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FERMIONIC FORMULAS FOR LEVEL-RESTRICTED GENERALIZED KOSTKA POLYNOMIALS AND COSET BRANCHING FUNCTIONS

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ABSTRACT. Level-restricted paths play an important rôle in crystal theory. They correspond to certain highest weight vectors of modules of quantum affine algebras. We show that the recently established bijection between Littlewood–Richardson tableaux and rigged configurations is well-behaved with respect to level-restriction and give an explicit characterization of level-restricted rigged configurations. As a consequence a new general fermionic formula for the level-restricted generalized Kostka polynomial is obtained. Some coset branching functions of type A are computed by taking limits of these fermionic formulas.

1. INTRODUCTION

Generalized Kostka polynomials [26, 33, 35, 36, 37, 38] are q-analogues of the tensor product multiplicity

$$c_R^\lambda = \dim \mathrm{Hom}_{sl_n}(V^\lambda, V^{R_1}\otimes\cdots\otimes V^{R_L}), \tag{1.1}$$

where λ is a partition, $R=(R_1,\ldots,R_L)$ is a sequence of rectangles and V^λ is the irreducible integrable highest weight module of highest weight λ over the quantized enveloping algebra $U_q(sl_n)$. The generalized Kostka polynomials can be expressed as generating functions of classically restricted paths [30, 33, 37]. In terms of the theory of $U_q(sl_n)$-crystals [16, 17] these paths correspond to the highest weight vectors of tensor products of perfect crystals. The statistic is given by the energy function on paths.

The $U_q(sl_n)$-crystal structure can be extended to a $U_q(\widehat{sl}'_n)$-crystal structure [18]. For particular weights, the highest weight vectors of the $U_q(\widehat{sl}'_n)$-modules correspond to level-restricted paths. Hence it is natural to consider the generating functions of level-restricted paths, giving rise to level-restricted generalized Kostka polynomials which will take a lead rôle in this paper. The notion of level-restriction is also very important in the context of restricted-solid-on-solid (RSOS) models in statistical mechanics [3] and

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fusion models in conformal field theory [39]. The one-dimensional configuration sums of RSOS models are generating functions of level-restricted paths (see for example [2, 9, 14]). The structure constants of the fusion algebras of Wess–Zumino–Witten conformal field theories are exactly the level-restricted analogues of the Littlewood–Richardson coefficients in (1.1) as shown by Kac [15, Exercise 13.35] and Walton [40, 41]. q-Analogues of these level-restricted Littlewood–Richardson coefficients in terms of ribbon tableaux were proposed in ref. [10].

The generalized Kostka polynomial admits a fermionic (or quasi-particle) formula [25]. Fermionic formulas originate from the Bethe Ansatz [4] which is a technique to construct eigenvectors and eigenvalues of row-to-row transfer matrices of statistical mechanical models. Under certain assumptions (the string hypothesis) it is possible to count the solutions of the Bethe equations resulting in fermionic expressions which look like sums of products of binomial coefficients. The Kostka numbers arise in the study of the XXX model in this way [22, 23, 24]. Fermionic formulas are of interest in physics since they reflect the particle structure of the underlying model [20, 21] and also reveal information about the exclusion statistics of the particles [5, 6, 7].

The fermionic formula of the Kostka polynomial can be combinatorialized by taking a weighted sum over sets of rigged configurations [22, 23, 24]. In ref. [25] the fermionic formula for the generalized Kostka polynomial was proven by establishing a statistic-preserving bijection between Littlewood–Richardson tableaux and rigged configurations. In this paper we show that this bijection is well-behaved with respect to level-restriction and we give an explicit characterization of level-restricted rigged configurations (see Definition 5.5 and Theorem 8.2). This enables us to obtain a combinatorial formula for the level-restricted generalized Kostka polynomials as the generating function of level-restricted rigged configurations (see Theorem 5.7). As an immediate consequence this proves a new general fermionic formula for the level-restricted generalized Kostka polynomial (see Theorem 6.2 and Eq. (6.7)). Special cases of this formula were conjectured in refs. [8, 12, 13, 27, 33, 42]. As opposed to some definitions of "fermionic formulas" the expression of Theorem 6.2 involves in general explicit negative signs. However, we would like to point out that because of the equivalent combinatorial formulation in terms of rigged configurations as given in Theorem 5.7 the fermionic sum is manifestly positive (i.e., a polynomial with positive coefficients).

The branching functions of type A can be described in terms of crystal graphs of irreducible integrable highest weight $U_q(\widehat{sl}_n)$-modules. For certain triples of weights they can be expressed as limits of level-restricted generalized Kostka polynomials. The structure of the rigged configurations allows one to take this limit, thereby yielding a fermionic formula for the corresponding branching functions (see Eq. (7.10)). The derivation of this formula requires the knowledge of the ground state energy, which is obtained from the explicit construction of certain local isomorphisms of perfect crystals (see Theorem 7.3). A more complete set of branching functions can be obtained by considering "skew" level-restricted generalized Kostka polynomials. We conjecture that rigged configurations are also well-behaved with respect to skew shapes (see Conjecture 8.3).

The paper is structured as follows. Section 2 sets out notation used in the paper. In Section 3 we review some crystal theory, in particular the definition of level-restricted paths, which are used to define the level-restricted generalized Kostka polynomials. Littlewood–Richardson tableaux and their level-restricted counterparts are defined in Section 4. The formulation of the generalized Kostka polynomials in terms of Littlewood–Richardson tableaux with charge statistic is necessary for the proof of the fermionic formula which makes use of the bijection between Littlewood–Richardson tableaux and rigged configurations. The latter are subject of Section 5 which also contains the new definition of level-restricted rigged configurations and our main Theorem 5.7. The proof of this theorem is reserved for Section 8. The fermionic formulas for the level-restricted Kostka polynomial and the type A branching functions are given in Sections 6 and 7, respectively.

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2. Notation

All partitions are assumed to have n parts, some of which may be zero. Let $R = (R_1, R_2, \dots, R_L)$ be a sequence of partitions whose Ferrers diagrams are rectangles. Let R_j have μ_j columns and η_j rows for $1 \leq j \leq L$. We adopt the English notation for partitions and tableaux. Unless otherwise specified, all tableaux are assumed to be column-strict (that is, the entries in each row weakly increase from left to right and in each column strictly increase from top to bottom).

3. Paths

The main goal of this section is to define the level-restricted generalized Kostka polynomials. These polynomials are defined in terms of certain finite $U_q(\widehat{sl}_n)$-crystal graphs whose elements are called paths. The theory of crystal graphs was invented by Kashiwara [16], who showed that the quantized universal enveloping algebras of Kac-Moody algebras and their integrable highest weight modules admit special bases whose structure at $q = 0$ is specified by a colored graph known as the crystal graph. The crystal graphs for the finite-dimensional irreducible modules for the classical Lie algebras were computed explicitly by Kashiwara and Nakashima [17]. The theory of perfect crystals gave a realization of the crystal graphs of the irreducible integrable highest weight modules for affine Kac-Moody algebras, as certain eventually periodic sequences of elements taken from finite crystal graphs

[19]. This realization is used for the main application, some new explicit formulas for coset branching functions of type A.

3.1. **Crystal graphs.** Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra for the Kac-Moody algebra $\mathfrak{g}$. Let I be an indexing set for the Dynkin diagram of $\mathfrak{g}$, P the weight lattice of $\mathfrak{g}$, P^* the dual lattice, $\{\alpha_i \mid i \in I\}$ the (not necessarily linearly independent) simple roots, $\{h_i \mid i \in I\}$ the simple coroots, and $\{\Lambda_i \mid i \in I\}$ the fundamental weights. Let $\langle \cdot\,, \cdot \rangle$ denote the natural pairing of P^* and P.

Suppose V is a $U_q(\mathfrak{g})$-module with crystal graph $\mathcal{B}$. Then $\mathcal{B}$ is a directed graph whose vertex set (also denoted $\mathcal{B}$) indexes a basis of weight vectors of V, and has directed edges colored by the elements of the set I. The edges may be viewed as a combinatorial version of the action of Chevalley generators. This graph has the property that for every $b \in \mathcal{B}$ and $i \in I$, there is at most one edge colored i entering (resp. leaving) b. If there is an edge $b \to b'$ colored i, denote this by $f_i(b) = b'$ and $e_i(b') = b$. If there is no edge colored i leaving b (resp. entering b') then say that $f_i(b)$ (resp. $e_i(b')$) is undefined. The f_i and e_i are called Kashiwara lowering and raising operators. Define $\phi_i(b)$ (resp. $\epsilon_i(b)$) to be the maximum $m \in \mathbb{N}$ such that $f_i^m(b)$ (resp. $e_i^m(b)$) is defined. There is a weight function $\mathrm{wt} : \mathcal{B} \to P$ that satisfies the following properties:

$$\begin{aligned} \mathrm{wt}(f_i(b)) &= \mathrm{wt}(b) - \alpha_i, \\ \mathrm{wt}(e_i(b)) &= \mathrm{wt}(b) + \alpha_i, \\ \langle\, h_i\,, \mathrm{wt}(b)\,\rangle &= \phi_i(b) - \epsilon_i(b). \end{aligned} \tag{3.1}$$

$\mathcal{B}$ is called a P-weighted I-crystal.

Let $P^+ = \{\Lambda \in P \mid \langle\, h_i\,, \Lambda\,\rangle \geq 0,\ \forall i \in I\}$ be the set of dominant integral weights. For $\Lambda \in P^+$ denote by $\mathbb{V}(\Lambda)$ the irreducible integrable highest weight $U_q(\mathfrak{g})$-module of highest weight Λ. Let $\mathbb{B}(\Lambda)$ be its crystal graph.

Say that an element $b \in \mathcal{B}$ of the P-weighted I-crystal $\mathcal{B}$ is a highest weight vector if $\epsilon_i(b) = 0$ for all $i \in I$.

Let u_Λ be the highest weight vector in $\mathbb{B}(\Lambda)$. By (3.1), for all $i \in I$,

$$\begin{aligned} \epsilon_i(u_\Lambda) &= 0, \\ \phi_i(u_\Lambda) &= \langle\, h_i\,, \Lambda\,\rangle. \end{aligned} \tag{3.2}$$

Let $\mathcal{B}'$ be the crystal graph of a $U_q(\mathfrak{g})$-module V'. A morphism of P-weighted I-crystals is a map $\tau : \mathcal{B} \to \mathcal{B}'$ such that $\mathrm{wt}(\tau(b)) = \mathrm{wt}(b)$ and $\tau(f_i(b)) = f_i(\tau(b))$ for all $b \in \mathcal{B}$ and $i \in I$. In particular $f_i(b)$ is defined if and only if $f_i(\tau(b))$ is.

Suppose V and V' are $U_q(\mathfrak{g})$-modules with crystal graphs $\mathcal{B}$ and $\mathcal{B}'$ respectively. Then $V \otimes V'$ admits a crystal graph denoted $\mathcal{B} \otimes \mathcal{B}'$ which is equal to the direct product $\mathcal{B} \times \mathcal{B}'$ as a set. We use the opposite of the convention used in the literature. Define

$$(3.3)\qquad f_i(b\otimes b') = \begin{cases} b\otimes f_i(b') & \text{if } \phi_i(b') > \epsilon_i(b),\\ f_i(b)\otimes b' & \text{if } \phi_i(b') \le \epsilon_i(b) \text{ and } \phi_i(b) > 0,\\ \text{undefined} & \text{otherwise.}\end{cases}$$

Equivalently,

$$(3.4)\qquad e_i(b\otimes b') = \begin{cases} e_i(b)\otimes b' & \text{if } \phi_i(b') < \epsilon_i(b),\\ b\otimes e_i(b') & \text{if } \phi_i(b') \ge \epsilon_i(b) \text{ and } \epsilon_i(b') > 0,\\ \text{undefined} & \text{otherwise.}\end{cases}$$

One has

$$(3.5)\qquad \begin{aligned} \phi_i(b\otimes b') &= \phi_i(b) + \max\{0, \phi_i(b') - \epsilon_i(b)\},\\ \epsilon_i(b\otimes b') &= \max\{0, \epsilon_i(b) - \phi_i(b')\} + \epsilon_i(b').\end{aligned}$$

Finally $\mathrm{wt} : \mathcal{B}\otimes\mathcal{B}' \to P$ is defined by $\mathrm{wt}(b\otimes b') = \mathrm{wt}_{\mathcal{B}}(b) + \mathrm{wt}_{\mathcal{B}'}(b')$ where $\mathrm{wt}_{\mathcal{B}} : \mathcal{B} \to P$ and $\mathrm{wt}_{\mathcal{B}'} : \mathcal{B}' \to P$ are the weight functions for $\mathcal{B}$ and $\mathcal{B}'$.

This construction is "associative", that is, the P-weighted I-crystals form a tensor category.

Remark 3.1. It follows from (3.4) that if $b = b_L \otimes \cdots \otimes b_1$ and $e_i(b)$ is defined, then $e_i(b) = b_L \otimes \cdots \otimes b_{j+1} \otimes e_i(b_j) \otimes b_{j-1} \otimes \cdots \otimes b_1$ for some $1 \le j \le L$.

3.2. $U_q(sl_n)$**-crystal graphs on tableaux.** Let $J = \{1, 2, \dots, n-1\}$ be the indexing set for the Dynkin diagram of type A_{n-1}, with weight lattice P_{fin}, simple roots $\{\overline{\alpha}_i \mid i \in J\}$, fundamental weights $\{\overline{\Lambda}_i \mid i \in J\}$, and simple coroots $\{h_i \mid i \in J\}$.

Let $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n) \in \mathbb{N}^n$ be a partition. There is a natural projection $\mathbb{Z}^n \to P_{\mathrm{fin}}$ denoted $\lambda \mapsto \overline{\lambda} = \sum_{i=1}^{n-1}(\lambda_i - \lambda_{i+1})\overline{\Lambda}_i$. Let $V(\overline{\lambda})$ be the irreducible integrable highest weight module of highest weight $\overline{\lambda}$ over the quantized universal enveloping algebra $U_q(sl_n)$ [17]. By abuse of notation we shall write $V^\lambda = V(\overline{\lambda})$ and denote the crystal graph of V^λ by $\mathcal{B}_\lambda$.

As a set $\mathcal{B}_\lambda$ may be realized as the set of tableaux of shape λ over the alphabet $\{1, 2, \dots, n\}$. Define the content of $b \in \mathcal{B}_\lambda$ by $\mathrm{content}(b) = (c_1, \dots, c_n) \in \mathbb{N}^n$ where c_j is the number of times the letter j appears in b. The weight function $\mathrm{wt} : \mathcal{B}_\lambda \to P_{\mathrm{fin}}$ is given by sending b to the image of $\mathrm{content}(b)$ under the projection $\mathbb{Z}^n \to P_{\mathrm{fin}}$. The row-reading word of b is defined by $\mathrm{word}(b) = \cdots w_2 w_1$ where w_r is the word obtained by reading the r-th row of b from left to right. This definition is useful even in the context that b is a skew tableau.

The edges of $\mathcal{B}_\lambda$ are given as follows. First let v be a word in the alphabet $\{1, 2, \dots, n\}$. View each letter i (resp. $i+1$) of v as a closing (resp. opening) parenthesis, ignoring other letters. Now iterate the following step: declare each adjacent pair of matched parentheses to be invisible. Repeat this until there are no matching pairs of visible parentheses. At the end the result must be a sequence of closing parentheses (say p of them) followed by a sequence of opening parentheses (say q of them). The unmatched (visible) subword is of the form $i^p(i+1)^q$. If $p > 0$ (resp. $q > 0$) then $f_i(v)$ (resp. $e_i(v)$) is obtained from v by replacing the unmatched subword $i^p(i+1)^q$ by $i^{p-1}(i+1)^{q+1}$ (resp. $i^{p+1}(i+1)^{q-1}$). Then $\phi_i(v) = p$, $\epsilon_i(v) = q$, and $f_i(v)$ (resp. $e_i(v)$) is defined if and only if $p > 0$ (resp. $q > 0$).

For the tableau $b \in \mathcal{B}_\lambda$, let $f_i(b)$ be undefined if $f_i(\mathrm{word}(b))$ is; otherwise define $f_i(b)$ to be the unique (not necessarily column-strict) tableau of shape λ such that $\mathrm{word}(f_i(b)) = f_i(\mathrm{word}(b))$. It is easy to verify that when defined, $f_i(b)$ is a column-strict tableau. Consequently $\phi_i(b) = \phi_i(\mathrm{word}(b))$. The operator e_i and the quantity $\epsilon_i(b)$ are defined similarly.

3.3. $U_q(\widehat{sl}'_n)$**-crystal structure on rectangular tableaux.** There is an inclusion of algebras $U_q(sl_n) \subset U_q(\widehat{sl}'_n)$ where $U_q(\widehat{sl}'_n)$ is the quantized universal enveloping algebra corresponding to the derived subalgebra $\widehat{sl}'_n$ of the affine Kac-Moody algebra $\widehat{sl}_n$ [15]. Let $I = \{0, 1, 2, \dots, n-1\}$ be the index set for the Dynkin diagram of $A^{(1)}_{n-1}$. Let P_{cl} be the weight lattice of $\widehat{sl}'_n$, with (linearly dependent) simple roots $\{\alpha^{\mathrm{cl}}_i \mid i \in I\}$, simple coroots $\{h_i \mid i \in I\}$, and fundamental weights $\{\Lambda^{\mathrm{cl}}_i \mid i \in I\}$. The simple roots satisfy the relation $\alpha^{\mathrm{cl}}_0 = -\sum_{i \in J} \alpha^{\mathrm{cl}}_i$. There is a natural projection $P_{\mathrm{cl}} \to P_{\mathrm{fin}}$ with kernel $\mathbb{Z}\Lambda_0$ such that $\Lambda^{\mathrm{cl}}_i \mapsto \overline{\Lambda}_i$ for $i \in J$ and $\Lambda^{\mathrm{cl}}_0 \mapsto 0$. Let $\mathrm{cl} : P_{\mathrm{fin}} \to P_{\mathrm{cl}}$ be the section of the above projection defined by $\mathrm{cl}(\overline{\Lambda}_i) = \Lambda^{\mathrm{cl}}_i - \Lambda^{\mathrm{cl}}_0$ for $i \in J$. Let $c \in \widehat{sl}'_n$ be the canonical central element. The level of a weight $\Lambda \in P_{\mathrm{cl}}$ is defined by $\langle c, \Lambda \rangle$. Let $(P^+_{\mathrm{cl}})_\ell = \{\Lambda \in P^+_{\mathrm{cl}} \mid \langle c, \Lambda \rangle = \ell\}$.

Suppose V is a finite-dimensional $U_q(\widehat{sl}'_n)$-module that has a crystal graph $\mathcal{B}$ (not all do); $\mathcal{B}$ is a P_{cl}-weighted I-crystal. A weight function $\mathrm{wt}_{\mathrm{cl}} : \mathcal{B} \to P_{\mathrm{cl}}$ may be given by $\mathrm{wt}_{\mathrm{cl}}(b) = \mathrm{cl}(\mathrm{wt}(b))$ where $\mathrm{wt} : \mathcal{B} \to P_{\mathrm{fin}}$ is the weight function on the set $\mathcal{B}$ viewed as a $U_q(sl_n)$-crystal graph. In addition to being a $U_q(sl_n)$-crystal graph, $\mathcal{B}$ also has some edges colored 0. The action of $U_q(sl_n)$ on V^λ extends to an action of $U_q(\widehat{sl}'_n)$ which admits a crystal structure, if and only if the partition λ is a rectangle [18, 30]. If λ is the rectangle with k rows and m columns, then write $V^{k,m}$ for the $U_q(\widehat{sl}'_n)$-module with $U_q(sl_n)$-structure V^λ and denote its crystal graph by $\mathcal{B}^{k,m}$. If one of m or k is 1, then it is easy to give e_0 and f_0 explicitly on $\mathcal{B}^{k,m}$, for in this case the weight spaces of $V^{k,m}$ are one-dimensional, and the zero edges can be deduced from (3.1) [18]. The general case is given as follows [37].

We shall first define a content-rotating bijection $\psi^{-1} : \mathcal{B}^{k,m} \to \mathcal{B}^{k,m}$. Let $b \in \mathcal{B}^{k,m}$ be a tableau, say of content $(c_1, c_2, \dots, c_n)$. $\psi^{-1}(b)$ will have content $(c_2, c_3, \dots, c_n, c_1)$. Remove all the letters 1 from b, leaving a vacant horizontal strip of size c_1 in the northwest corner of b. Compute Schensted's P tableau [34] of the row-reading word of this skew subtableau. It can be shown that this yields a tableau of the shape obtained by removing c_1 cells from the last row of the rectangle (m^k). Subtract one from the value of each entry of this tableau, and then fill in the c_1 vacant cells in the last row of the rectangle (m^k) with the letter n. It can be shown that ψ^{-1} is a well-defined bijection, whose inverse ψ can be given by a similar algorithm. Then

$$
\begin{aligned}
f_i &= \psi^{-1} \circ f_{i+1} \circ \psi, \\
e_i &= \psi^{-1} \circ e_{i+1} \circ \psi
\end{aligned}
\tag{3.6}
$$

for all i where indices are taken modulo n; in particular for $i = 0$ this defines explicitly the operators e_0 and f_0.

3.4. **Sequences of rectangular tableaux.** For a sequence of rectangles R, consider the tensor product $V^{R_L} \otimes \cdots \otimes V^{R_1}$. Its $U_q(\widehat{sl}'_n)$-crystal graph has underlying set $\mathcal{P}_R = \mathcal{B}_{R_L} \otimes \cdots \otimes \mathcal{B}_{R_1}$, where the tensor symbols denote the Cartesian product of sets. A typical element of $\mathcal{P}_R$ is called a path and is written $b = b_L \otimes \cdots \otimes b_2 \otimes b_1$ where $b_j \in \mathcal{B}_{R_j}$ is a tableau of shape R_j.

The edges of the crystal graph $\mathcal{P}_R$ are given explicitly as follows. Define the word of a path b by

$$\mathrm{word}(b) = \mathrm{word}(b_L) \cdots \mathrm{word}(b_2)\mathrm{word}(b_1).$$

Then for $i = 1, 2, \dots, n-1$ (as in the definition of f_i for $b \in \mathcal{B}_\lambda$), if $f_i(\mathrm{word}(b))$ is undefined, let $f_i(b)$ be undefined; otherwise it not hard to see that there is a unique path $f_i(b) \in \mathcal{P}_R$ such that $\mathrm{word}(f_i(b)) = f_i(\mathrm{word}(b))$. To define f_0, let $\psi(b) = \psi(b_L) \otimes \cdots \otimes \psi(b_1)$ and $f_0 = \psi^{-1} \circ f_1 \circ \psi$. This definition is equivalent to that given by taking the above definition of f_i on the crystals $\mathcal{B}_{R_j}$ and then applying the rule for lowering operators on tensor products (3.3). The action of e_i for $i \in I$ is defined analogously.

3.5. **Integrable affine crystals.** Consider the affine Kac-Moody algebra $\widehat{sl}_n$, with weight lattice P_{af}, independent simple roots $\{\alpha_i \mid i \in I\}$, simple coroots $\{h_i \mid i \in I\}$, and fundamental weights $\{\Lambda_i \mid i \in I\}$. Let $\delta \in P_{\mathrm{af}}$ be the null root. There is a natural projection which we shall by abuse of notation also call $\mathrm{cl} : P_{\mathrm{af}} \to P_{\mathrm{cl}}$ such that $\mathrm{cl}(\delta) = 0$ and $\mathrm{cl}(\Lambda_i) = \Lambda_i^{\mathrm{cl}}$ for $i \in I$. Write $\mathrm{af} : P_{\mathrm{cl}} \to P_{\mathrm{af}}$ for the section of cl given by $\mathrm{af}(\Lambda_i^{\mathrm{cl}}) = \Lambda_i$ for $i \in I$.

Let $\Lambda \in P_{\mathrm{cl}}^+$ be a dominant integral weight and $\mathbb{B}(\Lambda)$ the crystal graph of the irreducible integrable highest weight $U_q(\widehat{sl}'_n)$-module of highest weight Λ. If $\Lambda \neq 0$ then $\mathbb{B}(\Lambda)$ is infinite. The set of weights in P_{af} that project by cl to Λ are given by $\mathrm{cl}^{-1}(\Lambda) = \{\mathrm{af}(\Lambda) + j\delta \mid j \in \mathbb{Z}\}$. Now fix j. The irreducible integrable highest weight $U_q(\widehat{sl}_n)$-crystal graph $\mathbb{B}(\mathrm{af}(\Lambda) + j\delta)$ may be identified with $\mathbb{B}(\Lambda)$ as sets and as I-crystals (independent of j). The weight functions for $\mathbb{B}(\mathrm{af}(\Lambda) + j\delta)$ and $\mathbb{B}(\mathrm{af}(\Lambda))$ differ by the global constant $j\delta$. The weight function $\mathbb{B}(\Lambda) \to \mathbb{Z}$ is obtained by composing the weight function for $\mathbb{B}(\mathrm{af}(\Lambda) + j\delta)$, with the projection $\mathrm{cl} : P_{\mathrm{af}} \to P_{\mathrm{cl}}$.

The set $\mathbb{B}(\Lambda)$ is then endowed with an induced $\mathbb{Z}$-grading $E : \mathbb{B}(\Lambda) \to \mathbb{N}$ defined by $E(b) = \langle\, d\,,\, \mathrm{wt}(b)\,\rangle$ where $\mathbb{B}(\Lambda)$ is identified with $\mathbb{B}(\mathrm{af}(\Lambda))$, $\mathrm{wt} : \mathbb{B}(\mathrm{af}(\Lambda)) \to P_{\mathrm{af}}$ is the weight function and $d \in P_{\mathrm{af}}^*$ is the degree generator.

The map $\langle d, \cdot \rangle$ takes the coefficient of the element δ of an element in P_{af} when written in the basis $\{\Lambda_i \mid i \in I\} \cup \{\delta\}$.

3.6. **Energy function on finite paths.** The set of paths $\mathcal{P}_R$ has a natural statistic called the energy function. The definitions here follow [30].

Consider first the case that $R = (R_1, R_2)$ is a sequence of two rectangles. Let $\mathcal{B}_j = \mathcal{B}_{R_j}$ for $1 \leq j \leq 2$. Since $\mathcal{B}_2 \otimes \mathcal{B}_1$ is a connected crystal graph, there is a unique $U_q(\widehat{sl}'_n)$-crystal graph isomorphism

$$\sigma : \mathcal{B}_2 \otimes \mathcal{B}_1 \cong \mathcal{B}_1 \otimes \mathcal{B}_2. \tag{3.7}$$

This is called the local isomorphism (see Section 4.4 for an explicit construction). Write $\sigma(b_2 \otimes b_1) = b'_1 \otimes b'_2$. Then there is a unique (up to a global additive constant) map $H : \mathcal{B}_2 \otimes \mathcal{B}_1 \to \mathbb{Z}$ such that

(3.8)
$$H(e_i(b_2 \otimes b_1)) = H(b_2 \otimes b_1) + \begin{cases} -1 & \text{if } i = 0,\ e_0(b_2 \otimes b_1) = e_0 b_2 \otimes b_1 \\ & \text{and } e_0(b'_1 \otimes b'_2) = e_0 b'_1 \otimes b'_2, \\ 1 & \text{if } i = 0,\ e_0(b_2 \otimes b_1) = b_2 \otimes e_0 b_1 \\ & \text{and } e_0(b'_1 \otimes b'_2) = b'_1 \otimes e_0 b'_2, \\ 0 & \text{otherwise.} \end{cases}$$

This map is called the local energy function. By definition it is invariant under the local isomorphism and under f_i and e_i for $i \in J$. Let us normalize it by the condition that $H(u_2 \otimes u_1) = |R_1 \cap R_2|$ where u_j is the $U_q(sl_n)$ highest weight vector of $\mathcal{B}_j$ for $1 \leq j \leq 2$, $R_1 \cap R_2$ is the intersection of the Ferrers diagrams of R_1 and R_2, and $|R_1 \cap R_2|$ is the number of cells in this intersection. Explicitly $|R_1 \cap R_2| = \min\{\eta_1, \eta_2\} \min\{\mu_1, \mu_2\}$. If $\eta_1 + \eta_2 \leq n$ then the local energy function attains precisely the values from 0 to $|R_1 \cap R_2|$.

Now let $R = (R_1, \dots, R_L)$ be a sequence of rectangles and $b = b_L \otimes \cdots \otimes b_1 \in \mathcal{P}_R$. For $1 \leq p \leq L-1$ let σ_p denote the local isomorphism that exchanges the tensor factors in the p-th and $(p+1)$-th positions. For $1 \leq i < j \leq L$, let $b_j^{(i+1)}$ be the $(i+1)$-th tensor factor in $\sigma_{i+1}\sigma_{i+2}\dots\sigma_{j-1}(b)$. Then define the energy function

$$E(b) = \sum_{1 \leq i < j \leq L} H(b_j^{(i+1)} \otimes b_i). \tag{3.9}$$

The value of the energy function is unchanged under local isomorphisms and under e_i and f_i for $i \in J$, since the local energy function has this property.

The next lemma follows from the definition of the local energy function.

Lemma 3.2. *Suppose $b = b_L \otimes \cdots \otimes b_1 \in \mathcal{P}_R$ is such that $e_0(b)$ is defined and for any image $b' = b'_L \otimes \cdots \otimes b'_1$ of b under a composition of local isomorphisms, $e_0(b') = b'_L \otimes \cdots \otimes b'_{j+1} \otimes e_0(b'_j) \otimes b'_{j-1} \otimes \cdots \otimes b'_1$ where $j \neq 1$. Then $E(e_0(b)) = E(b) - 1$.*

If all rectangles R_j are the same then each of the local isomorphisms is the identity and

$$E(b) = \sum_{1 \le i \le L-1} (L-i) H(b_{i+1} \otimes b_i). \tag{3.10}$$

Say that $b \in \mathcal{P}_R$ is *classically restricted* if it is an sl_n-highest weight vector, that is, $\epsilon_i(b) = 0$ for all $i \in J$. Equivalently, word(b) is a (reverse) lattice permutation (every final subword has partition content). Let $\mathcal{P}_{\Lambda R}$ be the set of classically restricted paths in $\mathcal{P}_R$ of weight $\Lambda \in P_{\mathrm{cl}}$.

It was shown in [37] that the generalized Kostka polynomial (which was originally defined in terms of Littlewood–Richardson tableaux; see (4.3)) can be expressed as

$$K_{\lambda R}(q) = \sum_{b \in \mathcal{P}_{\mathrm{cl}(\overline{\lambda})R}} q^{E(b)}. \tag{3.11}$$

This extends the path formulation of the Kostka polynomial by Nakayashiki and Yamada [30].

3.7. **Level-restricted paths.** Let $\mathcal{B}$ be any P_{cl}-weighted I-crystal and $\Lambda \in P_{\mathrm{cl}}^+$. Say that $b \in \mathcal{B}$ is Λ-restricted if $b \otimes u_\Lambda$ is a highest weight vector in the P_{cl}-weighted I-crystal $\mathcal{B} \otimes \mathbb{B}(\Lambda)$, that is, $\epsilon_i(b \otimes u_\Lambda) = 0$ for all $i \in I$. Equivalently $\epsilon_i(b) \le \langle h_i , \Lambda \rangle$ for all $i \in I$ by (3.5) and (3.2). Denote by $\mathcal{H}(\Lambda, \mathcal{B})$ the set of elements $b \in \mathcal{B}$ that are Λ-restricted. If $\Lambda' \in P_{\mathrm{cl}}^+$ has the same level as Λ, define $\mathcal{H}(\Lambda, \mathcal{B}, \Lambda')$ to be the set of $b \in \mathcal{H}(\Lambda, \mathcal{B})$ such that $\mathrm{wt}(b) = \Lambda' - \Lambda \in P_{\mathrm{cl}}$, that is, the set of $b \in \mathcal{B}$ such that $b \otimes u_\Lambda$ is a highest weight vector of weight Λ'. Say that the element b is restricted of level ℓ if it is $(\ell\Lambda_0)$-restricted. Such paths are also classically restricted since $\langle h_i , \ell\Lambda_0 \rangle = 0$ for $i \in J$. Let $\mathcal{P}^\ell_{\Lambda R}$ denote the set of paths in $\mathcal{P}_{\Lambda R}$ that are restricted of level ℓ. Letting $\mathcal{B} = \mathcal{P}_R$, this is the same as saying $\mathcal{P}^\ell_{\Lambda R} = \mathcal{H}(\ell\Lambda_0, \mathcal{B}, \Lambda + \ell\Lambda_0)$.

Define the level-restricted generalized Kostka polynomial by

$$K^\ell_{\lambda R}(q) = \sum_{b \in \mathcal{P}^\ell_{\mathrm{cl}(\overline{\lambda})R}} q^{E(b)}. \tag{3.12}$$

3.8. **Perfect crystals.** This section is needed to compute the coset branching functions in Section 7. We follow [19], stating the definitions in the case of $\widehat{sl}'_n$. For any $U_q(\widehat{sl}'_n)$-crystal $\mathcal{B}$, define $\epsilon, \phi : \mathcal{B} \to P_{\mathrm{cl}}$ by $\epsilon(b) = \sum_{i \in I} \epsilon_i(b)\Lambda_i$ and $\phi(b) = \sum_{i \in I} \phi_i(b)\Lambda_i$.

Now let ℓ be a positive integer and $\mathcal{B}$ the crystal graph of a finite dimensional irreducible $U_q(\widehat{sl}'_n)$-module V. Say that $\mathcal{B}$ is perfect of level ℓ if

1. $\mathcal{B} \otimes \mathcal{B}$ is connected.
2. There is a weight $\Lambda' \in P_{\mathrm{cl}}$ such that $\mathcal{B}$ has a unique vector of weight Λ' and all other vectors in $\mathcal{B}$ have lower weight in the Chevalley order, that is, $\mathrm{wt}(\mathcal{B}) \subset \Lambda' - \sum_{i \in J} \mathbb{N}\alpha_i$.

