} \rangle} \\
\hline
\bar{\nu}^0 & \pi_{\alpha_4, \alpha_1}(\bar{\nu}) & \pi_{\alpha_4, \alpha_3}(\bar{\nu}) & \bar{\nu} & & \bar{E}_\kappa & \bar{E}_{\alpha_1} & \bar{E}_{\alpha_2} & \bar{E}_{\alpha_3} & \bar{E}_{\alpha_4}
\end{array}$$

FIGURE 4. An Example of 1-point extension

Definition 4.7. Let $p, q \in P_{\bar{E}}$. We say that p is an n-point extension of q ($p \leq^n q$) if there are $p^n, \ldots, p^0$ such that

$$p = p^n \leq^1 \cdots \leq^1 p^0 = q.$$

Definition 4.8. Let $p, q \in P_{\bar{E}}$. We say that p is an extension of q ($p \leq q$) if there is n such that $p \leq^n q$.

Later on by $P_{\bar{E}}$ we mean $\langle P_{\bar{E}}, \leq \rangle$.

Note 4.9. When $\mathrm{l}(\bar{E}) = 1$ the forcing $P_{\bar{E}}$ is the Gitik-Magidor forcing from section 1 of [3]. When $\mathrm{l}(\bar{E}) < \kappa$ the forcing $P_{\bar{E}}$ is similar to the forcing defined in [8].

In several places we want to prevent enlargment of the support of a condition. This makes all the conditions which are stronger than some condition but with the same support resemble Radin forcing. The following definition catches the meaning of not enlarging the support. The 'resemblence' we look for is 5.3.

Definition 4.10. Let $p, q \in P_{\bar{E}}$. We say that $p \leq^*_R q$ if
1. $p \leq^* q$,
2. $\operatorname{supp} p = \operatorname{supp} q$.

Definition 4.11. Let $p, q \in P_{\bar{E}}$. We say that $p \leq^1_R q$ if
1. $p \leq^1 q$,
2. In the definition of $\leq^1$ we can substitute $\leq^*$ by $\leq^*_R$.

Definition 4.12. Let $p, q \in P_{\bar{E}}$. We say that $p \leq^n_R q$ if there are $p^n, \ldots, p^0$ such that

$$p = p^n \leq^1_R \cdots \leq^1_R p^0 = q.$$

Definition 4.13. Let $p, q \in P_{\bar{E}}$. We say that $p \leq_R q$ if there is n such that $p \leq^n_R q$.

Note 4.14. The above definitions imply that if $q \leq p$ then there is r such that $q \leq^* r \leq_R p$.

Definition 4.15. Let $\bar{\epsilon}$ be an extender sequence such that $\kappa^0(\bar{\epsilon}) < \kappa^0(\bar{E})$.

$$P_{\bar{E}}/P_{\bar{\epsilon}} = \left\{ p \mid q \in P_{\bar{\epsilon}}, q \frown p \in P_{\bar{E}} \right\}.$$

5. Basic Properties of $P_{\bar{E}}$

Claim 5.1. $P_{\bar{E}}$ *satisfies* κ^{++}-*c.c.*

Proof. The usual Δ-lemma argument on the support will do.   □

Claim 5.2. *Let* $p \in P_{\bar{E}}$, $P^* = \{q \leq_R p \mid p \in P_{\bar{E}}\}$. *Then*
1. $\langle P^*, \leq_R \rangle$ *satisfies* κ^+-*c.c.*,
2. $\langle P^*, \leq_R \rangle$ *is sub-forcing of* $\langle P_{\bar{E}}/p, \leq \rangle$.

Proof. Showing κ^+-c.c. is trivial.

Showing that P^* is sub-forcing of $P_{\bar{E}}/p$ amounts to showing that any maximal anti-chain of P^* is also a maximal anti-chain of $P_{\bar{E}}/p$.

Let A be a maximal anti-chain of P^*. Let $q \in P_{\bar{E}}/p$. As $q \leq p$, there is $r' \in P^*$ such that $q \leq^* r' \leq_R p$. Assume that $r' = r'_n \frown \cdots \frown r'_0$. Then also $q = q_n \frown \cdots \frown q_0$. Let r_i be r'_i with $T^{r'_i}$ substituted by $T^{r'_i} \cap \pi_{\mathrm{mc}(q_i),\mathrm{mc}(r'_i)}(T^{q_i})$ and $r = r_n \frown \cdots \frown r_0$. As $r \in P^*$ and A is a maximal anti-chain there is $a \in A$ such that $a \parallel r$. Take $s \leq_R a, r$. Considering how we constructed r from r' we must have $t \leq^* s$ such that $t \leq q$. Hence $q \parallel a$. So we get that A is a maximal anti-chain of $P_{\bar{E}}/p$. $\qquad\square$

Claim 5.3. *Let $p \in P_{\bar{E}}$, $P^* = \{q \leq_R p \mid p \in P_{\bar{E}}\}$. Then there is $r \in R_{\mathrm{mc}(p)}$ such that $P^* \simeq R_{\mathrm{mc}(p)}/r$*

Proof. For simplicity assume that $p = p_0$. Then we set $r = \langle \langle \mathrm{mc}(p_0), p_0^{\mathrm{mc}} \rangle, T^p \rangle$.

We give the isomorphism: The image of $q \in P^*$ is $s \in \langle R_{\mathrm{mc}(p)}/r, \leq \rangle$ such that

1. $q \leq^* s$,
2. $T^{s_i} = T^{q_i}$ where $s = s_n \frown \cdots \frown s_0$, $q = q_n \frown \cdots \frown q_0$.

$\qquad\square$

Let G be $P_{\bar{E}}$-generic.

Definition 5.4. $\bar{E}_G$ is the enumeration of $\{ \bar{E}(p_k) \mid p_n \frown \cdots \frown p_0 \in G \}$ ordered increasingly by $\kappa^0(\bar{E}(p_k))$.

Definition 5.5. Let $\zeta < \mathrm{otp}(\bar{E}_G)$. Then

1. $G\!\restriction\!\zeta = \{ p_n \frown \cdots \frown p_k \mid p_n \frown \cdots \frown p_k \frown \cdots \frown p_0 \in G, \bar{E}(p_k) = \bar{E}_G(\zeta) \}$,
2. $G \setminus \zeta = \{ p_{k-1} \frown \cdots \frown p_0 \mid p_n \frown \cdots \frown p_k \frown \cdots \frown p_0 \in G, \bar{E}(p_k) = \bar{E}_G(\zeta) \}$.

Definition 5.6.

$$M_G^{\bar{\alpha}} = \begin{cases} \bigcup \{ M_G^{\bar{\mu}} \mid p \in G, \bar{\mu} = p^{\bar{\alpha}} \} \cup \{ \bar{\alpha} \} & \exists p \in G \ \bar{\alpha} \in \mathrm{supp}\, p \\ \{ \bar{\alpha} \} & \text{otherwise} \end{cases}$$

$$C_G^{\bar{\alpha}} = \{ \kappa(\bar{\mu}) \mid \bar{\mu} \in M_G^{\bar{\alpha}} \}$$

Proposition 5.7.   1. $C_G^{\bar{E}_\kappa} \setminus \{ \kappa \}$ *is a club in* κ,

2. $C_G^{\bar{E}_\alpha} \setminus \{ \alpha \}$ *is unbounded in* κ,
3. $\bar{\alpha} \neq \bar{\beta} \implies C_G^{\bar{\alpha}} \neq C_G^{\bar{\alpha}}$.

Proof. The first two claims are immediate as these are sequences which are generated by Radin forcing.

The last is by density and noticing that when $p^{\bar{\alpha}}, p^{\bar{\beta}}$ are permitted for $\bar{\nu}$ we required $\pi_{\mathrm{mc}(p),\bar{\alpha}}(\bar{\nu}) \neq \pi_{\mathrm{mc}(p),\bar{\beta}}(\bar{\nu})$. $\qquad\square$

6. Homogeneity in Dense Open Subsets

Our aim in this section is to prove the following

Theorem 6.1. *Let $D \subseteq P_{\bar{E}}$ be dense open and $p = p_k \frown \cdots \frown p_0 \in P_{\bar{E}}$. Then there is $p^* \leq^* p$ such that*

$$\exists S^k \, \exists n_k \forall \langle \bar{\nu}_{k,1}, \ldots, \bar{\nu}_{k,n_k} \rangle \in S^k \ldots \exists S^0 \, \exists n_0 \forall \langle \bar{\nu}_{0,1}, \ldots, \bar{\nu}_{0,n_0} \rangle \in S^0$$
$$(p_k^*)_{\langle \bar{\nu}_{k,1}, \ldots, \bar{\nu}_{k,n_k} \rangle} \frown \cdots \frown (p_0^*)_{\langle \bar{\nu}_{0,1}, \ldots, \bar{\nu}_{0,n_0} \rangle} \in D$$

where

1. $S^i \subseteq T^{p_i^*} \restriction [V_\kappa]^{n_i}$,
2. $\forall l < n_i \, \forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_l \rangle \in S^i \, \exists \xi \ \mathrm{Suc}_{S^i}(\langle \bar{\nu}_1, \ldots, \bar{\nu}_l \rangle) \in E_{\mathrm{mc}(p_i^*)}(\xi)$.

The proof is done by a series of lemmas.

Definition 6.2. *Let $p \in P_{\bar{E}}^*$. Let s be a function such that $\mathrm{dom}\, s \subseteq \bar{E}$. and for all $\bar{\alpha}, \bar{\beta} \in \mathrm{dom}\, s$, $\bar{\alpha} \neq \bar{\beta}$*

1. $s(\bar{\alpha})$ is an extender sequence,
2. $\mathrm{l}(s(\bar{\alpha})) = \mathrm{l}(s(\bar{\beta}))$,
3. $\kappa^0(s(\bar{\alpha})) = \kappa^0(s(\bar{\beta}))$,
4. $s(\bar{\alpha}) \neq s(\bar{\beta})$.

We define $(p)_{\langle s \rangle}$ to be $p_1' \frown p_0'$ where

1. $\mathrm{supp}\, p_1' = \left\{ s(\bar{\alpha}) \mid \bar{\alpha} \in \mathrm{supp}\, p \cap \mathrm{dom}\, s, \, \kappa^0(p^{\bar{\alpha}}) < \kappa^0(s(\bar{\alpha})) \right\}$,
2. $\forall \bar{\alpha} \in \mathrm{supp}\, p_1' \ p_1'^{s(\bar{\alpha})} = p^{\bar{\alpha}}$,
3. If $s(\mathrm{mc}(p)) \in T^p$ then $T^{p_1'} = T^p(s(\mathrm{mc}(p)))$. Otherwise we leave $T^{p_1'}$ undefined,
4. $\mathrm{supp}\, p_0' = \mathrm{supp}\, p$,
5. $\forall \bar{\alpha} \in \mathrm{supp}\, p_0' \ p_0'^{\bar{\alpha}} = \begin{cases} s(\bar{\alpha}) & \bar{\alpha} \in \mathrm{dom}\, s \text{ and } \kappa^0(p^{\bar{\alpha}}) < \kappa^0(s(\bar{\alpha})) \\ p^{\bar{\alpha}} & \text{otherwise} \end{cases}$,
6. If $s(\mathrm{mc}(p)) \in T^p$ then $T^{p_0'} = T^p_{\langle s(\mathrm{mc}(p)) \rangle}$. Otherwise we leave $T^{p_0'}$ undefined.

Definition 6.3. *Let $p \in P_{\bar{E}}^*$. Let s be a function with $\mathrm{dom}\, s = 1, \ldots, n$ such that for all i $s(i)$ satisfies definition 6.2. Then we define $(p)_{\langle s \rangle}$ as p^n where p^n is defined by induction as follows:*

$$p^0 = p,$$
$$p^{i+1} = p_i^i \frown \cdots \frown p_1^i \frown (p_0^i)_{\langle s(i+1) \rangle}.$$

We note the following: If $\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in T^p$ and we set for all $1 \leq i \leq n$

$$s(i) = \left\{ \langle \bar{\alpha}, \pi_{\mathrm{mc}(p), \bar{\alpha}}(\bar{\nu}_i) \rangle \mid \bar{\alpha} \in \mathrm{supp}\, p \right\}$$

then

$$(p)_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle} = (p)_{\langle s \rangle}.$$

We use this operation also in cases where p is not strictly a condition. That is if $p \cup \{T\} \in P_{\bar{E}}$ we also use $(p)_{\langle s \rangle}$. In this case we ignore the trees in the definition.

