Regime Switching Volatility Calibration by the Baum-Welch Method

by

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Abstract

Regime switching volatility models provide a tractable method of modelling stochastic volatility. Currently the most popular method of regime switching calibration is the Hamilton filter. We propose using the Baum-Welch algorithm, an established technique from Engineering, to calibrate regime switching models instead. We demonstrate the Baum-Welch algorithm and discuss the significant advantages that it provides compared to the Hamilton filter. We provide computational results of calibrating the Baum-Welch filter to S&P 500 data and validate its performance in and out of sample.

Key words: Regime switching, stochastic volatility, calibration, Hamilton filter, Baum-Welch.

1. Introduction and Outline

Regime switching (also known as hidden Markov models (HMM)) volatility models provide a tractable method of modelling stochastic volatility. Currently the most popular method of regime switching calibration is the Hamilton filter. However, regime switching calibration has been tackled in engineering (particularly for speech processing) for some time using the Baum-Welch algorithm (BW), where it is the most popular and standard method of HMM calibration. A review of the Baum-Welch algorithm can be found in [Lev05],[JR91]. The BW algorithm is increasingly being applied beyond engineering applications (for instance in bioinformatics [BEDE04]) but has been hardly applied to financial modelling, especially to regime switching stochastic volatility models.

Unlike the Hamilton filter, the BW algorithm is capable of determining the entire set of HMM parameters from a sequence of observation data. Furthermore, BW is a complete estimation method since it also provides the required optimisation method to determine the parameters by MLE.

The outline of the paper is as follows. Firstly, we introduce regime switching volatility models and the Hamilton filter. In the next section we introduce the Baum-Welch method, describing the algorithm even for multivariate Gaussian mixture observations. We then conduct a numerical experiment to verify the Baum-Welch method's to detect regimes for the S&P 500 index. We finally end with a conclusion.

2. Regime Switching Volatility Model and Calibration

2.1. Regime Switching Volatility

Wiener process driven stochastic volatility models capture price and volatility dynamics more successfully compared to previous volatility models. Specifically, such models successfully capture the short term volatility dynamics. However, for longer term dynamics and fundamental economic changes (e.g. "credit crunch"), no mechanism existed to address the change in volatility dynamics and it has been empirically shown that volatility is related to long term and fundamental conditions. Bekaert in [BHL06] claims that volatility changes are caused by economic reforms, for example on Black Wednesday the pound sterling was withdrawn from the ERM (European Exchange Rate Mechanism), causing a sudden change in value of the pound sterling [BR02]. Schwert [Sch89] empirically shows that volatility increases during financial crises.

A class of models that address fundamental and long term volatility modelling is the regime switching model (or hidden Markov model) e.g. as discusssed in [Tim00],[EvdH97]. In fact, Schwert suggests in [Sch89] that volatility changes during the Great Depression can be accounted for by a regime change such as in Hamilton's regime switching model [Ham89]. Regime switching is considered a tractable method of modelling price dynamics and does not violate Fama's "Efficient Market Hypothesis"[Fam65], which claims that price processes must follow a Markov process. Hamilton [Ham89] was the first to introduce regime switching models, which was applied to specifically model fundamental economic changes.

For regime switching models, generally the return distribution rather than the continuous time process is specified. A typical example of a regime switching model is Hardy's model [Har01]:

$$log((X(t+1)/X(t))|i) \sim \mathcal{N}(u_i, \varphi_i), i \in \{1, .., R\}, \tag{1}$$

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where

- φ_i and u_i are constant for the duration of the regime;

- i denotes the current regime (also called the Markov state or hidden Markov state);

- R denotes the total number of regimes;

- a transition matrix **A** is specified.

For Hardy's model the regime changes discretely in monthly time steps but stochastically, according to a Markov process.