3. $\ell = \min_{b\in\mathcal{B}} \langle\, c\,,\, \epsilon(b)\,\rangle$.
4. The maps ϵ and ϕ restrict to bijections $\mathcal{B}_{\min} \to (P^+_{\mathrm{cl}})_\ell$ where $\mathcal{B}_{\min} \subset \mathcal{B}$ is the set of $b \in \mathcal{B}$ achieving the minimum in 3.

For $\widehat{sl}'_n$ the perfect crystals of level ℓ are precisely those of the form $\mathcal{B}^{k,\ell}$ for $1 \le k \le n-1$ [18, 30]. Let $\mathcal{B} = \mathcal{B}^{k,\ell}$. The weight Λ' can be taken to be $\ell(\Lambda^{\mathrm{cl}}_k - \Lambda^{\mathrm{cl}}_0)$.

Example 3.3. We describe the bijections $\epsilon, \phi : \mathcal{B}_{\min} \to (P^+_{\mathrm{cl}})_\ell$ in this example. Let $\mathcal{B} = \mathcal{B}^{k,\ell}$. For this example let $n = 6$, $k = 3$, $\ell = 5$, and consider the weight $\Lambda = 2\Lambda_0 + \Lambda_1 + \Lambda_2 + \Lambda_4$. As usual subscripts are identified modulo n. The unique tableau $b \in \mathcal{B}^{k,\ell}$ such that $\phi(b) = \Lambda$ is constructed as follows. First let T be the following tableau of shape (ℓ^k). Its bottom row contains $\langle\, h_i\,,\, \Lambda\,\rangle$ copies of the letter i for $1 \le i \le n$ (here it is 12466 since the sequence of $\langle\, h_i\,,\, \Lambda\,\rangle$ for $1 \le i \le 6$ is $(1,1,0,1,0,2)$). Let every letter in T have value one smaller than the letter directly below it. Here we have

$$T = \begin{matrix} -1 & 0 & 2 & 4 & 4 \\ 0 & 1 & 3 & 5 & 5 \\ 1 & 2 & 4 & 6 & 6 \end{matrix}.$$

Let T_- be the subtableau of T consisting of the entries that are nonpositive and T_+ the rest. Say T_- has shape ν (here $\nu = (2,1)$). Let $\widetilde{\nu} = (\ell^k) - (\nu_k, \nu_{k-1}, \dots, \nu_1)$ (here $\widetilde{\nu} = (5,4,3)$). The desired tableau b is defined as follows. The restriction of b to the shape $\widetilde{\nu}$ is $P(T_+)$, or equivalently, the tableau obtained by taking the skew tableau T_+ and first pushing all letters straight upwards to the top of the bounding rectangle (ℓ^k), and then pushing all letters straight to the left inside (ℓ^k). The restriction of b to $(\ell^k)/\widetilde{\nu}$ is the tableau of that skew shape in the alphabet $\{1,2,\dots,n\}$ with maximal entries, that is, its bottom row is filled with the letter n, the next-to-bottom row is filled with the letter $n-1$, etc. In the example,

$$b = \begin{matrix} 1 & 1 & 2 & 4 & 4 \\ 2 & 3 & 5 & 5 & 5 \\ 4 & 6 & 6 & 6 & 6 \end{matrix}.$$

To construct the unique element $b' \in \mathcal{B}^{k,\ell}$ such that $\epsilon(b') = \Lambda$, let U be the tableau whose first row has $\langle\, h_i\,,\, \Lambda\,\rangle$ copies of the letter $i+1$ for $1 \le i \le n$, again identifying subscripts modulo n; here U has first row 11235. Now let the rest of U be defined by letting each entry have value one greater than the entry above it. So

$$U = \begin{matrix} 1 & 1 & 2 & 3 & 5 \\ 2 & 2 & 3 & 4 & 6 \\ 3 & 3 & 4 & 5 & 7 \end{matrix}.$$

Let U_- be the subtableau of U consisting of the values that are at most n. Let μ be the shape of U_- and $\widetilde{\mu} = (\ell^k) - (\mu_k, \mu_{k-1}, \dots, \mu_1)$. Here $\mu = (5,5,4)$ and $\widetilde{\mu} = (1,0,0)$. The element b' is defined as follows. Its restriction to the skew shape $(\ell^k)/\widetilde{\mu}$ is the unique skew tableau V of that shape such that $P(V) = U_-$, or equivalently, this restriction is obtained by taking the tableau U_-, pushing all letters directly down within the rectangle (ℓ^k) and then pushing all letters to the right within (ℓ^k). The restriction of b' to the shape $\widetilde{\mu}$ is filled with the smallest letters possible, so that the first row of this subtableau consists of ones, the second row consists of twos, etc. Here

$$b' = \begin{matrix} 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 3 & 4 & 5. \\ 3 & 3 & 4 & 5 & 6 \end{matrix}$$

The main theorem for perfect crystals is:

Theorem 3.4. [19] *Let $\mathcal{B}$ be a perfect crystal of level ℓ' and $\Lambda \in (P^+_{\mathrm{cl}})_\ell$ with $\ell \geq \ell'$. Then there is an isomorphism of $U_q(\widehat{sl}'_n)$-crystals*

$$\mathcal{B} \otimes \mathbb{B}(\Lambda) \cong \bigoplus_{b \in \mathcal{H}(\Lambda,\mathcal{B})} \mathbb{B}(\Lambda + \mathrm{wt}(b)). \tag{3.13}$$

Suppose now that $\mathcal{B}$ is perfect of level ℓ and $\Lambda \in (P^+_{\mathrm{cl}})_\ell$. Write $b(\Lambda)$ for the unique element of $\mathcal{B}$ such that $\phi(b(\Lambda)) = \Lambda$. Theorem 3.4 (with Λ therein replaced by $\Lambda' = \epsilon(b(\Lambda))$) says that $\mathcal{B} \otimes \mathbb{B}(\epsilon(b(\Lambda))) \cong \mathbb{B}(\Lambda)$ with corresponding highest weight vectors $b(\Lambda) \otimes u_{\epsilon(b(\Lambda))} \mapsto u_\Lambda$. This isomorphism can be iterated. Let $\sigma : \mathcal{B}_{\min} \to \mathcal{B}_{\min}$ be the unique bijection defined by $\phi \circ \sigma = \epsilon$. Then there are isomorphisms $\mathcal{B}^{\otimes N} \otimes \mathbb{B}(\phi(\sigma^N(b(\Lambda)))) \cong \mathbb{B}(\Lambda)$ such that the highest weight vector of the left-hand side is given by $b(\Lambda) \otimes \sigma(b(\Lambda)) \otimes \sigma^2(b(\Lambda)) \otimes \cdots \otimes \sigma^{N-1}(b(\Lambda)) \otimes u_{\phi(\sigma^N(b(\Lambda)))}$. For the $U_q(\widehat{sl}'_n)$ perfect crystals $\mathcal{B}^{k,\ell}$, it can be shown that the map σ is none other than the power ψ^{-k} of the content rotating map ψ. Moreover if σ is extended to a bijection $\sigma : \mathcal{B}^{k,\ell} \to \mathcal{B}^{k,\ell}$ by defining $\sigma = \psi^{-k}$, then the extended function also satisfies $\phi(\sigma(b)) = \epsilon(b)$ for all $b \in \mathcal{B}^{k,\ell}$ not just for $b \in \mathcal{B}_{\min}$. Since the bijection ψ on $\mathcal{B}^{k,\ell}$ has order n, the bijection σ has order $n/\gcd(n,k)$. The ground state path for the pair $(\Lambda, \mathcal{B})$ is by definition the infinite periodic sequence $\overline{b} = \overline{b}_1 \otimes \overline{b}_2 \otimes \dots$ where $\overline{b}_i = \sigma^{i-1}(b(\Lambda))$.

Let $\mathbb{P}(\Lambda, \mathcal{B})$ be the set of all semi-infinite sequences $b = b_1 \otimes b_2 \otimes \dots$ of elements in $\mathcal{B}$ such that b eventually agrees with the ground state path $\overline{b}$ for $(\Lambda, \mathcal{B})$. Then the set $\mathbb{P}(\Lambda, \mathcal{B})$ has the structure of the crystal $\mathbb{B}(\Lambda)$ with highest weight vector $u_\Lambda = \overline{b}$ and weight function $\mathrm{wt}(b) = \sum_{i \geq 1} (\mathrm{wt}(b_i) - \mathrm{wt}(\overline{b}_i))$. To recover the weight function of the $U_q(\widehat{sl}_n)$-crystal $\mathbb{B}(\mathrm{af}(\Lambda))$, define the energy function on $\mathbb{P}(\Lambda, \mathcal{B})$ by

$$E(b) = \sum_{i \geq 1} i(H(b_i \otimes b_{i+1}) - H(\overline{b}_i \otimes \overline{b}_{i+1})) \tag{3.14}$$

and define the map $\mathbb{B}(\mathrm{af}(\lambda)) \to P_{\mathrm{af}}$ by $b \mapsto \mathrm{wt}(b) - E(b)\delta$ where $\mathrm{wt} : \mathbb{B}(\Lambda) \to P_{\mathrm{cl}}$.

$\mathbb{P}(\Lambda, \mathcal{B})$ can be regarded as a direct limit of the finite crystals $\mathcal{B}^{\otimes N}$. Define the embedding $i_N : \mathcal{B}^{\otimes N} \to \mathbb{P}(\Lambda, \mathcal{B})$ by

$$b_1 \otimes \cdots \otimes b_N \mapsto b_1 \otimes b_2 \otimes b_N \otimes \overline{b}_{N+1} \otimes \overline{b}_{N+2} \otimes \ldots$$

Define $E_N : \mathcal{B}^{\otimes N} \to \mathbb{Z}$ by $E_N(b_1 \otimes \cdots \otimes b_N) = E(b_1 \otimes \cdots \otimes b_N \otimes \overline{b}_{N+1})$ where the E on the right hand side is the energy function for the finite path space $\mathcal{B}^{\otimes N+1}$. By definition for all $p = b_1 \otimes \cdots \otimes b_N \in \mathcal{B}^{\otimes N}$, $E(i_N(p)) = E_N(p) - E_N(\overline{b}_1 \otimes \cdots \otimes \overline{b}_N)$. Note that the last fixed step $\overline{b}_{N+1}$ is necessary to make the energy function on the finite paths stable under the embeddings into $\mathbb{P}(\Lambda, \mathcal{B})$.

3.9. **Standardization embeddings.** We require certain embeddings of finite path spaces. Given a sequence of rectangles R, let $\mathrm{r}(R)$ denote the sequence of rectangles given by splitting the rectangles of R into their constituent rows. For example, if $R = ((1), (2, 2))$ then $\mathrm{r}(R) = ((1), (2), (2))$. There is a unique embedding

$$i_R : \mathcal{P}_R \hookrightarrow \mathcal{P}_{\mathrm{r}(R)} \tag{3.15}$$

defined as follows. Its explicit computation is based on transforming R into $\mathrm{r}(R)$ using two kinds of steps.

1. Suppose R_1 has more than one row ($\eta_1 > 1$). Then use the transformation $R \to R^< = ((\mu_1), (\mu_1^{\eta_1 - 1}), R_2, R_3, \ldots, R_L)$. Informally, $R^<$ is obtained from R by splitting off the first row of R_1. There is an associated embedding of $U_q(\widehat{sl}'_n)$-crystal graphs $i_R^< : \mathcal{P}_R \to \mathcal{P}_{R^<}$ defined by the property that $\mathrm{word}(i^<(b)) = \mathrm{word}(b)$ for all $b \in \mathcal{P}_R$. Here it is crucial that the rectangle being split horizontally, is the first one, for otherwise the embedding does not preserve the edges labeled by 0.
2. If $\eta_1 = 1$, then use a transformation of the form $R \to s_p R$ for some p. Here $s_p R$ denotes the sequence of rectangles obtained by exchanging the p-th and $(p+1)$-th rectangles in R. The associated isomorphism of $U_q(\widehat{sl}'_n)$-crystal graphs is the local isomorphism $\sigma_p : \mathcal{P}_R \to \mathcal{P}_{s_p R}$ defined before.

It is clear that one can transform R into $\mathrm{r}(R)$ using these two kinds of steps. Now fix one such sequence of steps leading from R to $\mathrm{r}(R)$, say $R = R^{(0)} \to R^{(1)} \to \cdots \to R^{(N)} = \mathrm{r}(R)$ where each $R^{(m)}$ is a sequence of rectangles and each step $R^{(m-1)} \to R^{(m)}$ is one of the two types defined above. Define the map $i^{(m)} : \mathcal{P}_{R^{(m-1)}} \hookrightarrow \mathcal{P}_{R^{(m)}}$ by $i^{(m)} = i^<_{R^{(m-1)}}$ if the step is of the first kind, and by $i^{(m)} = \sigma_p$ if it is of the second kind. Let $i_R : \mathcal{P}_R \to \mathcal{P}_{\mathrm{r}(R)}$ be the composition $i_R = i^{(N)} \circ \cdots \circ i^{(1)}$. It can be shown that the map i_R does not depend on the sequence of the $R^{(m)}$; this is proven in the equivalent language of Littlewood–Richardson tableaux in [36].

4. Littlewood–Richardson tableaux

We now review some formulations of type A tensor product multiplicities that use tableaux. These tableaux, which we call Littlewood–Richardson (LR) tableaux, are the intermediate combinatorial objects between paths and rigged configurations, which give rise to fermionic expressions. For the most part, the material in this section is taken from [33, 35, 36, 37].

4.1. **Three formulations.** Let $I_1, I_2, \dots, I_L$ be intervals of integers such that if $i < j$, $x \in I_i$ and $y \in I_j$, then $x < y$. Set $I = \bigcup_{j=1}^{L} I_j$. For each $1 \le j \le L$, fix a tableau Z_j of shape R_j in the alphabet I_j. Define the set $\mathrm{SLR}(\lambda; Z)$ to be the set of tableaux Q of shape λ in the alphabet I such that $P(Q|_{I_j}) = Z_j$ for all j, where $Q|_{I_j}$ denotes the skew subtableau of Q obtained by restricting to the alphabet I_j, and $P(S)$ denotes the Schensted P-tableau [34] of the row-reading word of the skew tableau S. It is well-known that $|\mathrm{SLR}(\lambda; Z)| = c^{\lambda}_{R}$, where c^{λ}_{R} was defined in (1.1).

We shall define three kinds of LR tableaux given by $\mathrm{SLR}(\lambda; Z)$ for various choices of intervals I_j and tableaux Z_j.

1. $\mathrm{LR}(\lambda; R)$: Define the set of intervals of integers $I_j = A_j = [\eta_1 + \cdots + \eta_{j-1} + 1, \eta_1 + \cdots + \eta_{j-1} + \eta_j]$. Let $Z_j = Y_j$ be the tableau of shape R_j whose r-th row is filled with copies of the r-th largest letter of A_j, namely, $\eta_1 + \cdots + \eta_{j-1} + r$. Define $\mathrm{LR}(\lambda; R) := \mathrm{SLR}(\lambda; Y)$. When R consists of single rows (that is, $\eta_j = 1$ for all j), then $\mathrm{LR}(\lambda; R) = \mathrm{CST}(\lambda; \mu)$, the (column-strict) tableaux of shape λ and content μ.
2. $\mathrm{CLR}(\lambda; R)$ (Columnwise LR): Let ZC_1 be the standard tableau of shape R_1 obtained by placing the numbers 1 through η_1 down the first column, the next η_1 numbers down the second column, etc. Continue this process to obtain ZC_2, starting with the next available number, namely, $\eta_1\mu_1 + 1$. Explicitly, for $1 \le j \le L$, the (r, c)-th entry in the j-th tableau ZC_j is equal to $\eta_1\mu_1 + \cdots + \eta_{j-1}\mu_{j-1} + (c-1)\eta_j + r$. Let B_j be the interval consisting of the entries of the tableau ZC_j. Define $\mathrm{CLR}(\lambda; R) := \mathrm{SLR}(\lambda; ZC)$.
3. $\mathrm{RLR}(\lambda; R)$ (Rowwise LR): Define this similarly to $\mathrm{CLR}(\lambda; R)$ but label by rows, so that the (r, c)-th entry of ZR_j is $\eta_1\mu_1 + \cdots + \eta_{j-1}\mu_{j-1} + (r-1)\mu_j + c$. Then let $\mathrm{RLR}(\lambda; R) := \mathrm{SLR}(\lambda; ZR)$.

Example 4.1. Let $R = ((1), (2, 2))$ and $\lambda = (3, 2)$. Here $A_1 = \{1\}$, $A_2 = \{2, 3\}$, and

$$Y_1 = 1 \quad \text{and} \quad Y_2 = \begin{matrix} 2 & 2 \\ 3 & 3 \end{matrix}.$$

We have $B_1 = \{1\}$, $B_2 = \{2, 3, 4, 5\}$,

$$ZC_1 = 1 \quad \text{and} \quad ZC_2 = \begin{matrix} 2 & 4 \\ 3 & 5 \end{matrix}$$

and

$$ZR_1 = 1 \quad \text{and} \quad ZR_2 = \begin{matrix} 2 & 3 \\ 4 & 5 \end{matrix}.$$

Observe that

$$T = \begin{matrix} 1 & 2 & 4 \\ 3 & 5 & \end{matrix}$$

is in $\mathrm{CLR}(\lambda; R)$ since $P(T|_{B_1}) = 1 = ZC_1$ and $P(T|_{B_2}) = ZC_2$. On the other hand $T = \begin{matrix} 1 & 3 & 5 \\ 2 & 4 & \end{matrix}$ is not in $\mathrm{CLR}(\lambda; R)$ since $P(T|_{B_2}) = \begin{matrix} 2 & 3 & 5 \\ 4 & & \end{matrix} \neq ZC_2$.

4.2. **Obvious bijections among the various LR tableaux.** There are trivial relabeling bijections between the various kinds of LR tableaux defined above. We give them explicitly here for later use.

1. The bijection $\gamma_R : \mathrm{CLR}(\lambda; R) \to \mathrm{RLR}(\lambda; R)$ is given by the following relabeling. Consider an entry x in a standard tableau $S \in \mathrm{CLR}(\lambda; R)$. Then x appears in one of the ZC tableaux, say, it is the (r, c)-th entry of ZC_j. Let y be the (r, c)-th entry of the rowwise tableau ZR_j. Then replace x by y in S. Performing all such replacements simultaneously yields $\gamma_R(S) \in \mathrm{RLR}(\lambda; R)$.
2. The bijection $\mathrm{std} : \mathrm{LR}(\lambda; R) \to \mathrm{RLR}(\lambda; R)$ is given by Schensted's standardization map [34]. Let $Q \in \mathrm{LR}(\lambda; R)$ and i be some entry in Q. Suppose i is the r-th largest value in the subinterval A_j. Replace the occurrences of the letter i in Q from left to right by the consecutive integers given by the r-th row of ZR_j. The result of these substitutions is $\mathrm{std}(Q) \in \mathrm{RLR}(\lambda; R)$.
3. Define a bijection $\beta_R : \mathrm{LR}(\lambda; R) \to \mathrm{CLR}(\lambda; R)$ by $\gamma_R^{-1} \circ \mathrm{std}$.
4. Observe that ordinary transposition of standard tableaux restricts to a bijection $\mathrm{tr} : \mathrm{RLR}(\lambda; R) \leftrightarrow \mathrm{CLR}(\lambda^t; R^t)$ where λ^t denotes the transpose partition of λ and $R^t = (R_1^t, R_2^t, \dots, R_L^t)$.
5. There is a bijection $\mathrm{tr}_{\mathrm{LR}} : \mathrm{CLR}(\lambda; R) \to \mathrm{CLR}(\lambda^t; R^t)$ defined by $\mathrm{tr}_{\mathrm{LR}} = \mathrm{tr} \circ \gamma_R$.

4.3. **Paths to tableau pairs.** The Robinson–Schensted–Knuth correspondence allows one to pass from paths to pairs of tableaux. This bijection gives a combinatorial decomposition of the crystal graph of $\mathcal{P}_R$ into $U_q(sl_n)$ irreducible components and encodes the energy function in the recording tableau.

The column insertion version of the Robinson–Schensted–Knuth correspondence, restricts to a bijection

$$\mathrm{RSK} : \mathcal{P}_R \to \bigcup_\lambda \mathrm{CST}(\lambda; \cdot) \times \mathrm{LR}(\lambda; R) \tag{4.1}$$

as follows. Let $b = b_L \otimes \cdots \otimes b_2 \otimes b_1 \in \mathcal{P}_R$. Define $P(b) := P(\mathrm{word}(b))$. This can be computed by the column insertion of $\mathrm{word}(b)$ starting from the right end. Recall that b_j and Y_j are column-strict tableaux of shape R_j.

Let $Q(b)$ be the tableau obtained by recording the insertion of a letter in b_j by the letter in the corresponding position in Y_j. It can be shown that $Q(b) \in \mathrm{LR}(\lambda; R)$, and that the map (4.1) given by $b \mapsto (P(b), Q(b))$ is a bijection.

Remark 4.2.

1. This bijection is a morphism of $U_q(sl_n)$-crystal graphs in the sense that $P(e_i(b)) = e_i(P(b))$ for $i \in J$. In particular, $b \in \mathcal{P}_R$ is classically restricted if and only if $P(b)$ is a Yamanouchi tableau, that is, its r-th row is filled with copies of the letter r for all $1 \leq r \leq n$.
2. The energy function on paths can be transferred easily to a statistic on $\mathrm{LR}(\lambda; R)$ called the generalized charge (written c_R) such that $c_R(Q(b)) = E(b)$. The generalized charge is defined explicitly in (4.2) below.

Example 4.3. Let $R = ((1), (2, 2))$ and $b \in \mathcal{P}_R$ given by

$$b = \begin{matrix} 1 & 1 \\ 2 & 2 \end{matrix} \otimes 1.$$

Then $\mathrm{word}(b) = 2211\,1$ and

$$P(b) = \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & \end{matrix} \qquad Q(b) = \begin{matrix} 1 & 2 & 2 \\ 3 & 3 & \end{matrix}.$$

4.4. **Generalized Automorphisms of Conjugation.** For the moment let $R = (R_1, R_2)$ and $\mathcal{B}_j = \mathcal{B}_{R_j}$ for $1 \leq j \leq 2$. Recall that the local isomorphism (3.7) is the unique isomorphism of $U_q(\widehat{sl}'_n)$-crystal graphs $\mathcal{B}_2 \otimes \mathcal{B}_1 \to \mathcal{B}_1 \otimes \mathcal{B}_2$ or equivalently $\mathcal{P}_{(R_1,R_2)} \to \mathcal{P}_{(R_2,R_1)}$. Let us make this more explicit. By Remark 4.2 we have a commutative diagram of bijections

$$\begin{array}{ccc} \mathcal{P}_{(R_1,R_2)} & \xrightarrow{\mathrm{RSK}} & \bigcup_\lambda \mathrm{CST}(\lambda) \times \mathrm{LR}(\lambda; (R_1, R_2)) \\ \sigma \big\downarrow & & \big\downarrow \bigcup 1 \times s \\ \mathcal{P}_{(R_2,R_1)} & \xrightarrow[\mathrm{RSK}]{} & \bigcup_\lambda \mathrm{CST}(\lambda) \times \mathrm{LR}(\lambda; (R_2, R_1)) \end{array}$$

such that $P(\sigma(b)) = P(b)$. This induces a bijection $s : \mathrm{LR}(\lambda; (R_1, R_2)) \to \mathrm{LR}(\lambda; (R_2, R_1))$ for each λ. The tensor product $V^{R_2} \otimes V^{R_1}$ is multiplicity-free. Therefore the domain and codomain of s are both empty or both singletons. Hence the bijection s is unique and can be computed from the definition of the set LR. Then $\sigma(b)$ can be computed by applying RSK to obtain $(P(b), Q(b))$, then applying s to get $(P(b), s(Q(b)))$, and finally, the inverse of RSK to obtain $\sigma(b)$.

The local energy function is recovered using only the shape of the tableau pair. For a tableau $Q \in \mathrm{LR}(\lambda; (R_1, R_2))$ let $d(Q)$ be the number of cells in Q that lie strictly to the right of the $\max\{\mu_1, \mu_2\}$-th column, or equivalently, strictly to the right of the shape $R_1 \cup R_2$. Then $H(b) = d(Q(b))$.

Then the $U_q(\widehat{sl}'_n)$-crystal graph isomorphism $\sigma_p : \mathcal{P}_R \to \mathcal{P}_{s_pR}$ induces involutions $s_p : \mathrm{LR}(\lambda; R) \to \mathrm{LR}(\lambda; s_pR)$ such that the diagram commutes:

$$\begin{array}{ccc} \mathcal{P}_R & \xrightarrow{\text{RSK}} & \bigcup_\lambda \mathrm{CST}(\lambda) \times \mathrm{LR}(\lambda; R) \\ {\scriptstyle \sigma_p}\downarrow & & \downarrow{\scriptstyle \bigcup 1\times s_p} \\ \mathcal{P}_{s_pR} & \xrightarrow[\text{RSK}]{} & \bigcup_\lambda \mathrm{CST}(\lambda) \times \mathrm{LR}(\lambda; s_pR). \end{array}$$

The map s_p is computed explicitly as follows [37]. Let $Q \in \mathrm{LR}(\lambda; R)$ and A_j be the alphabets as in the definition of $\mathrm{LR}(\lambda; R)$. Remove the skew subtableau $U = Q|_{A_p \cup A_{p+1}}$. Use the usual column insertion of its row reading word, obtaining a pair of tableaux (P', Q') where $P' \in \mathrm{LR}(\rho; (R_p, R_{p+1}))$ for some partition ρ and Q' is the standard column insertion tableau. Next replace P' by $s(P')$ where s is the unique bijection $\mathrm{LR}(\rho; (R_p, R_{p+1})) \to \mathrm{LR}(\rho; (R_{p+1}, R_p))$. Finally, pull back the pair of tableaux $(s(P'), Q')$ under column insertion to obtain a word which turns out to be the row reading word of a skew column-strict tableau V of the same shape as U. Then $s_p(Q)$ is obtained by replacing U by V.

The bijections s_p specialize to the automorphisms of conjugation of Lascoux and Schützenberger [29] in the case that R consists of single rows.

It is shown in [37] that the bijections σ_p and s_p define an action of the symmetric group S_L on paths and LR tableaux respectively. Specifically, for $w \in S_L$ let $w = s_{i_1} s_{i_2} \dots s_{i_N}$ be any factorization of w into adjacent transpositions $s_i = (i, i+1)$. For $b \in \mathcal{P}_R$, define $wb = \sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_N} b \in \mathcal{P}_{wR}$. For $Q \in \mathrm{LR}(\lambda; R)$ define $wQ = s_{i_1}s_{i_2}\dots s_{i_N} Q \in \mathrm{LR}(\lambda; wR)$.

4.5. **Generalized charge.** The generalized charge on $Q \in \mathrm{LR}(\lambda; R)$ is defined by [35, 33]

$$c_R(Q) = \frac{1}{L!} \sum_{w \in S_L} \sum_{i=1}^{L-1} (L-i) d_{i,wR}(wQ). \tag{4.2}$$

where $d_{i,R}(Q) = d(P(\mathrm{word}(Q|_{A_i \cup A_{i+1}})))$ where d is understood to be the function $d : \mathrm{LR}(\rho; (R_i, R_{i+1})) \to \mathbb{N}$.

It was shown in [33, Section 6] and [35] that $\mathrm{LR}(R) = \cup_\lambda \mathrm{LR}(\lambda; R)$ has the structure of a graded poset with covering relation given by the R-cocyclage and grading function given by the generalized charge. The generalized Kostka polynomial is by definition the generating function of LR tableaux with the charge statistic [33, 35]

$$K_{\lambda R}(q) = \sum_{T \in \mathrm{LR}(\lambda; R)} q^{c_R(T)}. \tag{4.3}$$

This extends the charge representation of the Kostka polynomial $K_{\lambda\mu}(q)$ of Lascoux and Schützenberger [28, 29].

For a path $b \in \mathcal{P}_R$ one has $E(b) = c_R(Q(b))$ [37], so the formulas (3.11) and (4.3) are equivalent.

4.6. **Embeddings of LR tableaux.** The embeddings (3.15) of sets of paths, induce embeddings

$$i_R : \mathrm{LR}(\lambda; R) \hookrightarrow \mathrm{LR}(\lambda; \mathrm{r}(R)) \tag{4.4}$$

via RSK. These maps are defined in [33, 36]. In the notation of [25, Section 8.4] they are denoted $\theta_R^{\mathrm{r}(R)}$. They are given by compositions of the generalized automorphisms of conjugation s_p and by the embeddings of the form $i_R^< : \mathrm{LR}(\lambda; R) \to \mathrm{LR}(\lambda; R^<)$ (which is just the inclusion map). These embeddings preserve the R-cocyclage poset structure and the generalized charge, since they are induced by maps that preserve the $U_q(\widehat{sl}'_n)$-crystal graph structure.

4.7. **Level-restricted LR tableaux.** Say that a tableau $Q \in \mathrm{LR}(\lambda; R)$ is restricted of level ℓ if there is a level-restricted path $b \in \mathcal{P}^\ell_{\lambda R}$ such that $Q = Q(b)$. Denote the set of such tableaux by $\mathrm{LR}^\ell(\lambda; R)$.

Example 4.4. Suppose each rectangle is a single row so that $\mathrm{LR}(\lambda; R) = \mathrm{CST}(\lambda; \mu)$. In this case let us write $\mathrm{CST}^\ell(\lambda; \mu) = \mathrm{LR}^\ell(\lambda; R)$. The following explicit rule appears in [11]. Let $Q \in \mathrm{CST}(\lambda; \mu)$. The tableau Q may be viewed as a sequence of shapes $\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(L)} = \lambda$ where $\lambda^{(j)}$ is the shape of $Q|_{[1,j]}$. Then Q is restricted of level ℓ if

$$\lambda_1^{(j)} - \lambda_n^{(j-1)} \leq \ell \qquad \text{for all } 1 \leq j \leq L. \tag{4.5}$$

In the further special case that $R_j = (1)$ for all j, write $\mathrm{ST}(\lambda) = \mathrm{LR}(\lambda; R)$ for the set of standard tableaux of shape λ and write $\mathrm{ST}^\ell(\lambda) = \mathrm{LR}^\ell(\lambda; R)$ for the level-restricted subset. For $S \in \mathrm{ST}(\lambda)$, associate the chain of shapes $\lambda^{(j)}$ as above. Since passing from $\lambda^{(j-1)}$ to $\lambda^{(j)}$ adds only one additional cell, the condition (4.5) simplifies to

$$\lambda_1^{(j)} - \lambda_n^{(j)} \leq \ell \qquad \text{for all } 1 \leq j \leq L. \tag{4.6}$$

For general R it is possible to transfer the condition of level-restriction on paths to an explicit condition on LR tableaux. However for our purposes it is more convenient to use the following description of $\mathrm{LR}^\ell(\lambda; R)$. Since the embedding (4.4) is induced by the embedding (3.15) that preserves the $U_q(\widehat{sl}'_n)$-crystal graph structure, it follows that

$$\mathrm{LR}^\ell(\lambda; R) = \{Q \in \mathrm{LR}(\lambda; R) \mid i_R(Q) \in \mathrm{CST}^\ell(\lambda; \mathrm{r}(R))\}. \tag{4.7}$$

Hence an expression for the level-restricted generalized Kostka polynomials equivalent to (3.12) is

$$K^\ell_{\lambda R}(q) = \sum_{T \in \mathrm{LR}^\ell(\lambda; R)} q^{c_R(T)}.$$

5. Rigged configurations

This section follows [25, Section 2.2], with the notational difference that here R_j is a rectangle with μ_j columns and η_j rows. The reason for this is that here we work with $\mathrm{RC}(\lambda; R)$ rather than $\mathrm{RC}(\lambda^t; R^t)$ as in [25].

5.1. **Review of definitions.** A $(\lambda; R)$-configuration is a sequence of partitions $\nu = (\nu^{(1)}, \nu^{(2)}, \dots)$ with the size constraints

$$|\nu^{(k)}| = \sum_{j>k} \lambda_j - \sum_{a=1}^{L} \mu_a \max\{\eta_a - k, 0\} \tag{5.1}$$

for $k \geq 0$ where by convention $\nu^{(0)}$ is the empty partition. If λ has at most n parts all partitions $\nu^{(k)}$ for $k \geq n$ are empty. For a partition ρ, define $m_i(\rho)$ to be the number of parts equal to i and

$$Q_i(\rho) = \rho_1^t + \rho_2^t + \cdots + \rho_i^t = \sum_{j \geq 1} \min\{i, \rho_j\},$$

the size of the first i columns of ρ. Let $\xi^{(k)}(R)$ be the partition whose parts are the widths of the rectangles in R of height k. The vacancy numbers for the $(\lambda; R)$-configuration ν are the numbers (indexed by $k \geq 1$ and $i \geq 0$) defined by

$$P_i^{(k)}(\nu) = Q_i(\nu^{(k-1)}) - 2Q_i(\nu^{(k)}) + Q_i(\nu^{(k+1)}) + Q_i(\xi^{(k)}(R)). \tag{5.2}$$

In particular $P_0^{(k)}(\nu) = 0$ for all $k \geq 1$. The $(\lambda; R)$-configuration ν is said to be admissible if $P_i^{(k)}(\nu) \geq 0$ for all $k, i \geq 1$, and the set of admissible $(\lambda; R)$-configurations is denoted by $\mathrm{C}(\lambda; R)$. Following [26, (3.2)], set

$$\mathrm{cc}(\nu) = \sum_{k,i \geq 1} \alpha_i^{(k)} (\alpha_i^{(k)} - \alpha_i^{(k+1)})$$

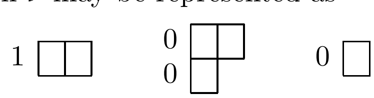
where $\alpha_i^{(k)}$ is the size of the i-th column in $\nu^{(k)}$. Define the charge $c(\nu)$ of a configuration $\nu \in \mathrm{C}(\lambda; R)$ by

$$c(\nu) = ||R|| - \mathrm{cc}(\nu) - |P|$$

$$\text{with} \quad ||R|| = \sum_{1 \leq i < j \leq L} |R_i \cap R_j| \quad \text{and} \quad |P| = \sum_{k,i \geq 1} m_i(\nu) P_i^{(k)}(\nu).$$

Observe that $c(\nu)$ depends on both ν and R but $\mathrm{cc}(\nu)$ depends only on ν.