This definition is used in the proof of the homogeneity for the following reason: Beforehand we do not know what a legitimate extension is. By checking with all the possible μ's we check on all legitimate conditions which *might* be extensions.

Claim 6.4. *Let D be dense open in $P_{\bar{E}}/P_{\bar{\epsilon}}$, $p = p_0 \in P_{\bar{E}}/P_{\bar{\epsilon}}$, $0 < n < \omega$. Then there is $p^* \leq^* p$ such that one and only one of the following is true:*

1. *There is $S \subseteq T^{p^*} \restriction [V_\kappa]^n$ such that*
   (1.1) $\forall k < n \, \exists \xi < \mathrm{l}(\bar{E}) \ \mathrm{Suc}_S(\langle \bar{\nu}_1, \ldots, \bar{\nu}_k \rangle) \in E_{\mathrm{mc}(p^*)}(\xi)$,

(1.2) $\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in S$ $(p^*)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle} \in D.$

2. $\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in T^{p^*} \forall q \leq^* (p^*)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle}$ $q \notin D.$

Proof. We give the proof for $n = 1$. Adapting the proof for higher n's require that whenever we enumerate singletons we should enumerate n-tuples and when we use j we should use j_n.

We start an induction on ξ in which we build

$$\langle \bar{\alpha}^\xi, u^\xi \mid \xi < \kappa \rangle.$$

We start by setting

$$u^0 = p_0 \setminus \{T^{p_0}\},$$
$$\bar{\alpha}^0 = \mathrm{mc}(p_0),$$
$$T^0 = T^{p_0} \restriction \pi_{\bar{\alpha}^0, 0}^{-1} \left\{ \bar{\nu} \mid \kappa^0(\bar{\nu}) \text{ is inaccessible} \right\},$$

and taking an increasing enumeration

$$\left\{ \kappa^0(\bar{\nu}) \mid \langle \bar{\nu} \rangle \in T^0 \right\} = \langle \tau_\xi \mid \xi < \kappa \rangle.$$

Assume that we have constructed

$$\langle \bar{\alpha}^\xi, u^\xi \mid \xi < \xi_0 \rangle.$$

We have 2 cases: ξ_0 is limit: Choose $\bar{\alpha}^{\xi_0} > \bar{\alpha}^\xi$ for all $\xi < \xi_0$ and set

$$u^{\xi_0} = \bigcup_{\xi < \xi_0} u^\xi \cup \left\{ \langle \bar{\alpha}^{\xi_0}, t \rangle \right\} \text{ where } \kappa^0(t) = \tau_{\xi_0}.$$

$\xi_0 = \xi + 1$: For each $\bar{\nu}$ such that $\kappa^0(\bar{\nu}) = \tau_\xi$ we set

$$S(\bar{\nu}) = \Big(\prod_{\substack{\bar{\alpha} \in \mathrm{supp}\, u^\xi \\ \kappa((u^\xi)^{\bar{\alpha}}) < \tau_\xi}} \left\{ \bar{\mu} \mid \kappa^0(\bar{\mu}) = \kappa^0(\bar{\nu}) \right\} \Big) \times \left\{ \langle \bar{\nu} \rangle \right\}.$$

Let

$$S = \bigcup_{\kappa^0(\bar{\nu}) = \tau_\xi} S(\bar{\nu})$$

and set enumeration of S

$$S = \langle s^{\xi_0, \rho} \mid \rho < \tau_{\xi_0} \rangle.$$

There are fewer than τ_{ξ_0} elements in S. We use τ_{ξ_0} as this is the maximum size S can have which is not 'killing' the induction.

We do induction on ρ which builds

$$\langle \bar{\alpha}^{\xi_0, \rho}, u_0^{\xi_0, \rho}, T_0^{\xi_0, \rho}, u_1^{\xi_0, \rho}, T_1^{\xi_0, \rho} \mid \rho < \tau_{\xi_0} \rangle$$

from which we build $\langle \bar{\alpha}^{\xi_0}, u^{\xi_0} \rangle$. Set

$$\bar{\alpha}^{\xi_0, 0} = \bar{\alpha}^\xi,$$
$$u_0^{\xi_0, 0} = u_0^\xi.$$

Assume we have constructed $\langle \bar{\alpha}^{\xi_0, \rho}, u_0^{\xi_0, \rho}, T_0^{\xi_0, \rho} \mid \rho < \rho_0 \rangle$.
We have 2 cases:

ρ_0 is limit: Set

$$\forall \rho < \rho_0 \ \bar{\alpha}^{\xi_0,\rho_0} > \bar{\alpha}^{\xi_0,\rho},$$

$$u^{\xi_0,\rho_0} = \bigcup_{\rho < \rho_0} u^{\xi_0,\rho} \cup \left\{ \langle \bar{\alpha}^{\xi_0,\rho_0}, t \rangle \right\} \ \text{ where } \kappa^0(t) = \tau_\xi.$$

We set $T_0^{\xi_0,\rho_0}$, $T_1^{\xi_0,\rho_0}$ to anything we like. We do not use them later.

$\rho_0 = \rho + 1$: Let $\langle \bar{\nu} \rangle = s^{\xi_0,\rho}(2)$. set

$$u'' = u_1'' \frown u_0'' = (u_0^{\xi_0,\rho})_{\langle s^{\xi_0,\rho} \rangle},$$

$$T_0'' = \pi_{\bar{\alpha}^\xi, \bar{\alpha}^0}^{-1}(T_{\langle \bar{\nu} \rangle}^0),$$

$$T_1'' = \pi_{\mathrm{mc}(u_1''), \bar{\nu}}^{-1}(T^0(\bar{\nu})).$$

If there are

$$q_1' \leq^* u_1'' \cup \{T_1''\},$$

$$q_0' \leq^* u_0'' \cup \{T_0''\},$$

such that

$$q_1' \frown q_0' \in D$$

then set

$$\bar{\alpha}^{\xi_0,\rho_0} = \mathrm{mc}(q_0'),$$

$$u_0^{\xi_0,\rho_0} = u^{\xi_0,\rho} \cup (q_0' \setminus (u_0'' \cup \left\{ T^{q_0'} \right\})),$$

$$T_0^{\xi_0,\rho_0} = T^{q_0'},$$

$$u_1^{\xi_0,\rho_0} = q_1' \setminus (u_1'' \cup \left\{ T^{q_1'} \right\}),$$

$$T_1^{\xi_0,\rho_0} = T^{q_1'},$$

otherwise set

$$\bar{\alpha}^{\xi_0,\rho_0} = \bar{\alpha}^{\xi_0,\rho},$$

$$u_0^{\xi_0,\rho_0} = u^{\xi_0,\rho},$$

$$T_0^{\xi_0,\rho_0} = T_0'',$$

$$u_1^{\xi_0,\rho_0} = \emptyset,$$

$$T_1^{\xi_0,\rho_0} = T_1''.$$

When the induction on ρ terminates we have $\langle \bar{\alpha}^{\xi_0,\rho}, u_0^{\xi_0,\rho}, T_0^{\xi_0,\rho}, u_1^{\xi_0,\rho}, T_1^{\xi_0,\rho} \mid \rho < \tau_{\xi_0} \rangle$. We continue with the induction on ξ. We set

$$\forall \rho < \tau_{\xi_0} \ \bar{\alpha}^{\xi_0} > \bar{\alpha}^{\xi_0,\rho},$$

$$u_0^{\xi_0} = \bigcup_{\rho < \tau_{\xi_0}} u_0^{\xi_0,\rho} \cup \left\{ \langle \bar{\alpha}^{\xi_0}, t \rangle \right\} \ \text{ where } \max \kappa^0(t) = \tau_\xi.$$

When the induction on ξ terminates we have $\langle \bar{\alpha}^\xi, u_0^\xi \mid \xi < \kappa \rangle$. Let

$$\forall \xi < \kappa \ \bar{\alpha}^{*\prime} > \bar{\alpha}^\xi,$$

$$p_0^{*\prime} = \bigcup_{\xi < \kappa} u_0^\xi \cup \{ \langle \bar{\alpha}^{*\prime}, t \rangle \} \ \text{ where } \max \kappa^0(t) = \max p_0^0.$$

We set

$$\mathrm{Lev}_0(T^{p_0^{*\prime}}) = \pi_{\bar{\alpha}^{*\prime},\bar{\alpha}^0}^{-1} \mathrm{Lev}_0(T^0).$$

Let us consider $\langle\bar{\nu}\rangle \in \mathrm{Lev}_0(T^{p_0^{*\prime}})$. There is ξ such that $\kappa^0(\bar{\nu}) = \tau_\xi$. We set

$$s(1) = \left\{\langle\bar{\alpha}, \pi_{\bar{\alpha}^{*\prime},\bar{\alpha}}(\bar{\nu})\rangle \mid \bar{\alpha} \in \mathrm{supp}\, p_0^*\right\},$$
$$s(2) = \left\{\langle\pi_{\bar{\alpha}^{*\prime},\bar{\alpha}^0}(\bar{\nu})\rangle\right\}.$$

Let $\xi_0 = \xi + 1$. By our construction there is ρ such that

$$(u_0^{\xi_0,\rho_0})_{\langle s\rangle} = (u_0^{\xi_0,\rho_0})_{\langle s^{\xi_0,\rho}\rangle}$$

where $\rho_0 = \rho + 1$. We set

$$T_{\langle\bar{\nu}\rangle}^{p_0^{*\prime}} = \pi_{\bar{\alpha}^{*\prime},\bar{\alpha}^{\xi_0,\rho_0}}^{-1}(T^{\xi_0,\rho_0}) \cap \pi_{\bar{\alpha}^{*\prime},\bar{\alpha}^0}^{-1}(T^0_{\langle\pi_{\bar{\alpha}^*,\bar{\alpha}^0}(\bar{\nu})\rangle}),$$
$$T^{p_0^{*\prime}}(\bar{\nu}) = \pi_{\bar{\nu},\mathrm{mc}(u_1^{\xi_0,\rho_0})}^{-1} T_1^{\xi_0,\rho_0},$$
$$p_1'(\bar{\nu}) = u_1^{\xi_0,\rho_0}.$$

Let us show that $p_0^{*\prime}$ approximates the p^* we look for. So let $\langle\bar{\nu}\rangle \in T^{p_0^{*\prime}}$ and assume

(6.4.1a) $$q_1' \leq^* p_1'(\bar{\nu}) \cup ((p_0^{*\prime})_{\langle\bar{\nu}\rangle})_1,$$

(6.4.1b) $$q_0' \leq^* ((p_0^{*\prime})_{\langle\bar{\nu}\rangle})_0,$$

(6.4.1c) $$q_1' {}^\frown q_0' \in D.$$

Let ξ be such that $\kappa^0(\bar{\nu}) = \tau_\xi$. Set s as

$$s(1) = \left\{\langle\bar{\alpha}, \pi_{\bar{\alpha}^{*\prime},\bar{\alpha}}(\bar{\nu})\rangle \mid \bar{\alpha} \in \mathrm{supp}\, p_0^*\right\},$$
$$s(2) = \left\{\langle\pi_{\bar{\alpha}^{*\prime},\bar{\alpha}^0}(\bar{\nu})\rangle\right\},$$

where $\xi_0 = \xi + 1$, $\rho_0 = \rho + 1$. By our construction there is ρ such that

$$(u^{\xi_0,\rho_0})_{\langle s^{\xi_0,\rho}\rangle} = (u^{\xi_0,\rho_0})_{\langle s\rangle}.$$

Let us set

$$r = \left((u_{\langle s^{\xi_0,\rho}\rangle}^{\xi_0,\rho_0})_1 \cup \left\{T_1^{\xi_0,\rho_0}\right\}\right) {}^\frown \left((u_{\langle s^{\xi_0,\rho}\rangle}^{\xi_0,\rho_0})_0 \cup \left\{T_0^{\xi_0,\rho_0}\right\}\right).$$