Due to the ability of regime switching models to capture long term and fundamental changes, regime switching models are primarily focussed on modelling the long term behaviour, rather than the continuous time dynamics. Therefore regime switching models switch regimes over time periods of months, rather than switching in continuous time. Examples of regime switching models that model dynamics over shorter time periods are Valls-Pereira et al.[VPHS04], who propose a regime switching GARCH process, while Hamilton and Susmel [HS94] give a regime switching ARCH process. Note that economic variables other than stock returns, such as inflation, can also be modelled using regime switching models.

Regime switching has been developed by various researchers. For example, Kim and Yoo [KY95] develop a multivariate regime switching model for coincident economic indicators. Honda [Hon03] determines the optimal portfolio choice in terms of utility for assets following GBM but with continuous time regime switching mean returns. Alexander and Kaeck [AK08] apply regime switching to credit default swap spreads, Durland and McCurdy [DM94] propose a model with a transition matrix that specifies state durations.

The theory of Markov models (MM) and Hidden Markov models (HMM) are methods of mathematically modelling time varying dynamics of certain statistical processes, requiring a weak set of assumptions yet allow us to deduce a significant number of properties. MM and HMM model a stochastic process (or any system) as a set of states with each state possessing a set of signals or observations. The models have been used in diverse applications such as economics [SSS02], queuing theory [SF06], engineering [TG01] and biological modelling [MGPG06]. Following Taylor [TK84] we define a Markov model:

Definition 1. *A Markov model is a stochastic process $X(t)$ with a countable set of states and possesses the Markov property:*

$$p(q_{t+1} = j \mid q_1, q_2, .., q_t = i) = p(q_{t+1} = j \mid q_t = i), \tag{2}$$

where

- q_t *is the Markov state (or regime) at time t of $X(t)$;*

- *i and j are specific Markov states.*

As time passes the process may remain or change to another state (known as state transition). The state transition probability matrix (also known as the *transition kernel or stochastic matrix*) **A**, with elements a_{ij}, tells us the probability of the process changing to state j given that we are now in state i, that is $a_{ij} = p(q_{t+1} = j \mid q_t = i)$. Note that a_{ij} is subject to the standard probability constraints:

$$0 \leq a_{ij} \leq 1, \forall i, j, \tag{3}$$

$$\sum_{j=1}^{\infty} a_{ij} = 1, \forall i. \tag{4}$$

We assume that all probabilities are stationary in time. From the definition of a MM the following proposition follows:

Proposition 1. *A Markov model is completely defined once the following parameters are known:*

- R, *the total number of regimes or (hidden) states;*

- *state transition probability matrix **A** of size $R \times R$. Each element is $a_{ij} = p(q_{t+1} = j|q_t = i)$, where i refers to the matrix row number and j to the column number of **A**;*

- *initial (t=1) state probabilities $\pi_i = p(q_1 = i), \forall i$.*

A hidden Markov model is simply a Markov model where we assume (as a modeller) we do not observe the Markov states. Instead of observing the Markov states (as in standard Markov models) we detect observations or time series data where each observation is assumed to be a function of the hidden Markov state, thus enabling statistical inferences about the HMM. Note that in a HMM it is the states which must be governed by a Markov process, not the observations and throughout the thesis we will assume one observation occurs after one state transition.

Proposition 2. *A hidden Markov model is fully defined when the parameter set $\{\mathbf{A}, \mathbf{B}, \pi\}$ are known:*

- R, *the total number of (hidden) states or regimes;*

- ***A**, the (hidden) state transition matrix of size $R \times R$. Each element is $a_{ij} = p(q_{t+1} = j|q_t = i)$;*

- *initial (t=1) state probabilities $\pi_i = p(q_1 = i), \forall i$;*

- ***B**, the observation matrix, where each entry is $b_j(O_t) = p(O_t|j)$ for observation O_t. For $b_j(O_t)$ is typically defined to follow some continuous distribution e.g. $b_j(O_t) \sim \mathcal{N}(u_j, \varphi_j)$.*

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2.2. Current Calibration Method: Hamilton Filter

In financial mathematics or economic literature the standard calibration method for regime switching models is the Hamilton filter [Ham89], which works by maximum likelihood estimation (MLE). MLE is a method of estimating a set of parameters of a statistical model (Θ) given some time series or empirical observations $O_1, O_2, ..., O_T$. MLE determines Θ by firstly determining the likelihood function $\mathcal{L}(\Theta)$, then maximising $\mathcal{L}(\Theta)$ by varying Θ through a search or an optimisation method.