Example 5.1. Let $\lambda = (3, 2, 2, 1)$ and $R = ((2), (2, 2), (1, 1))$. Then $\nu = ((2), (2, 1), (1))$ is a $(\lambda; R)$-configuration with $\xi^{(1)}(R) = (2)$ and $\xi^{(2)}(R) = (2, 1)$. The configuration ν may be represented as



where the vacancy numbers are indicated to the left of each part. In addition $\mathrm{cc}(\nu) = 3$, $\|R\| = 5$, $|P| = 1$ and $c(\nu) = 1$.

Define the q-binomial by

$$\begin{bmatrix} m+p \\ m \end{bmatrix} = \frac{(q)_{m+p}}{(q)_m (q)_p}$$

for $m, p \in \mathbb{N}$ and zero otherwise where $(q)_m = (1-q)(1-q^2)\cdots(1-q^m)$. The following fermionic or quasi-particle expression of the generalized Kostka polynomials, is a variant of [25, Theorem 2.10].

Theorem 5.2. *For λ a partition and R a sequence of rectangles*

$$K_{\lambda R}(q) = \sum_{\nu \in \mathrm{C}(\lambda;R)} q^{c(\nu)} \prod_{k,i \geq 1} \begin{bmatrix} P_i^{(k)}(\nu) + m_i(\nu^{(k)}) \\ m_i(\nu^{(k)}) \end{bmatrix}. \tag{5.3}$$

Expression (5.3) can be reformulated as the generating function over rigged configurations. To this end we need to define certain labelings of the rows of the partitions in a configuration. For this purpose one should view a partition as a multiset of positive integers. A rigged partition is by definition a finite multiset of pairs (i, x) where i is a positive integer and x is a nonnegative integer. The pairs (i, x) are referred to as strings; i is referred to as the length of the string and x as the label or quantum number of the string. A rigged partition is said to be a rigging of the partition ρ if the multiset consisting of the lengths of the strings, is the partition ρ. So a rigging of ρ is a labeling of the parts of ρ by nonnegative integers, where one identifies labelings that differ only by permuting labels among equal-sized parts of ρ.

A rigging J of the $(\lambda; R)$-configuration ν is a sequence of riggings of the partitions $\nu^{(k)}$ such that for every part of $\nu^{(k)}$ of length i and label x,

$$0 \leq x \leq P_i^{(k)}(\nu). \tag{5.4}$$

The pair (ν, J) is called a rigged configuration. The set of riggings of admissible $(\lambda; R)$-configurations is denoted by $\mathrm{RC}(\lambda; R)$. Let $(\nu, J)^{(k)}$ be the k-th rigged partition of (ν, J). A string $(i, x) \in (\nu, J)^{(k)}$ is said to be singular if $x = P_i^{(k)}(\nu)$, that is, its label takes on the maximum value.

Observe that the definition of the set $\mathrm{RC}(\lambda; R)$ is completely insensitive to the order of the rectangles in the sequence R. However the notation involving the sequence R is useful when discussing the bijection between LR tableaux and rigged configurations, since the ordering on R is essential in the definition of LR tableaux.

Define the cocharge and charge of $(\nu, J) \in \mathrm{RC}(\lambda; R)$ by

$$\begin{aligned} \mathrm{cc}(\nu, J) &= \mathrm{cc}(\nu) + |J| \\ c(\nu, J) &= c(\nu) + |J| \\ |J| &= \sum_{k,i \geq 1} |J_i^{(k)}| \end{aligned}$$

where $J_i^{(k)}$ is the partition inside the rectangle of height $m_i(\nu^{(k)})$ and width $P_i^{(k)}(\nu)$ given by the labels of the parts of $\nu^{(k)}$ of size i.

Since the q-binomial $\begin{bmatrix} m+p \\ m \end{bmatrix}$ is the generating function of partitions with at most m parts each not exceeding p [1, Theorem 3.1], Theorem 5.2 is equivalent to the following theorem.

Theorem 5.3. *For λ a partition and R a sequence of rectangles*

$$K_{\lambda R}(q) = \sum_{(\nu,J)\in \mathrm{RC}(\lambda;R)} q^{c(\nu,J)}. \tag{5.5}$$

5.2. **Switching between quantum and coquantum numbers.** Let $\theta_R : \mathrm{RC}(\lambda;R) \to \mathrm{RC}(\lambda;R)$ be the involution that complements quantum numbers. More precisely, for $(\nu,J) \in \mathrm{RC}(\lambda;R)$, replace every string $(i,x) \in (\nu,J)^{(k)}$ by $(i, P_i^{(k)}(\nu) - x)$. The notation here differs from that in [25], in which θ_R is an involution on $\mathrm{RC}(\lambda^t;R^t)$.

Lemma 5.4. $c(\theta_R(\nu,J)) = ||R|| - \mathrm{cc}(\nu,J)$ *for all* $(\nu,J) \in \mathrm{RC}(\lambda;R)$.

Proof. Let $\theta_R(\nu,J) = (\nu',J')$. It follows immediately from the definitions that $\nu' = \nu$. In particular ν and ν' have the same vacancy numbers and $|J'| = |P| - |J|$. Then

$$\begin{aligned} c(\theta_R(\nu,J)) &= c(\nu',J') = ||R|| - \mathrm{cc}(\nu') - |P| + |J'| \\ &= ||R|| - \mathrm{cc}(\nu) - |J| = ||R|| - \mathrm{cc}(\nu,J). \end{aligned}$$

□

There is a bijection $\mathrm{tr}_{\mathrm{RC}} : \mathrm{RC}(\lambda;R) \to \mathrm{RC}(\lambda^t;R^t)$ that has the property

$$\mathrm{cc}(\mathrm{tr}_{\mathrm{RC}}(\nu,J)) = ||R|| - \mathrm{cc}(\nu,J) \tag{5.6}$$

for all $(\nu,J) \in \mathrm{RC}(\lambda;R)$; see the proof of [26, Proposition 11].

5.3. **RC's and level-restriction.** Here we introduce the most important new definition in this paper, namely, that of a level-restricted rigged configuration.

Say that a partition λ is restricted of level ℓ if $\lambda_1 - \lambda_n \le \ell$, recalling that it is assumed that all partitions have at most n parts, some of which may be zero. Fix a shape λ and a sequence of rectangles R that are all restricted of level ℓ. Define $\widetilde{\ell} = \ell - (\lambda_1 - \lambda_n)$, which is nonnegative by assumption.

Set $\lambda' = (\lambda_1 - \lambda_n, \dots, \lambda_{n-1} - \lambda_n)^t$ and denote the set of all column-strict tableaux of shape λ' over the alphabet $\{1,2,\dots,\lambda_1 - \lambda_n\}$ by $\mathrm{CST}(\lambda')$. Define a table of modified vacancy numbers depending on $\nu \in \mathrm{C}(\lambda;R)$ and $t \in \mathrm{CST}(\lambda')$ by

(5.7)

$$P_i^{(k)}(\nu,t) = P_i^{(k)}(\nu) - \sum_{j=1}^{\lambda_k-\lambda_n} \chi(i \ge \widetilde{\ell} + t_{j,k}) + \sum_{j=1}^{\lambda_{k+1}-\lambda_n} \chi(i \ge \widetilde{\ell} + t_{j,k+1})$$

for all $i, k \geq 1$, where $\chi(S) = 1$ if the statement S is true and $\chi(S) = 0$ otherwise, and $t_{j,k}$ is the (j,k)-th entry of t. Finally let $x_i^{(k)}$ be the largest part of the partition $J_i^{(k)}$; if $J_i^{(k)}$ is empty set $x_i^{(k)} = 0$.

Definition 5.5. Say that $(\nu, J) \in \mathrm{RC}(\lambda; R)$ is restricted of level ℓ provided that

1. $\nu_1^{(k)} \leq \ell$ for all k.
2. There exists a tableau $t \in \mathrm{CST}(\lambda')$, such that for every $i, k \geq 1$,
$$x_i^{(k)} \leq P_i^{(k)}(\nu, t).$$

Let $\mathrm{C}^\ell(\lambda; R)$ be the set of all $\nu \in \mathrm{C}(\lambda; R)$ such that the first condition holds, and denote by $\mathrm{RC}^\ell(\lambda; R)$ the set of $(\nu, J) \in \mathrm{RC}(\lambda; R)$ that are restricted of level ℓ.

Note in particular that the second condition requires that $P_i^{(k)}(\nu, t) \geq 0$ for all $i, k \geq 1$.

Example 5.6. Let us consider Definition 5.5 for two classes of shapes λ more closely:

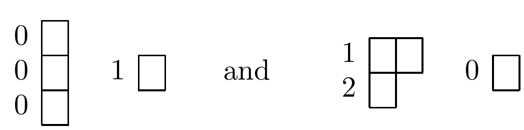
1. Vacuum case: Let $\lambda = (a^n)$ be rectangular with n rows. Then $\lambda' = \emptyset$ and $P_i^{(k)}(\nu, \emptyset) = P_i^{(k)}(\nu)$ for all $i, k \geq 1$ so that the modified vacancy numbers are equal to the vacancy numbers.
2. Two-corner case: Let $\lambda = (a^\alpha, b^\beta)$ with $\alpha + \beta = n$ and $a > b$. Then $\lambda' = (\alpha^{a-b})$ and there is only one tableau t in $\mathrm{CST}(\lambda')$, namely the Yamanouchi tableau of shape λ'. Since $t_{j,k} = j$ for $1 \leq k \leq \alpha$ we find that
$$P_i^{(k)}(\nu, t) = P_i^{(k)}(\nu) - \delta_{k,\alpha} \max\{i - \widetilde{\ell}, 0\}$$
for $1 \leq i \leq \ell$ and $1 \leq k < n$. We wish to thank Anatol Kirillov for communicating this formula to us [27].

Our main result is the following formula for the level-restricted generalized Kostka polynomial:

Theorem 5.7. *Let ℓ be a positive integer. For λ a partition and R a sequence of rectangles both restricted of level ℓ,*
$$K^\ell_{\lambda R}(q) = \sum_{(\nu,J) \in \mathrm{RC}^\ell(\lambda; R)} q^{c(\nu,J)}.$$

The proof of this theorem is given in Section 8.

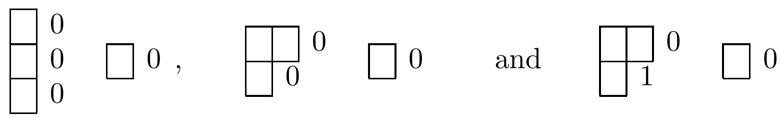
Example 5.8. Consider $n = 3$, $\ell = 2$, $\lambda = (3, 2, 1)$ and $R = ((2), (1)^4)$. Then

(5.8) 

are in $\mathrm{C}^\ell(\lambda; R)$ where again the vacancy numbers are indicated to the left of each part. The set $\mathrm{CST}(\lambda')$ consists of the two elements

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}.$$

Since $\widetilde{\ell} = 0$ the three rigged configurations



are restricted of level 2 with charges $2, 3, 4$, respectively. The riggings are given on the right of each part. Hence $K^2_{\lambda R}(q) = q^2 + q^3 + q^4$.

In contrast to this, the Kostka polynomial $K_{\lambda\mu}(q)$ is obtained by summing over both configurations in (5.8) with all possible riggings below the vacancy numbers. This amounts to $K_{\lambda\mu}(q) = q^2 + 2q^3 + 2q^4 + 2q^5 + q^6$.

In Section 7 we will use Theorem 5.7 to obtain explicit expressions for type A branching functions. The results suggest that it is also useful to consider the following sets of rigged configurations with imposed minima on the set of riggings.

Let $\rho \subset \lambda$ be a partition and $R_\rho = ((1^{\rho^t_1}), (1^{\rho^t_2}), \ldots, (1^{\rho^t_n}))$, the sequence of single columns of height ρ^t_i. Set $\rho' = (\rho_1 - \rho_n, \ldots, \rho_{n-1} - \rho_n)^t$ and

$$M_i^{(k)}(t) = \sum_{j=1}^{\rho_k - \rho_n} \chi(i \le \rho_1 - \rho_n - t_{j,k}) - \sum_{j=1}^{\rho_{k+1} - \rho_n} \chi(i \le \rho_1 - \rho_n - t_{j,k+1})$$

for all $t \in \mathrm{CST}(\rho')$. Then define $\mathrm{RC}^\ell(\lambda, \rho; R)$ to be the set of all $(\nu, J) \in \mathrm{RC}^\ell(\lambda; R_\rho \cup R)$ such that there exists a $t \in \mathrm{CST}(\rho')$ such that $M_i^{(k)}(t) \le x$ for $(i, x) \in (\nu, J)^{(k)}$ and $M_i^{(k)}(t) \le P_i^{(k)}(\nu)$ for all $i, k \ge 1$. Note that the second condition is obsolete if i occurs as a part in $\nu^{(k)}$ since by definition $M_i^{(k)}(t) \le x \le P_i^{(k)}(\nu)$ for all $(i, x) \in (\nu, J)^{(k)}$.

Conjecture 8.3 asserts that the set $\mathrm{RC}^\ell(\lambda, \rho; R)$ corresponds to the set of all level-ℓ restricted Littlewood–Richardson tableaux with a fixed subtableaux of shape ρ.

6. Fermionic expression of level-restricted generalized Kostka polynomials

6.1. **Fermionic expression.** Similarly to the Kostka polynomial case, one can rewrite the expression of the level-restricted generalized Kostka polynomials of Theorem 5.7 in fermionic form.

Lemma 6.1. *For all $\nu \in \mathrm{C}^\ell(\lambda, R)$, $t \in \mathrm{CST}(\lambda')$ and $1 \le k < n$, we have $P_i^{(k)}(\nu, t) = 0$ for $i \ge \ell$.*

Proof. Since $\nu_1^{(k)} \leq \ell$ it follows from [26, (11.2)] that $P_i^{(k)}(\nu) = \lambda_k - \lambda_{k+1}$ for $i \geq \ell$. Since t is over the alphabet $\{1, 2, \ldots, \lambda_1 - \lambda_n\}$ this implies for $i \geq \ell$

$$\begin{aligned} P_i^{(k)}(\nu, t) =& P_i^{(k)}(\nu) - \sum_{j=1}^{\lambda_k - \lambda_n} \chi(i \geq \widetilde{\ell} + t_{j,k}) + \sum_{j=1}^{\lambda_{k+1} - \lambda_n} \chi(i \geq \widetilde{\ell} + t_{j,k+1}) \\ =& \lambda_k - \lambda_{k+1} - (\lambda_k - \lambda_n) + (\lambda_{k+1} - \lambda_n) = 0. \end{aligned}$$

□

Let $\mathrm{SCST}(\lambda')$ be the set of all nonempty subsets of $\mathrm{CST}(\lambda')$. Furthermore set $P_i^{(k)}(\nu, S) = \min\{P_i^{(k)}(\nu, t) | t \in S\}$ for $S \in \mathrm{SCST}(\lambda')$. Then by inclusion-exclusion the set of allowed rigging for a given configuration $\nu \in \mathrm{C}^\ell(\lambda; R)$ is given by

$$\sum_{S \in \mathrm{SCST}(\lambda')} (-1)^{|S|+1} \{J | x_i^{(k)} \leq P_i^{(k)}(\nu, S)\}.$$

Since the q-binomial $\left[\begin{smallmatrix} m+p \\ m \end{smallmatrix}\right]$ is the generating function of partitions with at most m parts each not exceeding p and since $P_\ell^{(k)}(\nu, S) = 0$ by Lemma 6.1 the level-ℓ restricted generalized Kostka polynomials has the following fermionic form.

Theorem 6.2.

$$K_{\lambda R}^\ell(q) = \sum_{S \in \mathrm{SCST}(\lambda')} (-1)^{|S|+1} \sum_{\nu \in \mathrm{C}^\ell(\lambda; R)} q^{c(\nu)} \prod_{i=1}^{\ell-1} \prod_{k=1}^{n-1} \begin{bmatrix} m_i(\nu^{(k)}) + P_i^{(k)}(\nu, S) \\ m_i(\nu^{(k)}) \end{bmatrix}.$$

In Section 7 we will derive new expressions for branching functions of type A as limits of the level-restricted generalized Kostka polynomials. To this end we need to reformulate the fermionic formula of Theorem 6.2 in terms of a so-called $(\boldsymbol{m}, \boldsymbol{n})$-system. Set

$$\begin{aligned} m_i^{(a)} &= P_i^{(a)}(\nu, S) = P_i^{(a)}(\nu) + f_i^{(a)}(S), \\ n_i^{(a)} &= m_i(\nu^{(a)}), \end{aligned}$$

and $L_i^{(a)} = \sum_{j=1}^{L} \chi(i = \mu_j)\chi(a = \eta_j)$ for $1 \le i \le \ell$ and $1 \le a \le n$ which is the number of rectangles in R of shape (i^a). Then

$$\begin{aligned}
&- m_{i-1}^{(a)} + 2m_i^{(a)} - m_{i+1}^{(a)} - n_i^{(a-1)} + 2n_i^{(a)} - n_i^{(a+1)} \\
=&(\alpha_i^{(a-1)} - 2\alpha_i^{(a)} + \alpha_i^{(a+1)}) - (\alpha_{i+1}^{(a-1)} - 2\alpha_{i+1}^{(a)} + \alpha_{i+1}^{(a+1)}) \\
&+ \sum_{k=1}^{L} \delta_{a,\eta_k}(-\min\{i-1,\mu_k\} + 2\min\{i,\mu_k\} - \min\{i+1,\mu_k\}) \\
&- f_{i-1}^{(a)}(S) + 2f_i^{(a)}(S) - f_{i+1}^{(a)}(S) \\
&- (\alpha_i^{(a-1)} - \alpha_{i+1}^{(a-1)}) + 2(\alpha_i^{(a)} - \alpha_{i+1}^{(a)}) - (\alpha_i^{(a+1)} - \alpha_{i+1}^{(a+1)}) \\
=&L_i^{(a)} - f_{i-1}^{(a)}(S) + 2f_i^{(a)}(S) - f_{i+1}^{(a)}(S).
\end{aligned}$$

At this stage it is convenient to introduce vector notation. For a matrix $v_i^{(a)}$ with indices $1 \le i \le \ell - 1$ and $1 \le a \le n - 1$ define

$$\boldsymbol{v} = \sum_{i=1}^{\ell-1}\sum_{a=1}^{n-1} v_i^{(a)} \boldsymbol{e}_i \otimes \boldsymbol{e}_a,$$

where $\boldsymbol{e}_i$ and $\boldsymbol{e}_a$ are the canonical basis vectors of $\mathbb{Z}^{\ell-1}$ and $\mathbb{Z}^{n-1}$, respectively. Define

$$u_i^{(a)}(S) = -f_{i-1}^{(a)}(S) + 2f_i^{(a)}(S) - f_{i+1}^{(a)}(S)$$

which in vector notation reads

$$\boldsymbol{u}(S) = (C \otimes I)\boldsymbol{f}(S) + \sum_{a=1}^{n-1}(\lambda_a - \lambda_{a+1})\boldsymbol{e}_{\ell-1} \otimes \boldsymbol{e}_a, \tag{6.1}$$

where C is the Cartan matrix of type A and I is the identity matrix. Since $n_i^{(0)} = n_i^{(n)} = m_0^{(k)} = 0$ and $m_\ell^{(k)} = 0$ by Lemma 6.1 it follows that

$$(C \otimes I)\boldsymbol{m} + (I \otimes C)\boldsymbol{n} = \boldsymbol{L} + \boldsymbol{u}(S). \tag{6.2}$$

In terms of the new variables the condition (5.1) on $|\nu^{(a)}|$ becomes

$$n_\ell^{(a)} = -\boldsymbol{e}_{\ell-1} \otimes \boldsymbol{e}_a (C^{-1} \otimes I)\boldsymbol{n} - \frac{1}{\ell}\sum_{j=1}^{a} \lambda_j + \frac{1}{\ell}\sum_{i=1}^{\ell}\sum_{b=1}^{n} i \min\{a,b\} L_i^{(b)}, \tag{6.3}$$

where we used $C_{ij}^{-1} = \min\{i,j\} - ij/\ell$ if C is $(\ell-1)\times(\ell-1)$-dimensional and $\sum_{b=1}^{n}\sum_{i=1}^{\ell} ibL_i^{(b)} = |\lambda|$.

Lemma 6.3. *In terms of the above $(\boldsymbol{m},\boldsymbol{n})$-system*

$$\begin{aligned}
c(\nu) = \frac{1}{2}\boldsymbol{m}(C \otimes C^{-1})\boldsymbol{m} - \boldsymbol{m}(I \otimes C^{-1})\boldsymbol{u}(S) \\
+ \frac{1}{2}\boldsymbol{u}(S)(C^{-1} \otimes C^{-1})\boldsymbol{u}(S) + g(R,\lambda)
\end{aligned} \tag{6.4}$$

where

$$g(R,\lambda)=\|R\|-\frac{1}{2}\sum_{a,b=1}^{n-1}\sum_{j=1}^{\ell}C_{ab}^{-1}L_j^{(a)}\overline{L}_j^{(b)}+\frac{1}{2\ell}\sum_{j=1}^{n}(\lambda_j-\frac{1}{n}|\lambda|)^2$$

and $\overline{L}_i^{(a)}=\sum_{j=1}^{\ell}\min\{i,j\}L_j^{(a)}$.

Proof. By definition $c(\nu)=\|R\|-\mathrm{cc}(\nu)-|P|$. Note that

$$\begin{aligned}|P|&=\sum_{i=1}^{\ell}\sum_{k=1}^{n-1}m_i(\nu^{(k)})P_i^{(k)}(\nu)\\&=\sum_{i=1}^{\ell}\sum_{k=1}^{n-1}(\alpha_i^{(k)}-\alpha_{i+1}^{(k)})(\sum_{j=1}^{i}(\alpha_j^{(k-1)}-2\alpha_j^{(k)}+\alpha_j^{(k+1)})+\overline{L}_i^{(k)})\\&=-2\mathrm{cc}(\nu)+\sum_{i=1}^{\ell}\sum_{k=1}^{n-1}n_i^{(k)}\overline{L}_i^{(k)}.\end{aligned}$$

Hence eliminating $\mathrm{cc}(\nu)$ in favor of $|P|$ yields

$$c(\nu)=\|R\|-\frac{1}{2}|P|-\frac{1}{2}\sum_{i=1}^{\ell}\sum_{k=1}^{n-1}n_i^{(k)}\overline{L}_i^{(k)}.$$

On the other hand, using $n_i^{(k)}=m_i(\nu^{(k)})$ and $P_\ell^{(k)}(\nu)=\lambda_k-\lambda_{k+1}$,

$$|P|=\boldsymbol{n}(I\otimes I)\boldsymbol{P}(\nu)+\sum_{k=1}^{n-1}n_\ell^{(k)}(\lambda_k-\lambda_{k+1})$$

so that

$$c(\nu)=\|R\|-\frac{1}{2}\boldsymbol{n}(I\otimes I)(\boldsymbol{P}(\nu)+\overline{\boldsymbol{L}})-\frac{1}{2}\sum_{k=1}^{n-1}n_\ell^{(k)}(\lambda_k-\lambda_{k+1}+\overline{L}_\ell^{(k)}). \tag{6.5}$$

Eliminating $\boldsymbol{n}$ in favor of $\boldsymbol{m}$ using (6.2) and substituting $\boldsymbol{P}(\nu)=\boldsymbol{m}-\boldsymbol{f}(S)$ yields

$$\begin{aligned}-\frac{1}{2}\boldsymbol{n}(I\otimes I)(\boldsymbol{P}(\nu)+\overline{\boldsymbol{L}})=\frac{1}{2}\boldsymbol{m}\{C\otimes C^{-1}(\boldsymbol{m}+\overline{\boldsymbol{L}}-\boldsymbol{f}(S))-I\otimes C^{-1}(\boldsymbol{L}+\boldsymbol{u}(S))\}\\-\frac{1}{2}(\boldsymbol{L}+\boldsymbol{u}(S))(I\otimes C^{-1})(\overline{\boldsymbol{L}}-\boldsymbol{f}(S)).\end{aligned}$$

Similarly, replacing $\boldsymbol{n}$ by $\boldsymbol{m}$ in (6.3) we obtain

$$\begin{aligned}n_\ell^{(a)}=\boldsymbol{e}_{\ell-1}\otimes\boldsymbol{e}_a(I\otimes C^{-1}\boldsymbol{m}-C^{-1}\otimes C^{-1}\boldsymbol{u}(S))\\-\frac{1}{\ell}\sum_{j=1}^{a}(\lambda_j-\frac{1}{n}|\lambda|)+\sum_{b=1}^{n-1}C_{ab}^{-1}L_\ell^{(b)}.\end{aligned} \tag{6.6}$$

Inserting these equations into (6.5), trading $\boldsymbol{f}(S)$ for $\boldsymbol{u}(S)$ by (6.1) and using

$$(C\otimes I)\overline{\boldsymbol{L}}-\boldsymbol{L}-\sum_{a=1}^{n-1}\boldsymbol{e}_{\ell-1}\otimes\boldsymbol{e}_a\overline{L}_\ell^{(a)}=0$$

results in the claim of the lemma. □

As a corollary of Lemma 6.3 and Theorem 6.2 we obtain the following expression for the level-restricted generalized Kostka polynomial

$$(6.7)\quad K^\ell_{\lambda R}(q)=q^{g(R,\lambda)}\sum_{S\in\mathrm{SCST}(\lambda')}(-1)^{|S|+1}q^{\frac{1}{2}\boldsymbol{u}(S)C^{-1}\otimes C^{-1}\boldsymbol{u}(S)}\times\sum_{\boldsymbol{m}}q^{\frac{1}{2}\boldsymbol{m}C\otimes C^{-1}\boldsymbol{m}-\boldsymbol{m}I\otimes C^{-1}\boldsymbol{u}(S)}\begin{bmatrix}\boldsymbol{m}+\boldsymbol{n}\\ \boldsymbol{m}\end{bmatrix}$$

where $\boldsymbol{n}$ is determined by (6.2), the sum over $\boldsymbol{m}$ is such that

$$\boldsymbol{e}_{\ell-1}\otimes\boldsymbol{e}_a(I\otimes C^{-1}\boldsymbol{m}-C^{-1}\otimes C^{-1}\boldsymbol{u}(S))-\frac{1}{\ell}\sum_{j=1}^{a}(\lambda_j-\frac{1}{n}|\lambda|)+\sum_{b=1}^{n-1}C^{-1}_{ab}L_\ell^{(b)}\in\mathbb{Z},$$

for all $1\leq a\leq n-1$ and $\begin{bmatrix}\boldsymbol{m}+\boldsymbol{n}\\ \boldsymbol{m}\end{bmatrix}=\prod_{i=1}^{\ell-1}\prod_{k=1}^{n-1}\begin{bmatrix}m_i^{(k)}+n_i^{(k)}\\ m_i^{(k)}\end{bmatrix}$.

Now consider the second case of Example 5.6, namely $\lambda=(a^\alpha,b^\beta)$ with $a>b$ and $\alpha+\beta=n$. Then $\mathrm{SCST}(\lambda')$ only contains the element $S=\{t\}$ where t is the Yamanouchi tableau of shape λ' and $\boldsymbol{u}(S)=\boldsymbol{e}_{\tilde{\ell}}\otimes\boldsymbol{e}_\alpha$. In the vacuum case, that is, when $\lambda=((\frac{|\lambda|}{n})^n)$, the set $\mathrm{SCST}(\lambda')$ only contains $S=\{\emptyset\}$ and $\boldsymbol{u}(S)=\boldsymbol{f}(S)=0$. In this case (6.7) simplifies to

$$K^\ell_{\lambda R}(q)=q^{g(R,\lambda)}\sum_{\boldsymbol{m}}q^{\frac{1}{2}\boldsymbol{m}C\otimes C^{-1}\boldsymbol{m}}\begin{bmatrix}\boldsymbol{m}+\boldsymbol{n}\\ \boldsymbol{m}\end{bmatrix}.$$

When R is a sequence of single boxes this proves [8, Theorem 1][1]. When R is a sequence of single rows or single columns this settles [12, Conjecture 4.7].

6.2. **Polynomial Rogers–Ramanujan-type identities.** Let $\overline{W}$ be the Weyl group of sl_n, $M=\{\beta\in\mathbb{Z}^n|\sum_{i=1}^n\beta_i=0\}$ be the root lattice, ρ the half-sum of the positive roots, and $(\cdot|\cdot)$ the standard symmetric bilinear form. Recall the energy function (3.9). It was shown in [31] that

$$(6.8)\quad K^\ell_{\lambda R}(q)=\sum_{\tau\in\overline{W}}\sum_{\beta\in M}\sum_{\substack{b\in\mathcal{P}_R\\ \mathrm{wt}(b)=-\rho+\tau^{-1}(\overline{\lambda}-(\ell+n)\beta+\rho)}}(-1)^\tau q^{-\frac{1}{2}(\ell+n)(\beta|\beta)+(\overline{\lambda}+\rho|\beta)+E(b)}.$$

[1] We believe that the proof given in [8] is incomplete.

Equating (6.7) and (6.8) gives rise to polynomial Rogers–Ramanujan-type identities. For the vacuum case, that is, when the partition λ is rectangular with n rows, this proves [33, Eq. (9.2)][2].

7. New expressions for type A branching functions

The coset branching functions $b^{\Lambda}_{\Lambda'\Lambda''}$ labeled by the three weights $\Lambda, \Lambda', \Lambda''$ have a natural finitization in terms of $(\Lambda'+\Lambda'')$-restricted crystals. For certain triples of weights these can be reformulated in terms of level-restricted paths, which in turn yield an expression of the type A branching functions as a limit of the level-restricted generalized Kostka polynomials. Together with the results of the last section this implies new fermionic expressions for type A branching functions at certain triples of weights.

7.1. **Branching function in terms of paths.** Let $\Lambda, \Lambda', \Lambda'' \in P_{\mathrm{cl}}$ be dominant integral weights of levels ℓ, ℓ', and ℓ'' respectively, where $\ell = \ell' + \ell''$. The branching function $b^{\Lambda}_{\Lambda'\Lambda''}(z)$ is the formal power series defined by

$$b^{\Lambda}_{\Lambda'\Lambda''}(z) = \sum_{m\geq 0} z^m c^{\mathrm{af}(\Lambda)-m\delta}_{\mathrm{af}(\Lambda'),\mathrm{af}(\Lambda'')}$$

where $c^{\mathrm{af}(\Lambda)-m\delta}_{\mathrm{af}(\Lambda'),\mathrm{af}(\Lambda'')}$ is the multiplicity of the irreducible integrable highest weight $U_q(\widehat{sl}_n)$-module $\mathbb{V}(\mathrm{af}(\Lambda)-m\delta)$ in the tensor product $\mathbb{V}(\mathrm{af}(\Lambda')) \otimes \mathbb{V}(\mathrm{af}(\Lambda''))$.

The desired multiplicity is equal to the number of $\widehat{sl}_n$-highest weight vectors of weight $\mathrm{af}(\Lambda)-m\delta$ in the tensor product $\mathbb{B}(\mathrm{af}(\Lambda')) \otimes \mathbb{B}(\mathrm{af}(\Lambda''))$, that is, the number of elements $b' \otimes b'' \in \mathbb{B}(\mathrm{af}(\Lambda')) \otimes \mathbb{B}(\mathrm{af}(\Lambda''))$ such that $\mathrm{wt}(b'\otimes b'') = \mathrm{af}(\Lambda)-m\delta$ and $\epsilon_i(b'\otimes b'')=0$ for all $i \in I$. By (3.5), $b''=u_{\Lambda''}$, b' is Λ''-restricted, and $\mathrm{wt}(b') = \mathrm{af}(\Lambda-\Lambda'')-m\delta$.

Let $\mathcal{B}$ be a perfect crystal of level ℓ'. Using the isomorphism $\mathbb{B}(\Lambda') \cong \mathbb{P}(\Lambda',\mathcal{B})$ let $b' = b'_1 \otimes b'_2 \otimes \cdots$ and $\overline{b} \in \mathbb{P}(\Lambda',\mathcal{B})$ be the ground state path. Suppose N is such that for all $j>N$, $b'_j = \overline{b}_j$. Write $b = b'_1 \otimes \cdots \otimes b'_N$. In type $A^{(1)}_{n-1}$ the period of the ground state path $\overline{b}$ always divides n. Choose N to be a multiple of n, so that $b' = b \otimes \overline{b}$ and $\overline{b}_{N+1} = \overline{b}_1$.

Then the above desired highest weight vectors have the form $b'\otimes b'' = (b\otimes u_{\Lambda'}) \otimes u_{\Lambda''} \in \mathcal{B}^{\otimes N} \otimes \mathbb{B}(\mathrm{af}(\Lambda')) \otimes \mathbb{B}(\mathrm{af}(\Lambda''))$. But there is an embedding $\mathbb{B}(\mathrm{af}(\Lambda'+\Lambda'')) \hookrightarrow \mathbb{B}(\mathrm{af}(\Lambda')) \otimes \mathbb{B}(\mathrm{af}(\Lambda''))$ defined by $u_{\Lambda'+\Lambda''} \to u_{\Lambda'} \otimes u_{\Lambda''}$. With this rephrasing of the conditions on b and taking limits, we have

$$b^{\Lambda}_{\Lambda'\Lambda''}(z) = \lim_{\substack{N\to\infty \\ N\in n\mathbb{Z}}} z^{-E_N(\overline{b}_1\otimes\cdots\otimes\overline{b}_N)} \sum_{b\in\mathcal{H}(\Lambda'+\Lambda'',\mathcal{B}^{\otimes N},\Lambda)} z^{E_N(b)} \tag{7.1}$$

where $E_N : \mathcal{B}^{\otimes N} \to \mathbb{Z}$ is given by $E_N(b) = E(b\otimes \overline{b}_{N+1}) = E(b \otimes \overline{b}_1)$ and E is the energy function on finite paths.