By construction we have

$$\left(p_1'(\bar{\nu}) \cup ((p_0^{*\prime})_{\langle\bar{\nu}\rangle})_1\right) {}^\frown ((p_0^{*\prime})_{\langle\bar{\nu}\rangle})_0 \leq^* r.$$

So what we have is

$$D \ni q_1' {}^\frown q_0' \leq^* r.$$

This is a positive answer to the question in the induction, hence

$$r \in D,$$

which gives us, by openness of D, that

(6.4.2) $$\left(p_1'(\bar{\nu}) \cup ((p_0^{*\prime})_{\langle\bar{\nu}\rangle})_1\right) {}^\frown ((p_0^{*\prime})_{\langle\bar{\nu}\rangle})_0 \in D.$$

Having proved this approximation property of $p_0^{*\prime}$, let us consider the set

$$B = \left\{\langle\bar{\nu}\rangle \in T^{p_0^{*\prime}} \mid \exists q \leq^* \left(p_1'(\bar{\nu}) \cup ((p_0^{*\prime})_{\langle\bar{\nu}\rangle})_1\right) {}^\frown ((p_0^{*\prime})_{\langle\bar{\nu}\rangle})_0 \; q \in D\right\}.$$

Let $\bar{\alpha}^{*\prime} = \mathrm{mc}(p_0^{*\prime})$. There are 2 cases here:

1. $\exists \zeta < \mathrm{l}(\bar{E}) \; B \in E_{\bar{\alpha}^{*\prime}}(\zeta)$.
   Let us set

$$p_1^{\zeta} = j(p_1')(\bar{E}_{\bar{\alpha}^{*\prime}} {\upharpoonright} \zeta),$$
$$\bar{\beta}^{\zeta\prime} = \mathrm{mc}(p_1^{\zeta}),$$
$$A^{\zeta} = \left\{ \langle \bar{\nu} \rangle \mid \left((p_1^{\zeta})_{\langle \bar{\nu} \rangle} \right)_1 = p_1'(\pi_{\bar{\beta}^{\zeta\prime}, \bar{\alpha}^{*\prime}}(\bar{\nu})) \right\}.$$

Clearly

$$A^{\zeta} \in E_{\bar{\beta}^{\zeta\prime}}(\zeta).$$

Let $\bar{\beta}^{\zeta}$ be the larger of $\bar{\beta}^{\zeta\prime}$, $\bar{\alpha}^{*\prime}$. Set

$$T^{\zeta} = (\pi_{\bar{\beta}^{\zeta}, \bar{\alpha}^{*\prime}})^{-1}(T^{p_0^{*\prime}}),$$
$$p^{\zeta} = p_1^{\zeta} \cup p_0^{*\prime} \cup \left\{ T^{\zeta} \right\}.$$

The nice property of p^{ζ} is that when $\langle \bar{\nu} \rangle \in T^{\zeta} {\upharpoonright} (\pi_{\bar{\beta}^{\zeta}, \bar{\beta}^{\zeta\prime}})^{-1}(A^{\zeta})$ we get that

$$(p^{\zeta})_{\langle \bar{\nu} \rangle} \leq^* \left(p_1'(\pi_{\bar{\beta}^{\zeta}, \bar{\beta}'}(\bar{\nu})) \cup ((p_0^{*\prime})_{\langle \pi_{\bar{\beta}^{\zeta}, \bar{\alpha}^{*\prime}}(\bar{\nu}) \rangle})_1 \right) {}^\frown ((p_0^{*\prime})_{\langle \pi_{\bar{\beta}^{\zeta}, \bar{\alpha}^{*\prime}}(\bar{\nu}) \rangle})_0.$$

We set

$$p^* = p^{\zeta},$$
$$A = \mathrm{Lev}_0(T^{\zeta}) \cap (\pi_{\bar{\beta}^{\zeta}, \bar{\beta}^{\zeta\prime}})^{-1} A^{\zeta},$$

and show that the claim is satisfied: Assume that

$$\langle \bar{\nu} \rangle \in A,$$
$$q_1' \leq^* ((p^*)_{\langle \bar{\nu} \rangle})_1,$$
$$q_0' \leq^* ((p^*)_{\langle \bar{\nu} \rangle})_0,$$
$$q_1' {}^\frown q_0' \in D.$$

Note that

$$((p^*)_{\langle \bar{\nu} \rangle})_1 \leq^* p_1'(\pi_{\bar{\beta}^{\zeta}, \bar{\alpha}^{*\prime}}) \cup ((p_0^{*\prime})_{\langle \pi_{\bar{\beta}^{\zeta}, \bar{\alpha}^{*\prime}}(\bar{\nu}) \rangle})_1,$$
$$((p^*)_{\langle \bar{\nu} \rangle})_0 \leq^* ((p_0^{*\prime})_{\langle \pi_{\bar{\beta}^{\zeta}, \bar{\alpha}^{*\prime}}(\bar{\nu}) \rangle})_0.$$

Hence, we know that

$$q_1' \leq^* p_1'(\pi_{\bar{\beta}^{\zeta}, \bar{\alpha}^{*\prime}}(\bar{\nu})) \cup ((p_0^{*\prime})_{\langle \pi_{\bar{\beta}^{\zeta}, \bar{\alpha}^{*\prime}}(\bar{\nu}) \rangle})_1,$$
$$q_0' \leq^* ((p_0^{*\prime})_{\langle \pi_{\bar{\beta}^{\zeta}, \bar{\alpha}^{*\prime}}(\bar{\nu}) \rangle})_0,$$
$$q_1' {}^\frown q_0' \in D.$$

This is the assumption (6.4.1). So from (6.4.2) we know that

$$\left(p_1'(\pi_{\bar{\beta}^{\zeta}, \bar{\alpha}^{*\prime}}(\bar{\nu})) \cup ((p_0^{*\prime})_{\langle \pi_{\bar{\beta}^{\zeta}, \bar{\alpha}^{*\prime}}(\bar{\nu}) \rangle})_1 \right) {}^\frown ((p_0^{*\prime})_{\langle \pi_{\bar{\beta}^{\zeta}, \bar{\alpha}^{*\prime}}(\bar{\nu}) \rangle})_0 \in D$$

hence by openness of D

$$(p^*)_{\langle \bar{\nu} \rangle} \in D,$$

2. $\forall \zeta < l(\bar{E})\ B \notin E_{\bar{\alpha}^{*\prime}}(\zeta)$ which is the same as saying that

$$\left\{ \langle \bar{\nu} \rangle \in T^{p_0^{*\prime}} \mid \forall q \leq^* \left(p_1'(\bar{\nu}) \cup ((p_0^{*\prime})_{\langle \bar{\nu} \rangle})_1 \right) ^\frown ((p_0^{*\prime})_{\langle \bar{\nu} \rangle})_0\ q \notin D \right\} \in \bar{E}_{\bar{\alpha}^{*\prime}}.$$

In fact from the construction we can see that

$$\left\{ \langle \bar{\nu} \rangle \in T^{p_0^{*\prime}} \mid p_1'(\bar{\nu}) = \emptyset \right\} \in \bar{E}_{\bar{\alpha}^{*\prime}}.$$

So we really have

$$A = \left\{ \langle \bar{\nu} \rangle \in T^{p_0^{*\prime}} \mid \forall q \leq^* (p_0^{*\prime})_{\langle \bar{\nu} \rangle}\ q \notin D \right\} \in \bar{E}_{\bar{\alpha}^{*\prime}}$$

and the completion is quite easy now, we set

$$T^{p^*} = T^{p^{*\prime}} \restriction A,$$
$$p^* = p_0^{*\prime} \cup \left\{ T^{p^*} \right\}.$$

$\square$

Claim 6.5. *Let D be dense open in $P_{\bar{E}}/P_{\bar{\epsilon}}$, $p = p_0 \in P_{\bar{E}}/P_{\bar{\epsilon}}$. Then there is $p^* \leq^* p$ such that one and only one of the following is true:*
1. *There are $n < \omega$, $S \subseteq T^{p^*} \restriction [V_\kappa]^n$ such that*
   (1.1) *$\forall k < n \exists \xi < l(\bar{E})\ \mathrm{Suc}_S(\langle \bar{\nu}_1, \ldots, \bar{\nu}_k \rangle) \in E_{\mathrm{mc}(p^*)}(\xi)$,*
   (1.2) *$\forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in S\ (p^*)_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle} \in D$,*
2. *$\forall n < \omega\ \forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in T^{p^*} \forall q \leq^* (p^*)_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle}\ q \notin D$.*

Proof. Let $p^0 = p$.
Generate $p^{n+1} \leq^* p^n$ by invoking 6.4 for $n + 1$ levels.
Take $\forall n < \omega\ p^* \leq p^n$. $\square$

Claim 6.6. *Let D be dense open in $P_{\bar{E}}/P_{\bar{\epsilon}}$, $p = p_0 \in P_{\bar{E}}/P_{\bar{\epsilon}}$. Then there are $n < \omega$, $p^* \leq^* p$, $S \subseteq T^{p^*} \restriction [V_\kappa]^n$ such that*
1. *$\forall k < n \exists \xi < l(\bar{E})\ \mathrm{Suc}_S(\langle \bar{\nu}_1, \ldots, \bar{\nu}_k \rangle) \in E_{\mathrm{mc}(p^*)}(\xi)$,*
2. *$\forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in S\ (p^*)_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle} \in D$.*

Proof. Towards a contradiction, let us assume that the conclusion is false. That means that for all $p^* \leq^* p$, for all $n < \omega$, for all $S \subseteq T^{p^*} \restriction [V_\kappa]^n$ such that

$$\forall k < n \forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_k \rangle \in S \exists \xi < l(\bar{E})\ \mathrm{Suc}_S(\langle \bar{\nu}_1, \ldots, \bar{\nu}_k \rangle) \in E_{\mathrm{mc}(p^*)}(\xi)$$

we have

$$\exists \langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in S\ (p^*)_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle} \notin D.$$

We construct a $\leq^*$-decreasing sequence as follows: We set $p^0 = p$. We construct p^{n+1} from p^n using 6.4 for $n + 1$ levels. Due to our assumption we get

$$\forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in T^{p^n} \forall q \leq^* (p^n)_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle}\ q \notin D.$$

We choose $p^{*\prime}$ such that $\forall n < \omega\ p^{*\prime} \leq^* p^n$ we get

$$\forall n < \omega \forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in T^{p_0^{*\prime}} \forall q \leq^* (p^{*\prime})_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle}\ q \notin D.$$

Construct tree T from $T^{p^{*\prime}}$ using 3.10. Let us call p^* the condition $p^{*\prime}$ with T substituted for $T^{p^{*\prime}}$. Now if we have

$$q \leq p^*$$

then there is $\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in T^{p^{*'}}$ such that

$$q \leq^* (p^{*'})_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle}.$$

Hence

$$q \notin D.$$

However, D is dense. Contradiction.   □

Claim 6.7. *Let D be dense open in $P_{\bar{E}}$, $p = p_1 \frown p_0 \in P_{\bar{E}}$. Then there is $p^* \leq^* p$ such that*

$$\exists S^1 \ \exists n_1 \forall \langle \bar{\nu}_{1,1}, \ldots, \bar{\nu}_{1,n_1} \rangle \in S^1 \ \ldots \exists S^0 \ \exists n_0 \forall \langle \bar{\nu}_{0,1}, \ldots, \bar{\nu}_{0,n_0} \rangle \in S^0$$
$$(p_1^*)_{\langle \bar{\nu}_{1,1}, \ldots, \bar{\nu}_{1,n_1} \rangle} \frown \cdots \frown (p_0^*)_{\langle \bar{\nu}_{0,1}, \ldots, \bar{\nu}_{0,n_0} \rangle} \in D$$

where

1. $S^i \subseteq T^{p_i^*} \restriction [V_\kappa]^{n_i}$,
2. $\forall l < n_i \, \forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_l \rangle \in S^i \, \exists \xi \ \mathrm{Suc}_{S^i}(\langle \bar{\nu}_1, \ldots, \bar{\nu}_l \rangle) \in E_{\mathrm{mc}(p_i^*)}(\xi)$.