A statistical model with known parameter values can determine the probability of an observation sequence $O = O_1 O_2 ... O_T$. MLE does the opposite; we numerically maximise the parameter values of our model Θ such that we maximise the probability of the observation sequence $O = O_1 O_2 ... O_T$. To achieve this the MLE method makes two assumptions:

1. In maximising $\mathcal{L}(\Theta)$ the local optimum is also the global optimum (although this is generally not true in reality). The optimal values for Θ are in a search space of the same dimensions as Θ. Hamilton in [Ham94] gives a survey of various MLE maximisation techniques such as the Newton-Raphson method;

2. The observations $O_1, O_2, ..., O_T$ are statistically independent. Note that for Markov models we assume the conditional observations $(O_t|O_{t-1}), (O_{t-1}|O_{t-2}), (O_{t-2}|O_{t-3})...$ are independent.

For a regime switching process the general likelihood function $\mathcal{L}(\Theta)$ is:

$$\begin{aligned} \mathcal{L}(\Theta) \quad = \quad & f(O_1|\Theta) f(O_2|\Theta, O_1) f(O_3|\Theta, O_1, O_2) \\ & \cdots \quad f(O_T|\Theta, O_1, O_2, ..., O_{T-1}), \end{aligned}$$

where $f(O_{(.)}|\Theta)$ is the probability of $O_{(.)}$, given model parameters Θ. Now by properties of logarithms we have:

$$\begin{aligned} log(\mathcal{L}(\Theta)) \quad = \quad & log(f(O_1|\Theta)) + log(f(O_2|\Theta, O_1)) + \cdots && (5) \\ & + \quad log(f(O_T|\Theta, O_1, O_2, ..., O_{T-1})). && (6) \end{aligned}$$

Hamilton proposes a likelihood function for regime switching models, the Hamilton filter. As an example, if we assume we have a two regime model with each regime having a lognormal return distribution, we wish to determine parameters $\Theta = \{u_1, u_2, \varphi_1, \varphi_2, a_{12}, a_{21}\}$. Note that in this simple HMM $a_{22} = 1 - a_{12}$ and $a_{11} = 1 - a_{21}$ therefore we do not need to estimate a_{22}, a_{11} in Θ.

To obtain $f(O_t|\Theta)$ in equation (6) for $t > 1$, Hamilton showed it could be calculated by a recursive filter. We observe the relation:

$$f(O_t|\Theta, O_1, O_2, ..., O_{t-1}) = \sum_{q_t=1}^{2} \sum_{q_{t-1}=1}^{2} f(q_t, q_{t-1}, O_t|\Theta, O_1, ..., O_{t-1}). \tag{7}$$

Now using the relation:[1]

$$p(O, Q|\Theta) = p(O|\Theta, Q)p(Q|\Theta), \tag{8}$$

where $Q = q_1 q_2...$ represents some arbitrary state sequence, we make the substitution $f(q_t, q_{t-1}, O_t|\Theta, O_1, ..., O_{t-1})$

$$= p(q_{t-1}|\Theta, O_1, ..., O_{t-1}) \times p(q_t|q_{t-1}, \Theta) \times f(O_t|q_t, \Theta). \tag{9}$$