[2]The definition of level-restricted path as given in [33, p. 394] only works when R (or μ therein) consists of single rows; otherwise the description of Section 3.7 should be used.

Our goal is to express (7.1) in terms of level-restricted generalized Kostka polynomials. We find that this is possible for certain triples of weights. Using the results of Section 6 this provides explicit formulas for the branching functions.

7.2. **Reduction to level-restricted paths.** The first step in the transformation of (7.1) is to replace the condition of $(\Lambda' + \Lambda'')$-restrictedness by level ℓ restrictedness. This is achieved at the cost of appending a fixed inhomogeneous path.

Consider any tensor product $\mathcal{B}''$ of perfect crystals each of which has level at most ℓ'' (the level of Λ''), such that there is an element $y'' \in \mathcal{H}(\ell''\Lambda_0, \mathcal{B}'', \Lambda'')$. We indicate how such a $\mathcal{B}''$ and y'' can be constructed explicitly. Let λ be the partition with strictly less than n rows with $\langle h_i \, , \, \Lambda'' \rangle$ columns of length i for $1 \le i \le n-1$. Let Y_λ be the Yamanouchi tableau of shape λ. Then any factorization (in the plactic monoid) of Y_λ into a sequence of rectangular tableaux, yields such a $\mathcal{B}''$ and y''.

Example 7.1. Let $n = 6$, $\ell'' = 5$, $\Lambda'' = \Lambda_0 + 2\Lambda_2 + \Lambda_3 + \Lambda_4$. Then $\lambda = (4, 4, 2, 1)$ (its transpose is $\lambda^t = (4, 3, 2, 2)$) and

$$Y_\lambda = \begin{matrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & & \\ 4 & & & \end{matrix} .$$

One way is to factorize into single columns: $\mathcal{B}'' = \mathcal{B}^{2,1} \otimes \mathcal{B}^{2,1} \otimes \mathcal{B}^{3,1} \otimes \mathcal{B}^{4,1}$ and $y'' = y_4 \otimes y_3 \otimes y_2 \otimes y_1$ where each y_j is an sl_n highest weight vector, namely, the j-th column of Y_λ. Another way is to factorize into the minimum number of rectangles by slicing Y_λ vertically. This yields $\mathcal{B}'' = \mathcal{B}^{2,2} \otimes \mathcal{B}^{3,1} \otimes \mathcal{B}^{4,1}$; again the factors of $y'' = y_3 \otimes y_2 \otimes y_1$ are the sl_n highest weight vectors, namely,

$$y_3 = \begin{matrix} 1 & 1 \\ 2 & 2 \end{matrix}, \qquad y_2 = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}, \qquad y_1 = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} .$$

Consider also a tensor product $\mathcal{B}'$ of perfect crystals such that there is an element $y' \in \mathcal{H}(\ell'\Lambda_0, \mathcal{B}', \Lambda')$. Then $y = y' \otimes y'' \in \mathcal{H}(\ell\Lambda_0, \mathcal{B}' \otimes \mathcal{B}'', \Lambda' + \Lambda'')$. Instead of $b \in \mathcal{H}(\Lambda' + \Lambda'', \mathcal{B}^{\otimes N}, \Lambda)$, we work with $b \otimes y$ where $b \otimes y$ is restricted of level ℓ.

This trick doesn't help unless one can recover the correct energy function directly from $b \otimes y$. Let p be the first N steps of the ground state path $\overline{b} \in \mathbb{P}(\Lambda', \mathcal{B})$. Define the normalized energy function on $\mathcal{B}^{\otimes N}$ by $\overline{E}(b) = E(b \otimes y') - E(p \otimes y')$. A priori it depends on Λ', $\mathcal{B}$, and y'. The energy function occurring in the branching function is $E'(b) = E(b \otimes \overline{b}_1) - E(p \otimes \overline{b}_1)$.

Lemma 7.2. $\overline{E} = E'$.

Proof. It suffices to show that the function $\mathcal{B}^{\otimes N} \to \mathbb{Z}$ given by $b \mapsto E(b \otimes y') - E(b \otimes \overline{b}_1)$ is constant. Using the definition (3.9) and the fact that b is homogeneous of length N, we have

$$E(b \otimes y') = E(b) + NE(b_N \otimes y') - (N-1)E(y').$$

Similarly $E(b\otimes \overline{b}_1) = E(b) + NE(b_N \otimes \overline{b}_1)$. Therefore $E(b\otimes y') - E(b\otimes \overline{b}_1) = N(E(b_N \otimes y') - E(b_N \otimes \overline{b}_1)) - (N-1)E(y')$. Thus it suffices to show that the function $\mathcal{B} \to \mathbb{Z}$ given by $b' \mapsto E(b' \otimes y') - E(b' \otimes \overline{b}_1)$ is a constant function.

Suppose first that $\epsilon_i(b') > \langle h_i , \Lambda' \rangle$ for some $1 \leq i \leq n-1$. By the construction of y' and $\overline{b}_1$, $\phi_i(y') = \langle h_i , \Lambda' \rangle = \phi_i(\overline{b}_1)$ for $1 \leq i \leq n-1$, since $\phi(\overline{b}_1) = \Lambda'$. Then $e_i(b' \otimes y') = e_i(b') \otimes y'$ and $e_i(b' \otimes \overline{b}_1) = e_i(b') \otimes \overline{b}_1$ by (3.4). Passing from b' to $e_i(b')$ repeatedly, the values of the energy functions are constant, so it may be assumed that $b' \otimes y'$ is a sl_n highest weight vector; in particular, $\epsilon_i(b') \leq \langle h_i , \Lambda' \rangle$ for all $1 \leq i \leq n-1$.

Next suppose that $\epsilon_0(b') > \langle h_0 , \Lambda' \rangle$. Now $\phi_0(y') = 0$ and $\phi_0(\overline{b}_1) = \langle h_0 , \Lambda' \rangle$. By (3.4) $e_0(b' \otimes \overline{b}_1) = e_0(b') \otimes \overline{b}_1$ and $e_0(b' \otimes y') = e_0(b') \otimes y'$. By (3.8) and the fact that the local isomorphism on $\mathcal{B} \otimes \mathcal{B}$ is the identity, we have $E(e_0(b' \otimes \overline{b}_1)) = E(b' \otimes \overline{b}_1) - 1$.

To show that $E(e_0(b' \otimes y')) = E(b' \otimes y') - 1$ we check the conditions of Lemma 3.2. By (3.1) $\epsilon_0(y') = \phi_0(y') - \langle h_0 , \mathrm{wt}(y') \rangle = 0 - \langle h_0 , \Lambda' - \ell'\Lambda_0 \rangle = \ell' - \langle h_0 , \Lambda' \rangle$. Also by (3.5), since $\phi_0(y') = 0$, we have $\epsilon_0(b' \otimes y') = \epsilon_0(b') + \epsilon_0(y') > \langle h_0 , \Lambda' \rangle + \ell' - \langle h_0 , \Lambda' \rangle = \ell'$. Let $z \otimes x$ be the image of $b' \otimes y'$ under an arbitrary composition of local isomorphisms. Since $b' \otimes y'$ is an sl_n highest weight vector, so is $z \otimes x$ and x. Now x is the sl_n-highest weight vector in a perfect crystal of level at most ℓ', so $\phi_0(x) = 0$ and $\epsilon_0(x) \leq \ell'$. But $\ell' < \epsilon_0(b' \otimes y') = \epsilon_0(z \otimes x) = \epsilon_0(z) + \epsilon_0(x)$ so that $\epsilon_0(z) > 0$. By (3.4) $e_0(z \otimes x) = e_0(z) \otimes x$. So $E(e_0(b' \otimes y')) = E(b' \otimes y') - 1$ by Lemma 3.2.

By induction we may now assume that $\epsilon_0(b') \leq \langle h_0 , \Lambda' \rangle$. But then $\sum_i \epsilon_i(b') \leq \sum_i \langle h_i , \Lambda' \rangle$, or $\langle c , \epsilon(b') \rangle \leq \langle c , \Lambda' \rangle = \ell'$. Since $b' \in \mathcal{B}$ and $\mathcal{B}$ is a perfect crystal of level ℓ', b' must be the unique element of $\mathcal{B}$ such that $\epsilon(b') = \Lambda'$. Thus the function $\mathcal{B} \to \mathbb{Z}$ given by $b' \mapsto E(b' \otimes y') - E(b' \otimes \overline{b}_1)$ is constant on $\mathcal{B}$ if it is constant on the singleton set $\{\epsilon^{-1}(\Lambda')\}$, which it obviously is. □

7.3. **Explicit ground state energy.** To go further, an explicit formula for the value $E(p \otimes y')$ is required. This is achieved in (7.2). The derivation makes use of the following explicit construction of the local isomorphism.

Theorem 7.3. *Let $\mathcal{B} = \mathcal{B}^{k,\ell}$ be a perfect crystal of level ℓ, $\Lambda, \Lambda' \in (P^+_{\mathrm{cl}})_\ell$, $\mathcal{B}'$ a perfect crystal of level $\ell' \leq \ell$, and $b \in \mathcal{H}(\Lambda', \mathcal{B}', \Lambda)$. Let $x \in \mathcal{B}$ (resp. $y \in \mathcal{B}$) be the unique element such that $\epsilon(x) = \Lambda$ (resp. $\epsilon(y) = \Lambda'$). Then under the local isomorphism $\mathcal{B} \otimes \mathcal{B}' \cong \mathcal{B}' \otimes \mathcal{B}$, we have $x \otimes b \cong \psi^k(b) \otimes y$.*

The proof requires several technical lemmas and is given in the next section.

Example 7.4. Let $n = 5$, $\ell = 4$, $k = 2$, $\Lambda' = \Lambda_0 + \Lambda_1 + \Lambda_3 + \Lambda_4$, $\Lambda = \Lambda_0 + \Lambda_1 + \Lambda_2 + \Lambda_4$, $\ell' = 2$, $\mathcal{B}' = \mathcal{B}^{2,2}$. Here the set $\mathcal{H}(\Lambda', \mathcal{B}', \Lambda)$ consists of two elements, namely,

$$\begin{matrix} 1 & 2 \\ 4 & 5 \end{matrix} \quad \text{and} \quad \begin{matrix} 1 & 4 \\ 2 & 5 \end{matrix}.$$

Let b be the second tableau. The theorem says that

$$\begin{matrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 \end{matrix} \otimes \begin{matrix} 1 & 4 \\ 2 & 5 \end{matrix} \cong \begin{matrix} 1 & 3 \\ 2 & 4 \end{matrix} \otimes \begin{matrix} 1 & 1 & 2 & 4 \\ 2 & 3 & 5 & 5 \end{matrix}.$$

Proposition 7.5. *Let $\Lambda \in (P^+_{\mathrm{cl}})_\ell$, $\mathcal{B} = \mathcal{B}^{k,\ell}$ a perfect crystal of level ℓ, $\overline{b} \in \mathbb{P}(\Lambda, \mathcal{B})$ the ground state path, p a finite path (say of length N where N is a multiple of n) such that $p \otimes \overline{b} = \overline{b}$, $\mathcal{B}'$ the tensor product of perfect crystals each of level at most ℓ, and $y \in \mathcal{H}(\ell\Lambda_0, \mathcal{B}', \Lambda)$. Let p' be the path of length N such that $p' \otimes \overline{b}' = \overline{b}'$ where $\overline{b}' \in \mathbb{P}(\ell\Lambda_0, \mathcal{B})$ is the ground state path. Then under the composition of local isomorphisms $\mathcal{B}^{\otimes N} \otimes \mathcal{B}' \cong \mathcal{B}' \otimes \mathcal{B}^{\otimes N}$ we have $p \otimes y \cong y \otimes p'$.*

Proof. Induct on the length of the path y. Suppose $\mathcal{B}' = \mathcal{B}_1 \otimes \mathcal{B}_2$ and $y = y_1 \otimes y_2$ where $y_j \in \mathcal{B}_j$ and $\mathcal{B}_j$ is a perfect crystal. Let $\Lambda' = \Lambda - \mathrm{wt}(y_1)$. By the definitions $y_2 \in \mathcal{H}(\ell\Lambda_0, \mathcal{B}_2, \Lambda')$. By induction the first N steps p'' of the ground state path of $\mathbb{P}(\Lambda', \mathcal{B})$ satisfy $p'' \otimes y_2 \cong y_2 \otimes p'$ under the composition of local isomorphisms $\mathcal{B}^{\otimes N} \otimes \mathcal{B}_2 \cong \mathcal{B}_2 \otimes \mathcal{B}^{\otimes N}$. Tensoring on the left with y_1, it remains to show that $p \otimes y_1 \cong y_1 \otimes p''$ under the composition of local isomorphisms $\mathcal{B}^{\otimes N} \otimes \mathcal{B}_1 \cong \mathcal{B}_1 \otimes \mathcal{B}^{\otimes N}$. Now $p_N \in \mathcal{B}$ and $p''_N \in \mathcal{B}$ are the unique elements such that $\epsilon(p_N) = \Lambda$ and $\epsilon(p''_N) = \Lambda'$. Applying Theorem 7.3 we obtain $p_N \otimes y_1 \cong \psi^k(y_1) \otimes p''_N$. Now $p_N \otimes y_1 \in \mathcal{H}(\Lambda', \mathcal{B} \otimes \mathcal{B}_1, \phi(p_N))$ so that $\psi^k(y_1) \otimes p''_N \in \mathcal{H}(\Lambda', \mathcal{B}_1 \otimes \mathcal{B}, \phi(p_N))$. This implies that $\psi^k(y_1) \in \mathcal{H}(\phi(p''_N), \mathcal{B}_1, \phi(p_N))$. Now by definition $\epsilon(p''_{N-1}) = \phi(p''_N)$ and $\epsilon(p_{N-1}) = \phi(p_N)$. Applying Theorem 7.3 we obtain $p_{N-1} \otimes \psi^k(y_1) \cong \psi^{2k}(y_1) \otimes p''_{N-1}$. Continuing in this manner it follows that $p_{N-j} \otimes \psi^{jk}(y_1) \cong \psi^{(j+1)k}(y_1) \otimes p''_{N-j}$ for $0 \le j \le N-1$. Composing these local isomorphisms it follows that $p \otimes y_1 \cong \psi^{Nk}(y_1) \otimes p''$. But ψ^N is the identity since the order of ψ divides n which divides N. Therefore $p \otimes y_1 \cong y_1 \otimes p''$ under the composition of local isomorphisms and we are done. ∎

In the notation in the previous section, $E(p \otimes y') = E(y' \otimes p')$ where p' is the first N steps of the ground state path of $\mathbb{P}(\ell'\Lambda_0, \mathcal{B})$. Write $N = nM$ and $\mathcal{B} = \mathcal{B}^{k,\ell'}$. Then using the generalized cocyclage one may calculate explicitly the generalized charge of the LR tableau corresponding to the level ℓ' restricted (and hence classically restricted) path $y' \otimes p'$. Let $|y'|$ denote the total number of cells in the tableaux comprising y'. Then

$$E(y' \otimes p') = E(y') + |y'|kM + n\ell' \binom{kM}{2}. \tag{7.2}$$

Example 7.6. Let $n = 5$, $\ell' = 3$, $\Lambda' = \Lambda_0 + \Lambda_3 + \Lambda_4$, $k = 2$ and $M = 1$. Then p' is the path

$$\begin{matrix} 4 & 4 & 4 \\ 5 & 5 & 5 \end{matrix} \otimes \begin{matrix} 2 & 2 & 2 \\ 3 & 3 & 3 \end{matrix} \otimes \begin{matrix} 1 & 1 & 1 \\ 5 & 5 & 5 \end{matrix} \otimes \begin{matrix} 3 & 3 & 3 \\ 4 & 4 & 4 \end{matrix} \otimes \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{matrix}.$$

The element y' can be taken to be the tensor product

$$\begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \otimes \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}.$$

Let $\lambda = (8, 8, 8, 7, 6)$. Then the tableau $Q \in \mathrm{LR}(\lambda; R)$ (resp. Y) that records the path $y' \otimes p'$ (resp. y') is given by

$$Q = \begin{matrix} 1 & 1 & 1 & 5 & 5 & 5 & 11 & 15 \\ 2 & 2 & 2 & 7 & 7 & 7 & 12 & 16 \\ 3 & 3 & 3 & 8 & 8 & 8 & 13 & 17 \\ 4 & 4 & 4 & 9 & 9 & 9 & 14 & \\ 6 & 6 & 6 & 10 & 10 & 10 & & \end{matrix}, \qquad Y = \begin{matrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & \end{matrix}$$

with $R = ((3,3),(3,3),(3,3),(3,3),(3,3),(1,1,1,1),(1,1,1))$ and subalphabets $\{1,2\}$, $\{3,4\}$, $\{5,6\}$, $\{7,8\}$, $\{9,10\}$, $\{11,12,13,14\}$, $\{15,16,17\}$. The generalized charge $c_R(Q)$ is equal to the energy $E(y' \otimes p')$ [37, Theorem 23]. Here the widest rectangle in the path is of width ℓ'. For any tableau $T \in \mathrm{LR}(\rho; R)$ for some partition ρ, define $V(T) = P((w_0^R T_e)(w_0^R T_w))$ where P is the Schensted P tableau, w_0^R is the automorphism of conjugation that reverses each of the subalphabets, and T_w and T_e are the west and east subtableaux obtained by slicing T between the ℓ'-th and $(\ell'+1)$-th columns. It can be shown that there is a composition of $|T_e|$ generalized R-cocyclages leading from T to $V(T)$ where $|T_e|$ denotes the number of cells in T_e. It follows from the ideas in [35, Section 3] and the intrinsic characterization of c_R in [35, Theorem 21] that

$$c_R(T) = c_R(V(T)) + |T_e|. \tag{7.3}$$

For the above tableau Q we have

$$Q_w = \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 4 & 4 & 4 \\ 6 & 6 & 6 \end{matrix} \qquad w_0^R Q_w = \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 4 & 4 & 4 \\ 5 & 5 & 5 \end{matrix}$$

and

$$Q_e = \begin{matrix} 5 & 5 & 5 & 11 & 15 \\ 7 & 7 & 7 & 12 & 16 \\ 8 & 8 & 8 & 13 & 17 \\ 9 & 9 & 9 & 14 & \\ 10 & 10 & 10 & & \end{matrix} \qquad w_0^R Q_e = \begin{matrix} 6 & 6 & 6 & 11 & 15 \\ 7 & 7 & 7 & 12 & 16 \\ 8 & 8 & 8 & 13 & 17 \\ 9 & 9 & 9 & 14 & \\ 10 & 10 & 10 & & \end{matrix}.$$

Then

$$V(Q) = \begin{matrix} 1 & 1 & 1 & 11 & 15 \\ 2 & 2 & 2 & 12 & 16 \\ 3 & 3 & 3 & 13 & 17 \\ 4 & 4 & 4 & 14 & \\ 5 & 5 & 5 & & \\ 6 & 6 & 6 & & \\ 7 & 7 & 7 & & \\ 8 & 8 & 8 & & \\ 9 & 9 & 9 & & \\ 10 & 10 & 10 & & \end{matrix} \qquad \text{and} \qquad V(V(Q)) = \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 4 & 4 & 4 \\ 5 & 5 & 5 \\ 6 & 6 & 6 \\ 7 & 7 & 7 \\ 8 & 8 & 8 \\ 9 & 9 & 9 \\ 10 & 10 & 10 \\ 11 & 15 & \\ 12 & 16 & \\ 13 & 17 & \\ 14 & & \end{matrix}.$$

We have $c_R(V(V(Q))) = c_R(Y) = E(y')$ by [35, Theorem 21] and $c_R(Q) = c_R(V(Q))+|Q_e| = c_R(V(Q))+\ell' n+|Y|$, and $c_R(V(Q)) = c_R(V(V(Q)))+|Y|$ by (7.3). This implies $c_R(Q) = \ell' n + E(y') + 2|Y|$.

7.4. **Proof of Theorem 7.3.** The proof of Theorem 7.3 requires several lemmas.

Words of length L in the alphabet $\{1, 2, \dots, n\}$ are identified with the elements of the crystal basis of the L-fold tensor product $(\mathcal{B}^{1,1})^{\otimes L}$.

Lemma 7.7. *Let u and v be words such that uv is an A_{n-1} highest weight vector. Then v is an A_{n-1} highest weight vector and $\epsilon_j(u) \le \phi_j(v)$ for all $1 \le j \le n-1$.*

Proof. Let uv be an A_{n-1} highest weight vector and $1 \le j \le n-1$. By (3.5)

$$0 = \epsilon_j(uv) = \epsilon_j(v) + \max\{0, \epsilon_j(u) - \phi_j(v)\}.$$

Since both summands on the right hand side are nonnegative and sum to zero they must both be zero. □

Lemma 7.8. *Let w be a word in the alphabet $\{1, 2\}$ and $\widehat{w}$ a word obtained by removing a letter i of w. Then*

1. $\epsilon_1(\widehat{w}) \le \epsilon_1(w) + 1$ *with equality only if* $i = 1$.
2. $\epsilon_1(w) \le \epsilon_1(\widehat{w}) + 1$ *with equality only if* $i = 2$.

Proof. Write $w = uiv$ and $\widehat{w} = uv$. By (3.5)

$$\begin{aligned} \epsilon_1(ui) &= \epsilon_1(i) + \max\{0, \epsilon_1(u) - \phi_1(i)\} \\ &= \begin{cases} \max\{0, \epsilon_1(u) - 1\} & \text{if } i = 1 \\ 1 + \epsilon_1(u) & \text{if } i = 2. \end{cases} \end{aligned} \tag{7.4}$$

In particular $\epsilon_1(ui) \geq \epsilon_1(u) - 1$. Applying (3.5) to both $\epsilon_1(uv)$ and $\epsilon_1(uiv)$ and subtracting, we obtain

$$\begin{aligned}\epsilon_1(uv) - \epsilon_1(uiv) &= \max\{0, \epsilon_1(u) - \phi_1(v)\} - \max\{0, \epsilon_1(ui) - \phi_1(v)\} \\ &\leq \max\{0, \epsilon_1(u) - \phi_1(v)\} - \max\{0, \epsilon_1(u) - 1 - \phi_1(v)\} \\ &\leq 1.\end{aligned}$$

Moreover if $\epsilon_1(uv) - \epsilon_1(uiv) = 1$ then all of the inequalities are equalities. In particular it must be the case that $\epsilon_1(ui) = \epsilon_1(u) - 1$, which by (7.4) implies that $i = 1$, proving the first assertion.

On the other hand, (7.4) also implies $\epsilon_1(ui) \leq 1 + \epsilon_1(u)$. Subtracting $\epsilon_1(uv)$ from $\epsilon_1(uiv)$ and computing as before, the second part follows. □

Say that w is an almost highest weight vector with defect i if there is an index $1 \leq i \leq n-1$ such that $\epsilon_j(w) = \delta_{ij}$ for $1 \leq j \leq n-1$, and also $\epsilon_{i-1}(e_i(w)) = 0$ if $i > 1$.

Lemma 7.9. *Let w be an almost highest weight vector with defect i for $1 \leq i \leq n-1$. Then $e_i(w)$ is either an A_{n-1} highest weight vector or an almost highest weight vector of defect $i+1$.*

Proof. For $j \notin \{i-1, i, i+1\}$, the restriction of the words w and $e_i(w)$ to the alphabet $\{j, j+1\}$ are identical, so that $\epsilon_j(e_i(w)) = \epsilon_j(w) = 0$ by the definition of an almost highest weight vector. Also $\epsilon_i(w) = 1$ implies that $\epsilon_i(e_i(w)) = 0$. Again by the definition of an almost highest weight vector, $\epsilon_{i-1}(e_i(w)) = 0$.

If $i = n-1$ we have shown that $e_i(w)$ is an A_{n-1} highest weight vector. So it may be assumed that $i < n-1$. It is enough to show that one of the two following possibilities occurs.

1. $\epsilon_{i+1}(e_i(w)) = 0$.
2. $\epsilon_{i+1}(e_i(w)) = 1$ and $\epsilon_i(e_{i+1}e_i(w)) = 0$.

Recall that $e_i(w)$ is obtained from w by changing an $i+1$ into an i. Write $w = u(i+1)v$ such that $e_i(w) = uiv$. In this notation we have $\phi_i(v) = 0$ and $\epsilon_i(u) = 0$. By Lemma 7.8 point 1 with $\{1, 2\}$ replaced by $\{i+1, i+2\}$ and using that w is an almost highest weight vector of defect i, we have $\epsilon_{i+1}(e_i(w)) \leq \epsilon_{i+1}(w)+1 = 1$. It is now enough to assume that $\epsilon_{i+1}(e_i(w)) = 1$ and to show that $\epsilon_i(e_{i+1}e_i(w)) = 0$. By (3.5)

$$\begin{aligned}0 = \epsilon_{i+1}(w) &= \epsilon_{i+1}(u(i+1)v) \\ &= \epsilon_{i+1}(v) + \max\{0, \epsilon_{i+1}(u) - \phi_{i+1}((i+1)v)\}.\end{aligned}$$

In particular $\epsilon_{i+1}(v) = 0$. Hence $e_{i+1}(e_i(w)) = e_{i+1}(uiv) = e_{i+1}(u)iv$. Similar computations starting with $\epsilon_i(w) = 1$ and which use the fact that

$\epsilon_i(u) = \phi_i(v) = 0$, yield $\epsilon_i(v) = 0$. We have

$$\begin{aligned}\epsilon_i(e_{i+1}e_i(w)) &= \epsilon_i(e_{i+1}(u)iv) \\ &= \epsilon_i(iv) + \max\{0, \epsilon_i(e_{i+1}(u)) - \phi_i(iv)\} \\ &= 0 + \max\{0, \epsilon_i(e_{i+1}(u)) - 1\}.\end{aligned}$$

But $\epsilon_i(u) = 0$ and in passing from u to $e_{i+1}(u)$ an $i+2$ is changed into an $i+1$. By Lemma 7.8 point 2 applied to the restriction of u to the alphabet $\{i, i+1\}$, we have $\epsilon_i(e_{i+1}(u)) \leq \epsilon_i(u) + 1 = 1$. It follows that $\epsilon_i(e_{i+1}e_i(w)) = 0$, and that $e_i(w)$ is an almost highest weight vector of defect $i + 1$. □

Lemma 7.10. *Suppose w is an A_{n-1} highest weight vector and $\widehat{w}$ is a word obtained by removing a letter (say i) from w. Then there is an index r such that $i \leq r \leq n$ and $e_{r-1}e_{r-2}\cdots e_i(\widehat{w})$ is an A_{n-1} highest weight vector.*

Proof. By Lemma 7.9 it suffices to show that $\widehat{w}$ is either an A_{n-1} highest weight vector or an almost highest weight vector of defect i.

First it is shown that $\epsilon_j(\widehat{w}) = 0$ for $j \neq i$. For $j \notin \{i-1, i\}$, the restrictions of w and $\widehat{w}$ to the alphabet $\{j, j+1\}$ are the same, so that $\epsilon_j(\widehat{w}) = \epsilon_j(w) = 0$. For $j = i - 1$, by Lemma 7.8 point 1 and the assumption that w is an A_{n-1} highest weight vector, it follows that $\epsilon_{i-1}(\widehat{w}) \leq \epsilon_{i-1}(w) + 1 = 1$. But equality cannot hold since the removed letter is i as opposed to $i - 1$. Thus $\epsilon_{i-1}(\widehat{w}) = 0$.

Next we observe that $\epsilon_i(\widehat{w}) \leq \epsilon_i(w) + 1 = 1$ by Lemma 7.8 point 1 and the fact that w is an A_{n-1} highest weight vector.

If $\epsilon_i(\widehat{w}) = 0$ then $\widehat{w}$ is an A_{n-1} highest weight vector. So it may be assumed that $\epsilon_i(\widehat{w}) = 1$. It suffices to show that $\epsilon_{i-1}(e_i(\widehat{w})) = 0$. Write $w = uiv$ and $\widehat{w} = uv$. Now $\epsilon_j(v) = 0$ for all $1 \leq j \leq n - 1$ by Lemma 7.7 since w is an A_{n-1} highest weight vector. In particular $\epsilon_i(v) = 0$ so that $e_i(\widehat{w}) = e_i(uv) = e_i(u)v$. We have

$$\begin{aligned}\epsilon_{i-1}(e_i(\widehat{w})) &= \epsilon_{i-1}(e_i(u)v) \\ &= \epsilon_{i-1}(v) + \max\{0, \epsilon_{i-1}(e_i(u)) - \phi_{i-1}(v)\} \\ &= \max\{0, \epsilon_{i-1}(e_i(u)) - \phi_{i-1}(v)\}\end{aligned}$$

since $\epsilon_{i-1}(v) = 0$ by Lemma 7.7. It is enough to show that $\epsilon_{i-1}(e_i(u)) \leq \phi_{i-1}(v)$. But

$$\epsilon_{i-1}(e_i(u)) \leq \epsilon_{i-1}(u) + 1 = \epsilon_{i-1}(ui) \leq \phi_{i-1}(v).$$

The first inequality holds by an application of Lemma 7.8 point 2 since the restrictions of u and $e_i(u)$ to the alphabet $\{i - 1, i\}$ differ by inserting a letter i. The last inequality holds by Lemma 7.7 since $w = uiv$ is an A_{n-1} highest weight vector. □

Lemma 7.11. *Let $\mathcal{B} = \mathcal{B}^{k,\ell'}$ be a perfect crystal of level $\ell' \leq \ell$, $\Lambda \in (P^+_{\mathrm{cl}})_\ell$, $\mathcal{B}'$ a finite (possibly empty) tensor product of perfect crystals of level at most ℓ, $x \in \mathcal{B}'$ and $b \in \mathcal{B}$ such that $x \otimes b \in \mathcal{H}(\Lambda, \mathcal{B}' \otimes \mathcal{B})$. Let $i \in J$ such that*

$\langle h_i , \Lambda \rangle > 0$ *and set* $\Lambda' = \Lambda - \Lambda_i + \Lambda_{i-1}$. *Then there is an index* $0 \le s \le k$ *such that*

$$e_{i+s-1} \cdots e_{i+1} e_i (x \otimes b) = x \otimes e_{i+s-1} \cdots e_{i+1} e_i (b) \tag{7.5}$$

and $e_{i+s-1} \cdots e_i(b) \in \mathcal{H}(\Lambda', \mathcal{B})$ *where the subscripts are taken modulo* n. *Moreover if* $\ell' = \ell$ *then* $s = k$.

Proof. Since the Dynkin diagram $A^{(1)}_{n-1}$ has an automorphism given by rotation, it may be assumed that $i = 1$. Let λ be the partition of length less than n, given by $\langle h_j , \Lambda \rangle = \lambda_j - \lambda_{j+1}$ for $1 \le j \le n-1$ and $\lambda_n = 0$. Since $\langle h_1 , \Lambda \rangle > 0$ it follows that λ has a column of size 1. Let $m = \lambda_1$ and y_i be the A_{n-1}-highest weight vector in $\mathcal{B}^{\lambda^t_j,1}$ for $1 \le j \le m$. Write $y = y_m \otimes \cdots \otimes y_1$ and $\widehat{y} = y_{m-1} \otimes \cdots \otimes y_1$. Observe that $y \otimes u_{\ell\Lambda_0}$ is an affine highest weight vector in $\mathcal{B}^{\lambda^t_m,1} \otimes \cdots \otimes \mathcal{B}^{\lambda^t_1,1} \otimes \mathbb{B}(\ell\Lambda_0)$ and has weight Λ so its connected component is isomorphic to $\mathbb{B}(\Lambda)$. A similar statement holds for $\widehat{y} \otimes u_{\ell\Lambda_0}$ and $\mathbb{B}(\Lambda')$. In particular, $b \otimes y$ is an A_{n-1} highest weight vector. The map $x \otimes b \otimes y \mapsto \mathrm{word}(x)\mathrm{word}(b)\mathrm{word}(y)$ gives an embedding of A_{n-1}-crystals into a tensor product of crystals $\mathcal{B}^{1,1}$. By Lemma 7.10, there exists an index $1 \le r \le n$ such that $e_{r-1} e_{r-2} \cdots e_1(\mathrm{word}(x)\mathrm{word}(b)\mathrm{word}(\widehat{y}))$ is an A_{n-1} highest weight vector. Since $\widehat{y}$ is an A_{n-1} highest weight vector it follows that $e_{r-1} \cdots e_1(\mathrm{word}(x)\mathrm{word}(b)\mathrm{word}(\widehat{y})) = e_{r-1} \cdots e_1(\mathrm{word}(x)\mathrm{word}(b))\mathrm{word}(\widehat{y})$.

Let p_j be the position of the letter in $e_{j-1} \ldots e_1(\mathrm{word}(x)\mathrm{word}(b))$ that changes from a $j+1$ to j upon the application of e_j, for $1 \le j \le r-1$. It follows from the proof of Lemma 7.9 that

$$p_{r-1} < p_{r-2} < \cdots < p_2 < p_1. \tag{7.6}$$

Let s be the maximal index such that p_s is located in $\mathrm{word}(b)$. Write $b' = e_s \cdots e_1(b)$. It follows that $e_s e_{s-1} \cdots e_1(x \otimes b) = x \otimes b'$ and that $b' \otimes \widehat{y}$ is an A_{n-1} highest weight vector.

It remains to show that

$$\epsilon_0(b' \otimes \widehat{y} \otimes u_{\ell\Lambda_0}) = 0 \tag{7.7}$$

and that $s \le k$ with equality if $\ell' = \ell$.