Proof. Let $\bar{\epsilon}$ be such that $p_1 \in P_{\bar{\epsilon}}$. We prove that there are $n < \omega$, $p_0^* \leq^* p_0$, $q_1 \leq p_1$, $S \subseteq T^{p^*} \restriction [V_\kappa]^n$ such that

1. $\forall k < n \, \exists \xi < l(\bar{E}) \ \mathrm{Suc}_S(\langle \bar{\nu}_1, \ldots, \bar{\nu}_k \rangle) \in E_{\mathrm{mc}(p^*)}(\xi)$,
2. $\forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in S \ q_1 \frown (p^*)_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle} \in D$.

Set

$$E = \left\{ r \in P_{\bar{E}} \mid \exists q_1 \ q_1 \frown r \in D, q_1 \leq p_1 \right\}.$$

This E is dense open in $P_{\bar{E}}/P_{\bar{\epsilon}}$: Let $r \in P_{\bar{E}}/P_{\bar{\epsilon}}$. Then $p_1 \frown r \in P_{\bar{E}}$. By density of D, there is $q_1 \frown s \in D$ such that $q_1 \leq p_1$, $s \leq r$. By the definition of E, $s \in E$. Hence E is dense. Openness of E is immediate from openness of D.

By 6.6 there are $p_0^* \leq^* p_0$, S', $n < \omega$ such that

1. $\forall k < n \, \exists \xi < l(\bar{E}) \ \mathrm{Suc}_{S'}(\langle \bar{\nu}_1, \ldots, \bar{\nu}_k \rangle) \in E_{\mathrm{mc}(p^*)}(\xi)$,
2. $\forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in S' \ (p_0^*)_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle} \in E$.

This means that $\forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in S'$ there is $q_1(\bar{\nu}_1, \ldots, \bar{\nu}_n) \leq p_1$ such that

$$\forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in S' \ q_1(\bar{\nu}_1, \ldots, \bar{\nu}_n) \frown (p_0^*)_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle} \in D.$$

As $|P_{\bar{\epsilon}}| < \kappa$, $q_1(\bar{\nu}_1, \ldots, \bar{\nu}_n)$ is in fact almost always constant. Hence, by shrinking S' to S and letting q_1 be this constant value, we get

$$\forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in S \ q_1 \frown (p_0^*)_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle} \in D.$$

With this, we finished the first part of the proof. We use this claim for all conditions in $P_{\bar{\epsilon}}$.

Let $P_{\bar{\epsilon}} = \left\{ p_1^\zeta \mid \zeta < \lambda \right\}$ where $\lambda < \kappa$.

We construct by induction a $\leq^*$-decreasing sequence $\langle p_0^\zeta \mid \zeta < \lambda \rangle$. Set

$$p_0^0 = p_0.$$

Assume we have constructed $\langle p_0^\zeta \mid \zeta < \zeta_0 \rangle$.

ζ_0 is limit: Choose $p_0^{\zeta_0} \leq^* p_0^\zeta$ for all $\zeta < \zeta_0$.

$\zeta_0 = \zeta + 1$: Use the first part of the proof on $p_1^\zeta \frown p_0^\zeta$ to construct $p_0^{\zeta_0}$.

When the induction terminates we have $\langle p_0^\zeta \mid \zeta < \lambda \rangle$. Choose

$$\forall \zeta < \lambda \ p_0^* \leq^* p_0^\zeta.$$

Let

$$D_{\bar{\epsilon}} = \left\{ q_1 \in P_{\bar{\epsilon}} \mid \exists n\, \exists S\, \forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in S\ q_1 \frown (p_0^*)_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle} \in D \right\}.$$

$D_{\bar{\epsilon}}$ is dense open: Let $q_1 \in P_{\bar{\epsilon}}$. Then there is ζ such that $q_1 = p_1^{\zeta}$. By the induction we have that there are n, S, $r_1 \leq q_1$ such that

$$\forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in S\ r_1 \frown (p_0^{\zeta_0})_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle} \in D.$$

By openness of D we get

$$\forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in S\ r_1 \frown (p_0^*)_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle} \in D.$$

Hence

$$r_1 \in D_{\bar{\epsilon}}.$$

As $D_{\bar{\epsilon}}$ is dense open we can use 6.6. Hence there are $p_1^* \leq p_1$, S^1, n_1 such that

$$\forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_{n_1} \rangle \in S^1\ (p_1^*)_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_{n_1} \rangle} \in D_{\bar{\epsilon}}.$$

This means that

$$\forall \langle \bar{\nu}_{1,1}, \ldots, \bar{\nu}_{1,n_1} \rangle \in S^1\ \exists S\, \exists n\, \forall \langle \bar{\nu}_{0,1}, \ldots, \bar{\nu}_{0,n} \rangle \in S$$
$$(p_1^*)_{\langle \bar{\nu}_{1,1}, \ldots, \bar{\nu}_{1,n_1} \rangle} \frown (p_0^*)_{\langle \bar{\nu}_{0,1}, \ldots, \bar{\nu}_{0,n} \rangle} \in D,$$

which is what we need to prove. □

Finally, we add the last touch.

Proof of 6.1. The proof is done by induction on k. The case $k = 1$ is 6.7. We assume, then, that the theorem is proved for k and prove it for $k + 1$.

Then $p = p_{k+1} \frown p_k \frown \cdots p_0$. Let $\bar{\epsilon}$ be such that $p_{k+1} \in P_{\bar{\epsilon}}$. We just repeat the proof of 6.7 with $P_{\bar{\epsilon}}$ and use the induction hypothese to conclude the proof. □

7. Prikry's condition

Theorem 7.1. *Let $p \in P_{\bar{E}}$ and σ a formula in the forcing language. Then there is $p^* \leq p$ such that $p^* \parallel \sigma$.*

Proof. The set $\{ q \in P_{\bar{E}} \mid q \parallel \sigma \}$ is dense open. Assuming $p = p_k \frown \cdots \frown p_0$ and using 6.1 we get that there is $q \leq^* p$ such that

$$\exists S^k\, \exists n_k \forall \langle \bar{\nu}_{k,1}, \ldots, \bar{\nu}_{k,n_k} \rangle \in S^k \ldots \exists S^0\, \exists n_0 \forall \langle \bar{\nu}_{0,1}, \ldots, \bar{\nu}_{0,n_0} \rangle \in S^0$$
$$(p_k^*)_{\langle \bar{\nu}_{k,1}, \ldots, \bar{\nu}_{k,n_k} \rangle} \frown \cdots \frown (p_0^*)_{\langle \bar{\nu}_{0,1}, \ldots, \bar{\nu}_{0,n_0} \rangle} \parallel \sigma.$$

Recall that we really should write

$$S^{k-1}(\bar{\nu}_{k,1}, \ldots, \bar{\nu}_{k,n_k}),$$
$$S^{k-2}(\bar{\nu}_{k,1}, \ldots, \bar{\nu}_{k,n_k}, \bar{\nu}_{k-1,1}, \ldots, \bar{\nu}_{k-1,n_k}),$$
$$\vdots$$

In order to avoid (too much) clutter, we use the following convention in the proof. When we write

$$\vec{\nu} \in \prod_{1 \leq l \leq k} S^l$$

we mean that

$$\langle \bar{\nu}_{k,1}, \ldots, \bar{\nu}_{k,n_k} \rangle \in S^k,$$

$$\vdots$$

$$\langle \bar{\nu}_{1,1}, \ldots, \bar{\nu}_{1,n_1} \rangle \in S^1,$$

and $r(\vec{\nu})$ is

$$(q_k)_{\langle \bar{\nu}_{k,1}, \ldots, \bar{\nu}_{k,n_k} \rangle} \frown \cdots \frown (q_1)_{\langle \bar{\nu}_{1,1}, \ldots, \bar{\nu}_{1,n_1} \rangle}.$$

We start by naming q_0 as $q_0^{n_0}$ and T^{q_0} as T^{0,n_0}. For $\langle \bar{\nu}_1, \ldots, \bar{\nu}_{n_0-1} \rangle \in S^0$ set

$$A^0 = \left\{ \langle \bar{\nu}_{n_0} \rangle \in \mathrm{Suc}_{T^{0,n_0}}(\langle \bar{\nu}_1, \ldots, \bar{\nu}_{n-1} \rangle) \mid r(\vec{\nu}) \frown (q_0^{n_0})_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_{n_0} \rangle} \Vdash \sigma \right\},$$

$$A^1 = \left\{ \langle \bar{\nu}_{n_0} \rangle \in \mathrm{Suc}_{T^{0,n_0}}(\langle \bar{\nu}_1, \ldots, \bar{\nu}_{n_0-1} \rangle) \mid r(\vec{\nu}) \frown (q_0^{n_0})_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_{n_0} \rangle} \Vdash \neg\sigma \right\}.$$

Note that

$$\mathrm{Suc}_{S^0}(\langle \bar{\nu}_1, \ldots, \bar{\nu}_{n_0-1} \rangle) \subseteq A^0 \cup A^1,$$

$$A^0 \cap A^1 = \emptyset$$

Hence, there is $\xi < \mathrm{l}(\bar{E})$ such that one and only one of the following is true:

1. $A^0 \in E_{\mathrm{mc}(q_0^{n_0})}(\xi)$,
2. $A^1 \in E_{\mathrm{mc}(q_0^{n_0})}(\xi)$.

In either case, using 3.8, we can shrink $T^{0,n_0}_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_{n_0-1} \rangle}$ and get a condition q_0' such that $r(\vec{\nu}) \frown q'_{0\langle \bar{\nu}_1, \ldots, \bar{\nu}_{n_0-1} \rangle} \parallel \sigma$.

So we shrink now $T^{0,n_0}_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_{n_0-1} \rangle}$ for all $\langle \bar{\nu}_1, \ldots, \bar{\nu}_{n_0-1} \rangle \in S^0$ and we call this tree T^{0,n_0-1}. The name of the condition $q_0^{n_0}$ with T^{0,n_0-1} substituted for T^{0,n_0} is $q_0^{n_0-1}$.
$q_0^{n_0-1}$ satisfies

$$\forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_{n_0-1} \rangle \in S^0 \ r(\vec{\nu}) \frown (q_0^{n_0-1})_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_{n_0-1} \rangle} \parallel \sigma.$$

We are now in the same position as we were when setting $q_0^{n_0}$. So by repeating the above arguments we get

$$p_0 \geq^* q_0 = q_0^{n_0} \geq^* q_0^{n_0-1} \geq^* \cdots \geq^* q_0^1 \geq^* q_0^0$$

such that for each $l = n_0, n_0 - 1, \ldots, 1, 0$

$$\forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_l \rangle \in S^0 \ r(\vec{\nu}) \frown (q_0^l)_{\langle \bar{\nu}_1, \ldots, \bar{\nu}_l \rangle} \parallel \sigma.$$

Specifically we get

$$r(\vec{\nu}) \frown (q_0^0) \parallel \sigma.$$

Of course q_0^0 depends on $\vec{\nu}$. Note that we got from $q_0^{n_0}$ to q_0^0 only by shrinking the trees. So, we repeat this process for all $\vec{\nu}$ calling the resulting condition $q_0^0(\vec{\nu})$. So we have

$$\forall \vec{\nu} \in \prod_{1 \leq l \leq k} S^l \ r(\vec{\nu}) \frown q_0^0(\vec{\nu}) \parallel \sigma.$$

By setting

$$T^{p_0^*} = \bigcap_{\vec{\nu}} T^{q_0^0(\vec{\nu})}$$

and letting p_0^* be q_0 with $T^{p_0^*}$ substituted for T^{q_0} we get

$$\forall \vec{\nu} \in \prod_{1 \le l \le k} S^l \ r(\vec{\nu}) \frown p_0^* \parallel \sigma.$$

We are in the same position as in the beginning of the proof. So we can generate in the same way p_1^* from p_1 and so on until we have

$$p_k^* \frown \cdots \frown p_0^* \parallel \sigma.$$

$\square$

8. Properness

The notions $\langle N, P \rangle$-generic and properness are due to Shelah [9].