Therefore
$f(O_t|\Theta, O_1, O_2, ..., O_{t-1})$

$$= \sum_{q_t=1}^{2} \sum_{q_{t-1}=1}^{2} p(q_{t-1}|\Theta, O_1, ..., O_{t-1}) \times p(q_t|q_{t-1}, \Theta) \times f(O_t|q_t, \Theta), \tag{10}$$

where

- $p(q_t|q_{t-1}, \Theta) = p(q_t = j|q_{t-1} = i, \Theta)$ represents the transition probability a_{ij} we wish to estimate;

- $f(O_t|q_t = i, \Theta) = p_i(O_t)$ where $p_i(\cdot) \sim \mathcal{N}(u_i, \varphi_i)$ the Gaussian probability density function for state i, whose parameters u_i, φ_i we wish to estimate.

The parameters $\Theta = \{u_1, u_2, \varphi_1, \varphi_2, a_{12}, a_{21}\}$ are obtained by maximising the likelihood function using some chosen search method.

To calculate $f(O_t|\Theta, O_1, O_2, ..., O_{t-1})$ we require the probability $p(q_{t-1}|\Theta, O_1, O_2, ..., O_{t-1})$ in equation (10) (summed over two different values of q_{t-1} in the summations in equation (10)). This can be achieved through recursion, that is the probability $p(q_{t-1}|\Theta, O_1, .., O_{t-1})$ can be obtained from $p(q_{t-2}|\Theta, O_1, .., O_{t-2})$:

$$p(q_{t-1}|\Theta, O_1, O_2, ..., O_{t-1}) = \frac{\left(\sum_{i=1}^{2} f(q_{t-1}, q_{t-2} = i, O_{t-1}|\Theta, O_1, ..., O_{t-2}) \right)}{f(O_{t-1}|\Theta, O_1, ..., O_{t-2})}. \tag{11}$$

The denominator of equation (11) is obtained from the previous period of $f(O_t|\Theta, O_1, O_2, ..., O_{t-1})$ (in other words $f(O_{t-1}|\Theta, O_1, O_2, ..., O_{t-2})$) so by inspecting

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[1]Note: following discussions with Prof.Rabiner [Rab08] on the equation for $p(O, Q|\Theta)$ it was concluded that the equation for $p(O, Q|\Theta)$ in Rabiner's paper [Rab89] is incorrect.

equation (10) we can see it is a function of $p(q_{t-2}|\Theta, O_1, ..., O_{t-2})$. The numerator of equation (11) is obtained from calculating equation (9) for the previous time period, which is also a function of $p(q_{t-2}|\Theta, O_1, ..., O_{t-2})$.

To start the recursion of equation (11) at $p(q_1 = i|O_1, \Theta)$ we require $f(O_1|\Theta)$. Hamilton assumes the Markov chain has been running sufficiently long enough so that we can make the following assumption about our observations $O_1, O_2, ..., O_T$. Technically, Hamilton assumes the observations $O_1, O_2, ..., O_T$ are all drawn from the Markov chain's invariant distribution. If a Markov chain has been running for a sufficiently long time, the following property of Markov chains can be applied:

$$\eta_j = \lim_{t \to \infty} p(q_t = j|q_1 = i), \qquad \forall i, j = 1, 2, .., R, \tag{12}$$

$$\text{where } \sum_{j=1}^{R} \eta_j = 1, \eta_j > 0. \tag{13}$$

The probability η_j tells us in the long run $(t \to \infty)$ the (unconditional) probability of being in state j and this probability is independent of the initial state (at time t=1). An important interpretation of η_j is as the fraction of time spent in state j in the long run. Therefore the probability of state j is simply η_j and so:

$$f(O_1|\Theta) = f(q_1 = 1, O_1|\Theta) + f(q_1 = 2, O_1|\Theta), \tag{14}$$

$$\text{where } f(q_1 = i, O_1|\Theta) = \eta_i p_i(O_1). \tag{15}$$

We can therefore calculate $p(q_1 = i|O_1, \Theta)$:

$$p(q_1 = i|O_1, \Theta) = \frac{f(q_1 = i, O_1|\Theta)}{f(O_1|\Theta)}, \tag{16}$$