Consider the corresponding positions in the tableau b. Since $b \mapsto \mathrm{word}(b)$ is an A_{n-1}-crystal morphism, $e_s \cdots e_1(\mathrm{word}(b)) = \mathrm{word}(e_s \cdots e_1(b))$. Let (i_1, j_1) be the position in the tableau b corresponding to the position p_1 in $\mathrm{word}(b)$, and analogously define (i_2, j_2), (i_3, j_3), and so on. Since the rows of all tableaux (and in particular b, $e_1(b)$, $e_2 e_1(b)$, etc.) are weakly increasing and (7.6) holds, it follows that $i_1 < i_2 < i_3 < \cdots < i_s$. But b has k rows, so $s \le k$.

The next goal is to prove (7.7). Suppose first that $s < n-1$. In this case the letters 1 and n are undisturbed in passing from $e_1(b)$ to $e_s \cdots e_1(b)$.

Using this and the Dynkin diagram rotation it follows that

$$\begin{aligned}\epsilon_0(e_s\cdots e_2e_1(b)\otimes\widehat{y}\otimes u_{\ell\Lambda_0}) &= \epsilon_0(e_1(b)\otimes u_{\Lambda'})\\ &= \max\{0,\epsilon_0(e_1(b))-\phi_0(u_{\Lambda'})\}\\ &= \max\{0,\epsilon_0(e_1(b))-\phi_0(u_\Lambda)-1\}.\end{aligned}\tag{7.8}$$

But $\phi_0(u_\Lambda)\geq\epsilon_0(b)\geq\epsilon_0(e_1(b))-1$ by the fact that $\epsilon_0(b\otimes u_\Lambda)=0$ and Lemma 7.8 point 2 applied after rotation of the Dynkin diagram. By (7.8) the desired result (7.7) follows.

Otherwise assume $s=n-1$. Here $k=n-1$ since $s\leq k<n$ with the inequality holding by the perfectness of $\mathcal{B}$. By (7.6) and the fact that b is a tableau, it must be the case that e_1 acting on b changes a 2 in the first row of b into a 1, e_2 acting on $e_1(b)$ changes a 3 in the second row of $e_1(b)$ into a 2, etc. Since b is a tableau with $n-1$ rows with entries between 1 and n, there are integers $0\leq\nu_{n-1}\leq\nu_{n-2}\leq\cdots\leq\nu_1<\ell'$ such that the i-th row of b consists of ν_i copies of the letter i and $\ell'-\nu_i$ copies of the letter $i+1$. For tableaux b of this very special form, the explicit formula for e_0 in [37, (3.11)] yields $\epsilon_0(b)=\ell'-m_n(b)$ where $m_n(b)$ is the number of occurrences of the letter n in b. Since $b'=e_{n-1}\cdots e_1(b)$ also has the same form (with ν_i replaced by ν_i+1 for $1\leq i\leq n-1$) and $m_n(b')=m_n(b)-1$, it follows that $\epsilon_0(b')=\epsilon_0(b)+1$. We have

$$\begin{aligned}\epsilon_0(b'\otimes\widehat{y}\otimes u_{\ell\Lambda_0}) &= \epsilon_0(b'\otimes u_{\Lambda'})\\ &= \max\{0,\epsilon_0(b')-\phi_0(u_{\Lambda'})\}\\ &= \max\{0,\epsilon_0(b)+1-(\phi_0(u_\Lambda)+1)\}=0\end{aligned}$$

since $b\in\mathcal{H}(\Lambda,\mathcal{B})$.

Finally, assuming $\ell'=\ell$, it must be shown that $s=k$. Since the level of $\mathcal{B}$ is the same as that of the weights Λ and Λ', it follows from the perfectness of $\mathcal{B}$ that both b and b' are uniquely defined by the property that $\epsilon(b)=\Lambda$ and $\epsilon(b')=\Lambda'$. Let $\Lambda=\sum_{i=0}^{n-1}z_i\Lambda_i$. By the explicit construction of b in Example 3.3

$$\mathrm{wt}(b)=\sum_{j=1}^{k}\sum_{i=0}^{n-1}z_i(\Lambda_{i+j}-\Lambda_{i+j-1})=\sum_{i=0}^{n-1}z_i(\Lambda_{i+k}-\Lambda_i)$$

with indices taken modulo n. Subtracting the analogous formula for $\mathrm{wt}(b')$, $\mathrm{wt}(b)-\mathrm{wt}(b')=-\sum_{j=1}^{k}\alpha_j$. Using (3.1) it follows that $k=s$. □

Proof of Theorem 7.3. First observe that $x\otimes b\in\mathcal{H}(\Lambda',\mathcal{B}\otimes\mathcal{B}',\phi(x))$ by (3.1), $b\in\mathcal{H}(\Lambda',\mathcal{B}',\Lambda)$, and $\epsilon(x)=\Lambda$. Let $c\in\mathcal{B}'$ and $z\in\mathcal{B}$ be such that $x\otimes b\cong c\otimes z$ under the local isomorphism. Then $c\otimes z\in\mathcal{H}(\Lambda',\mathcal{B}'\otimes\mathcal{B},\phi(x))$ which means that z is Λ'-restricted. Hence $z\in\mathcal{H}(\Lambda',\mathcal{B},\phi(z))$ and $c\in\mathcal{H}(\phi(z),\mathcal{B}',\phi(x))$. The former together with the perfectness of $\mathcal{B}$ implies that $y=z$. From the latter it follows that $\psi^{-k}(c)\in\mathcal{H}(\Lambda',\mathcal{B}',\Lambda)$. However the set $\mathcal{H}(\Lambda',\mathcal{B}',\Lambda)$ might have multiplicities so it is not obvious why $b=\psi^{-k}(c)$ or equivalently $c=\psi^k(b)$.

The proof proceeds by an induction that changes the weight Λ' to a weight $\widehat{\Lambda}'$ that is "closer to" $\ell\Lambda_0$. Suppose first that there is a root direction $i \neq 0$ such that $\langle h_i , \Lambda' \rangle > 0$ and $\widehat{\Lambda}' = \Lambda' - \Lambda_i + \Lambda_{i-1}$. By Lemma 7.11 applied for the weight Λ', simple root α_i, and element $x \otimes b \in \mathcal{H}(\Lambda', \mathcal{B} \otimes \mathcal{B}')$, there is an $0 \leq s < n$ such that $\widehat{b} = e_{i+s-1} \cdots e_{i+1} e_i(b) \in \mathcal{H}(\widehat{\Lambda}', \mathcal{B}', \widehat{\Lambda})$ where $\widehat{\Lambda} = \Lambda - \Lambda_{s+i} + \Lambda_{s+i-1}$ and $e_{i+s-1} \cdots e_i(x \otimes b) = x \otimes \widehat{b}$. Applying Lemma 7.11 with Λ, α_{s+i}, and $x \in \mathcal{H}(\Lambda, \mathcal{B})$, it follows that $\widehat{x} = e_{k+s+i-1} \cdots e_{s+i}(x) \in \mathcal{H}(\widehat{\Lambda}, \mathcal{B})$.

The above computations imply $e_{k+s+i-1} \cdots e_i(x \otimes b) = \widehat{x} \otimes \widehat{b} \in \mathcal{H}(\widehat{\Lambda}', \mathcal{B} \otimes \mathcal{B}')$.

We have $e_{k+s+i-1} \cdots e_{i+1} e_i(c \otimes y) \in \mathcal{H}(\widehat{\Lambda}', \mathcal{B}' \otimes \mathcal{B})$ since $x \otimes b \mapsto c \otimes y$ under the local isomorphism. It must be seen which of these raising operators act on the tensor factor in $\mathcal{B}'$ and which act in $\mathcal{B}$. By Lemma 7.11 applied with Λ', α_i, and $c \otimes y \in \mathcal{H}(\Lambda', \mathcal{B}' \otimes \mathcal{B})$, it follows that $\widehat{y} = e_{k+i-1} \cdots e_i(y) \in \mathcal{H}(\widehat{\Lambda}', \mathcal{B})$ and that $e_{k+i-1} \cdots e_i(c \otimes y) = c \otimes \widehat{y}$. Since $\widehat{y} \otimes u_{\widehat{\Lambda}'}$ is an $A^{(1)}_{n-1}$ highest weight vector, the rest of the raising operators $e_{s+k-1} \cdots e_{k+i}$ must act on the first tensor factor. Let $\widehat{c} = e_{k+s+i-1} \cdots e_{k+i}(c)$. Then $e_{k+s+i-1} \cdots e_i(c \otimes y) = \widehat{c} \otimes \widehat{y}$. But the local isomorphism is a crystal morphism so it sends $\widehat{x} \otimes \widehat{b} \mapsto \widehat{c} \otimes \widehat{y}$. By induction $\widehat{c} = \psi^k(\widehat{b})$. By (3.6) it follows that $c = \psi^k(b)$.

Otherwise there is no index $i \neq 0$ such that $\langle h_i , \Lambda' \rangle > 0$. This means $\Lambda' = \ell\Lambda_0$. But the sets $\mathcal{H}(\ell\Lambda_0, \mathcal{B}, \Lambda)$ and $\mathcal{H}(\ell\Lambda_0, \mathcal{B}', \phi(y))$ are singletons whose lone elements are given by the A_{n-1} highest weight vectors in $\mathcal{B}$ and $\mathcal{B}'$ respectively. Since $\mathcal{B} \otimes \mathcal{B}'$ is A_{n-1} multiplicity-free it follows that the sets $\mathcal{H}(\phi(y), \mathcal{B}', \phi(x))$ and $\mathcal{H}(\Lambda, \mathcal{B}, \phi(x))$ are singletons. In this case it follows directly that $c = \psi^k(b)$ since both c and $\psi^k(b)$ are elements of the set $\mathcal{H}(\phi(y), \mathcal{B}', \phi(x))$. ◻

7.5. **Branching function by restricted generalized Kostka polynomials.** The appropriate map from LR tableaux to rigged configurations, sends the generalized charge of the LR tableau to the charge of the rigged configuration. Unfortunately in general it is not clear what happens when one uses the statistic coming from the energy function $E(b \otimes y')$ but using the path $b \otimes y' \otimes y''$. It is only known that the statistic $E(b \otimes y' \otimes y'')$ on the path $b \otimes y' \otimes y''$, is well-behaved. So to continue the computation we require that $y'' = \emptyset$. This is achieved when $\Lambda'' = \ell''\Lambda_0$. So let us assume this.

The other problem is that we do not consider all paths in $\mathcal{H}(\ell\Lambda_0, \mathcal{B}^{\otimes N} \otimes \mathcal{B}', \Lambda)$, but only those of the form $b \otimes y'$ where $y' \in \mathcal{B}'$ is a fixed path. Passing to LR tableaux, this is equivalent to imposing an additional condition that the subtableaux corresponding to the first several rectangles, must be in fixed positions. Conjecture 8.3 asserts that the corresponding sets of rigged configurations are well-behaved.

The special case that requires no extra work, is when $\mathcal{B}'$ consists of a single perfect crystal. This is achievable when Λ' has the form $\Lambda' = r\Lambda_s + (\ell' - r)\Lambda_0$; in this case $\mathcal{B}' = \mathcal{B}^{s,r}$ and y' is the sl_n-highest weight element of $\mathcal{B}^{s,r}$. This is the same as requiring that the first subtableau of the LR tableau be fixed.

But this is always the case. Let $R^{(M)}$ consist of the single rectangle (r^s) followed by $N = Mn$ copies of the rectangle $({\ell'}^k)$ where $\mathcal{B} = \mathcal{B}^{k,\ell'}$. Let $\lambda^{(M)}$ be the partition of the same size as the total size of $R^{(M)}$, such that $\lambda^{(M)}$ projects to $\Lambda - \ell\Lambda_0$. Then the set of paths $\mathcal{H}(\ell\Lambda_0, \mathcal{B}^{\otimes N} \otimes \mathcal{B}^{s,r}, \Lambda)$ is equal to $\mathcal{P}^{\ell}_{\Lambda-\ell\Lambda_0,R^{(M)}}$. This is summarized by

$$b^{\Lambda}_{\Lambda'\Lambda''}(q) = \lim_{M\to\infty} q^{-rskM-n\ell'\binom{kM}{2}} K^{\ell}_{\lambda^{(M)},R^{(M)}}(q), \tag{7.9}$$

where Λ is arbitrary, $\Lambda' = r\Lambda_s + (\ell' - r)\Lambda_0$, and $\Lambda'' = \ell''\Lambda_0$.

Inserting expression (6.7) for the generalized Kostka polynomial in (7.9) and taking the limit yields the following fermionic expression for the branching function

(7.10)

$$b^{\Lambda}_{\Lambda'\Lambda''}(q) = q^{\frac{rs(s-n)}{2n}+\frac{1}{2\ell}\sum_{j=1}^{n}(\lambda_j-\frac{|\lambda|}{n})^2} \sum_{S\in\mathrm{SCST}(\lambda')} (-1)^{|S|+1} q^{\frac{1}{2}\boldsymbol{u}(S)C^{-1}\otimes C^{-1}\boldsymbol{u}(S)}$$
$$\times \sum_{\boldsymbol{m}} q^{\frac{1}{2}\boldsymbol{m}C\otimes C^{-1}\boldsymbol{m}-\boldsymbol{m}I\otimes C^{-1}\boldsymbol{u}(S)} \Big(\prod_{\substack{i=1\\ i\neq\ell'}}^{\ell-1}\prod_{a=1}^{n-1} \begin{bmatrix} m_i^{(a)}+n_i^{(a)} \\ m_i^{(a)} \end{bmatrix}\Big)\Big(\prod_{a=1}^{n-1}\frac{1}{(q)_{m_{\ell'}^{(a)}}}\Big),$$

where λ is any partition which projects to $\Lambda - \ell\Lambda_0$ and $\boldsymbol{u}(S)$ as defined in (6.1). The sum over $\boldsymbol{m}$ runs over all $\boldsymbol{m} = \sum_{i=1}^{\ell-1}\sum_{a=1}^{n-1} m_i^{(a)} \boldsymbol{e}_i \otimes \boldsymbol{e}_a$ such that $m_i^{(a)} \in \mathbb{Z}$ and

$$\boldsymbol{e}_{\ell-1} \otimes \boldsymbol{e}_a(I \otimes C^{-1}\boldsymbol{m} - C^{-1} \otimes C^{-1}\boldsymbol{u}(S)) - \frac{1}{\ell}\sum_{j=1}^{a}(\lambda_j - \frac{1}{n}|\lambda|) \in \mathbb{Z}$$

for all $1 \le a \le n-1$. The variables $n_i^{(a)}$ are given by

$$n_i^{(a)} = \boldsymbol{e}_i \otimes \boldsymbol{e}_a\big\{-C \otimes C^{-1}\boldsymbol{m} + I \otimes C^{-1}(\boldsymbol{u}(s) + \boldsymbol{e}_r \otimes \boldsymbol{e}_s)\big\}$$

for all $1 \le a < n$ and $1 \le i < \ell$, $i \neq \ell'$.

8. Proof of Theorem 5.7

To prove Theorem 5.7 it clearly suffices to show that there is a bijection $\overline{\psi}_R : \mathrm{RLR}^{\ell}(\lambda; R) \to \mathrm{RC}^{\ell}(\lambda; R)$ that is charge-preserving, that is, $c_R(T) = c(\overline{\psi}_R(T))$ for all $T \in \mathrm{RLR}^{\ell}(\lambda; R)$. Here we identify $\mathrm{LR}(\lambda; R)$ with $\mathrm{RLR}(\lambda; R)$ via the standardization bijection std. Also define $c'_R : \mathrm{CLR}(\lambda; R) \to \mathbb{N}$ by $c'_R = c_R \circ \gamma_R$ where $c_R : \mathrm{RLR}(\lambda; R) \to \mathbb{N}$. It will be shown that one of the standard bijections $\overline{\psi}_R : \mathrm{RLR}(\lambda; R) \to \mathrm{RC}(\lambda; R)$ is charge-preserving, and that it restricts to a bijection $\mathrm{RLR}^{\ell}(\lambda; R) \to \mathrm{RC}^{\ell}(\lambda; R)$.

With this in mind let us review the bijections from LR tableaux to rigged configurations.

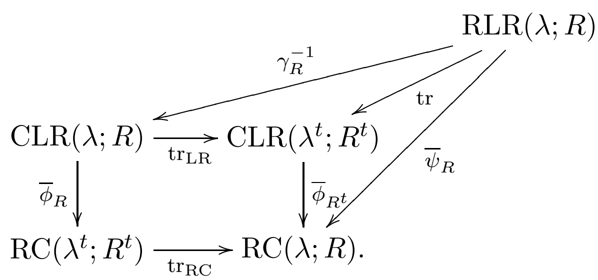
8.1. **Bijections from LR tableaux to rigged configurations.** A bijection $\overline{\phi}_R : \mathrm{CLR}(\lambda; R) \to \mathrm{RC}(\lambda^t; R^t)$ was defined recursively in [25, Definition-Proposition 4.1]. It is one of four natural bijections from LR tableaux to rigged configurations.

1. Column index quantum: $\overline{\phi}_R : \mathrm{CLR}(\lambda; R) \to \mathrm{RC}(\lambda^t; R^t)$.
2. Column index coquantum: $\widetilde{\phi}_R : \mathrm{CLR}(\lambda; R) \to \mathrm{RC}(\lambda^t; R^t)$, defined by $\widetilde{\phi}_R = \theta_{R^t} \circ \overline{\phi}_R$.
3. Row index quantum: $\overline{\psi}_R : \mathrm{RLR}(\lambda; R) \to \mathrm{RC}(\lambda; R)$, defined by $\overline{\psi}_R = \overline{\phi}_{R^t} \circ \mathrm{tr}$, and
4. Row index coquantum: $\widetilde{\psi}_R : \mathrm{RLR}(\lambda; R) \to \mathrm{RC}(\lambda; R)$, defined by $\widetilde{\psi}_R = \theta_R \circ \overline{\psi}_R$.

Of these four, the one that is compatible with level-restriction is $\overline{\psi}$. First we show that it is charge-preserving. This fact is a corollary of the difficult result [25, Theorem 9.1].

Proposition 8.1. *$c(\overline{\psi}_R(T)) = c_R(T)$ for all $T \in \mathrm{RLR}(\lambda; R)$.*

Proof. Consider the following diagram, which commutes by the definitions and [25, Theorem 7.1]



In particular $\overline{\psi}_R = \mathrm{tr}_{\mathrm{RC}} \circ \overline{\phi}_R \circ \gamma_R^{-1}$. Let $T \in \mathrm{RLR}(\lambda; R)$ and $Q = \gamma_R^{-1}(T)$. Then, using $\mathrm{tr}_{\mathrm{RC}} \circ \theta_{R^t} = \theta_R \circ \mathrm{tr}_{\mathrm{RC}}$,

$$\overline{\psi}_R(T) = \theta_R(\mathrm{tr}_{\mathrm{RC}}(\widetilde{\phi}_R(Q))).$$

Let $(\nu, J) = \mathrm{tr}_{\mathrm{RC}}(\widetilde{\phi}_R(Q))$. Then

$$\begin{aligned} c(\overline{\psi}_R(T)) &= c(\theta_R(\nu, J)) = ||R|| - \mathrm{cc}(\nu, J) \\ &= ||R|| - \mathrm{cc}(\mathrm{tr}_{\mathrm{RC}}(\widetilde{\phi}_R(Q))) = \mathrm{cc}(\widetilde{\phi}_R(Q)) = c'_R(Q) = c_R(T) \end{aligned}$$

by Lemma 5.4, (5.6) and [25, Theorem 9.1] to pass from cc to c'_R. □

In light of Proposition 8.1, to prove Theorem 5.7 it suffices to establish the following result.

Theorem 8.2. *The bijection $\overline{\psi}_R : \mathrm{RLR}(\lambda; R) \to \mathrm{RC}(\lambda; R)$ restricts to a well-defined bijection $\overline{\psi}_R : \mathrm{RLR}^\ell(\lambda; R) \to \mathrm{RC}^\ell(\lambda; R)$.*

Computer data suggests that the bijection $\overline{\psi}_R$ is not only well-behaved with respect to level-restriction, but also with respect to fixing certain subtableaux. It was argued in Section 7.5 that the branching functions can be expressed in terms of generating functions of tableaux with certain fixed subtableaux.

Let $\rho \subset \lambda$ be partitions, $R_\rho = ((1^{\rho^t_1}), \dots, (1^{\rho^t_n}))$ and T_ρ the unique tableau in $\mathrm{RLR}(\rho; R_\rho)$. Define $\mathrm{RLR}^\ell(\lambda, \rho; R)$ to be the set of tableaux $T \in \mathrm{RLR}^\ell(\lambda; R_\rho \cup R)$ such that T restricted to shape ρ equals T_ρ. Recall the set of rigged configurations $\mathrm{RC}^\ell(\lambda, \rho; R)$ defined in Section 5.3.

Conjecture 8.3. *The bijection* $\overline{\psi}_R : \mathrm{RLR}(\lambda; R) \to \mathrm{RC}(\lambda; R)$ *restricts to a well-defined bijection* $\overline{\psi}_R : \mathrm{RLR}^\ell(\lambda, \rho; R) \to \mathrm{RC}^\ell(\lambda, \rho; R)$.

8.2. **Reduction to single rows.** In this section it is shown that to prove Theorem 8.2 it suffices to consider the case where R consists of single rows.

Recall the nontrivial embedding $i_R : \mathrm{LR}(\lambda; R) \hookrightarrow \mathrm{LR}(\lambda; \mathrm{r}(R))$. We identify $\mathrm{LR}(\lambda; R)$ and $\mathrm{RLR}(\lambda; R)$ via std, and therefore have an embedding $i_R : \mathrm{RLR}(\lambda; R) \hookrightarrow \mathrm{RLR}(\lambda; \mathrm{r}(R))$.

Define a map $j_R : \mathrm{RC}(\lambda; R) \to \mathrm{RC}(\lambda; \mathrm{r}(R))$ as follows. Let $(\nu, J) \in \mathrm{RC}(\lambda; R)$. For each rectangle of R having k rows and m columns, add $k - j$ strings $(m, 0)$ of length m and label zero to the rigged partition $(\nu, J)^{(j)}$ for $1 \le j \le k-1$. The resulting rigged configuration is $j_R(\nu, J)$.

Proposition 8.4. *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{RLR}(\lambda; R) & \xrightarrow{i_R} & \mathrm{RLR}(\lambda; \mathrm{r}(R)) \\ {\scriptstyle \overline{\psi}_R}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\psi}_{\mathrm{r}(R)}} \\ \mathrm{RC}(\lambda; R) & \xrightarrow[j_R]{} & \mathrm{RC}(\lambda; \mathrm{r}(R)). \end{array}$$

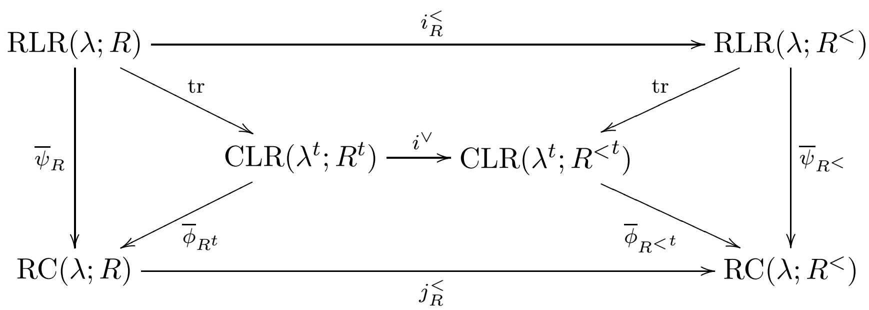
It must be shown that similar diagrams commute in which i_R is replaced by either $i_R^<$ or s_p, the maps that occur in the definition of i_R.

Let $j_R^< : \mathrm{RC}(\lambda; R) \to \mathrm{RC}(\lambda; R^<)$ be defined by adding a string $(\mu_1, 0)$ to each of the first $\eta_1 - 1$ rigged partitions in $(\nu, J) \in \mathrm{RC}(\lambda; R)$.

Lemma 8.5. $j_R^<$ *is well-defined and the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{RLR}(\lambda; R) & \xrightarrow{i_R^<} & \mathrm{RLR}(\lambda; R^<) \\ {\scriptstyle \overline{\psi}_R}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\psi}_{R^<}} \\ \mathrm{RC}(\lambda; R) & \xrightarrow[j_R^<]{} & \mathrm{RC}(\lambda; R^<). \end{array}$$

Proof. Consider the following diagram.

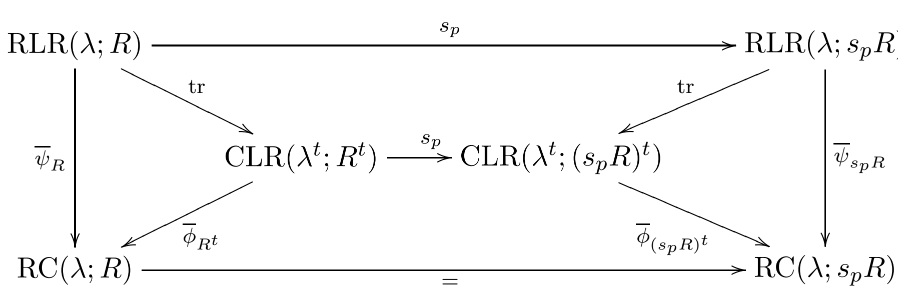


Let us view this diagram as a prism in which the large rectangular face is the front, and the other faces with four sides are the top and bottom, and the faces with three sides are the left and right. We want to show that the front face commutes. For this it suffices to show that all other faces commute. The left and right faces commute by the definition of $\overline{\psi}$. Let us define $i^\vee : \mathrm{CLR}(\lambda^t; R^t) \to \mathrm{CLR}(\lambda^t; R^{<t})$ so that the top face commutes. It suffices to show the bottom face commutes. Observe that $i^\vee$ is the embedding for CLR that splits off the first column of the first rectangle in R^t. But then the bottom face commutes by [25, Lemma 5.4] applied to R^t in place of R. ☐

Lemma 8.6. *The following diagram commutes:*

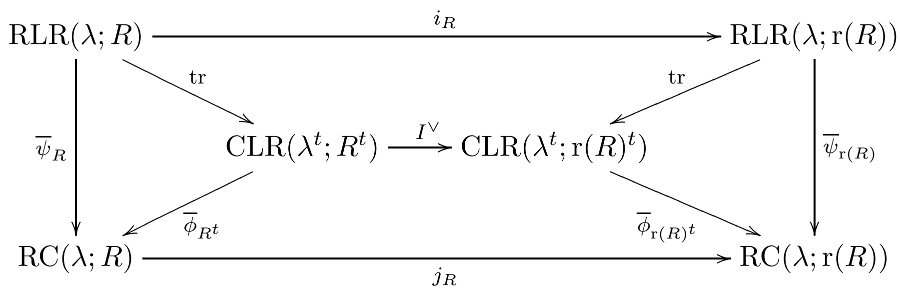
$$\begin{array}{ccc} \mathrm{RLR}(\lambda; R) & \xrightarrow{s_p} & \mathrm{RLR}(\lambda; s_p) \\ {\scriptstyle \overline{\psi}_R}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\psi}_{s_pR}} \\ \mathrm{RC}(\lambda; R) & = & \mathrm{RC}(\lambda; s_pR). \end{array}$$

Proof. We use the same kind of diagram as in the previous lemma. Of course $(s_pR)^t = s_p(R^t)$.



We argue as in the previous lemma. The left and right faces commute by the definition of $\overline{\psi}$, the top face commutes by [36, Proposition 32], and the bottom face commutes by [25, Lemma 8.5]. ☐

Proof of Proposition 8.4. Consider the diagram



The left and right faces commute by the definition of $\overline{\psi}$. Let us define $I^\vee$ so that the top face commutes. It suffices to show the bottom face commutes. By the previous two lemmas, the bottom face commutes if j_R is given by the composition of maps of the form $j_R^{\lessdot}$ and the identity map, corresponding to the way that i_R was computed. But it is easy to see that the effect of this composition of maps is precisely j_R. ☐

By the definition of j_R and Definition 5.5 of the level-restriction for rigged configurations, we have

$$\mathrm{RC}^\ell(\lambda; R) = \{(\nu, J) \in \mathrm{RC}(\lambda; R) \mid j_R(\nu, J) \in \mathrm{RC}^\ell(\lambda; \mathrm{r}(R))\}. \tag{8.1}$$

We now show that Theorem 8.2 follows from the special case when R consists of single rows. The proof is a diagram chase using the the commutative diagram in Proposition 8.4. Since $\mathrm{r}(R)$ consists of single rows, it is assumed that $\overline{\psi}_{\mathrm{r}(R)} : \mathrm{LR}(\lambda; \mathrm{r}(R)) \to \mathrm{RC}(\lambda; \mathrm{r}(R))$ restricts to a bijection $\mathrm{LR}^\ell(\lambda; \mathrm{r}(R)) \to \mathrm{RC}^\ell(\lambda; \mathrm{r}(R))$. In particular $\overline{\psi}_{\mathrm{r}(R)}(\mathrm{LR}^\ell(\lambda; \mathrm{r}(R))) = \mathrm{RC}^\ell(\lambda; \mathrm{r}(R))$. Since $\overline{\psi}_R : \mathrm{LR}(\lambda; R) \to \mathrm{RC}(\lambda; R)$ is a bijection, it is enough to show that $\overline{\psi}_R(\mathrm{LR}^\ell(\lambda; R)) = \mathrm{RC}^\ell(\lambda; R)$. For the inclusion $\overline{\psi}_R(\mathrm{LR}^\ell(\lambda; R)) \subset \mathrm{RC}^\ell(\lambda; R)$, suppose that $x \in \mathrm{LR}^\ell(\lambda; R)$. By (4.7) $i_R(x) \in \mathrm{LR}^\ell(\lambda; \mathrm{r}(R))$. By assumption, $\overline{\psi}_{\mathrm{r}(R)}(i_R(x)) \in \mathrm{RC}^\ell(\lambda; \mathrm{r}(R))$. But $\overline{\psi}_{\mathrm{r}(R)} \circ i_R = j_R \circ \overline{\psi}_R$ by Proposition 8.4, so $j_R(\overline{\psi}_R(x)) \in \mathrm{RC}^\ell(\lambda; \mathrm{r}(R))$. By (8.1), $\overline{\psi}_R(x) \in \mathrm{RC}^\ell(\lambda; R)$. For the other inclusion, suppose $y \in \mathrm{RC}^\ell(\lambda; R)$. Let $x \in \mathrm{LR}(\lambda; R)$ be the unique element such that $\overline{\psi}_R(x) = y$. Now $\overline{\psi}_{\mathrm{r}(R)}(i_R(x)) = j_R(\overline{\psi}_R(x)) = j_R(y)$. By (8.1) $j_R(y) \in \mathrm{RC}^\ell(\lambda; \mathrm{r}(R))$. By assumption $i_R(x) \in \mathrm{LR}^\ell(\lambda; \mathrm{r}(R))$. By (4.7) $x \in \mathrm{LR}^\ell(\lambda; R)$, that is, $y \in \overline{\psi}_R(\mathrm{LR}^\ell(\lambda; R))$.

8.3. **Single row quantum number bijection.** We must prove Theorem 8.2 when R consists of single rows. For the rest of the paper we shall assume this is the case. Then $\eta_j = 1$ for all j, $R_j = (\mu_j)$ for $1 \le j \le L$, $\mathrm{LR}(\lambda; R) = \mathrm{CST}(\lambda; \mu)$, $\mathrm{LR}^\ell(\lambda; R) = \mathrm{CST}^\ell(\lambda; \mu)$, and R^t consists of single columns. We also write $\overline{\psi}_\mu$ for $\overline{\psi}_R$ in this case. Again using std we identify $\mathrm{LR}(\lambda; R)$ with $\mathrm{RLR}(\lambda; R)$, and $\mathrm{LR}^\ell(\lambda; R)$ with its image in $\mathrm{RLR}(\lambda; R)$ under std. Now [25, Section 4.2] gives a direct description of $\overline{\phi}_{R^t}$ that is particularly simple when R^t consists of single columns. This is easily translated to the following algorithm to compute the bijection $\overline{\psi}_R : \mathrm{RLR}(\lambda; R) \to \mathrm{RC}(\lambda; R)$. First, $\nu \in \mathrm{C}(\lambda; R)$ requires that

$$|\nu^{(k)}| = \sum_{j>k} \lambda_j$$

for $k \geq 1$. The vacancy numbers may be given by

$$P_i^{(k)}(\nu) = Q_i(\nu^{(k-1)}) - 2Q_i(\nu^{(k)}) + Q_i(\nu^{(k+1)})$$

where $\nu^{(0)} = \mu$ and (since μ is not necessarily a partition)

$$Q_i(\mu) := \sum_j \min\{\mu_j, i\}.$$

Now let us describe the bijection $\overline{\psi}_R : \mathrm{RLR}(\lambda; R) \to \mathrm{RC}(\lambda; R)$. Start with $T \in \mathrm{RLR}(\lambda; R)$. Write T^- for the tableau obtained by removing the largest letter from T (which occurs in row r, say) and λ^- for the shape of T^-. Let $R^- = ((\mu_1), (\mu_2), \ldots, (\mu_{L-1}), (\mu_L - 1))$. Since $T^- \in \mathrm{RLR}(\lambda^-; R^-)$, by induction $\overline{\psi}_R(T^-) = (\overline{\nu}, \overline{\mathrm{J}})$ is defined. Let $s^{(r)} = \infty$. For $k = r-1$ down to 1, select the longest singular string in $(\overline{\nu}, \overline{\mathrm{J}})^{(k)}$ of length $s^{(k)}$ (possibly of zero length) such that $s^{(k)} \leq s^{(k+1)}$. With the convention $s^{(0)} = \mu_L - 1$, it can be shown that $s^{(0)} \leq s^{(1)}$ as well. Then $\overline{\psi}_R(T) := (\nu, J)$ is obtained from $(\overline{\nu}, \overline{\mathrm{J}})$ by lengthening each of the selected strings by one, and resetting their labels to make them singular with respect to the vacancy numbers in the definition of $\mathrm{RC}(\lambda; R)$, and leaving all other strings unchanged. Denote the transformation $(\overline{\nu}, \overline{\mathrm{J}}) \to (\nu, J)$ by $\overline{\delta}^{-1}$.