Definition 8.1. Let $N \prec H_\chi$ such that

1. $|N| = \kappa$,
2. $N \supseteq V_\kappa$,
3. $N \supseteq N^{<\kappa}$,
4. $P \in N$.

Then $p \in P$ is called $\langle N, P \rangle$-generic if

$$p \Vdash \ulcorner \forall D \in \widehat{N} \ D \text{ is dense open in } \widehat{P} \implies D \cap \widetilde{G} \cap \widehat{N} \ne \emptyset \urcorner.$$

Definition 8.2. A forcing notion P is called proper if for all $N \prec H_\chi$ such that

1. $|N| = \kappa$,
2. $N \supseteq V_\kappa$,
3. $N \supseteq N^{<\kappa}$,
4. $P \in N$,

and for all $q \in P \cap N$ there is $p \le q$ which is $\langle N, P \rangle$-generic.

Claim 8.3. *Let $p \in P_{\bar{E}}$, $N \prec H_\chi$ such that*

1. $|N| = \kappa$,
2. $N \supseteq V_\kappa$,
3. $N \supseteq N^{<\kappa}$,
4. $P_{\bar{E}} \in N$,
5. $p \in P_{\bar{E}} \cap N$.

Then there is $p^* \le^* p$ such that p^* is $\langle N, P_{\bar{E}} \rangle$-generic

Proof. Let $p = p_{k(p)} \frown \cdots \frown p_1 \frown p_0$.

Let $\langle D_\xi \mid \xi < \kappa \rangle$ be an enumeration of all dense open subsets of $P_{\bar{E}}$ which are in N. Note that for $\xi_0 < \kappa$ we have that $\langle D_\xi \mid \xi < \xi_0 \rangle \in N$.

We start now an induction on ξ in which we build

$$\langle \bar{\alpha}^\xi, u^\xi \mid \xi < \kappa \rangle.$$

The construction is done ensuring that $\langle \bar{\alpha}^\xi, u^\xi \mid \xi < \xi_0 \rangle \in N$ for all $\xi_0 < \kappa$. We start by setting

$$u^0 = p_0 \setminus \{T^{p_0}\},$$
$$\bar{\alpha}^0 = \mathrm{mc}(p_0),$$
$$T^0 = T^{p_0} {\restriction} \pi_{\bar{\alpha}^0, 0}^{-1} \left\{ \bar{\nu} \mid \kappa^0(\bar{\nu}) \text{ is inaccessible} \right\},$$

and taking an increasing enumeration in N

$$\left\{\kappa^0(\bar{\nu}) \mid \langle\bar{\nu}\rangle \in T^0\right\} = \langle\tau_\xi \mid \xi < \kappa\rangle.$$

Assume then that we have

$$\langle\bar{\alpha}^\xi, u^\xi \mid \xi < \xi_0\rangle.$$

The constructions splits now according to wether ξ_0 is limit or successor. In both cases the work is done inside N.

ξ_0 is limit: Choose $\bar{\alpha}^{\xi_0} > \bar{\alpha}^\xi$ for all $\xi < \xi_0$ and set

$$u^{\xi_0} = \bigcup_{\xi<\xi_0} u^\xi \cup \left\{\langle\bar{\alpha}^{\xi_0}, t\rangle\right\} \text{ where } \kappa^0(t) = \tau_{\xi_0}.$$

$\xi_0 = \xi + 1$: For each $\bar{\nu}_1, \ldots, \bar{\nu}_n$ such that $\kappa^0(\bar{\nu}_1) < \cdots < \kappa^0(\bar{\nu}_n) = \tau_\xi$ we set

$$S(\bar{\nu}_1, \ldots, \bar{\nu}_n) = \left(\prod_{\bar{\alpha}\in\operatorname{supp} u^\xi} \left\{\bar{\mu}_1 \mid \kappa^0(\bar{\mu}_1) = \kappa^0(\bar{\nu}_1)\right\}\right) \times$$

$$\left(\prod_{\bar{\alpha}\in\operatorname{supp} u^\xi} \left\{\bar{\mu}_2 \mid \kappa^0(\bar{\mu}_2) = \kappa^0(\bar{\nu}_2)\right\}\right) \times$$

$$\vdots$$

$$\left(\prod_{\bar{\alpha}\in\operatorname{supp} u^\xi} \left\{\bar{\mu}_n \mid \kappa^0(\bar{\mu}_n) = \kappa^0(\bar{\nu}_n)\right\}\right) \times$$

$$\left\{\langle\bar{\nu}_1, \ldots, \bar{\nu}_n\rangle\right\}.$$

Let

$$S = \bigcup_{\kappa^0(\bar{\nu}_1)<\cdots<\kappa^0(\bar{\nu}_n)=\tau_\xi} S(\bar{\nu}_1, \ldots, \bar{\nu}_n)$$

and set enumeration of S

$$S = \langle s^{\xi_0,\rho} \mid \rho < \tau_{\xi_0}\rangle.$$

We do induction on ρ which builds

$$\langle\bar{\alpha}^{\xi_0,\rho}, u_0^{\xi_0,\rho}, T_0^{\xi_0,\rho} \mid \rho < \tau_{\xi_0}\rangle,$$

from which we build $\langle\bar{\alpha}^{\xi_0}, u^{\xi_0}\rangle$. Set

$$\bar{\alpha}^{\xi_0,0} = \bar{\alpha}^\xi,$$
$$u_0^{\xi_0,0} = u_0^\xi.$$

Assume we have constructed $\langle\bar{\alpha}^{\xi_0,\rho}, u_0^{\xi_0,\rho}, T_0^{\xi_0,\rho} \mid \rho < \rho_0\rangle$.

ρ_0 is limit: Set

$$\forall\rho < \rho_0\ \bar{\alpha}^{\xi_0,\rho_0} > \bar{\alpha}^{\xi_0,\rho},$$
$$u^{\xi_0,\rho_0} = \bigcup_{\rho<\rho_0} u^{\xi_0,\rho} \cup \left\{\langle\bar{\alpha}^{\xi_0,\rho_0}, t\rangle\right\} \text{ where } \kappa^0(t) = \tau_{\xi_0}.$$

We set T^{ξ_0,ρ_0} to anything we like as we do not use it later.

$\rho_0 = \rho + 1$: Let $s^{\xi_0,\rho}(n(s) + 1) = \langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle$.

$$u'' = (u_0^{\xi_0,\rho})_{\langle s^{\xi_0,\rho} \rangle},$$
$$T_0'' = \pi_{\bar{\alpha}^\xi,\bar{\alpha}^0}^{-1}(T_{\langle \bar{\nu}_1,\ldots,\bar{\nu}_n \rangle}^0),$$
$$T_1'' = \pi_{\mathrm{mc}(u_1''),\bar{\nu}_n}^{-1}(T_{\langle \bar{\nu}_1,\ldots,\bar{\nu}_{n-1} \rangle}^0(\bar{\nu}_n)),$$
$$\vdots$$
$$T_{n-1}'' = \pi_{\mathrm{mc}(u_{n-1}''),\bar{\nu}_2}^{-1}(T_{\langle \bar{\nu}_1 \rangle}^0(\bar{\nu}_2)),$$
$$T_n'' = \pi_{\mathrm{mc}(u_n''),\bar{\nu}_1}^{-1}(T^0(\bar{\nu}_1)).$$

Take enumeration

$$\{D_\sigma \mid \sigma < \tau_\xi\} \times$$
$$\left\{ q \mid q \le p_{k(p)} \frown \cdots \frown p_1 \frown u_{n(s)}'' \cup \{T_n''\} \frown \cdots \frown u_1'' \cup \{T_1''\} \right\} =$$
$$\langle \langle E^{\xi_0,\rho_0,\zeta}, q^{\xi_0,\rho_0,\zeta} \rangle \mid \zeta < \tau_{\xi_0} \rangle.$$

We start induction on ζ. Set

$$\bar{\alpha}^{\xi_0,\rho_0,0} = \bar{\alpha}^{\xi_0,\rho},$$
$$u_0^{\xi_0,\rho_0,0} = u^{\xi_0,\rho}.$$

Assume we have constructed $\langle \bar{\alpha}^{\xi_0,\rho_0,\zeta}, u_0^{\xi_0,\rho_0,\zeta}, T_0^{\xi_0,\rho_0,\zeta} \mid \zeta < \zeta_0 \rangle$.
ζ_0 is limit:

$$\forall \zeta < \zeta_0 \ \bar{\alpha}^{\xi_0,\rho_0,\zeta_0} > \bar{\alpha}^{\xi_0,\rho_0,\zeta},$$
$$u_0^{\xi_0,\rho_0,\zeta_0} = \bigcup_{\zeta < \zeta_0} u_0^{\xi_0,\rho_0,\zeta} \cup \left\{ \langle \bar{\alpha}^{\xi_0,\rho_0,\zeta_0}, t \rangle \right\} \ \text{ where } \kappa^0(t) = \tau_{\xi_0}.$$

We set $T^{\xi_0,\rho_0,\zeta}$ to whatever we want as no use of it is made later.
$\zeta_0 = \zeta + 1$: We set

$$u'' = (u_0^{\xi_0,\rho_0,\zeta})_{s^{\xi_0,\rho}},$$
$$T_0'' = \pi_{\bar{\alpha}^{\xi_0,\rho_0,\zeta},\bar{\alpha}^0}^{-1} T_{\langle \bar{\nu}_1,\ldots,\bar{\nu}_n \rangle}^0.$$

If there is

$$u_0' \le^* u_0'' \cup \{T_0''\}$$

such that

$$q^{\xi_0,\rho_0,\zeta} \cup u_0' \in E^{\xi_0,\rho_0,\zeta}$$

then set

$$\bar{\alpha}^{\xi_0,\rho_0,\zeta_0} = \mathrm{mc}(u_0'),$$
$$u_0^{\xi_0,\rho_0,\zeta_0} = u_0^{\xi_0,\rho_0,\zeta} \cup (u_0' \setminus (u_0'' \cup \left\{ T^{u_0'} \right\})),$$
$$T^{\xi_0,\rho_0,\zeta_0} = T^{u_0'},$$

otherwise set

$$\bar{\alpha}^{\xi_0,\rho_0,\zeta_0} = \bar{\alpha}^{\xi_0,\rho_0,\zeta},$$
$$u_0^{\xi_0,\rho_0,\zeta_0} = u_0^{\xi_0,\rho_0,\zeta},$$
$$T_0^{\xi_0,\rho_0,\zeta_0} = T_0''.$$

When the induction on ζ terminates we have $\langle \bar{\alpha}^{\xi_0,\rho_0,\zeta}, u_0^{\xi_0,\rho_0,\zeta}, T^{\xi_0,\rho_0,\zeta} \mid \zeta < \tau_{\xi_0} \rangle$

We continue with the induction on ρ. We set

$$\forall \zeta < \tau_{\xi_0} \ \bar{\alpha}^{\xi_0,\rho_0} > \bar{\alpha}^{\xi_0,\rho_0,\zeta},$$
$$u_0^{\xi_0,\rho_0} = \bigcup_{\zeta < \tau_{\xi_0}} u_0^{\xi_0,\rho_0,\zeta} \cup \left\{ \langle \bar{\alpha}^{\xi_0,\rho_0}, t \rangle \right\} \ \text{where } \kappa^0(t) = \tau_{\xi_0}.$$

When the induction on ρ terminates we have $\langle \bar{\alpha}^{\xi_0,\rho}, u_0^{\xi_0,\rho}, T^{\xi_0,\rho} \mid \rho < \tau_{\xi_0} \rangle$. We continue with the induction on ξ. We set

$$\forall \rho < \tau_{\xi_0} \ \bar{\alpha}^{\xi_0} > \bar{\alpha}^{\xi_0,\rho},$$
$$u_0^{\xi_0} = \bigcup_{\rho < \tau_{\xi_0}} u_0^{\xi_0,\rho} \cup \left\{ \langle \bar{\alpha}^{\xi_0}, t \rangle \right\} \ \text{where } \kappa^0(t) = \tau_{\xi_0}.$$