$$= \frac{\eta_i p_i(O_1)}{\eta_1 p_1(O_1) + \eta_2 p_2(O_1)}. \tag{17}$$

Furthermore it can be proved for a two state HMM that:

$$\eta_1 = a_{21}/(a_{12} + a_{21}),$$

$$\eta_2 = 1 - \eta_1.$$

Therefore $p(q_1 = i|O_1, \Theta)$ can be obtained from estimating the parameter set $\Theta = \{u_1, u_2, \varphi_1, \varphi_2, a_{12}, a_{21}\}$, which is obtained by a chosen search method.

The advantages of Hamilton's filter method are firstly we do not need to specify or determine the initial probabilities, therefore there are fewer parameters to estimate (compared to the alternative Baum-Welch method). Therefore the MLE parameter optimisation will be over a lower dimension search space. Secondly, the MLE equation is simpler to understand and so easier to implement compared to other calibration methods.

3. Baum-Welch Algorithm

The Baum-Welch (BW) is a complete estimation method since it also provides the required optimisation method to determine the parameters by MLE. We will now explain the BW algorithm and to do so we must first explain the forward algorithm, which we will do now.

3.1. Forward Algorithm

The forward algorithm calculates $p(O|M)$, the probability of a fixed or observed sequence $O=O_1O_2...O_T$, given all the HMM parameters denoted by $M = \{\mathbf{A}, \mathbf{B}, \pi\}$. We recall from the definition of HMM that the probability of each observation $p(O_t)$ will change depending on the state at time t (q_t). Hence the most straightforward way to calculate $p(O|M)$ is:

$$
\begin{aligned}
p(O|M) &= \sum_{\text{all Q}} p(O, Q|M), && (18) \\
&= \sum_{\text{all Q}} p(O|M, Q).p(Q|M), && (19) \\
&= \sum_{\text{all Q}} \pi_{q_1} b_{q_1}(O_1).a_{q_1 q_2} b_{q_2}(O_2)....a_{q_{T-1} q_T} b_{q_T}(O_T), && (20)
\end{aligned}
$$
$$
\text{where } p(O|M, Q) = b_{q_1}(O_1).b_{q_2}(O_2).....b_{q_T}(O_T). \tag{21}
$$

Here "all Q" means all possible state sequences $q_1 q_2...q_T$ that could account for observation sequence O, $b_{(.)}(O_{(.)})$ is defined in proposition 2, $p(O|M, Q)$ is the probability of the observed sequence O, given it is along one single state sequence $Q = q_1 q_2...q_T$ and for HMM M. We must sum equation (20) over all possible Q state sequences, requiring R^T computations and so this is computationally infeasible even for small R and T.

To overcome the computational difficulty of calculating $p(O|M)$ in equation (20) we apply the forward algorithm, which uses recursion (dynamic programming). The forward algorithm only requires computations of the order $R^2 T$ and so is significantly faster than calculating equation (20) for large R and T.

Let us define the forward variable $\kappa_t(i)$:

$$
\kappa_t(i) = p(O_1 O_2...O_t, q_t = i|M). \tag{22}
$$

Given the HMM M, $\kappa_t(i)$ is the probability of the joint observation upto time t of $O_1 O_2...O_t$ and the state at time t is i i.e. $q_t = i$. If we can determine $\kappa_T(i)$ we can calculate $p(O|M)$ since:

$$
p(O|M) = \sum_{i=1}^{R} \kappa_T(i). \tag{23}
$$

Now $\kappa_{t+1}(j)$ can be expressed in terms of $\kappa_t(i)$, therefore we can calculate $\kappa_{t+1}(j)$ by recursion:

$$\kappa_{t+1}(j) = \left[\sum_{i=1}^{R} \kappa_t(i)a_{ij}\right] b_j(O_{t+1}), 1 \leq t \leq T - 1. \tag{24}$$

The variable $\kappa_{t+1}(j)$ in equation (24) can be understood as follows: $\kappa_t(j)a_{ij}$ is the probability of the joint event $O_1....O_t$ is observed, the state at time t is i and state j is reached at time t+1. If we sum this probability over all R possible states for i, we get the probability of j at t+1 accompanied with all previous observations from $O_1 O_2...O_t$ only. Thus to get $\kappa_{t+1}(j)$ we must multiply by $b_j(O_{t+1})$ so that we have all observations $O_1...O_{t+1}$.