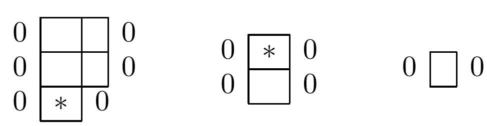
The inverse of $\overline{\delta}^{-1}$, denoted $\overline{\delta}$, is obtained as follows. Set $\ell^{(0)} = \mu_L$. Select inductively a singular string of length $\ell^{(k)}$ in $(\nu, J)^{(k)}$ with $\ell^{(k)}$ smallest such that $\ell^{(k)} \geq \ell^{(k-1)}$. If no such singular string exists set $\ell^{(k)} = \infty$. Then $(\overline{\nu}, \overline{\mathrm{J}})$ is obtained from (ν, J) by shortening all selected strings by one, making them singular again and leaving all other strings unchanged.

Remark 8.7. Up to the relabeling bijection std this is precisely the description of the bijection $\mathrm{CST}(\lambda; \mu) \to \mathrm{RC}(\lambda; (\mu_1), \ldots, (\mu_L))$ that was given in terms of the map called π_* in [24].

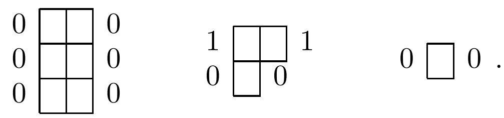
Example 8.8. Take $\mu = (2, 2, 2, 2, 1)$, $\lambda = (3, 3, 2, 1)$ and

$$T = \begin{matrix} 1 & 2 & 6 \\ 3 & 4 & 8 \\ 5 & 9 & \\ 7 & & \end{matrix} \quad \text{so that} \quad T^- = \begin{matrix} 1 & 2 & 6 \\ 3 & 4 & 8 \\ 5 & & \\ 7 & & \end{matrix}$$

and $r = 3$. The rigged configuration corresponding to T^- is



where the labels are written to the right of each part and the vacancy numbers to the left. The selected strings under $\overline{\delta}^{-1}$ with $r = 3$ are indicated by $*$. Hence the rigged configuration corresponding to T is



8.4. **Proof of the single row case.** Now we come to the proof of Theorem 8.2 when R is a sequence of single rows. More precisely we will prove the following theorem.

Theorem 8.9. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition of level ℓ and $\mu = (\mu_1, \mu_2, \ldots, \mu_L)$ an array of positive integers not exceeding ℓ. Then (ν, J) is in the image of $\mathrm{CST}^\ell(\lambda, \mu)$ under $\overline{\psi}_\mu$ if and only if*

1. *$\nu_1^{(k)} \leq \ell$ for all $1 \leq k < n$, and*
2. *there exists a column-strict tableau $t \in \mathrm{CST}(\lambda')$ such that*

$$x_i^{(k)} \leq P_i^{(k)}(\nu) - \sum_{j=1}^{\lambda_k - \lambda_n} \chi(i \geq \widetilde{\ell} + t_{j,k}) + \sum_{j=1}^{\lambda_{k+1} - \lambda_n} \chi(i \geq \widetilde{\ell} + t_{j,k+1}) \tag{8.2}$$

for all $1 \leq k < n$ and $1 \leq i \leq \ell$.

Remark 8.10. The first column of $t \in \mathrm{CST}(\lambda')$ has length $\lambda_1 - \lambda_n$. Since t is a column-strict tableau over the alphabet $\{1, 2, \ldots, \lambda_1 - \lambda_n\}$ this requires that $t_{j,1} = j$.

Remark 8.11. Since $t_{j,k} \leq t_{j,k+1}$ the bounds in (8.2) can be rewritten as

$$\begin{aligned} x_i^{(k)} &\leq P_i^{(k)}(\nu) \\ &- \sum_{j=1}^{\lambda_{k+1} - \lambda_n} \chi(\widetilde{\ell} + t_{j,k} \leq i < \widetilde{\ell} + t_{j,k+1}) - \sum_{j=\lambda_{k+1} - \lambda_n + 1}^{\lambda_k - \lambda_n} \chi(i \geq \widetilde{\ell} + t_{j,k}). \end{aligned} \tag{8.3}$$

For the proof of Theorem 8.9 it will be useful to have the following graphical description of (8.3) in mind. Consider $n - 1$ strips of length ℓ and height $\lambda_1 - \lambda_n$ arranged on top of each other. Assign the label k to the k-th strip from the top. Within each strip assign a height label with height 1 at the bottom of the strip and height $\lambda_1 - \lambda_n$ at the top of the strip. Call the coordinate along the horizontal axis the position. Then draw a horizontal line from position $\widetilde{\ell} + t_{j,k}$ to position $\widetilde{\ell} + t_{j,k+1}$ at height j in the k-th strip with a closed dot at position $\widetilde{\ell} + t_{j,k}$ and an open dot at position $\widetilde{\ell} + t_{j,k+1}$ to indicate that the first position belongs to the line, whereas the second one does not. If $t_{j,k} = t_{j,k+1}$ draw an open dot. If $t_{j,k+1}$ does not exist draw a horizontal line from position $\widetilde{\ell} + t_{j,k}$, indicated by a closed dot, to position ℓ. If there is an open dot at position $\widetilde{\ell} + t_{j,k+1}$ of height j in strip k, then there is also a dot at the same position and height in strip $k + 1$. Connect all such

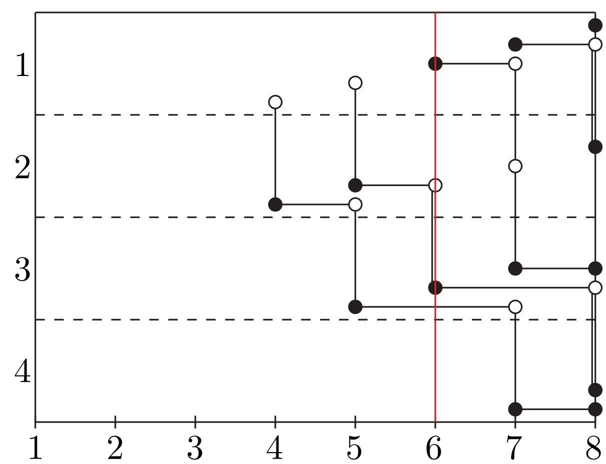


FIGURE 1. An example for nonintersecting paths illustrating (8.2)

dots by a vertical line. This way one obtains $\lambda_1 - \lambda_n$ paths which all end at position ℓ. The j-th path in strip 1 starts at position $\widetilde{\ell} + j$ by Remark 8.10. Furthermore, since $t_{j,k} < t_{j+1,k}$ the paths do not intersect. The k-th strip contains $\lambda_k - \lambda_n$ paths. The other $\lambda_1 - \lambda_k$ paths already ended at position ℓ in previous strips.

An example for a set of such nonintersecting paths is given in Figure 1. It corresponds to $n = 5$, $\ell = 8$, $\lambda = (6, 5, 4, 3, 1)$ and

$$t = \begin{matrix} 1 & 1 & 2 & 4 \\ 2 & 2 & 3 & 5 \\ 3 & 4 & 4 & \\ 4 & 5 & & \\ 5 & & & \end{matrix} \quad .$$

The dashed lines separate the various strips.

To read off the bound on $x_i^{(k)}$ from the picture, draw a vertical line at position i. Suppose that m paths cross this line horizontally in strip k (when the vertical line goes through a closed/open dot we consider this as crossing/not crossing). Then $P_i^{(k)}(\nu) - m$ is the maximal possible rigging for strings of length i in $(\nu, J)^{(k)}$. For example, the vertical line at position 6 in Figure 1 crosses one line in strip 1, no line in strip 2 and 4, and two lines in strip 3, so that $x_6^{(1)} \le P_6^{(1)}(\nu) - 1$, $x_6^{(2)} \le P_6^{(2)}(\nu)$, $x_6^{(3)} \le P_6^{(3)}(\nu) - 2$, and $x_6^{(4)} \le P_6^{(4)}(\nu)$.

Recall that the rigging of a singular string of length i in $(\nu, J)^{(k)}$ equals the vacancy number $P_i^{(k)}(\nu)$. Hence the above graphical description of the bounds shows that $(\nu, J)^{(k)}$ cannot contain singular strings of length i in the intervals

$$\begin{aligned} &\widetilde{\ell} + t_{j,k} \le i < \widetilde{\ell} + t_{j,k+1} && \text{for } 1 \le j \le \lambda_{k+1} - \lambda_n, \\ \text{and} \quad &\widetilde{\ell} + t_{\lambda_{k+1}-\lambda_n+1,k} \le i && \text{if } \lambda_{k+1} < \lambda_k, \end{aligned}$$

since in these intervals a vertical line at position i would cross at least one path. Conversely, if i is the length of a singular string in $(\nu, J)^{(k)}$ then it must be in the complements of these intervals, that is,

(8.4)

$$\begin{aligned} &1 \le i < \widetilde{\ell} + t_{1,k} \\ \text{or} \quad &\widetilde{\ell} + t_{j-1,k+1} \le i < \widetilde{\ell} + t_{j,k} && \text{for } 1 < j \le \lambda_{k+1} - \lambda_n, \\ \text{or} \quad &\widetilde{\ell} + t_{\lambda_{k+1}-\lambda_n,k+1} \le i < \widetilde{\ell} + t_{\lambda_{k+1}-\lambda_n+1,k} && \text{if } \lambda_{k+1} < \lambda_k, \\ \text{or} \quad &\widetilde{\ell} + t_{\lambda_{k+1}-\lambda_n,k+1} \le i \le \ell && \text{if } \lambda_{k+1} = \lambda_k. \end{aligned}$$

Since $t_{j,k} \le t_{j,k+1}$ these intervals are pairwise disjoint, but some of these intervals can of course be empty. Graphically the conditions in (8.4) require that i lies between two paths. More precisely, the first case in (8.4) states that i lies to the left of the first path, the second condition requires that i lies between the $(j-1)$-th and j-th path, and the third case applies if there are more than $\lambda_{k+1} - \lambda_n$ paths in the k-th strip in which case i lies between paths $\lambda_{k+1} - \lambda_n$ and $\lambda_{k+1} - \lambda_n + 1$. The last condition applies if there are exactly $\lambda_{k+1} - \lambda_n$ paths in strip k. None of these ends at ℓ in this strip and the condition implies that i lies to the right of the rightmost path.

Remark 8.12. We use the following conventions throughout the proof: $t_{0,k} = -\widetilde{\ell}$ and $t_{j,k} = \lambda_1 - \lambda_n + 1$ for $j > \lambda_k - \lambda_n$.

Without further ado we present the gory details of the proof of Theorem 8.9.

Proof of Theorem 8.9. We prove the theorem by induction on $|\lambda|$. The theorem is true for $\lambda = \emptyset$ since then $T = \emptyset$ and $(\nu, J) = \emptyset$. In this case T is of level $\ell \ge 0$ and conditions 1 and 2 are trivially satisfied.

Proof of the forward direction. Let $T \in \mathrm{CST}^{\ell}(\lambda, \mu)$ and $(\nu, J) = \overline{\psi}_{\mu}(T)$ its image under the row-wise quantum number bijection. Let T^- be the tableau obtained from T by removing the rightmost largest entry. Set $\overline{\lambda} = \mathrm{shape}(T^-)$, $(\overline{\nu}, \overline{\mathrm{J}}) = \overline{\delta}(\nu, J)$ and denote by r the row index of the cell $\lambda/\overline{\lambda}$.

Set $\lambda^0 = \mathrm{shape}(T^{-\mu_L})$. The tableau T^- is of level ℓ since $\overline{\lambda}_1 - \lambda_n^0 \le \lambda_1 - \lambda_n^0 \le \ell$ by the condition that T is of level ℓ. By induction the theorem holds for T^- so that $\overline{\nu}_1^{(k)} \le \ell$ and there exists a column-strict tableau $\overline{t}$ of shape $(\overline{\lambda}_1 - \overline{\lambda}_n, \dots, \overline{\lambda}_{n-1} - \overline{\lambda}_n)^t$ such that by (8.3)

$$(8.5)\quad \overline{x}_i^{(k)} \le P_i^{(k)}(\overline{\nu}) - \sum_{j=1}^{\overline{\lambda}_{k+1}-\overline{\lambda}_n} \chi(\overline{\overline{\ell}} + \overline{t}_{j,k} \le i < \overline{\overline{\ell}} + \overline{t}_{j,k+1}) - \sum_{j=\overline{\lambda}_{k+1}-\overline{\lambda}_n+1}^{\overline{\lambda}_k-\overline{\lambda}_n} \chi(i \ge \overline{\overline{\ell}} + \overline{t}_{j,k}).$$

for all $1 \le k < n$ and $1 \le i \le \ell$. Here $\overline{\overline{\ell}} = \ell - \overline{\lambda}_1 + \overline{\lambda}_n$, and $\overline{x}_i^{(k)}$ is the largest part of $\overline{\mathrm{J}}_i^{(k)}$ and zero if $\overline{\mathrm{J}}_i^{(k)}$ is empty. The aim is to show that conditions 1 and 2 of the theorem hold for (ν, J).

Denote by $s^{(k)}$ the length of the selected singular string in $(\overline{\nu}, \overline{\mathrm{J}})^{(k)}$ under $\overline{\delta}^{-1}$. By definition $\mu_L - 1 = s^{(0)} \le s^{(1)} \le \dots \le s^{(r-1)}$ and $s^{(k)} = \infty$ for $k \ge r$. We claim that there exist indices $j^{(k)}$ for $0 \le k < r$ such that

$$(8.6)\qquad \overline{\overline{\ell}} + \overline{t}_{j^{(k)}-1,k+1} \le s^{(k)} < \overline{\overline{\ell}} + \overline{t}_{j^{(k)},k} \qquad \text{for } 1 \le k < r$$

$$(8.7)\qquad \overline{\overline{\ell}} + \overline{t}_{j^{(0)}-1,1} \le s^{(0)} < \overline{\overline{\ell}} + \overline{t}_{j^{(0)},1}$$

and

$$(8.8)\qquad 1 \le j^{(0)} \le j^{(1)} \le \dots \le j^{(r-1)} \le \lambda_r - \lambda_n + \delta_{r,n},$$

where by definition $\overline{t}_{0,k} = -\overline{\overline{\ell}}$. The proof proceeds by descending induction on k for $1 \le k < r$. We make frequent use of (8.4) applied to $(\overline{\nu}, \overline{\mathrm{J}})$ where the first and third line are viewed as the cases $j = 1$ and $j = \overline{\lambda}_{k+1} - \overline{\lambda}_n + 1$ of the general interval appearing in the second line of (8.4). First assume $k = r - 1$. Note that $\overline{\lambda}_r < \overline{\lambda}_{r-1}$ since $\overline{\lambda}_r + 1 = \lambda_r \le \lambda_{r-1} = \overline{\lambda}_{r-1}$. Hence the existence of $j^{(r-1)}$ follows from (8.4) since the last case does not apply. In particular we have $j^{(r-1)} \le \overline{\lambda}_r - \overline{\lambda}_n + 1 = \lambda_r - \lambda_n + \delta_{r,n}$. Now consider $0 \le k < r - 1$ and assume that $\overline{\overline{\ell}} + \overline{t}_{j^{(k)}-1,k+1} \le s^{(k)}$ for some $j^{(k)} > j^{(k+1)}$. Then by induction and the column-strictness of $\overline{t}$

$$s^{(k+1)} < \overline{\overline{\ell}} + \overline{t}_{j^{(k+1)},k+1} \le \overline{\overline{\ell}} + \overline{t}_{j^{(k)}-1,k+1} \le s^{(k)}$$

which is a contradiction. Hence $j^{(k)} \le j^{(k+1)}$. Since $s^{(k)}$ is the length of a singular string and by induction $j^{(k)} \le \overline{\lambda}_r - \overline{\lambda}_n + 1 \le \overline{\lambda}_{k+1} - \overline{\lambda}_n$ for $1 \le k < r-1$, $s^{(k)}$ must be in the first or second set of the intervals in (8.4) with all quantities replaced by their barred counterparts which proves (8.6). Equation (8.7) follows since $\overline{t}_{j,1} = j$ by Remark 8.10.

Let us now prove condition 1 of the theorem. By construction (ν, J) is obtained from $(\overline{\nu}, \overline{\mathrm{J}})$ by increasing the length of the selected strings in $(\overline{\nu}, \overline{\mathrm{J}})^{(k)}$ by one for $1 \le k < r$, making them singular again and leaving all other strings unchanged. For $r = 1$ this means that $(\nu, J) = (\overline{\nu}, \overline{\mathrm{J}})$ so that condition 1 of the theorem is satisfied by induction. Now assume $r > 1$. Since $\overline{t}_{j^{(r-1)},k} \in \{1, 2, \dots, \overline{\lambda}_1 - \overline{\lambda}_n\}$ it follows from (8.6) with $k = r - 1$ that

$s^{(r-1)} < \overline{\widetilde{\ell}} + \overline{t}_{j^{(r-1)},r-1} \le \overline{\widetilde{\ell}} + \overline{\lambda}_1 - \overline{\lambda}_n = \ell$. Hence $\mu_L - 1 \le s^{(1)} \le \cdots \le s^{(r-1)} < \ell$ which ensures condition 1 of the theorem for $1 < r \le n$.

It remains to prove that the second condition of the theorem holds. The vacancy numbers of ν and $\overline{\nu}$ are related as follows

$$P_i^{(k)}(\overline{\nu}) = P_i^{(k)}(\nu) - \chi(s^{(k-1)} < i \le s^{(k)}) + \chi(s^{(k)} < i \le s^{(k+1)}).$$

By construction $x_i^{(k)} \le \overline{x}_i^{(k)}$ for $i \ne s^{(k)} + 1$ and $x_{s^{(k)}+1}^{(k)} = P_{s^{(k)}+1}^{(k)}(\nu)$ for $1 \le k < r$. Hence

$$(8.9) \quad x_i^{(k)} \le P_i^{(k)}(\nu) - \chi(s^{(k-1)} < i \le s^{(k)}) + \chi(s^{(k)} < i \le s^{(k+1)}) - \sum_{j=1}^{\overline{\lambda}_k - \overline{\lambda}_n} \chi(i \ge \overline{\widetilde{\ell}} + \overline{t}_{j,k}) + \sum_{j=1}^{\overline{\lambda}_{k+1} - \overline{\lambda}_n} \chi(i \ge \overline{\widetilde{\ell}} + \overline{t}_{j,k+1})$$

for $i \ne s^{(k)} + 1$. In the remainder of the proof of the forward direction it will be shown that (8.2) holds for $i = s^{(k)} + 1$ and that (8.9) implies (8.2) for $i \ne s^{(k)} + 1$. We distinguish the cases $r = 1$, $1 < r < n$, and $r = n$.

Case $r = 1$**.** In this case $\overline{\lambda}_1 = \lambda_1 - 1$, $\overline{\lambda}_k = \lambda_k$ for $1 < k \le n$, $\overline{\widetilde{\ell}} = \widetilde{\ell} + 1$ and $\overline{t}$ is a column-strict tableau over the alphabet $\{1, 2, \ldots, \lambda_1 - \lambda_n - 1\}$. Furthermore $s^{(0)} = \mu_L - 1$ and $s^{(k)} = \infty$ for $k \ge 1$. By (8.9) we have for all $1 \le k < n$ and $1 \le i \le \ell$

$$(8.10) \quad x_i^{(k)} \le P_i^{(k)}(\nu) - \chi(i \ge \mu_L)\delta_{k,1} - \sum_{j=1}^{\lambda_k - \lambda_n - \delta_{k,1}} \chi(i \ge \widetilde{\ell} + 1 + \overline{t}_{j,k}) + \sum_{j=1}^{\lambda_{k+1} - \lambda_n} \chi(i \ge \widetilde{\ell} + 1 + \overline{t}_{j,k+1}).$$

Remark 8.10 requires that $t_{j,1} = j$ for $1 \le j \le \lambda_1 - \lambda_n$. Hence

$$(8.11) \quad -\chi(i \ge \mu_L) - \sum_{j=1}^{\lambda_1 - \lambda_n - 1} \chi(i \ge \widetilde{\ell} + 1 + \overline{t}_{j,1}) \le -\sum_{j=1}^{\lambda_1 - \lambda_n} \chi(i \ge \widetilde{\ell} + t_{j,1}) + \chi(\widetilde{\ell} + 1 \le i < \mu_L).$$

If $\mu_L \le \widetilde{\ell}+1$ the term $\chi(\widetilde{\ell}+1 \le i < \mu_L)$ vanishes. In this case set $t_{j,k} = \overline{t}_{j,k}+1$ for $1 < k < n$ and $1 \le j \le \lambda_k - \lambda_n$ which defines a column-strict tableau of shape $(\lambda_1 - \lambda_n, \ldots, \lambda_{n-1} - \lambda_n)^t$ over $\{1, 2, \ldots, \lambda_1 - \lambda_n\}$. Then (8.10) implies (8.2).

The case $\mu_L > \widetilde{\ell} + 1$ is considerably harder to establish due to the extra term $\chi(\widetilde{\ell} + 1 \le i < \mu_L)$. Our strategy is as follows. The term $\chi(\widetilde{\ell} + 1 \le i < \mu_L)$ can be absorbed by defining $t_{j,2}$ appropriately except in certain cases. In general this introduces extra terms for the bounds at $k = 2$. These in turn can be absorbed by defining $t_{j,3}$ appropriately (except in certain cases) and so on. If all $t_{j,k}$ for $1 \le k < n$ can be defined and all bounds for $1 \le k < n$ are written in the form of (8.2) we are done. In the exceptional cases (when (8.10) does not imply (8.2)) it can be shown that the corresponding tableau T is not of level ℓ which contradicts the assumptions.

Let us now plunge into the details. Define $t_{j,1} = j$ for $1 \le j \le \lambda_1 - \lambda_n$ and set $d = \mu_L - \widetilde{\ell} - 1$. Since $r = 1$, we have $\overline{x}_i^{(k)} = x_i^{(k)}$ and $P_i^{(k)}(\overline{\nu}, \overline{t})$ equals the right-hand side of (8.10). Let $a_j^{(1)}$ for $1 \le j \le \lambda_2 - \lambda_n + 1$ be the minimal index $i \in [\overline{t}_{j-1,2} + 1, \overline{t}_{j,2}] \cap [1, d]$ such that $\overline{x}^{(1)}_{\widetilde{\ell}+i} = P^{(1)}_{\widetilde{\ell}+i}(\overline{\nu}, \overline{t})$, where $\overline{t}_{0,k} = -\overline{\widetilde{\ell}}$ and $\overline{t}_{j,k} = \overline{\lambda}_1 - \overline{\lambda}_n + 1$ for $j > \overline{\lambda}_k - \overline{\lambda}_n$. If no such i exists set $a_j^{(1)} = \overline{t}_{j,2} + 1$. By definition $\overline{x}^{(1)}_{\widetilde{\ell}+i} < P^{(1)}_{\widetilde{\ell}+i}(\overline{\nu}, \overline{t})$ for $\overline{t}_{j-1,2} < i < a_j^{(1)}$ and $1 \le i \le d$ so that we can sharpen the bounds in (8.10) for $k = 1$ by adding $-\sum_{j=1}^{\lambda_2-\lambda_n+1} \chi(\widetilde{\ell} + \overline{t}_{j-1,2} < i < \widetilde{\ell} + a_j^{(1)})\chi(\widetilde{\ell} + 1 \le i \le \widetilde{\ell} + d)$. Note that $\overline{t}_{\lambda_2-\lambda_n+1,2} + 1 = \overline{\lambda}_1 - \overline{\lambda}_n + 2 = \lambda_1 - \lambda_n + 1 = t_{\lambda_2-\lambda_n+1,2}$. The case $a^{(1)}_{\lambda_2-\lambda_n+1} < t_{\lambda_2-\lambda_n+1,2}$ will be dealt with later. Suppose that $a^{(1)}_{\lambda_2-\lambda_n+1} = t_{\lambda_2-\lambda_n+1,2}$. Then one finds using (8.11)

$$(8.12)\quad x_i^{(1)} \le P_i^{(1)}(\nu) - \sum_{j=1}^{\lambda_1-\lambda_n} \chi(i \ge \widetilde{\ell} + t_{j,1}) + \sum_{j=1}^{\lambda_2-\lambda_n} \chi(i \ge \widetilde{\ell} + 1 + \overline{t}_{j,2}) + \sum_{j=1}^{\lambda_2-\lambda_n} \chi(\widetilde{\ell} + a_j^{(1)} \le i \le \widetilde{\ell} + \overline{t}_{j,2}).$$

Define $t_{j,2} = \min\{a_j^{(1)}, t_{j+1,2} - 1\}$ recursively by descending $1 \le j \le \lambda_2 - \lambda_n$. From its definition it is clear that $a_j^{(1)}$ lies in the interval $[\overline{t}_{j-1,2} + 1, \overline{t}_{j,2} + 1]$. By descending induction on j it also follows that $t_{j,2} \in [\overline{t}_{j-1,2} + 1, \overline{t}_{j,2} + 1]$ and that either $t_{j,2} = a_j^{(1)}$ or $a_j^{(1)} - 1$. The latter case only occurs when $a_j^{(1)} = t_{j+1,2} = \overline{t}_{j,2} + 1$. In addition there must exist an index $j' > j$ such that $a_{j'}^{(1)} = \overline{t}_{j'-1,2} + 1$ if $t_{j,2} = a_j^{(1)} - 1$. This is because $t_{j+1,2} = \overline{t}_{j,2} + 1$ is at its lower bound in the interval $[\overline{t}_{j,2} + 1, \overline{t}_{j+1,2} + 1]$ and this can happen in only two ways; either $t_{j+1,2} = a_{j+1}^{(1)}$ which proves the assertion with $j' = j + 1$ or $t_{j+1,2} = a_{j+1}^{(1)} - 1$ in which case the assertion must be true by induction since the initial case is $t_{\lambda_2-\lambda_n+1,2} = a^{(1)}_{\lambda_2-\lambda_n+1}$. Note that it also follows by induction that $a_{j'}^{(1)} = a_j^{(1)} + j' - j - 1$ if j' is minimal. From its definition it follows that $t_{j,2} < t_{j+1,2}$ and furthermore $t_{j,2} \ge \overline{t}_{j-1,2} + 1 \ge \overline{t}_{j-1,1} + 1 = j$ which are the conditions for column-strictness for the first two columns of t. Hence (8.12) yields (8.2) for $k = 1$.

We proceed inductively on $1 < k < n$. Assume that by induction $t_{j,k'} \in [\overline{t}_{j-1,k'} + 1, \overline{t}_{j,k'} + 1]$ is already defined for $1 \le k' \le k$. In terms of $t_{j,k}$ (8.10) reads

(8.13)
$$x_i^{(k)} \leq P_i^{(k)}(\nu) - \sum_{j=1}^{\lambda_k-\lambda_n} \chi(i \geq \widetilde{\ell} + t_{j,k}) + \sum_{j=1}^{\lambda_{k+1}-\lambda_n} \chi(i \geq \widetilde{\ell} + 1 + \overline{t}_{j,k+1}) + \sum_{j=1}^{\lambda_k-\lambda_n} \chi(\widetilde{\ell} + t_{j,k} \leq i \leq \widetilde{\ell} + \overline{t}_{j,k}).$$

For $1 < k < n$ define $a_j^{(k)}$ for $1 \leq j \leq \lambda_{k+1} - \lambda_n + 1$ to be the minimal index $i \in [\overline{t}_{j-1,k+1} + 1, \overline{t}_{j,k+1}] \cap \bigcup_{h=1}^{\lambda_k-\lambda_n} [t_{h,k}, \overline{t}_{h,k}]$ such that $\overline{x}_{\widetilde{\ell}+i}^{(k)} = P_{\widetilde{\ell}+i}^{(k)}(\overline{\nu}, \overline{t})$. If no such i exists set $a_j^{(k)} = \overline{t}_{j,k+1} + 1$. Note that $\overline{t}_{\lambda_{k+1}-\lambda_n+1,k+1} + 1 = \lambda_1 - \lambda_n + 1 = t_{\lambda_{k+1}-\lambda_n+1,k+1}$. The case $a_{\lambda_{k+1}-\lambda_n+1}^{(k)} < t_{\lambda_{k+1}-\lambda_n+1,k+1}$ will be dealt with later. Now assume that $a_{\lambda_{k+1}-\lambda_n+1}^{(k)} = t_{\lambda_{k+1}-\lambda_n+1,k+1}$, and define recursively $t_{j,k+1} = \min\{a_j^{(k)}, t_{j+1,k+1} - 1\}$ on descending $1 \leq j \leq \lambda_{k+1} - \lambda_n$. By definition $a_j^{(k)} \in [\overline{t}_{j-1,k+1} + 1, \overline{t}_{j,k+1} + 1]$. As in the case $k = 1$ it follows by descending induction on j that $t_{j,k+1} \in [\overline{t}_{j-1,k+1} + 1, \overline{t}_{j,k+1} + 1]$ and that either $t_{j,k+1} = a_j^{(k)}$ or $a_j^{(k)} - 1$. By definition $t_{j,k+1} < t_{j+1,k+1}$. Let us now show that also $t_{j,k} \leq t_{j,k+1}$ which would prove the column-strictness of t. By definition $t_{h,k} \leq a_j^{(k)} \leq \overline{t}_{h,k}$ for some h. Assume $h < j$. Then $\overline{t}_{h,k} \geq a_j^{(k)} \geq \overline{t}_{j-1,k+1} + 1$ which violates the column-strictness of $\overline{t}$. Hence $h \geq j$. If $t_{j,k+1} = a_j^{(k)}$ then $t_{j,k+1} \geq t_{h,k} \geq t_{j,k}$ as desired. If $t_{j,k+1} = a_j^{(k)} - 1$ then a problem can only occur if $h = j$ and $a_j^{(k)} = t_{j,k}$. However in this case $a_j^{(k)} = t_{j,k+1} + 1 = t_{j+1,k+1} = \overline{t}_{j,k+1} + 1 > \overline{t}_{j,k} = \overline{t}_{h,k}$ which is a contradiction. This proves $t_{j,k} \leq t_{j,k+1}$. By the same arguments as in the case $k = 1$ there must exist an index $j' > j$ such that $a_{j'}^{(k)} = \overline{t}_{j'-1,k+1} + 1$ if $t_{j,k+1} = a_j^{(k)} - 1$. For minimal j' it follows again by induction that $a_{j'}^{(k)} = a_j^{(k)} + j' - j - 1$.

Since by definition $x_{\widetilde{\ell}+i}^{(k)} = \overline{x}_{\widetilde{\ell}+i}^{(k)} < P_{\widetilde{\ell}+i}^{(k)}(\overline{\nu}, \overline{t})$ for $\overline{t}_{j-1,k+1} < i < a_j^{(k)}$ and $t_{h,k} \leq i \leq \overline{t}_{h,k}$ for some $1 \leq h \leq \lambda_k - \lambda_n$, one can add

$$- \sum_{j=1}^{\lambda_{k+1}-\lambda_n+1} \chi(\widetilde{\ell} + \overline{t}_{j-1,k+1} < i < \widetilde{\ell} + a_j^{(k)}) \sum_{h=1}^{\lambda_k-\lambda_n} \chi(\widetilde{\ell} + t_{h,k} \leq i \leq \widetilde{\ell} + \overline{t}_{h,k})$$

to (8.13). If $a_{\lambda_{k+1}-\lambda_n+1}^{(k)} = t_{\lambda_{k+1}-\lambda_n+1,k+1}$ then the sum of this term and $\sum_{j=1}^{\lambda_k-\lambda_n} \chi(\widetilde{\ell} + t_{j,k} \leq i \leq \widetilde{\ell} + \overline{t}_{j,k})$ does not exceed $\sum_{j=1}^{\lambda_{k+1}-\lambda_n} \chi(\widetilde{\ell} + a_j^{(k)} \leq i \leq \widetilde{\ell} + \overline{t}_{j,k+1})$ and (8.2) is proven for $1 < k < n$.

It remains to treat the case when there exists a $1 \le k < n$ such that $a^{(k)}_{\lambda_{k+1}-\lambda_n+1} < \lambda_1 - \lambda_n + 1$. Let κ be minimal with this property. We will show that in this case T is not of level ℓ which contradicts the assumptions.

We claim that there exist indices h_k and j_k for $1 \le k \le \kappa$ such that

$$\overline{t}_{j_k-1,k+1} + 1 \le a^{(k)}_{j_k} \le \overline{t}_{j_k,k+1}, \tag{8.14}$$

$$t_{h_k,k} \le a^{(k)}_{j_k} \le \overline{t}_{h_k,k}, \tag{8.15}$$

$h_1 \ge h_2 \ge \cdots \ge h_\kappa$ and $h_k \ge j_k \ge h_{k+1}$. The inequalities (8.14) and (8.15) hold for $k = \kappa$ with $j_\kappa = \lambda_{\kappa+1} - \lambda_n + 1$ and some h_κ by the definition of κ. Now suppose that $k < \kappa$ and that $h_{k'}$ and $j_{k'}$ for $k < k' \le \kappa$ satisfying (8.14) and (8.15) have been defined by induction. Recall that either $t_{j,k+1} = a^{(k)}_j$ or $a^{(k)}_j - 1$. First assume that $t_{h_{k+1},k+1} = a^{(k)}_{h_{k+1}}$. This implies in particular that $a^{(k)}_{h_{k+1}} = t_{h_{k+1},k+1} \le \overline{t}_{h_{k+1},k+1}$ by (8.15) and hence $\overline{t}_{h_{k+1}-1,k+1} + 1 \le a^{(k)}_{h_{k+1}} \le \overline{t}_{h_{k+1},k+1}$ by the definition of $a^{(k)}_j$. Set $j_k = h_{k+1}$ and choose h_k such that (8.15) holds which must be possible by the definition of $a^{(k)}_j$. Also $h_k \ge j_k = h_{k+1}$ since otherwise $\overline{t}_{h_k,k} \le \overline{t}_{j_k-1,k+1}$ by the column-strictness of $\overline{t}$ which yields a contradiction since then (8.14) and (8.15) cannot hold simultaneously. Next assume $t_{h_{k+1},k+1} = a^{(k)}_{h_{k+1}} - 1$. Let $j_k > h_{k+1}$ be minimal such that $a^{(k)}_{j_k} = \overline{t}_{j_k-1,k+1} + 1 \le \overline{t}_{j_k,k+1}$; the existence of j_k was proved before. In addition it was shown that $a^{(k)}_{j_k} = a^{(k)}_{h_{k+1}} + j_k - h_{k+1} - 1$. The existence of h_k follows again from the definition of $a^{(k)}_j$. As before $h_k \ge j_k \ge h_{k+1}$.