When the induction on ξ terminates we have $\langle \bar{\alpha}^{\xi}, u_0^{\xi} \mid \xi < \kappa \rangle$. We note that this sequence is not in N. Let

$$\forall \xi < \kappa \ \bar{\alpha}^* > \bar{\alpha}^{\xi},$$
$$p_0^* = \bigcup_{\xi < \kappa} u_0^{\xi} \cup \left\{ \langle \bar{\alpha}^*, t \rangle \right\} \ \text{where } \kappa^0(t) = \max p_0^0.$$

We construct a series of trees, R^n, and $T^{p_0^*}$ is $\bigcap_{n < \omega} R^n$.

$$\mathrm{Lev}_0(R^0) = \pi_{\bar{\alpha}^*,\bar{\alpha}^0}^{-1} \mathrm{Lev}_0(T^0).$$

Let us consider $\langle \bar{\nu}_1 \rangle \in \mathrm{Lev}_0(R^0)$. There is ξ such that $\kappa^0(\bar{\nu}_1) = \tau_{\xi}$. We set

$$s(0) = \left\{ \langle \bar{\alpha}, \pi_{\bar{\alpha}^*,\bar{\alpha}}(\bar{\nu}_1) \rangle \mid \bar{\alpha} \in \mathrm{supp} \, p_0^* \right\},$$
$$s(1) = \left\{ \langle \pi_{\bar{\alpha}^*,\bar{\alpha}^0}(\bar{\nu}_1) \rangle \right\}.$$

Let $\xi_0 = \xi + 1$. By our construction there is ρ such that

$$(u_0^{\xi_0,\rho_0})_{\langle s \rangle} = (u_0^{\xi_0,\rho_0})_{\langle s^{\xi_0,\rho} \rangle},$$

where $\rho_0 = \rho + 1$. We set

$$R^1_{\langle \bar{\nu}_1 \rangle} = \pi_{\bar{\alpha}^*,\bar{\alpha}^{\xi_0,\rho_0}}^{-1}(T^{\xi_0,\rho_0}) \cap \pi_{\bar{\alpha}^*,\bar{\alpha}^0}^{-1}(T^0_{\langle \pi_{\bar{\alpha}^*,\bar{\alpha}^0}(\bar{\nu}_1) \rangle}).$$

Assume that we have constructed R^n. We set the first n levels of R^{n+1} to be the same as the first n levels of R^n and we complete the tree as follows. Let us consider $\langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in R^n$. There is ξ such that $\kappa^0(\bar{\nu}_n) = \tau_{\xi}$. We set s as folows

$$\forall 1 \le k \le n \ s(k) = \left\{ \langle \bar{\alpha}, \pi_{\bar{\alpha}^*,\bar{\alpha}}(\bar{\nu}_k) \rangle \mid \bar{\alpha} \in \mathrm{supp} \, p_0^* \right\},$$
$$s(n+1) = \left\{ \langle \pi_{\bar{\alpha}^*,\bar{\alpha}^0}(\bar{\nu}_1), \ldots, \pi_{\bar{\alpha}^*,\bar{\alpha}^0}(\bar{\nu}_n) \rangle \right\}.$$

Let $\xi_0 = \xi + 1$. By our construction there is ρ such that

$$(u_0^{\xi_0,\rho_0})_{\langle s \rangle} = (u_0^{\xi_0,\rho_0})_{\langle s^{\xi_0,\rho} \rangle},$$

where $\rho_0 = \rho + 1$. We set

$$R^{n+1}_{\langle \bar{\nu}_1,\ldots,\bar{\nu}_n \rangle} = \pi^{-1}_{\bar{\alpha}^*,\bar{\alpha}\xi_0,\rho_0}(T^{\xi_0,\rho_0}) \cap \pi^{-1}_{\bar{\alpha}^*,\bar{\alpha}^0}(T^0_{\langle \pi_{\bar{\alpha}^*,\bar{\alpha}^0}(\bar{\nu}_1),\ldots,\pi_{\bar{\alpha}^*,\bar{\alpha}^0}(\bar{\nu}_n) \rangle}).$$

After ω stages we set

$$T^{p_0^*} = \bigcap_{n<\omega} R^n.$$

We finish the construction by setting

$$p^* = p_{k(p)} \frown \cdots \frown p_1 \frown p_0^*.$$

We show that p^* is as required.

Let G be $P_{\bar{E}}$-generic such that $p^* \in P_{\bar{E}}$. Let $D \in N$ be dense open in $P_{\bar{E}}$. We want to show that $D \cap G \cap N \neq \emptyset$.

Choose $q \frown r_0 \in D \cap G$ such that

$$r_0 \leq^* p_0'',$$
$$q \leq p_{k(p)} \frown \cdots \frown p_1 \frown p_n'' \frown \cdots \frown p_1'',$$

where

$$\langle \bar{\nu}_1,\ldots,\bar{\nu}_n \rangle \in \mathrm{dom}\, T^{p_0^*},$$
$$\kappa^0(\bar{\nu}_n) = \tau_\xi,$$
$$D \in \langle D_\zeta \mid \zeta < \tau_\xi \rangle,$$
$$p'' = (p_0^*)_{\langle \bar{\nu}_1,\ldots,\bar{\nu}_n \rangle}.$$

We set s to be

$$\forall 1 \leq k \leq n \; s(k) = \left\{ \langle \bar{\alpha}, \pi_{\bar{\alpha}^*,\bar{\alpha}}(\bar{\nu}_k) \rangle \mid \bar{\alpha} \in \mathrm{supp}\, p_0^* \right\},$$
$$s(n+1) = \left\{ \langle \pi_{\bar{\alpha}^*,\bar{\alpha}^0}(\bar{\nu}_1),\ldots,\pi_{\bar{\alpha}^*,\bar{\alpha}^0}(\bar{\nu}_n) \rangle \right\}.$$

We get that

$$(p_0^*)_{\langle \bar{\nu}_1,\ldots,\bar{\nu}_n \rangle} = (p_0^*)_{\langle s \rangle}.$$

We let $\xi_0 = \xi + 1$. Recall the enumeration of S in the construction. There is ρ such that

$$(p_0^* \setminus \{\langle \mathrm{mc}(p_0^*), (p_0^*)^{\mathrm{mc}} \rangle\})_{\langle s \rangle} \cup \{\langle \mathrm{mc}(p_0^*), (p_0^*)^{\mathrm{mc}} \rangle\} = (p_0^*)_{\langle s^{\xi_0,\rho} \rangle}.$$

We let $\rho_0 = \rho + 1$. Considering the construction of $T^{p_0^*}$, we see that

$$T^{p_0^*}_{\langle \bar{\nu}_1,\ldots,\bar{\nu}_n \rangle} \leq T^{\xi_0,\rho_0}$$

hence

$$(p_0^*)_{\langle s \rangle} \leq^* (u_0^{\xi_0,\rho_0})_{\langle s^{\xi_0,\rho_0} \rangle}.$$

We note that

$$\forall 1 \leq k \leq n \; p_k'' = ((p_0^*)_{\langle s \rangle})_k.$$

Recalling that q was chosen so that

$$q \leq p_{k(p)} \frown \cdots \frown p_1 \frown p_n'' \frown \cdots \frown p_1''$$

we conclude that there is ζ such that $q = q^{\xi_0,\rho_0,\zeta}$ and $D = E^{\xi_0,\rho_0,\zeta}$. That is

$$E^{\xi_0,\rho_0,\zeta} \ni q^{\xi_0,\rho_0,\zeta} \frown r_0 \leq^* q^{\xi_0,\rho_0,\zeta} \frown ((u^{\xi_0,\rho_0,\zeta})_{\langle s^{\xi_0,\rho} \rangle})_0.$$

Note that this is an answer to the question we asked in the construction. Hence, due to elementarity of N, there was such a condition in N. Hence

$$q^{\xi_0,\rho_0,\zeta} \frown ((u^{\xi_0,\rho_0,\zeta})_{\langle s^{\xi_0,\rho} \rangle})_0 \in D \cap N.$$

The last point to note is that

$$q^{\xi_0,\rho_0,\zeta} \frown ((u^{\xi_0,\rho_0,\zeta})_{\langle s^{\xi_0,\rho} \rangle})_0 \geq^* .q^{\xi_0,\rho_0,\zeta} \frown p_0'' \geq^* q \frown r_0 \in G$$

Hence

$$q^{\xi_0,\rho_0,\zeta} \frown ((u^{\xi_0,\rho_0,\zeta})_{\langle s^{\xi_0,\rho} \rangle})_0 \in G.$$

$\square$

Corollary 8.4. $P_{\bar{E}}$ is proper.

9. CARDINALS IN $V^{P_{\bar{E}}}$

Lemma 9.1. κ^+ remains a cardinal in $V^{P_{\bar{E}}}$.

Proof. The proof really has no connection to the specific structure of $P_{\bar{E}}$. It is an exercise in properness.

Let

$$p \Vdash \ulcorner \widetilde{f} {:} \widehat{\kappa} \to \widehat{\kappa^+} \urcorner.$$

Choose χ large enough so that H_χ contains everything we are interested in. Take $N \prec H_\chi$ such that

1. $p, P_{\bar{E}}, \widetilde{f} \in N$,
2. $|N| = \kappa$,
3. $N \supseteq V_\kappa$,
4. $N \supseteq N^{<\kappa}$.

By 8.4 there is $q \leq p$ which is $\langle N, P_{\bar{E}} \rangle$-generic. Let us set

$$\lambda = N \cap \kappa^+,$$

where λ is an ordinal $< \kappa^+$.

Let G be $P_{\bar{E}}$-generic with $q \in G$. The $\langle N, P_{\bar{E}} \rangle$-genericity ensures us that for all $\xi < \kappa$ $\widetilde{f}(\xi)^{N[G]} \in N$ and $\widetilde{f}(\xi)^{N[G]} = \widetilde{f}(\xi)^{H_\chi[G]}$. Hence ran $\widetilde{f}^{V[G]} \subseteq \lambda$. That is

$$q \Vdash \ulcorner \widetilde{f} \text{ is bounded in } \kappa^+ \urcorner.$$

$\square$

Lemma 9.2. No cardinals $> \kappa$ are collapsed by $P_{\bar{E}}$.

Proof. κ^+ is not collapsed by 9.1. No cardinals $\geq \kappa^{++}$ are collapsed as $P_{\bar{E}}$ satisfies κ^{++}-c.c. $\square$

Lemma 9.3. Let $\xi < \kappa$ and ζ the ordinal such that $\kappa^0(\bar{E}_G(\zeta)) \leq \xi < \kappa^0(\bar{E}_G(\zeta+1))$. Then $\mathcal{P}(\xi) \cap V[G] = \mathcal{P}(\xi) \cap V[G{\restriction}\zeta]$.

Proof. Take $p = p_n \frown \cdots \frown p_{k+1} \frown p_k \frown \cdots \frown p_0 \in G$ such that $\bar{E}(p_{k+1}) = \bar{E}_G(\zeta)$, $\bar{E}(p_k) = \bar{E}_G(\zeta+1)$. We know that $V[G] = V[G/p]$. So we work in $P_{\bar{E}}/p$. Set $p^l = p_n \frown \cdots \frown p_{k+1}$, $p^h = \{\langle \bar{E}_G(\zeta), \emptyset \rangle\} \frown p_k \frown \cdots \frown p_0$. Then $P_{\bar{E}}/p = P_{\bar{E}}{\restriction}p^l \times P_{\bar{E}}{\restriction}p^h$. Note that $\langle P_{\bar{E}}/p^h, \leq^* \rangle$ is $\kappa^0(\bar{E}_G(\zeta+1))$-closed. In particular it is ξ^+-closed.

Let $A \in V[G]$, $A \subseteq \xi$. Choose $\widetilde{A}$, a canonical $P_{\bar{E}}/p$-name for A. Let $q \in P_{\bar{E}}/p^h$. By induction we construct $\langle q_\tau \mid \tau < \xi \rangle$ satisfying

1. $\tau_0 < \tau_1 \implies q_{\tau_1} \leq^* q_{\tau_0}$,
2. $q_\tau \parallel \ulcorner \widehat{\tau} \in \widetilde{A} \urcorner$.