Therefore the recursive algorithm is as follows:

1. Initialisation:

$$\kappa_1(i) = \pi_i b_i(O_1), 1 \leq i \leq R. \tag{25}$$

2. Recursion:

$$\kappa_{t+1}(j) = \left[\sum_{i=1}^{R} \kappa_t(i)a_{ij}\right] b_j(O_{t+1}), 1 \leq t \leq T - 1. \tag{26}$$

3. Termination: t+1=T.

4. Final Output:

$$p(O|M) = \sum_{i=1}^{R} \kappa_T(i). \tag{27}$$

At t=1 no sequence exists but we initialise the recursion with π_i to determine $\kappa_1(i)$.

3.2. Baum-Welch Algorithm

Having explained the forward algorithm we can now explain the BW algorithm. Using observation sequence O, the BW algorithm iteratively calculates the HMM parameters $M = \{\mathbf{A}, \mathbf{B}, \pi\}$. Specifically, BW estimates $M = \{a_{ij}, b_j(\cdot), \pi_i\}\forall i, j$, denoted respectively by $\overline{M} = \{\overline{a}_{ij}, \overline{b}_j(\cdot), \overline{\pi}_i\}$, such that it maximises the likelihood of $p(O|\overline{M})$. No method of analytically finding the globally optimal $\overline{M}$ exists. However it has been theoretically proven BW is guaranteed to find the local optimum [Rab89].

Let us define $\psi_t(i, j)$:

$$\psi_t(i, j) = p(q_t = i, q_{t+1} = j \mid O, M). \tag{28}$$

The variable $\psi_t(i,j)$ is the probability of being in state i at time t and state j at time t+1, given the HMM parameters M and the observed observation sequence O. We can re-express $\psi_t(i,j)$ as:

$$\psi_t(i,j) = p(q_t = i, q_{t+1} = j \mid O, M), \tag{29}$$

$$= \frac{p(q_t = i, q_{t+1} = j, O \mid M)}{p(O|M)}. \tag{30}$$

Now we can re-express equation (30) using the forward variable $\kappa_t(i) = p(O_1 O_2 ... O_t, q_t = i|M)$ and using analogously the so called backward variable $\varrho_{t+1}(i)$:

$$\varrho_t(i) = p(O_{t+1} O_{t+2} ... O_T | q_t = i, M), \tag{31}$$

$$\text{so that } \varrho_{t+1}(i) = p(O_{t+2} O_{t+3} ... O_T | q_{t+1} = i, M). \tag{32}$$

The backward variable $\varrho_t(i)$ is the probability of the partial observed observation sequence from time t+1 to the end T, given M and the state at time t is i. It is calculated in a similar recursive method to the forward variable using the backward algorithm (see [Rab89] for more details). Hence we can rewrite $\psi_t(i,j)$ as

$$\psi_t(i,j) = \frac{\kappa_t(i) a_{ij} b_j(O_{t+1}) \varrho_{t+1}(j)}{p(O|M)}. \tag{33}$$

We can also rewrite the denominator $p(O|M)$ in terms of the forward and backward variables, so that $\psi_t(i,j)$ is entirely expressed in terms of $\kappa_t(i), a_{ij}, b_j(O_{t+1}), \varrho_{t+1}(j)$:

$$p(O|M) = \sum_{i=1}^{R} \sum_{j=1}^{R} \kappa_t(i) a_{ij} b_j(O_{t+1}) \varrho_{t+1}(j). \tag{34}$$