By definition $x^{(k)}_{\widetilde{\ell}+a^{(k)}_{j_k}} = \overline{x}^{(k)}_{\widetilde{\ell}+a^{(k)}_{j_k}} = P^{(k)}_{\widetilde{\ell}+a^{(k)}_{j_k}}(\overline{\nu}, \overline{t})$. Since $P^{(k)}_i(\overline{\nu}, \overline{t})$ is given by the right-hand side of (8.13), it follows from (8.14), (8.15) and the fact that $t_{j,k} \in [\overline{t}_{j-1,k} + 1, \overline{t}_{j,k} + 1]$ that

$$x^{(k)}_{\widetilde{\ell}+a^{(k)}_{j_k}} = P^{(k)}_{\widetilde{\ell}+a^{(k)}_{j_k}}(\nu) - h_k + j_k \qquad \text{for } 1 \le k \le \kappa. \tag{8.16}$$

Define $T^b = T^{-\mu_L - b}$ for $0 \le b \le \mu_L$ with corresponding rigged configurations $(\nu^b, J^b) = \overline{\delta}^{\mu_L - b}(\nu, J)$. Let r_b be the row index of the cell T^b/T^{b-1}. Denote the length of the selected string in $(\nu^b, J^b)^{(k)}$ under $\overline{\delta}$ by $\ell^{(k)}_b$. We claim that (8.16) implies

$$\ell^{(k)}_{\widetilde{\ell}+j_k} \le \widetilde{\ell} + a^{(k)}_{j_k} \quad \text{for } 1 \le k \le \kappa. \tag{8.17}$$

This is shown by induction on k. By construction $b = \ell^{(0)}_b \le \ell^{(1)}_b \le \ell^{(2)}_b \le \cdots \le \ell^{(n-1)}_b$ and $\ell^{(k)}_1 < \ell^{(k)}_2 < \ldots < \ell^{(k)}_{\mu_L}$. In addition

$$P^{(k)}_i(\nu^{b-1}) = P^{(k)}_i(\nu^b) - \chi(\ell^{(k-1)}_b \le i < \ell^{(k)}_b) + \chi(\ell^{(k)}_b \le i < \ell^{(k+1)}_b). \tag{8.18}$$

If $\ell^{(1)}_{\widetilde{\ell}+a^{(1)}_{j_1}} \le \widetilde{\ell} + a^{(1)}_{j_1}$ then (8.17) follows immediately since $j_1 \le a^{(1)}_{j_1}$ and $\ell^{(1)}_{b-1} < \ell^{(1)}_b$. Hence assume that $\ell^{(1)}_{\widetilde{\ell}+a^{(1)}_{j_1}} > \widetilde{\ell} + a^{(1)}_{j_1}$. Since $\ell^{(0)}_b = b$, the vacancy number at $i = \widetilde{\ell} + a^{(1)}_{j_1}$ is decreased by one with each application of $\overline{\delta}$ until $\ell^{(1)}_b \le \widetilde{\ell} + a^{(1)}_{j_1}$ because of (8.18) at $k = 1$. By (8.16) it takes $h_1 - j_1$ applications of $\overline{\delta}$ until there is a singular string of length $\widetilde{\ell} + a^{(1)}_{j_1}$ in the first rigged partition. Since $a^{(1)}_{j_1} = h_1$ by (8.15) and Remark 8.10 this means that the singular string occurs at $b = \widetilde{\ell} + a^{(1)}_{j_1} - h_1 + j_1 = \widetilde{\ell} + j_1$ which proves (8.17) at $k = 1$.

Now consider the cases $1 < k \le \kappa$ and assume that (8.17) holds for $k' < k$. First assume that $t_{h_k,k} = a^{(k-1)}_{h_k}$. In this case $j_{k-1} = h_k$ and by (8.15) $a^{(k-1)}_{j_{k-1}} = t_{h_k,k} \le a^{(k)}_{j_k}$ so that $\ell^{(k-1)}_{\widetilde{\ell}+j_{k-1}} \le \widetilde{\ell} + a^{(k)}_{j_k}$ by (8.17) at $k-1$. If $\ell^{(k)}_{\widetilde{\ell}+j_{k-1}} \le \widetilde{\ell} + a^{(k)}_{j_k}$ there is nothing to show since $\ell^{(k)}_{b-1} < \ell^{(k)}_b$ and $j_{k-1} \ge j_k$. Hence assume that $\ell^{(k)}_{\widetilde{\ell}+j_{k-1}} > \widetilde{\ell} + a^{(k)}_{j_k}$. Again by (8.16) and (8.18) it takes $h_k - j_k$ applications of $\overline{\delta}$ until there is a singular string of length $\widetilde{\ell} + a^{(k)}_{j_k}$ in the k-th rigged partition. Since $j_{k-1} = h_k$ the singular string occurs at $b = \widetilde{\ell} + j_{k-1} - (h_k - j_k) = \widetilde{\ell} + j_k$ which proves (8.17).

Next assume that $t_{h_k,k} = a^{(k-1)}_{h_k} - 1$. Then $a^{(k-1)}_{j_{k-1}} = a^{(k-1)}_{h_k} + j_{k-1} - h_k - 1 = t_{h_k,k} + j_{k-1} - h_k$ so that by (8.15) $a^{(k-1)}_{j_{k-1}} \le a^{(k)}_{j_k} + j_{k-1} - h_k$. Since $\ell^{(k-1)}_{\widetilde{\ell}+j_{k-1}} \le \widetilde{\ell} + a^{(k-1)}_{j_{k-1}}$ it takes at most $j_{k-1} - h_k$ applications of $\overline{\delta}$ before there is a singular string in the $(k-1)$-th rigged partition of length not exceeding $\widetilde{\ell} + a^{(k)}_{j_k}$. After that, by (8.16) and (8.18), it takes $h_k - j_k$ applications of $\overline{\delta}$ until there is a singular string of length $\widetilde{\ell} + a^{(k)}_{j_k}$ in the k-th rigged partition. Hence altogether the existence of a singular string of length $\widetilde{\ell} + a^{(k)}_{j_k}$ is assured at $b = \widetilde{\ell} + j_{k-1} - (j_{k-1} - h_k) - (h_k - j_k) = \widetilde{\ell} + j_k$ which concludes the proof of (8.17).

Recall that $j_\kappa = \lambda_{\kappa+1} - \lambda_n + 1$. Therefore (8.17) implies that $\ell^{(\kappa)}_b$ is finite for $1 \le b \le \widetilde{\ell} + \lambda_{\kappa+1} - \lambda_n + 1$. If $\ell^{(k)}_b$ is finite this means that $r_b > k$ so that $r_b > \kappa$ for $1 \le b \le \widetilde{\ell} + \lambda_{\kappa+1} - \lambda_n + 1$. Since at most $\lambda_{\kappa+1} - \lambda_n$ boxes can be removed from $T^{\widetilde{\ell}+\lambda_{\kappa+1}-\lambda_n+1}$ in rows with index $\kappa < r_b < n$ it follows that $r_{\widetilde{\ell}+1} = n$. This implies $\lambda^0_n \le \lambda_n - \widetilde{\ell} - 1$ where $\lambda^0 = \mathrm{shape}(T^0)$. Hence $\lambda_1 - \lambda^0_n \ge \ell + 1$ which contradicts the assumption that T is of level ℓ. This concludes the proof of the case $r = 1$.

Case $1<r<n$**.** In this case $\overline{\lambda}_r=\lambda_r-1$, $\overline{\lambda}_k=\lambda_k$ for $k\neq r$ and $\overline{\widetilde{\ell}}=\widetilde{\ell}$. It is convenient to introduce $p^{(1)}=j^{(1)}-1$,

$$(8.19)\qquad p^{(k)}=\begin{cases}\max\{j^{(k)}-1,p^{(k-1)}\} & \text{for } s^{(k)}<\widetilde{\ell}+\overline{t}_{j^{(k)},k-1}-1,\\ j^{(k)} & \text{for } s^{(k)}\geq\widetilde{\ell}+\overline{t}_{j^{(k)},k-1}-1,\end{cases}$$

for $1<k<r$ and $p^{(r)}=\lambda_r-\lambda_n$. Note that for $1\leq k<r$ either $p^{(k)}=j^{(k)}$ or $p^{(k)}=j^{(k)}-1$ and that $p^{(k)}\leq p^{(k+1)}$. Define $t_{j,1}=j$ for $1\leq j\leq\lambda_1-\lambda_n$,

(8.20)
$$t_{j,k}=\begin{cases}\overline{t}_{j,k} & \text{for } 1\leq j<j^{(k-1)} \text{ and } p^{(k)}<j\leq\lambda_k-\lambda_n,\\ s^{(k-1)}-\widetilde{\ell}+1 & \text{for } j=j^{(k-1)}=p^{(k-1)},\\ \max\{\overline{t}_{j-1,k},\overline{t}_{j,k-1}\} & \text{for } p^{(k-1)}<j\leq p^{(k)},\end{cases}$$

for $1<k\leq r$ and $t_{j,k}=\overline{t}_{j,k}$ for $r<k<n$ and $1\leq j\leq\lambda_k-\lambda_n$.

By (8.6) we have $\overline{t}_{j^{(k-1)}-1,k}<\overline{t}_{j^{(k-1)},k-1}$ for $1<k\leq r$ so that

$$(8.21)\qquad t_{j^{(k-1)},k}=\overline{t}_{j^{(k-1)},k-1}\qquad \text{for } p^{(k-1)}=j^{(k-1)}-1<p^{(k)}.$$

It needs to be shown that t indeed defines a column-strict tableau over the alphabet $\{1,2,\dots,\lambda_1-\lambda_n\}$. Since $\overline{t}_{j,k}\in\{1,2,\dots,\lambda_1-\lambda_n\}$ and $s^{(k)}<\ell$ for all $1\leq k<r$ the condition $t_{j,k+1}\in\{1,2,\dots,\lambda_1-\lambda_n\}$ might only be violated if $p^{(k)}=j^{(k)}$ and $s^{(k)}<\widetilde{\ell}$ for $1\leq k<r$. By (8.6) the latter condition requires $j^{(k)}=1$ so that $1=j^{(1)}=\cdots=j^{(k)}$ by (8.8). Since $0\leq s^{(1)}\leq\cdots\leq s^{(k)}<\widetilde{\ell}$ the first condition in (8.19) applies for $p^{(h)}$ for $2\leq h\leq k$. However, since $p^{(1)}=j^{(1)}-1=0$ this implies that $p^{(k)}=0$ which contradicts the requirement $j^{(k)}=p^{(k)}$. This shows that $t_{j,k+1}\in\{1,2,\dots,\lambda_1-\lambda_n\}$.

Next we check that t is column-strict. The condition $t_{j,k}<t_{j+1,k}$ only needs to be checked for $1<k\leq r$ and $j^{(k-1)}-1\leq j\leq p^{(k)}$ since in all other cases it automatically follows from the column-strictness of $\overline{t}$. First assume $p^{(k-1)}=j^{(k-1)}$. Then $t_{j^{(k-1)}-1,k}=\overline{t}_{j^{(k-1)}-1,k}<s^{(k-1)}-\widetilde{\ell}+1=t_{j^{(k-1)},k}$ by (8.6). Furthermore $t_{j^{(k-1)},k}=s^{(k-1)}-\widetilde{\ell}+1\leq\overline{t}_{j^{(k-1)},k-1}<t_{j^{(k-1)}+1,k}$ by (8.6) and (8.20). Next assume $p^{(k-1)}=j^{(k-1)}-1$. Then for $p^{(k-1)}<p^{(k)}$, $t_{j^{(k-1)}-1,k}=\overline{t}_{j^{(k-1)}-1,k}<\overline{t}_{j^{(k-1)},k-1}=t_{j^{(k-1)},k}$ by (8.6) and (8.21). For $p^{(k-1)}=p^{(k)}$ the column-strictness is trivial. Furthermore $\overline{t}_{j-1,k}<\overline{t}_{j,k}\leq\max\{\overline{t}_{j,k},\overline{t}_{j+1,k-1}\}$ and $\overline{t}_{j,k-1}<\overline{t}_{j+1,k-1}\leq\max\{\overline{t}_{j,k},\overline{t}_{j+1,k-1}\}$ so that $t_{j,k}<t_{j+1,k}$ for $p^{(k-1)}<j<p^{(k)}$. And finally $t_{p^{(k)},k}=\max\{\overline{t}_{p^{(k)}-1,k},\overline{t}_{p^{(k)},k-1}\}\leq\overline{t}_{p^{(k)},k}<\overline{t}_{p^{(k)}+1,k}=t_{p^{(k)}+1,k}$.

The conditions $t_{j,k}\leq t_{j,k+1}$ only need to be verified for $j^{(1)}\leq j\leq p^{(2)}$ and $k=1$, for $j^{(k-1)}\leq j\leq p^{(k+1)}$ and $1<k<r$, and for $j^{(r-1)}\leq j$ and $k=r$. First assume $k=1$. Then $t_{j,1}=j\leq\max\{\overline{t}_{j-1,2},\overline{t}_{j,1}\}=t_{j,2}$ for $j^{(1)}\leq j\leq p^{(2)}$. Now assume $1<k<r$. For $p^{(k-1)}=j^{(k-1)}<j^{(k)}$ we have $t_{j^{(k-1)},k}=s^{(k-1)}-\widetilde{\ell}+1\leq\overline{t}_{j^{(k-1)},k-1}\leq\overline{t}_{j^{(k-1)},k+1}=t_{j^{(k-1)},k+1}$ by (8.6). For $p^{(k-1)}=$

$j^{(k-1)} = j^{(k)}$ we have $t_{j^{(k-1)},k} = s^{(k-1)} - \widetilde{\ell} + 1 \le s^{(k)} - \widetilde{\ell} + 1 = t_{j^{(k)},k+1}$. For $p^{(k-1)} < j < j^{(k)}$ one obtains $t_{j,k} = \max\{\overline{t}_{j-1,k}, \overline{t}_{j,k-1}\} \le \overline{t}_{j,k+1} = t_{j,k+1}$. Next assume $p^{(k-1)} < p^{(k)} = j^{(k)}$. Then the second case of (8.19) applies so that $s^{(k)} - \widetilde{\ell} + 1 \ge \overline{t}_{j^{(k)},k-1}$. By (8.6) also $s^{(k)} - \widetilde{\ell} + 1 \ge \overline{t}_{j^{(k)}-1,k+1} \ge \overline{t}_{j^{(k)}-1,k}$. This implies $t_{j^{(k)},k} = \max\{\overline{t}_{j^{(k)}-1,k}, \overline{t}_{j^{(k)},k-1}\} \le s^{(k)} - \widetilde{\ell} + 1 = t_{j^{(k)},k+1}$. And finally for $p^{(k)} < j \le p^{(k+1)}$ we have $t_{j,k} = \overline{t}_{j,k} \le \max\{\overline{t}_{j-1,k+1}, \overline{t}_{j,k}\} = t_{j,k+1}$. In a similar fashion one shows that $t_{j,r} \le t_{j,r+1}$.

Hence t forms a column-strict tableau of shape $(\lambda_1 - \lambda_n, \dots, \lambda_{n-1} - \lambda_n)^t$.

We will now show that (8.2) holds with t as defined in (8.20). First assume that $i = s^{(k)} + 1$. In this case $x_i^{(k)} = P_i^{(k)}(\nu)$ if $1 \le k < r$. Hence it needs to be shown that in this case $P_i^{(k)}(\nu, t) = P_i^{(k)}(\nu)$. To this end it suffices to show that there exists an index j such that

$$\widetilde{\ell} + t_{j-1,k+1} \le s^{(k)} + 1 < \widetilde{\ell} + t_{j,k}. \tag{8.22}$$

Assume that $p^{(k)} = j^{(k)}$, so that $\widetilde{\ell} + t_{j^{(k)},k+1} = s^{(k)} + 1$. By (8.6) $s^{(k)} + 1 \le \widetilde{\ell} + \overline{t}_{j^{(k)},k} < \widetilde{\ell} + \overline{t}_{j^{(k)}+1,k} = \widetilde{\ell} + t_{j^{(k)}+1,k}$ so that (8.22) holds with $j = j^{(k)} + 1$. Next assume that $p^{(k)} = j^{(k)} - 1$. Then by (8.19), $s^{(k)} + 1 < \widetilde{\ell} + \overline{t}_{j^{(k)},k-1} \le \widetilde{\ell} + \overline{t}_{j^{(k)},k} = \widetilde{\ell} + t_{j^{(k)},k}$. Furthermore by (8.6), $\widetilde{\ell} + t_{j^{(k)}-1,k+1} = \widetilde{\ell} + \overline{t}_{j^{(k)}-1,k+1} \le s^{(k)} + 1$ which implies (8.22) with $j = j^{(k)}$. In summary (8.22) holds for $j = p^{(k)} + 1$.

It remains to show that for $i \ne s^{(k)} + 1$ the bounds (8.9) imply (8.2) with t as in (8.20). First assume $1 \le k < r$. For i such that $\widetilde{\ell} + \overline{t}_{j-1,k+1} \le i < \widetilde{\ell} + \overline{t}_{j,k}$ with $1 \le j \le \lambda_r - \lambda_n$ (8.5) simply reads $\overline{x}_i^{(k)} \le P_i^{(k)}(\overline{\nu})$. By construction there are no singular strings of length $s^{(k)} < i \le s^{(k+1)}$ in $(\overline{\nu}, \overline{\mathrm{J}})^{(k)}$. Hence, for $1 \le k \le r - 2$ we can sharpen the bounds in (8.5) and therefore also those in (8.9) by adding the terms

$$-\chi(s^{(k)} < i \le \min\{s^{(k+1)}, \widetilde{\ell} + \overline{t}_{j^{(k)},k} - 1\}) \tag{8.23}$$

if $j^{(k)} = j^{(k+1)}$ and

$$\begin{aligned} &- \chi(s^{(k)} < i < \widetilde{\ell} + \overline{t}_{j^{(k)},k}) - \sum_{j=j^{(k)}}^{j^{(k+1)}-2} \chi(\widetilde{\ell} + \overline{t}_{j,k+1} \le i < \widetilde{\ell} + \overline{t}_{j+1,k}) \\ &\qquad - \chi(\widetilde{\ell} + \overline{t}_{j^{(k+1)}-1,k+1} \le i \le \min\{s^{(k+1)}, \widetilde{\ell} + \overline{t}_{j^{(k+1)},k} - 1\}) \end{aligned} \tag{8.24}$$

if $j^{(k)} < j^{(k+1)}$. In terms of the paths, this corresponds to adding a horizontal line segment (which is equivalent to extra minus signs) in the k-th strip in the interval $s^{(k)} < i \le s^{(k+1)}$ whenever there is a horizontal gap between two neighboring paths. An example is given in Figure 2. It depicts the k-th strip and the zigzag lines correspond to the added line segments.

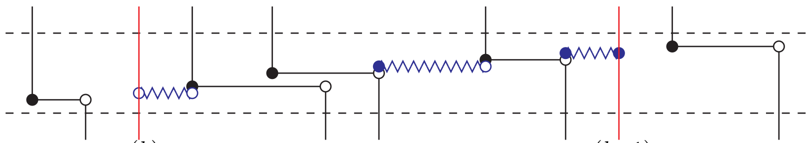


FIGURE 2. Illustration of the extra terms in (8.23) and (8.24)

The sum of extra terms (8.23) or (8.24) and $\chi(s^{(k)} < i \le s^{(k+1)})$ does not exceed

$$\sum_{j=j^{(k)}}^{p^{(k+1)}} \chi(\widetilde{\ell} + \max\{\overline{t}_{j-1,k+1}, \overline{t}_{j,k}\} \le i < \widetilde{\ell} + \overline{t}_{j,k+1}) \tag{8.25}$$
$$= \sum_{j=j^{(k)}}^{p^{(k+1)}} \chi(\widetilde{\ell} + t_{j,k+1} \le i < \widetilde{\ell} + \overline{t}_{j,k+1}) - \chi(s^{(k)} < i < \widetilde{\ell} + \overline{t}_{p^{(k)},k}).$$

To obtain the first line of (8.25) we have used $\overline{t}_{j^{(k)}-1,k+1} < \overline{t}_{j^{(k)},k}$ by (8.6) for the term $j = j^{(k)}$, the definition (8.19) of $p^{(k+1)}$ and $s^{(k+1)} < \widetilde{\ell} + \overline{t}_{j^{(k+1)},k+1}$ when $p^{(k+1)} = j^{(k+1)}$ which follows from (8.6). When $p^{(k)} = j^{(k)}$ the second line follows directly using (8.20). When $p^{(k)} = j^{(k)} - 1$ use that $\widetilde{\ell} + \overline{t}_{j^{(k)}-1,k} \le \widetilde{\ell} + \overline{t}_{j^{(k)}-1,k+1} \le s^{(k)}$ by (8.6) so that the last term vanishes.

Similarly for $k = r-1$ the bounds in (8.9) can be sharpened by adding

$$-\chi(s^{(r-1)} < i < \widetilde{\ell} + \overline{t}_{j^{(r-1)},r-1}) - \sum_{j=j^{(r-1)}}^{\lambda_r - \lambda_n - 1} \chi(\widetilde{\ell} + \overline{t}_{j,r} \le i < \widetilde{\ell} + \overline{t}_{j+1,r-1}).$$

Together with $\chi(s^{(r-1)} < i \le s^{(r)}) = \chi(s^{(r-1)} < i)$ this yields by similar reasons as before

$$\sum_{j=j^{(r-1)}}^{\lambda_r - \lambda_n - 1} \chi(\widetilde{\ell} + t_{j,r} \le i < \widetilde{\ell} + \overline{t}_{j,r}) + \chi(i \ge \widetilde{\ell} + t_{\lambda_r - \lambda_n, r}) \tag{8.26}$$
$$- \chi(s^{(r-1)} < i < \widetilde{\ell} + \overline{t}_{p^{(r-1)},r-1}).$$

Note that $\overline{t}_{j-1,k} \le t_{j,k} \le \overline{t}_{j,k}$ for $j^{(k-1)} \le j \le p^{(k)}$ and $1 \le k < r$. In addition $s^{(k-1)} < \widetilde{\ell} + t_{j^{(k-1)},k}$ for $1 \le k \le r$. For $k = 1$ this follows from (8.7) and Remark 8.10, and for $1 < k \le r$ and $p^{(k-1)} = j^{(k-1)}$ this follows from (8.20) and for $p^{(k-1)} = j^{(k-1)} - 1$ one exploits the first condition of (8.19) and (8.21). Also $s^{(k)} \ge \widetilde{\ell} + \overline{t}_{j^{(k)}-1,k}$ for $1 \le k < r$ thanks to (8.6).

This implies for $1 \le k < r$

$$(8.27) \quad -\chi(s^{(k-1)} < i \le s^{(k)}) \le -\sum_{j=j^{(k-1)}}^{p^{(k)}} \chi(\widetilde{\ell} + t_{j,k} \le i < \widetilde{\ell} + \overline{t}_{j,k}) + \chi(s^{(k)} < i < \widetilde{\ell} + \overline{t}_{p^{(k)},k}).$$

From (8.25) (or (8.26) for $k = r-1$) and (8.27) it is straightforward to see that (8.9) implies (8.2) for $1 \le k < r$.

Now consider $k = r$. Then $s^{(r)} = \infty$, $s^{(r-1)} < \widetilde{\ell} + t_{j^{(r-1)},r}$ as shown above (8.27) and $\overline{t}_{j,r} \le t_{j+1,r}$ for $j^{(r-1)} \le j < \lambda_r - \lambda_n$ so that

$$\begin{aligned}
&-\chi(s^{(r-1)} < i \le s^{(r)}) - \sum_{j=1}^{\overline{\lambda}_r - \overline{\lambda}_n} \chi(i \ge \widetilde{\ell} + \overline{t}_{j,r})\\
&\le -\chi(i \ge \widetilde{\ell} + t_{j^{(r-1)},r}) - \sum_{j=1}^{j^{(r-1)}-1} \chi(i \ge \widetilde{\ell} + t_{j,r}) - \sum_{j=j^{(r-1)}}^{\lambda_r - \lambda_n - 1} \chi(i \ge \widetilde{\ell} + t_{j+1,r})\\
&= -\sum_{j=1}^{\lambda_r - \lambda_n} \chi(i \ge \widetilde{\ell} + t_{j,r}).
\end{aligned}$$

For $r < k < n$ we have $s^{(k-1)} = s^{(k)} = \infty$ and $\overline{t}_{j,k} = t_{j,k}$ so that $-\chi(s^{(k-1)} < i \le s^{(k)}) = 0$ and $-\sum_{j=1}^{\overline{\lambda}_k - \overline{\lambda}_n} \chi(i \ge \overline{\widetilde{\ell}} + \overline{t}_{j,k}) = -\sum_{j=1}^{\lambda_k - \lambda_n} \chi(i \ge \widetilde{\ell} + t_{j,k})$.

For $r \le k < n$ we have $s^{(k)} = s^{(k+1)} = \infty$ and $\overline{t}_{j,k+1} = t_{j,k+1}$ so that $\chi(s^{(k)} < i \le s^{(k+1)}) = 0$ and $\sum_{j=1}^{\overline{\lambda}_{k+1} - \overline{\lambda}_n} \chi(i \ge \overline{\widetilde{\ell}} + \overline{t}_{j,k+1}) = \sum_{j=1}^{\lambda_{k+1} - \lambda_n} \chi(i \ge \widetilde{\ell} + t_{j,k+1})$.

This concludes the proof that (8.9) implies (8.2) for $1 < r < n$.

Case $r = n$. In this case $\overline{\lambda}_k = \lambda_k$ for $1 \le k < n$, $\overline{\lambda}_n = \lambda_n - 1$, $\overline{\widetilde{\ell}} = \widetilde{\ell} - 1$ and $\overline{t}$ is a tableau over the alphabet $\{1, 2, \dots, \lambda_1 - \lambda_n + 1\}$. It follows from (8.8) that $j^{(0)} = \cdots = j^{(n-1)} = 1$. Note that this requires in particular that $s^{(0)} = \mu_L - 1 < \overline{\widetilde{\ell}} + 1 = \widetilde{\ell}$. Define $t_{j,k} = \overline{t}_{j+1,k} - 1$ for $1 \le k < n$ and $1 \le j \le \lambda_k - \lambda_n$. Then by the column-strictness of $\overline{t}$ we have $t_{j,k} < t_{j+1,k}$ and $t_{j,k} \le t_{j,k+1}$. Note in particular that $t_{j,1} = \overline{t}_{j+1,1} - 1 = j$ so that t is a column-strict tableau over the alphabet $\{1, 2, \dots, \lambda_1 - \lambda_n\}$. In addition it follows from (8.6) that $0 \le s^{(k)} + 1 \le \overline{\widetilde{\ell}} + \overline{t}_{1,k} < \overline{\widetilde{\ell}} + \overline{t}_{2,k} = \widetilde{\ell} + t_{1,k}$ so that (8.22) holds for $j = 1$. This ensures (8.2) for $i = s^{(k)} + 1$. Using the fact that there are no singular strings of length $s^{(k)} < i < \overline{\widetilde{\ell}} + \overline{t}_{1,k}$ in $(\overline{\nu}, \overline{\mathrm{J}})^{(k)}$ and that $s^{(k+1)} < \overline{\widetilde{\ell}} + \overline{t}_{1,k+1}$ by (8.6) the term $\chi(s^{(k)} < i \le s^{(k+1)})$ in (8.9) can be safely replaced by $\chi(\overline{\widetilde{\ell}} + \overline{t}_{1,k} \le i < \overline{\widetilde{\ell}} + \overline{t}_{1,k+1})$. Furthermore dropping the term $-\chi(s^{(k-1)} < i \le s^{(k)})$ equation (8.9) becomes

$$x_i^{(k)} \le P_i^{(k)}(\nu) - \sum_{j=2}^{\lambda_k-\lambda_n+1} \chi(i \ge \overline{\overline{\ell}} + \overline{t}_{j,k}) + \sum_{j=2}^{\lambda_{k+1}-\lambda_n+1} \chi(i \ge \overline{\overline{\ell}} + \overline{t}_{j,k+1}).$$

Using $\overline{\overline{\ell}} = \widetilde{\ell} - 1$ and the definition of t this is exactly (8.2).

This concludes the proof of the forward direction of the theorem.

Proof of the reverse direction. Let us now prove the reverse direction. To this end consider a rigged configuration (ν, J) corresponding to a column-strict tableau T of shape λ and content μ under $\overline{\psi}_\mu$ which satisfies $\nu_1^{(k)} \le \ell$ for all $1 \le k < n$ and (8.2). We need to show that then T is of level ℓ. This is equivalent to showing that T^- is of level ℓ and that $\lambda_1 - \lambda_n^0 \le \ell$ if $r = 1$, where r is the row index of the cell $\lambda/\overline{\lambda}$, $\overline{\lambda} = \mathrm{shape}(T^-)$ and $\lambda^0 = \mathrm{shape}(T^{-\mu_L})$. By induction the statement that T^- is of level ℓ is equivalent to the statement that $\overline{\overline{\ell}} = \ell - \overline{\lambda}_1 + \overline{\lambda}_n \ge 0$ and $(\overline{\nu}, \overline{\mathrm{J}}) = \overline{\delta}(\nu, J)$ satisfies $\overline{\nu}_1^{(k)} \le \ell$ for all $1 \le k < n$ and

$$\overline{x}_i^{(k)} \le P_i^{(k)}(\overline{\nu}) - \sum_{j=1}^{\overline{\lambda}_k-\overline{\lambda}_n} \chi(i \ge \overline{\overline{\ell}} + \overline{t}_{j,k}) + \sum_{j=1}^{\overline{\lambda}_{k+1}-\overline{\lambda}_n} \chi(i \ge \overline{\overline{\ell}} + \overline{t}_{j,k+1}) \tag{8.28}$$

for some column-strict tableau $\overline{t}$ of shape $(\overline{\lambda}_1 - \overline{\lambda}_n, \dots, \overline{\lambda}_{n-1} - \overline{\lambda}_n)^t$. To prove $\overline{\overline{\ell}} \ge 0$ it suffices to show that $r = n$ cannot occur when $\widetilde{\ell} = 0$.

Let $\ell^{(k)}$ $(1 \le k < r)$ be the length of the selected singular string in $(\nu, J)^{(k)}$ under $\overline{\delta}$. By definition $\mu_L = \ell^{(0)} \le \ell^{(1)} \le \ell^{(2)} \le \cdots \le \ell^{(r-1)} \le \ell$, and the rigged configuration $(\overline{\nu}, \overline{\mathrm{J}})$ is obtained from (ν, J) by shortening the selected strings by one, making them singular again and leaving all other strings unchanged. Since $\nu_1^{(k)} \le \ell$ this immediately implies $\overline{\nu}_1^{(k)} \le \ell$ for all $1 \le k < n$.

The vacancy numbers are related by

$$P_i^{(k)}(\overline{\nu}) = P_i^{(k)}(\nu) - \chi(\ell^{(k-1)} \le i < \ell^{(k)}) + \chi(\ell^{(k)} \le i < \ell^{(k+1)}). \tag{8.29}$$

Furthermore $\overline{x}_i^{(k)} \le x_i^{(k)}$ for $i \ne \ell^{(k)}-1$ and $\overline{x}_{\ell^{(k)}-1}^{(k)} = P_{\ell^{(k)}-1}^{(k)}(\overline{\nu})$ for $1 \le k < r$ so that (8.2) implies

$$\begin{aligned} \overline{x}_i^{(k)} \le{}& P_i^{(k)}(\overline{\nu}) + \chi(\ell^{(k-1)} \le i < \ell^{(k)}) - \chi(\ell^{(k)} \le i < \ell^{(k+1)}) \\ &- \sum_{j=1}^{\lambda_k-\lambda_n} \chi(i \ge \widetilde{\ell} + t_{j,k}) + \sum_{j=1}^{\lambda_{k+1}-\lambda_n} \chi(i \ge \widetilde{\ell} + t_{j,k+1}) \end{aligned} \tag{8.30}$$

for $i \ne \ell^{(k)} - 1$.