Choose $q_\xi \leq^* q_\tau$ for all $\tau < \xi$.

By density argument we can construct, $\widetilde{B}$, a $P{\restriction}p^l$-name such that $A = \widetilde{A}[G] = \widetilde{B}[G{\restriction}p^l]$. $\square$

Corollary 9.4. *No carindals $\leq \kappa$ are collapsed by $P_{\bar{E}}$.*

Proof. Let $G \subseteq P_{\bar{E}}$ be generic. Assume $\lambda < \kappa$ is a collapsed cardinal. Let $\mu = |\lambda|^{V[G]}$. We have $\mu < \lambda$, and there is $A \in \mathcal{P}(\mu)^{V[G]}$ which codifies the order type λ. Let ζ be the unique ordinal such that $\kappa^0(\bar{E}_G(\zeta)) \leq \mu < \kappa^0(\bar{E}_G(\zeta + 1))$. By 9.3 $A \in V[G{\restriction}\zeta]$. Hence λ is collapsed already in $V[G{\restriction}\zeta]$. However, by 9.2, $P_{\bar{E}_G(\zeta)}$ collapses no cardinals above $\kappa^0(\bar{E}_G(\zeta))$. Contradiction.

So, no cardinal $< \kappa$ is collapsed. As κ is a limit of cardinals which are not collapsed, it is not collapsed. $\square$

We have just shown

Theorem 9.5. *No cardinals are collapsed in $V^{P_{\bar{E}}}$.*

10. PROPERTIES OF κ IN $V^{P_{\bar{E}}}$

Theorem 10.1. *If $\mathrm{l}(\bar{E}) = \kappa^+$ then $V^{P_{\bar{E}}} \vDash \ulcorner \kappa$ is regular$\urcorner$.*

Proof. Let $\lambda < \kappa$, $\widetilde{f}$ be such that

$$\Vdash_{P_{\bar{E}}} \ulcorner \widetilde{f}{:}\widehat{\lambda} \to \widehat{\kappa} \urcorner.$$

Let

$$D_0 = \left\{ p \mid \exists i \ p \Vdash \ulcorner \widetilde{f}(0) = \widehat{i} \urcorner \right\}.$$

As D_0 is a dense open set we can invoke 6.1 to get p'^0, n_0, $S^0 \subset T^{p'^0}$ such that

$$\forall k < n_0 \, \forall \langle \nu_1, \ldots, \nu_k \rangle \in S'^0 \, \exists \xi < \mathrm{l}(\bar{E}) \ \mathrm{Suc}_S(\langle \nu_1, \ldots, \nu_k \rangle) \in E_{\mathrm{mc}(p'^0)}(\xi),$$

$$\forall \langle \nu_1, \ldots, \nu_{n_0} \rangle \in S'^0 \ (p'^0)_{\langle \nu_1, \ldots, \nu_{n_0} \rangle} \in D_0.$$

Let us set

$$A'_0 = \left\{ (p'^0)_{\langle \nu_1, \ldots, \nu_{n_0} \rangle} \mid \langle \nu_1, \ldots, \nu_{n_0} \rangle \in S'^0 \right\}.$$

A_0 is an anti-chain. By shrinking $T^{p'^0}$ as was done in the proof of 7.1 we can make A_0 into a maximal anti-chain below p'^0. As $\lambda < \kappa$ and $\langle P_{\bar{E}}^*, \leq^* \rangle$ is κ-closed we can construct a $\leq^*$-decreasing sequence

$$p'^0 \geq^* p'^1 \geq^* \cdots \geq^* p'^\tau \geq^* \cdots \qquad \tau < \lambda.$$

and n_τ, $S'^\tau \subseteq T^{p'^\tau}$ such that

$$\forall k < n_\tau \, \forall \langle \nu_1, \ldots, \nu_k \rangle \in S'^\tau \, \exists \xi < \mathrm{l}(\bar{E}) \ \mathrm{Suc}_{S'^\tau}(\langle \nu_1, \ldots, \nu_k \rangle) \in E_{\mathrm{mc}(p'^\tau)}(\xi),$$

$$\forall \langle \nu_1, \ldots, \nu_{n_\tau} \rangle \in S'^\tau \, \exists i \ (p'^\tau)_{\langle \nu_1, \ldots, \nu_{n_\tau} \rangle} \Vdash \ulcorner \widetilde{f}(\widehat{\tau}) = \widehat{i} \urcorner,$$

and

$$A'_\tau = \left\{ (p'^\tau)_{\langle \nu_1, \ldots, \nu_{n_\tau} \rangle} \mid \langle \nu_1, \ldots, \nu_{n_\tau} \rangle \in S'^\tau \right\}$$

is a maximal anti-chain below p'^τ.

Let $p' \leq^* p'^\tau$ for all $\tau < \lambda$. We set $S^\tau = \pi^{-1}_{\mathrm{mc}(p'),\mathrm{mc}(p'^\tau)}(S'^\tau)$ and p^τ to be p' with $\pi^{-1}_{\mathrm{mc}(p'),\mathrm{mc}(p'^\tau)}(T^{p'^\tau})$ substituted for $T^{p'}$ and maybe shrunken a bit so that

$$A_\tau = \left\{ (p^\tau)_{\langle \nu_1,\ldots,\nu_{n_\tau}\rangle} \mid \langle \nu_1,\ldots,\nu_{n_\tau}\rangle \in S^\tau \right\}$$

is a maximal anti-chain below p^τ.

Let $p \leq^* p^\tau$ for all $\tau < \lambda$ and let $\widetilde{g}$ be the following $P_{\bar{E}}$-name

$$\widetilde{g} = \bigcup_{\tau < \lambda} \left\{ \langle \widehat{\langle \tau, i\rangle}, (p^\tau)_{\langle \nu_1,\ldots,\nu_{n_\tau}\rangle}\rangle \mid A_\tau \ni (p^\tau)_{\langle \nu_1,\ldots,\nu_{n_\tau}\rangle} \Vdash \ulcorner \widetilde{f}(\tau) = \widehat{i} \urcorner \right\}.$$

Then

$$p \Vdash \ulcorner \widetilde{f} = \widetilde{g} \urcorner.$$

Let P^* be the following forcing notion:

$$P^* = \left\{ q \leq_R p \mid q \in P_{\bar{E}} \right\}.$$

By 5.2 $\langle P^*, \leq_R\rangle$ is sub-forcing of $\langle P_{\bar{E}}/p, \leq\rangle$. Hence if G is $P_{\bar{E}}$-generic then $G^* = G \cap P^*$ is P^*-generic. $\widetilde{g}$ is in fact a P^*-name and as can be seen from its' definition $\widetilde{g}[G] = \widetilde{g}[G^*] \in V[G^*]$. So in order to complete the proof it is enough to show that $\Vdash_{P^*} \ulcorner \widetilde{g}$ is bounded $\urcorner$.

By 5.3 there is $r \in R_{\mathrm{mc}(p)}$ such that $P^* \simeq R_{\mathrm{mc}(p)}/r$. Now we use the following fact about Radin forcing: When the measure sequence is of length κ^+, κ is regular in the generic extension. Necessarily, $\Vdash_{P^*} \ulcorner \widetilde{g}$ is bounded $\urcorner$. $\qquad\square$

Definition 10.2. We say that $\tau < \mathrm{l}(\bar{E})$ is a repeat point of $\bar{E}$ if $P_{\bar{E}} = P_{\bar{E}\restriction\tau}$.

Note that if τ is a repeat point then $P_{\bar{E}\restriction\tau} \in M$.

Theorem 10.3. If $\bar{E}$ has a repeat point, $j'' \left\{ \widetilde{A}_\xi \mid \xi < \lambda \right\} \in M$ where $\left\{ \widetilde{A}_\xi \mid \xi < \lambda \right\}$ is an enumeration of all canonical $P_{\bar{E}}$-names of subsets of κ, then $V^{P_{\bar{E}}} \vDash \ulcorner \kappa$ is measurable $\urcorner$.

Proof. We use the usual method under these circumstances. Let τ be a repeat point of $\bar{E}$ and G be $P_{\bar{E}}$-generic over V. For the duration of this proof let us define:

- If $p = p_0 \in P^*_{\bar{E}}$ then

$$p\restriction\tau = \left\{ \langle \bar{E}_\alpha \restriction \tau, p^{\bar{E}_\alpha}\rangle \mid \bar{E}_\alpha \in \mathrm{supp}\, p \right\} \cup \{T^p\}.$$

- If $p = p_n \frown \cdots \frown p_1 \frown p_0$ then

$$p\restriction\tau = p_n \frown \cdots \frown p_1 \frown p_0\restriction\tau.$$

Let us set

$$G\restriction\tau = \{ p\restriction\tau \mid p \in G \}.$$

We note that

1. $\lambda < j(\kappa)$,
2. $M \vDash \ulcorner j(P^*_{\bar{E}})/P_{\bar{E}}$ is $j(k) - closed \urcorner$,
3. $G\restriction\tau$ is $P_{\bar{E}\restriction\tau}$-generic over M.

So, we can construct a $\leq^*$-decreasing sequence in $M[G{\restriction}\tau]$, $\langle p^\xi \mid \xi < \lambda \rangle$, such that
$$p^{\xi+1} \parallel \ulcorner \kappa \in j(\widetilde{A}_\xi) \urcorner.$$
We define an ultrafilter, U, in V by
$$\widetilde{A}_\xi[G] \in U \iff p_{\xi+1} \Vdash \ulcorner \kappa \in j(\widetilde{A}_\xi) \urcorner.$$

$\square$

The assumptions we used in this theorem are very strong. We believe that a repeat point is enough in order to get measurability.

11. What have we proved?

We can sum everything as follows:

We can control independently two properties of κ in a generic extension. The first is the size of 2^κ which is controlled by $|\bar{E}|$. The second is how 'big' we want κ to be which is controlled by $\mathrm{l}(\bar{E})$.

12. Generic by Iteration

Recall that if R is Radin forcing generated from $j{:}V \to M$ then there is τ and $G \in V$ such that G is $j_{0,\tau}(R)$-generic over M_τ.

Our original aim was to find some form of this claim for our forcing. We have a partial result in this direction. Namely, when $\mathrm{l}(\bar{E}) = 1$ we have a generic filter in V over an elementary submodel in M_ω.

In this section we assume that $\mathrm{l}(\bar{E}) = 1$.

Let us take an iteration of $j = j_{0,1}$
$$\langle \langle M_n \mid n < \omega \rangle, \langle j_{n,m} \mid n \leq m < \omega \rangle \rangle.$$
Choose χ large enough so that everything interesting is in H_χ (i.e. $P_{\bar{E}} \in H_\chi$) and set
$$\begin{aligned}
\kappa_n &= j_{0,n}(\kappa), \\
\bar{E}^n &= j_{0,n}(\bar{E}), \\
P^n &= j_{0,n}(P_{\bar{E}})(= P_{\bar{E}^n}), \\
\chi_n &= j_{0,n}(\chi).
\end{aligned}$$

Definition 12.1 (when $\mathrm{l}(\bar{E}) = 1$). We call $\langle N, p \rangle$ a k-pair if
1. $M_k \vDash \ulcorner N \prec H_{\chi_k}^{M_k} \urcorner$,
2. $M_k \vDash \ulcorner |N| = \kappa_k \urcorner$,
3. $M_k \vDash \ulcorner N \supseteq V_{\kappa_k}^{M_k} \urcorner$,
4. $M_k \vDash \ulcorner N \supseteq N^{<\kappa_k} \urcorner$,
5. $p \in P^{k+1} \cap j_{k,k+1}(N)$,
6. If $D \in N$, D is dense open in P^k then there are n, $S \leq T^p$ such that
$$\forall \langle \nu_1, \ldots, \nu_n \rangle \in S \ (p)_{\langle \nu_1, \ldots, \nu_n \rangle} \in j_{k,k+1}(D).$$

Claim 12.2. *Let $N \in M_k$, $k < \omega$ and $q \in j_{k,k+1}(N) \cap P^{k+1}$ such that M_k satisifes:*
1. $N \prec H_{\chi_k}^{M_k}$,
2. $|N| = \kappa_k$,
3. $N \supseteq V_{\kappa_k}^{M_k}$,

4. $N \supseteq N^{<\kappa_k}$,
5. $P^k \in N$.

Then there is $p \in j_{k,k+1}(N) \cap P^{k+1}$ *such that*

1. $p \leq^* q$,
2. $\langle N, p \rangle$ *is a* k-*pair*.

Proof. Let $\langle D_\xi^k \mid \xi < \kappa_k \rangle$ be an enumeration, in M_k, of all the dense open subsets of P^k which are in N. Set $N_{k+1} = j_{k,k+1}(N)$. As we have

$$\langle j_{k,k+1}(D_\xi^k) \mid \xi < \kappa_k \rangle \subseteq N_{k+1},$$
$$\langle j_{k,k+1}(D_\xi^k) \mid \xi < \kappa_k \rangle \in M_{k+1},$$
$$M_{k+1} \vDash \ulcorner N_{k+1} \supseteq N_{k+1}^{<\kappa_{k+1}} \urcorner,$$

we get that

$$\langle j_{k,k+1}(D_\xi^k) \mid \xi < \kappa_k \rangle \in N_{k+1}.$$

Starting with q we construct in N_{k+1} a $\leq^*$-decreasing sequence $\langle p_\xi \mid \xi < \kappa_k \rangle$ using 6.1. Note that we have no problem at the limit stages as $N_{k+1} \vDash \ulcorner \langle P^{k+1}, \leq^* \rangle$ is κ_{k+1}-closed$\urcorner$. Choose now $p \in P^{k+1} \cap N_{k+1}$ such that $\forall \xi < \kappa_k \; p \leq^* p_\xi$. We get that $\langle N, p \rangle$ is a k-pair. $\qquad \square$

Definition 12.3. We call $\langle \vec{N}, \vec{p} \rangle$ a P^ω-generic approximation sequence if

$$\langle \vec{N}, \vec{p} \rangle = \langle \langle N_k, p^{k+1} \rangle \mid k_0 \leq k < \omega \rangle$$

such that for all $k_0 \leq k < \omega$

1. $M_k \vDash \ulcorner N_k \prec H_{\chi_k}^{M_k} \urcorner$,
2. $M_k \vDash \ulcorner |N_k| = \kappa_k \urcorner$,
3. $M_k \vDash \ulcorner N_k \supseteq V_{\kappa_k}^{M_k} \urcorner$,
4. $M_k \vDash \ulcorner N_k \supseteq N^{<\kappa_k} \urcorner$,
5. $P^k \in N_k$,
6. $\langle N_k, p^{k+1} \rangle$ is a k-pair,
7. $j_{k_1,k_2}(N_{k_1}) = N_{k_2}$,
8. $p^{k+2} \leq^* (j_{k+1,k+2}(p^{k+1}))_{\langle \mathrm{mc}(p^{k+1}) \rangle}$.

Definition 12.4. Let $\langle \vec{N}, \vec{r} \rangle$ be a P^ω-generic approximating sequence. Then

$$G(\langle \vec{N}, \vec{r} \rangle) = \left\{ p \in P^\omega \mid \exists k \; j_{k,\omega}(r^k) \leq^* p \right\}.$$

Claim 12.5. Let $k_0 < \omega$, $q \in P^{k_0} \cap N_{k_0}$ and assume for all $k_0 \leq k < \omega$

1. $P^k \in N_k$,
2. $j_{k,k+1}(N_k) = N_{k+1}$,
3. $M_k \vDash \ulcorner N_k \prec H_{\chi_k}^{M_k} \urcorner$,
4. $M_k \vDash \ulcorner |N_k| = \kappa_k \urcorner$,
5. $M_k \vDash \ulcorner N_k \supseteq V_{\kappa_k}^{M_k} \urcorner$,
6. $M_k \vDash \ulcorner N_k \supseteq N_k^{<\kappa_k} \urcorner$.

Then there is a P^ω-*generic approximating sequence*

$$\langle \vec{N}, \vec{p} \rangle = \langle \langle N_k, p^{k+1} \rangle \mid k_0 \leq k < \omega \rangle$$

such that $p^{k_0+1} \leq^* j_{k_0,k_0+1}(q)$.

Proof. We construct the p^{k+1} by induction. We set $p^{k_0} = q$.

Assume that we have constructed $\langle p^{k'} \mid k_0 \leq k' \leq k \rangle$. We set $q^{k+1} = j_{k,k+1}(p^k)_{\langle \mathrm{mc}(p^k) \rangle}$. Invoke 12.2 to get $p^{k+1} \leq^* q^{k+1}$ such that $\langle N_k, p^{k+1} \rangle$ is a k-pair.

When the induction terminates we have $\langle \langle N_k, p^{k+1} \rangle \mid k_0 \leq k < \omega \rangle$ as required. $\square$

Claim 12.6. *Let $k_0 < \omega$ and assume*

1. $M_{k_0} \vDash \ulcorner N_{k_0} \prec H^{M_{k_0}}_{\chi_{k_0}} \urcorner$,
2. $M_{k_0} \vDash \ulcorner |N_{k_0}| = \kappa_{k_0} \urcorner$,
3. $M_{k_0} \vDash \ulcorner N_{k_0} \supseteq V^{M_{k_0}}_{\kappa_{k_0}} \urcorner$,
4. $M_{k_0} \vDash \ulcorner N_{k_0} \supseteq N^{<\kappa_{k_0}}_{k_0} \urcorner$,
5. $P^{k_0} \in N_{k_0}$,
6. $q \in P^{k_0} \cap N_{k_0}$.

Then there is a P^ω-generic approximating sequence

$$\langle \vec{N}, \vec{p} \rangle = \langle \langle N_k, p^{k+1} \rangle \mid k_0 \leq k < \omega \rangle$$

such that $p^{k_0+1} \leq^* j_{k_0,k_0+1}(q)$.

Proof. We set $N_k = j_{k_0,k}(N_{k_0})$ for all $k_0 < k < \omega$ and then we invoke 12.5. $\square$

Theorem 12.7. *Assume*

1. $M_\omega \vDash \ulcorner N \prec H^{M_\omega}_{\chi_\omega} \urcorner$,
2. $M_\omega \vDash \ulcorner |N| = \kappa_\omega \urcorner$,
3. $M_\omega \vDash \ulcorner N \supseteq V^{M_\omega}_{\kappa_\omega} \urcorner$,
4. $M_\omega \vDash \ulcorner N \supseteq N^{<\kappa_\omega} \urcorner$,
5. $P^\omega \in N$,
6. $q \in P^\omega \cap N$.

Then there is, in V, a filter $G \subseteq P^\omega$ such that

1. $q \in G$,
2. $\forall D \in N$ D *is dense open in* P^ω $\implies$ $G \cap D \cap N \neq \emptyset$.

Proof. We find, $\langle \vec{N}, \vec{p} \rangle$, a P^ω-generic approximating sequence. $G(\langle \vec{N}, \vec{p} \rangle)$ is the required filter.

Find k_0 and N_{k_0}, q^{k_0} such that

$$P^{k_0} \in N_{k_0},$$
$$j_{k_0,\omega}(N_{k_0}) = N,$$
$$j_{k_0,\omega}(q^{k_0}) = q.$$

Invoke 12.6 to get from N^{k_0}, q^{k_0} a generic approximating sequence $\langle \vec{N}, \vec{p} \rangle$.

Let $D \in N$ be dense open in P^ω.

Find $k \geq k_0$ and D^k such that $j_{k,\omega}(D^k) = D$. As usual we set $D^{k+l} = j_{k,k+l}(D^k)$. By construction there is n such that

$$\forall \langle \nu_1, \ldots, \nu_n \rangle \in T^{p^{k+1}} \ (p^{k+1})_{\langle \nu_1, \ldots, \nu_n \rangle} \in D^{k+1} \cap N_{k+1},$$

which means that

$$j_{k+1,k+1+n}(p^{k+1})_{\langle \mathrm{mc}(p^{k+1}), \ldots, j_{k+1,k+n}(\mathrm{mc}(p^{k+1})) \rangle} \in D^{k+1+n} \cap N_{k+1+n}.$$

Hence

$$(12.7.1) \qquad j_{k+1,\omega}(p^{k+1})_{\langle \mathrm{mc}(p^{k+1}),\dots,j_{k+1,k+n}(\mathrm{mc}(p^{k+1}))\rangle} \in D \cap N.$$

As $\langle \vec{N}, \vec{p}\rangle$ is a P^ω-generic approximation sequence it satisfies

$$p^{k+1+n} \leq^* j_{k+1,k+1+n}(p^{k+1})_{\langle \mathrm{mc}(p^{k+1}),\dots,j_{k+1,k+1+n}(\mathrm{mc}(p^{k+1}))\rangle}.$$

Hence

$$j_{k+1+n,\omega}(p^{k+1+n}) \leq^*$$

$$j_{k+1,\omega}(p^{k+1})_{\langle \mathrm{mc}(p^{k+1}),\dots,j_{k+1,k+n}(\mathrm{mc}(p^{k+1}))\rangle},$$

giving us that

$$(12.7.2) \qquad j_{k+1,\omega}(p^{k+1})_{\langle \mathrm{mc}(p^{k+1}),\dots,j_{k+1,k+n}(\mathrm{mc}(p^{k+1}))\rangle} \in G(\langle \vec{N}, \vec{p}\rangle),$$

so $G(\langle \vec{N}, \vec{p}\rangle) \cap D \cap N \neq \emptyset$ by 12.7.1 and 12.7.2.   □

13. Concluding Remarks

1. We believe a repeat point is enough in order to get measurability of κ. We use a much stronger assumption in our proof.
2. A definition of repeat point that depends only on the extender sequence and is equivalent to the one we gave (which mentions $P_{\bar{E}}$) will probably be useful.
3. It is not completely clear what $l(\bar{E})$ should be in order to make sure that $\bar{E}$ has a repeat point.
4. A finer analysis in the case of measurability and stronger properties is needed. For example, extending the elementary embedding to the generic extension, and not just constructing a normal ultrafilter.
5. We do not know how to get a generic by iteration when $l(\bar{E}) > 1$.
6. Making this forcing more 'precise' by adding 'gentle' collapses so we get a prescribed behaviour on *all* cardinals below κ in the generic extension is in preparation.

References

[1] J. Cummings, *A Model In Which GCH Holds At Successors But Fails At Limits*, Transactions of the American Mathematical Society 329(1992), Number 1, 1–39
[2] M. Magidor, *Changing Cofinality Of Cardinals*, Fundamenta Mathematicae 99 (1978), 61–71
[3] M. Gitik and M. Magidor, *The Singular Cardinal Hypothese Revisited*, in *Set Theory of the Continuum*, H. Judah, W. Just, H. Woodin, (Eds.), Springer-Verlarg (1992), 243–278
[4] M. Gitik, *Changing Cofinalities And The Non-Stationary Ideal*, Israel Journal of Mathematics 56 (1986), 280–314
[5] W. Mitchell, *How Weak is a Closed Unbounded Ultrafilter?*, in *Logic Colloquium '80*, D. van Dalen, D. Lascar, J. Smiley, (Eds.), North-Holland Publishing Company (1982), 209–230
[6] K. Prikry, *Changing measurables into accessibles*, Diss. Math. 68 (1970), 5–52
[7] L. B. Radin, *Adding Closed Cofinal Sequences To Large Cardinals*, Annals of Mathematical Logic 22(1982), 243–261
[8] M. Segal, *On Powers Of Singular Cardinals With Cofinality $> \omega$*, Master's Thesis, The Hebrew University of Jerusalem (1995)
[9] S. Shela, *Proper Forcing*, Lecture Notes in Mathematics, Vol. 940, Springer, Berlin, 1982
[10] H. Woodin and J. Cummings, *Chapters from an unpublished book on Radin forcing*

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