Since $\ell^{(k)}$ is the length of a singular string in $(\nu, J)^{(k)}$ it must be in one of the intervals in (8.4). Let $j^{(k)}$ for $1 \le k < r$ be the index such that

$$\widetilde{\ell} + t_{j^{(k)}-1,k+1} \le \ell^{(k)} < \widetilde{\ell} + t_{j^{(k)},k} \tag{8.31}$$

where recall that $t_{0,k+1} = -\widetilde{\ell}$ and $t_{j,k} = \lambda_1 - \lambda_n + 1$ for all $j > \lambda_k - \lambda_n$. By similar arguments as in the derivation of (8.8) one finds

$$1 \le j^{(1)} \le \cdots \le j^{(r-1)} \le \lambda_r - \lambda_n + 1. \tag{8.32}$$

Case $r = 1$. In this case $\overline{\lambda}_k = \lambda_k$ for $1 < k \le n$, $\overline{\lambda}_1 = \lambda_1 - 1$, $\widetilde{\ell} = \overline{\widetilde{\ell}} - 1$ and $t_{j,k} \in \{1, 2, \ldots, \lambda_1 - \lambda_n\} = \{1, 2, \ldots, \overline{\lambda}_1 - \overline{\lambda}_n + 1\}$. Let $a^{(k)}$ $(1 \le k < n)$ be maximal such that $t_{j,k} = j$ for all $1 \le j \le a^{(k)}$. It follows from Remark 8.10 and $t_{j,k} \le t_{j,k+1}$ that $0 \le a^{(n-1)} \le \cdots \le a^{(2)} \le a^{(1)} = \lambda_1 - \lambda_n$. Set $\overline{t}_{j,1} = j$ for $1 \le j \le \overline{\lambda}_1 - \overline{\lambda}_n$ and

$$\overline{t}_{j,k} = \begin{cases} j & \text{for } 1 \le j \le a^{(k)}, \\ t_{j,k} - 1 & \text{for } a^{(k)} < j \le \overline{\lambda}_k - \overline{\lambda}_n, \end{cases}$$

for $1 < k < n$. The definition of $a^{(k)}$ and column-strictness of t ensure the column-strictness of $\overline{t}$. Note that the terms $j = 1, 2, \ldots, a^{(k+1)}$ in the two sums in (8.30) cancel each other. Recall that $\ell^{(0)} = \mu_L$ and $\ell^{(k)} = \infty$ for $k \ge 1$. Assume $k = 1$. The term $\chi(\ell^{(0)} \le i < \ell^{(1)}) = \chi(i \ge \mu_L)$ in (8.30) can be replaced by $\chi(i \ge \widetilde{\ell} + t_{a^{(2)}+1,1}) = \chi(i \ge \widetilde{\ell} + a^{(2)} + 1)$. If $\mu_L \le \widetilde{\ell} + a^{(2)} + 1$ this follows from the fact that by construction there are no singular strings of length $\ge \mu_L$ in $(\nu, J)^{(1)}$. For $\mu_L > \widetilde{\ell} + a^{(2)} + 1$ we have $\chi(i \ge \mu_L) \le \chi(i \ge \widetilde{\ell} + a^{(2)} + 1)$. Hence using $\widetilde{\ell} = \overline{\widetilde{\ell}} - 1$

$$\begin{aligned} \overline{x}_i^{(1)} \le & P_i^{(1)}(\overline{\nu}) + \chi(i \ge \overline{\widetilde{\ell}} + a^{(2)}) - \sum_{j=a^{(2)}+1}^{\overline{\lambda}_1 - \overline{\lambda}_n + 1} \chi(i \ge \widetilde{\ell} + j) \\ & + \sum_{j=a^{(2)}+1}^{\overline{\lambda}_2 - \overline{\lambda}_n} \chi(i \ge \widetilde{\ell} + t_{j,2}) \\ = & P_i^{(1)}(\overline{\nu}) - \sum_{j=a^{(2)}+1}^{\overline{\lambda}_1 - \overline{\lambda}_n} \chi(i \ge \overline{\widetilde{\ell}} + j) + \sum_{j=a^{(2)}+1}^{\overline{\lambda}_2 - \overline{\lambda}_n} \chi(i \ge \overline{\widetilde{\ell}} + \overline{t}_{j,2}) \end{aligned}$$

which is (8.28) for $k=1$. Now assume $1<k<n$. Since $\ell^{(k)}=\infty$ for $1\le k\le n$ the terms involving $\ell^{(k)}$ in (8.30) vanish and

$$\begin{aligned}\overline{x}_i^{(k)} \le& P_i^{(k)}(\overline{\nu}) - \sum_{j=a^{(k+1)}+1}^{\overline{\lambda}_k-\overline{\lambda}_n} \chi(i\ge \widetilde{\ell}+t_{j,k}) + \sum_{j=a^{(k+1)}+1}^{\overline{\lambda}_{k+1}-\overline{\lambda}_n} \chi(i\ge \widetilde{\ell}+t_{j,k+1})\\ \le& P_i^{(k)}(\overline{\nu}) - \sum_{j=a^{(k+1)}+1}^{\overline{\lambda}_k-\overline{\lambda}_n} \chi(i\ge \overline{\widetilde{\ell}}+\overline{t}_{j,k}) + \sum_{j=a^{(k+1)}+1}^{\overline{\lambda}_{k+1}-\overline{\lambda}_n} \chi(i\ge \overline{\widetilde{\ell}}+\overline{t}_{j,k+1})\end{aligned}$$

which is (8.28) for $1<k<n$. This concludes the proof that (8.2) implies (8.28) for $r=1$.

Case $1<r<n$**.** Here $\overline{\lambda}_k=\lambda_k$ for $k\neq r$, $\overline{\lambda}_r=\lambda_r-1$ and $\overline{\widetilde{\ell}}=\widetilde{\ell}$. Set $p^{(r)}=j^{(r)}=\lambda_r-\lambda_n$ and

$$p^{(k)}=\begin{cases} j^{(k)}-1 & \text{for } \ell^{(k)}\le \widetilde{\ell}+t_{j^{(k)}-1,k+2},\\ \min\{j^{(k)},p^{(k+1)}\} & \text{for } \ell^{(k)}> \widetilde{\ell}+t_{j^{(k)}-1,k+2},\end{cases} \tag{8.33}$$

for $1\le k<r$ where recall that $t_{j,r+1}=\lambda_1-\lambda_n+1$ for $j>\lambda_{r+1}-\lambda_n$. Note that $p^{(k)}=j^{(k)}$ or $j^{(k)}-1$ and $p^{(k)}\le p^{(k+1)}$ due to (8.32). Define $\overline{t}_{j,1}=j$ for $1\le j\le \overline{\lambda}_1-\overline{\lambda}_n$,

(8.34)
$$\overline{t}_{j,k}=\begin{cases} t_{j,k} & \text{for } 1\le j<p^{(k-1)} \text{ and } j^{(k)}\le j\le \overline{\lambda}_k-\overline{\lambda}_n,\\ \min\{t_{j,k+1},t_{j+1,k}\} & \text{for } p^{(k-1)}\le j<p^{(k)},\\ \ell^{(k)}-\widetilde{\ell} & \text{for } j=p^{(k)}=j^{(k)}-1,\end{cases}$$

for $1<k\le r$ and $\overline{t}_{j,k}=t_{j,k}$ for $r<k<n$ and $1\le j\le \overline{\lambda}_k-\overline{\lambda}_n$. Recall that $t_{j,k}=\lambda_1-\lambda_n+1$ for $j>\lambda_k-\lambda_n$.

It needs to be shown that $\overline{t}$ is a tableau over the alphabet $\{1,2,\dots,\overline{\lambda}_1-\overline{\lambda}_n\}$. Since $t_{j,k}\in\{1,2,\dots,\lambda_1-\lambda_n\}=\{1,2,\dots,\overline{\lambda}_1-\overline{\lambda}_n\}$ the only problematic case is the third case in (8.34). Condition 1 of the theorem implies that $\ell^{(k)}\le \ell$ for $1\le k<r$ so that $\ell^{(k)}-\widetilde{\ell}\le \overline{\lambda}_1-\overline{\lambda}_n$. By (8.31) the condition $1\le \ell^{(k)}-\widetilde{\ell}$ can only be violated if $j^{(k)}=1$. Assume that $j^{(k)}=1$ for some $1\le k<r$. Let h be maximal such that $j^{(k)}=1$ for all $1\le k\le h$. Then $\ell^{(k)}>\widetilde{\ell}+t_{j^{(k)}-1,k+2}=0$ for all $1\le k\le h$ so that the second case in (8.33) applies. If $h<r-1$ we have $p^{(h+1)}\ge j^{(h+1)}-1\ge 1$ by the maximality of h. If $h=r-1$, $p^{(r)}=j^{(r)}=\lambda_r-\lambda_n\ge 1$. In both cases it follows that $p^{(k)}=j^{(k)}=1$ for all $1\le k\le h$. Hence by (8.34) the case $\overline{t}_{p^{(k)},k}=\ell^{(k)}-\widetilde{\ell}<1$ does not occur. This proves that $\overline{t}_{j,k}\in\{1,2,\dots,\overline{\lambda}_1-\overline{\lambda}_n\}$.

It remains to show that $\overline{t}$ is column-strict. The condition $\overline{t}_{j,k}<\overline{t}_{j+1,k}$ only needs to be considered for $p^{(k-1)}-1\le j<j^{(k)}$ and $1<k<r$ and for

$p^{(r-1)}-1 \le j \le \lambda_{r+1}-\lambda_n$ and $k=r$ by the column-strictness of t. In these cases $\overline{t}_{j,k} < \overline{t}_{j+1,k}$ can be deduced from the following inequalities:

$$\begin{aligned}
&(a) \quad t_{j,k} < \min\{t_{j+1,k+1}, t_{j+2,k}\},\\
&(b) \quad \min\{t_{j,k+1}, t_{j+1,k}\} \le t_{j+1,k} < t_{j+2,k},\\
&\qquad \min\{t_{j,k+1}, t_{j+1,k}\} \le t_{j,k+1} < t_{j+1,k+1},\\
&(c) \quad \min\{t_{j^{(k)}-2,k+1}, t_{j^{(k)}-1,k}\} \le t_{j^{(k)}-2,k+1} < t_{j^{(k)}-1,k+1} \le \ell^{(k)} - \widetilde{\ell},\\
&\qquad \ell^{(k)} - \widetilde{\ell} < t_{j^{(k)},k} \qquad \text{for } 1 \le k < r,\\
&(d) \quad \min\{t_{j^{(k)}-1,k+1}, t_{j^{(k)},k}\} = t_{j^{(k)}-1,k+1} < t_{j^{(k)},k} \qquad \text{for } 1 \le k < r,
\end{aligned}$$

where (8.31) was employed extensively. The condition $\overline{t}_{j,k} \le \overline{t}_{j,k+1}$ needs to be verified for $k=1$ and $p^{(1)} \le j < j^{(2)}$, for $1<k<r$ and $p^{(k-1)} \le j < j^{(k+1)}$ and for $k=r$ and $p^{(r-1)} \le j \le \lambda_{r+1}-\lambda_n$. In these cases $\overline{t}_{j,k} \le \overline{t}_{j,k+1}$ can be deduced from the following inequalities:

$$\begin{aligned}
&(a) \quad \min\{t_{j,k+1}, t_{j+1,k}\} \le t_{j,k+1},\\
&(b) \quad t_{j,k} \le \min\{t_{j,k+2}, t_{j+1,k+1}\},\\
&(c) \quad t_{j^{(k+1)}-1,k} \le t_{j^{(k+1)}-1,k+2} \le \ell^{(k+1)} - \widetilde{\ell},
\end{aligned}$$

where again (8.31) was employed. In addition for $1<k<r$ we have $\ell^{(k)}-\widetilde{\ell} \le \min\{t_{j^{(k)}-1,k+2}, t_{j^{(k)},k+1}\}$ if $\ell^{(k)} \le \widetilde{\ell}+t_{j^{(k)}-1,k+2}$. If $\ell^{(k)} > \widetilde{\ell}+t_{j^{(k)}-1,k+2}$ then $p^{(k)} = j^{(k)}-1$ is only possible if $p^{(k)} = p^{(k+1)} = j^{(k)}-1$ which implies that $j^{(k)} = j^{(k+1)}$. However in this case $\overline{t}_{j^{(k)}-1,k} = \ell^{(k)} - \widetilde{\ell} \le \ell^{(k+1)} - \widetilde{\ell} = \overline{t}_{j^{(k)}-1,k+1}$. This proves the column-strictness of $\overline{t}$.

By definition $\overline{x}^{(k)}_{\ell^{(k)}-1} = P^{(k)}_{\ell^{(k)}-1}(\overline{\nu})$ for $1 \le k < r$. Hence we need to check that $P^{(k)}_i(\overline{\nu},\overline{t}) = P^{(k)}_i(\overline{\nu})$ for $i = \ell^{(k)}-1$. It suffices to show that there exists an index j such that

$$\widetilde{\ell} + \overline{t}_{j-1,k+1} \le \ell^{(k)} - 1 < \widetilde{\ell} + \overline{t}_{j,k}. \tag{8.35}$$

Assume that $p^{(k)} = j^{(k)}-1$. Then $\widetilde{\ell} + \overline{t}_{j^{(k)}-1,k} = \ell^{(k)}$ and by (8.31) $\ell^{(k)} \ge \widetilde{\ell} + t_{j^{(k)}-1,k+1} > \widetilde{\ell} + t_{j^{(k)}-2,k+1} = \widetilde{\ell} + \overline{t}_{j^{(k)}-2,k+1}$ so that (8.35) holds for $j = j^{(k)}-1$. Now assume $p^{(k)} = j^{(k)}$. Then by (8.33), $\ell^{(k)} > \widetilde{\ell}+t_{j^{(k)}-1,k+2} \ge \widetilde{\ell} + t_{j^{(k)}-1,k+1} = \widetilde{\ell} + \overline{t}_{j^{(k)}-1,k+1}$. Furthermore by (8.31) $\ell^{(k)} < \widetilde{\ell} + t_{j^{(k)},k} = \widetilde{\ell} + \overline{t}_{j^{(k)},k}$ so that (8.35) holds for $j = j^{(k)}$.

For $i \ne \ell^{(k)}-1$ we need to show that (8.30) implies (8.28). Note that for $1 \le k < r$ we have $\overline{t}_{j,k+1} \le t_{j+1,k+1}$ for $p^{(k)} \le j < j^{(k+1)}$ since $\min\{t_{j,k+2}, t_{j+1,k+1}\} \le t_{j+1,k+1}$ and $\ell^{(k+1)} - \widetilde{\ell} < t_{j^{(k+1)},k+1}$ for $1 \le k < r-1$ by (8.31). In addition $\ell^{(k+1)} \ge \widetilde{\ell} + \overline{t}_{j^{(k+1)}-1,k+1}$. For $p^{(k+1)} = j^{(k+1)}-1$ this follows directly from (8.34), and for $p^{(k+1)} = j^{(k+1)}$ we have $\ell^{(k+1)} \ge \widetilde{\ell} + t_{j^{(k+1)}-1,k+2} \ge \widetilde{\ell} + \overline{t}_{j^{(k+1)}-1,k+1}$ by (8.31). Since furthermore $\ell^{(k)} <$

$\widetilde{\ell}+t_{j^{(k)},k+1}$ by (8.31) we have for $1 \le k < r$

$$(8.36)\quad \begin{aligned} &-\chi(\ell^{(k)} \le i < \ell^{(k+1)}) \le \chi(\widetilde{\ell}+t_{p^{(k)},k+1} \le i < \ell^{(k)}) \\ &-\sum_{j=p^{(k)}}^{j^{(k+1)}-1} \chi(\widetilde{\ell}+t_{j,k+1} \le i < \widetilde{\ell}+\overline{t}_{j,k+1}) - \delta_{k+1,r}\chi(i \ge \widetilde{\ell}+t_{\lambda_r-\lambda_n,r}),\end{aligned}$$

where the last term occurs since $\ell^{(r)}=\infty$. Observe that $t_{j,k+1} \le \overline{t}_{j,k+1}$ for $p^{(k)} \le j < j^{(k+1)}$ and $1 \le k < r$ since $\min\{t_{j,k+2}, t_{j+1,k+1}\} \ge t_{j,k+1}$ and $\ell^{(k+1)} - \widetilde{\ell} \ge t_{j^{(k+1)}-1,k+2} \ge t_{j^{(k+1)}-1,k+1}$ for $1 \le k < r-1$ by (8.31). Hence using (8.36) and (8.34) we have for $1 \le k < r$

$$(8.37)\quad \begin{aligned} &-\chi(\ell^{(k)} \le i < \ell^{(k+1)}) + \sum_{j=1}^{\lambda_{k+1}-\lambda_n} \chi(i \ge \widetilde{\ell}+t_{j,k+1}) \\ &\le \sum_{j=1}^{\overline{\lambda}_{k+1}-\overline{\lambda}_n} \chi(i \ge \widetilde{\ell}+\overline{t}_{j,k+1}) + \chi(\widetilde{\ell}+t_{p^{(k)},k+1} \le i < \ell^{(k)}).\end{aligned}$$

Since $t_{j,1}=j$ by Remark 8.10 and $t_{j,1} \le t_{j,2}$ equation (8.31) implies that either $\ell^{(0)} \le \ell^{(1)} < \widetilde{\ell}+1$ for $j^{(1)}=1$ or $\ell^{(1)} = \widetilde{\ell}+j^{(1)}-1$ and $t_{j,2}=j$ for $1 \le j < j^{(1)} \le \lambda_r-\lambda_n+1$. Note that in both cases (8.2) reads $x_i^{(1)} \le P_i^{(1)}(\nu)$ for $1 \le i < \ell^{(1)}$. Since by construction there are no singular strings of length $\ell^{(0)} \le i < \ell^{(1)}$ in $(\nu, J)^{(1)}$ we can add the term $-\chi(\ell^{(0)} \le i < \ell^{(1)})$ to (8.2) for $k=1$. This has the effect that the term $\chi(\ell^{(0)} \le i < \ell^{(1)})$ in (8.30) for $k=1$ can be dropped. Note that for $j^{(1)}=1$ we have $p^{(1)}=1$ so that the term $\chi(\widetilde{\ell}+t_{p^{(1)},2} \le i < \ell^{(1)})$ in (8.37) is zero. For $j^{(1)}>1$ this term is also zero since $t_{p^{(1)},2} \ge t_{j^{(1)}-1,2} = j^{(1)}-1$ and $\ell^{(1)} = \widetilde{\ell}+j^{(1)}-1$. Since in addition $-\sum_{j=1}^{\lambda_1-\lambda_n} \chi(i \ge \widetilde{\ell}+t_{j,1}) = -\sum_{j=1}^{\overline{\lambda}_1-\overline{\lambda}_n} \chi(i \ge \widetilde{\ell}+\overline{t}_{j,1})$, this proves that (8.30) implies (8.28) for $k=1$.

Now assume that $1 < k < r$. By construction there are no singular strings of length $\ell^{(k-1)} \le i < \ell^{(k)}$ in $(\nu, J)^{(k)}$. Therefore the bounds (8.2) and hence also the bounds in (8.30) can be sharpened by adding

$$-\chi(\max\{\ell^{(k-1)}, \widetilde{\ell}+t_{j^{(k)}-1,k+1}\} \le i < \ell^{(k)})$$

for $j^{(k-1)} = j^{(k)}$ and

$$\begin{aligned} &-\chi(\max\{\ell^{(k-1)}, \widetilde{\ell}+t_{j^{(k-1)}-1,k+1}\} \le i < \widetilde{\ell}+t_{j^{(k-1)},k}) \\ &-\sum_{j=j^{(k-1)}+1}^{j^{(k)}-1} \chi(\widetilde{\ell}+t_{j-1,k+1} \le i < \widetilde{\ell}+t_{j,k}) - \chi(\widetilde{\ell}+t_{j^{(k)}-1,k+1} \le i < \ell^{(k)})\end{aligned}$$

for $j^{(k-1)} < j^{(k)}$. Adding these to $\chi(\ell^{(k-1)} \le i < \ell^{(k)})$ does not exceed

$$\sum_{j=p^{(k-1)}}^{j^{(k)}-1} \chi(\widetilde{\ell} + t_{j,k} \le i < \widetilde{\ell} + \min\{t_{j,k+1}, t_{j+1,k}\})$$

$$= \sum_{j=p^{(k-1)}}^{j^{(k)}-1} \chi(\widetilde{\ell} + t_{j,k} \le i < \widetilde{\ell} + \overline{t}_{j,k}) - \chi(\widetilde{\ell} + t_{p^{(k)},k+1} \le i < \ell^{(k)}).$$

Using again that $t_{j,k} \le \overline{t}_{j,k}$ for $p^{(k-1)} \le j < j^{(k)}$ this can be combined with the term $-\sum_{j=1}^{\lambda_k-\lambda_n} \chi(i \ge \widetilde{\ell} + t_{j,k})$ of (8.30) to yield $-\sum_{j=1}^{\overline{\lambda}_k-\overline{\lambda}_n} \chi(i \ge \widetilde{\ell} + \overline{t}_{j,k}) - \chi(\widetilde{\ell} + t_{p^{(k)},k+1} \le i < \ell^{(k)})$. Together with (8.37) this proves that (8.30) implies (8.28) for $1 < k < r$.

Consider $k = r$. Recall that $\lambda_{r+1} < \lambda_r$ so that (8.2) implies $x_i^{(r)} \le P_i^{(r)}(\nu) - 1$ for $i \ge \widetilde{\ell} + t_{\lambda_{r+1}-\lambda_n+1,r}$. By construction there are no singular strings of length $i \ge \ell^{(r-1)}$ in $(\nu, J)^{(r)}$. Hence for $j^{(r-1)} \le \lambda_{r+1} - \lambda_n + 1$ the bounds in (8.2) and (8.30) can be sharpened by adding

$$- \chi(\max\{\ell^{(r-1)}, \widetilde{\ell} + t_{j^{(r-1)}-1,r+1}\} \le i < \widetilde{\ell} + t_{j^{(r-1)},r}) \\ - \sum_{j=j^{(r-1)}+1}^{\lambda_{r+1}-\lambda_n+1} \chi(\widetilde{\ell} + t_{j-1,r+1} \le i < \widetilde{\ell} + t_{j,r})$$

which added to $\chi(i \ge \ell^{(r-1)})$ does not exceed

(8.38)

$$\sum_{j=p^{(r-1)}}^{\lambda_{r+1}-\lambda_n} \chi(\widetilde{\ell} + t_{j,r} \le i < \widetilde{\ell} + \min\{t_{j,r+1}, t_{j+1,r}\}) + \chi(i \ge \widetilde{\ell} + t_{\lambda_{r+1}-\lambda_n+1,r})$$

$$= \sum_{j=p^{(r-1)}}^{\lambda_r-\lambda_n-1} \chi(\widetilde{\ell} + t_{j,r} \le i < \widetilde{\ell} + \min\{t_{j,r+1}, t_{j+1,r}\}) + \chi(i \ge \widetilde{\ell} + t_{\lambda_r-\lambda_n,r})$$

where in the last line we used that $\min\{t_{j,r+1}, t_{j+1,r}\} = t_{j+1,r}$ for $j > \lambda_{r+1} - \lambda_n$ since by definition $t_{j,r+1} = \lambda_1 - \lambda_n + 1$ in this case. The last line of (8.38) also makes sense for $j^{(r-1)} > \lambda_{r+1} - \lambda_n + 1$ since then $p^{(r-1)} = j^{(r-1)} - 1$ and $\widetilde{\ell} + t_{j^{(r-1)}-1,r} \le \ell^{(r-1)}$ by (8.31). The last line of (8.38) combined with $-\sum_{j=1}^{\lambda_r-\lambda_n} \chi(i \ge \widetilde{\ell} + t_{j,r})$ yields $-\sum_{j=1}^{\overline{\lambda}_r-\overline{\lambda}_n} \chi(i \ge \widetilde{\ell} + \overline{t}_{j,r})$ using (8.34). For $r < k < n$ the term $\chi(\ell^{(k-1)} \le i < \ell^{(k)})$ vanishes and $-\sum_{j=1}^{\lambda_k-\lambda_n} \chi(i \ge \widetilde{\ell} + t_{j,k}) = -\sum_{j=1}^{\overline{\lambda}_k-\overline{\lambda}_n} \chi(i \ge \widetilde{\ell} + \overline{t}_{j,k})$. Similarly $-\chi(\ell^{(k)} \le i < \ell^{(k+1)})$ is zero for $r \le k < n$ and $\sum_{j=1}^{\lambda_{k+1}-\lambda_n} \chi(i \ge \widetilde{\ell} + t_{j,k+1}) = \sum_{j=1}^{\overline{\lambda}_{k+1}-\overline{\lambda}_n} \chi(i \ge \widetilde{\ell} + \overline{t}_{j,k+1})$. Together these results prove (8.28) for $r \le k < n$.

This concludes the proof of the reverse direction of the theorem for $1 < r < n$.

Case $r = n$**.** In this case $\overline{\lambda}_k = \lambda_k$ for $1 \le k < n$, $\overline{\lambda}_n = \lambda_n - 1$ and $\overline{\widetilde{\ell}} = \widetilde{\ell} - 1$. Then by (8.32) it follows that $j^{(1)} = \cdots = j^{(n-1)} = 1$. In particular from (8.31), $\ell^{(1)} < \widetilde{\ell} + t_{1,1} = \widetilde{\ell} + 1$ which yields a contradiction when $\widetilde{\ell} = 0$ since by assumption $\ell^{(1)} \ge \mu_L \ge 1$. Hence the case $r = n$ cannot occur when $\widetilde{\ell} = 0$. Define $\overline{t}_{1,k} = 1$ and $\overline{t}_{j,k} = t_{j-1,k} + 1$ for $1 < j \le \overline{\lambda}_k - \overline{\lambda}_n$ for all $1 \le k < n$. Since $t_{j,k} \in \{1, 2, \ldots, \overline{\lambda}_1 - \overline{\lambda}_n - 1\}$ it follows that $\overline{t}_{j,k} \in \{1, 2, \ldots, \overline{\lambda}_1 - \overline{\lambda}_n\}$. The column-strictness of t immediately implies the column-strictness of $\overline{t}$.

Since $\mu_L \le \ell^{(k)} < \widetilde{\ell} + t_{1,k}$ and there are no singular strings of length $\ell^{(k-1)} \le i < \ell^{(k)}$ in $(\nu, J)^{(k)}$ we may drop the term $\chi(\ell^{(k-1)} \le i < \ell^{(k)})$ in (8.30). In addition dropping the term $-\chi(\ell^{(k)} \le i < \ell^{(k+1)})$ (8.30) implies for $i \ne \ell^{(k)} - 1$

$$
\begin{aligned}
\overline{x}_i^{(k)} \le & P_i^{(k)}(\overline{\nu}) - \sum_{j=1}^{\overline{\lambda}_k - \overline{\lambda}_n - 1} \chi(i \ge \overline{\widetilde{\ell}} + 1 + t_{j,k}) + \sum_{j=1}^{\overline{\lambda}_{k+1} - \overline{\lambda}_n - 1} \chi(i \ge \overline{\widetilde{\ell}} + 1 + t_{j,k+1}) \\
= & P_i^{(k)}(\overline{\nu}) - \sum_{j=2}^{\overline{\lambda}_k - \overline{\lambda}_n} \chi(i \ge \overline{\widetilde{\ell}} + \overline{t}_{j,k}) + \sum_{j=2}^{\overline{\lambda}_{k+1} - \overline{\lambda}_n} \chi(i \ge \overline{\widetilde{\ell}} + \overline{t}_{j,k+1}).
\end{aligned}
$$

The terms $j = 1$ can be added to both sums since they just cancel each other so that we have (8.28) for $i \ne \ell^{(k)} - 1$.

Finally consider the case $i = \ell^{(k)} - 1$. We have $\ell^{(k)} < \widetilde{\ell} + t_{1,k}$ for $1 \le k < n$ so that $\ell^{(k)} - 1 < \overline{\widetilde{\ell}} + \overline{t}_{2,k}$. Since the terms $j = 1$ in the two sums cancel, (8.28) for $i = \ell^{(k)} - 1$ reduces to $\overline{x}^{(k)}_{\ell^{(k)}-1} \le P^{(k)}_{\ell^{(k)}-1}(\overline{\nu})$, or equivalently $P^{(k)}_{\ell^{(k)}-1}(\overline{\nu}, \overline{t}) = P^{(k)}_{\ell^{(k)}-1}(\overline{\nu})$ as desired.

This concludes the proof that T^- is of level ℓ.

Zu guter Letzt. It remains to show that $\lambda_1 - \lambda_n^0 \le \ell$.

Define $T^b = T^{-\mu_L - b}$ for $0 \le b \le \mu_L$ with corresponding rigged configurations $(\nu^b, J^b) = \overline{\delta}^{\mu_L - b}(\nu, J)$, and $\lambda^b = \mathrm{shape}(T^b)$. Let $(x^b)_i^{(k)}$ be the largest rigging occurring for the strings of length i in (ν^b, J^b) and let r_b be the row index of the cell $\lambda^b / \lambda^{b-1}$ for $1 \le b \le \mu_L$. Then $n \ge r_1 \ge r_2 \ge \cdots \ge r_{\mu_L} \ge 1$. Denote the length of the selected string in $(\nu^b, J^b)^{(k)}$ under $\overline{\delta}$ by $\ell_b^{(k)}$. Let $1 \le \beta \le \mu_L$ be maximal such that $r_\beta = n$. If no such β exists set $\beta = 0$. Then $\lambda_n^0 = \lambda_n - \beta$. Hence proving $\lambda_1 - \lambda_n^0 \le \ell$ is equivalent to showing that $\beta \le \widetilde{\ell}$.

If $\mu_L \le \widetilde{\ell}$ then also $\beta \le \mu_L \le \widetilde{\ell}$. Hence assume that $\mu_L = \widetilde{\ell} + d$ with $d \ge 1$. For $b > \widetilde{\ell}$ set $\overline{b} = b - \widetilde{\ell}$. We will show by descending induction on

$\widetilde{\ell} < b \le \widetilde{\ell} + d$ that

$$(8.39)\quad (x^b)_i^{(k)} \le P_i^{(k)}(\nu^b) - \sum_{j=1}^{\min\{\overline{b},\lambda_k-\lambda_n\}} \chi(i \ge \widetilde{\ell} + t_{j,k}) + \sum_{j=1}^{\min\{\overline{b},\lambda_{k+1}-\lambda_n\}} \chi(i \ge \widetilde{\ell} + t_{j,k+1})$$

for all $1 \le k < n$ and $1 \le i < \widetilde{\ell} + t_{\overline{b},k+1}$ where recall that $t_{j,k} = \lambda_1 - \lambda_n + 1$ if $j > \lambda_k - \lambda_n$.

Let us first show that (8.39) holds for $b = \widetilde{\ell} + d$. This follows directly from (8.2) using that $-\sum_{j=1}^{\lambda_k-\lambda_n} \chi(i \ge \widetilde{\ell} + t_{j,k}) \le -\sum_{j=1}^{\min\{d,\lambda_k-\lambda_n\}} \chi(i \ge \widetilde{\ell} + t_{j,k})$ and $\sum_{j=1}^{\lambda_{k+1}-\lambda_n} \chi(i \ge \widetilde{\ell} + t_{j,k+1}) = \sum_{j=1}^{\min\{d,\lambda_{k+1}-\lambda_n\}} \chi(i \ge \widetilde{\ell} + t_{j,k+1})$ thanks to the fact that by assumption $1 \le i < \widetilde{\ell} + t_{d,k+1}$ and $t_{j,k+1} < t_{j+1,k+1}$.

Now assume (8.39) to be true for some $\widetilde{\ell} < b \le \mu_L$. We will prove that then $r_b < n$ and that (8.39) holds for $b-1$ if $b > \widetilde{\ell} + 1$. We claim that

$$(8.40)\qquad \ell_b^{(k)} \ge \widetilde{\ell} + t_{\overline{b},k+1}$$

for all $1 \le k < n$ where by definition $t_{j,n} = \lambda_1 - \lambda_n + 1$. Assume the opposite, namely let $1 \le \kappa < n$ be the smallest index such that $\ell_b^{(\kappa)} < \widetilde{\ell} + t_{\overline{b},\kappa+1}$. By (8.39) there are no singular strings of lengths $\widetilde{\ell} + t_{\overline{b},\kappa} \le i < \widetilde{\ell} + t_{\overline{b},\kappa+1}$ in $(\nu^b, J^b)^{(\kappa)}$ so that $\ell_b^{(\kappa)} < \widetilde{\ell} + t_{\overline{b},\kappa}$. By the minimality of κ and the fact that $\ell_b^{(0)} = b = \widetilde{\ell} + t_{\overline{b},1}$ this implies that $\ell_b^{(\kappa)} < \ell_b^{(\kappa-1)}$ which is a contradiction. This proves (8.40). Note that similar to Remark 8.11 equation (8.39) can be interpreted in terms of $\overline{b}$ non-intersecting paths which all end at position ℓ. In this language the condition (8.40) states that $\ell_b^{(k)}$ is to the right of the $\overline{b}$-th path. Since all paths end at ℓ and there are no parts of length greater than ℓ in $\nu_b^{(k)}$ this implies that $r_b < n$. More precisely, $r_b \le k$ if $\overline{b} > \lambda_{k+1} - \lambda_n$ for all $1 \le k < n$.

Let us now prove (8.39) at $b-1$. It follows from (8.29) that

$$P_i^{(k)}(\nu^b) = P_i^{(k)}(\nu^{b-1}) + \chi(\ell_b^{(k-1)} \le i < \ell_b^{(k)}) - \chi(\ell_b^{(k)} \le i < \ell_b^{(k+1)}).$$

By (8.40), $\ell_b^{(k-1)} \ge \widetilde{\ell} + t_{\overline{b},k}$. Hence $\chi(\ell_b^{(k-1)} \le i < \ell_b^{(k)}) \le \chi(\widetilde{\ell} + t_{\overline{b},k} \le i)$ so that for $\overline{b} \le \lambda_k - \lambda_n$

$$\chi(\ell_b^{(k-1)} \le i < \ell_b^{(k)}) - \sum_{j=1}^{\min\{\overline{b},\lambda_k-\lambda_n\}} \chi(i \ge \widetilde{\ell} + t_{j,k}) \le - \sum_{j=1}^{\min\{\overline{b}-1,\lambda_k-\lambda_n\}} \chi(i \ge \widetilde{\ell} + t_{j,k})$$

as desired. When $\overline{b} > \lambda_k - \lambda_n$ then $\ell_b^{(k-1)} = \infty$ so that the above inequality still holds. Since we only consider $1 \leq i < \widetilde{\ell} + t_{\overline{b}-1,k+1}$ the term $-\chi(\ell_b^{(k)} \leq i < \ell_b^{(k+1)})$ does not contribute by (8.40) and in addition $\overline{b}$ can be replaced by $\overline{b} - 1$ in $\sum_{j=1}^{\min\{\overline{b}, \lambda_{k+1} - \lambda_n\}} \chi(i \geq \widetilde{\ell} + t_{j,k+1})$. This proves (8.39) for $b - 1$. Since we have shown that (8.39) implies that $r_b < n$ for $\widetilde{\ell} < b \leq \mu_L$ it follows that $\beta \leq \widetilde{\ell}$.

This concludes the proof of Theorem 8.9.

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