Now let us define $\Gamma_t(i)$:

$$\Gamma_t(i) = p(q_t = i|O, M), \tag{35}$$

$$= \sum_{j=1}^{R} \psi_t(i,j). \tag{36}$$

Equation (36) can be understood from the definition of $\psi_t(i,j)$ in equation (29); summing $\psi_t(i,j)$ over all j must give $p(q_t = i|O, M)$, the probability in state i at time t, given the observation sequence O and model M. Now if we sum $\Gamma_t(i)$ from t=1 to T-1 it gives us $\Upsilon(i)$, the expected number of transitions made from state i:

$$\Upsilon(i) = \sum_{t=1}^{T-1} \Gamma_t(i). \tag{37}$$

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If we sum $\Gamma_t(i)$ from t=1 to T it gives us $\vartheta(i)$, the expected number of times state i is visited:

$$\vartheta(i) = \sum_{t=1}^{T} \Gamma_t(i). \tag{38}$$

We are now in a position to estimate $\overline{M}$. The variable $\overline{a}_{ij}$ is estimated as the expected number of transitions from state i to state j divided by the expected number of transitions from state i:

$$\overline{a}_{ij} = \frac{\sum_{t=1}^{T-1} \psi_t(i,j)}{\Upsilon(i)}. \tag{39}$$

The variable $\overline{\pi}_i$ is estimated as the expected number of times in state i at time t=1:

$$\overline{\pi}_i = \Gamma_1(i). \tag{40}$$

The variable $\overline{b}_j(\tilde{s})$ is estimated as the expected number of times in state j and observing a particular signal $\tilde{s}$, divided by the expected number of times in state j:

$$\overline{b}_j(\tilde{s}) = \frac{\sum_{t=1}^{T} \Gamma_t(j)'}{\vartheta(j)}, \tag{41}$$

where $\Gamma_t(j)'$ is $\Gamma_t(j)$ with condition $O_t = \tilde{s}$.

We can now describe our BW algorithm:

1. Initialisation:

   Input initial values of $\overline{M}$ (otherwise randomly initialise) and calculate $p(O|\overline{M})$ using the forward algorithm.

2. Estimate new values of $\overline{M}$:

   Iterate until convergence:

   (a) Using current $\overline{M}$ calculate variables $\overline{\kappa}_t(i), \overline{\varrho}_{t+1}(j)$ by the forward and backward algorithm and then calculate $\overline{\psi}_t(i,j)$ as in equation (33).

   (b) Using calculated $\overline{\psi}_t(i,j)$ in (a) determine new estimates of $\overline{M}$ using equations (36)-(41).

   (c) Calculate $p(O|\overline{M})$ with new $\overline{M}$ values using the forward algorithm.

   (d) Goto step 3 if two consecutive calculations of $p(O|\overline{M})$ are equal (or converge within a specified range). Otherwise repeat iterations: goto (a).

3. Output $\overline{M}$.

The BW algorithm is started with initial estimates of $\overline{M} = (\overline{\mathbf{A}}, \overline{\mathbf{B}}, \overline{\pi})$. These estimates in turn are used to calculate the right hand side of equations (39),(40) and (41) to give the next new estimate of $\overline{M} = (\overline{\mathbf{A}}, \overline{\mathbf{B}}, \overline{\pi})$. We consider the new estimate $\overline{M_n}$ to be a better estimate than the previous estimate $\overline{M_p}$, if $p(O|\overline{M_n}) > p(O|\overline{M_p})$, with both probabilities calculated via the forward algorithm. In other words, we prefer the $\overline{M}$ that increases the probability of observation O occurring.

If $p(O|\overline{M_n}) > p(O|\overline{M_p})$ then the iterative calculation is repeated with $\overline{M_n}$ as the input. Note that at the end of step two, if the algorithm re-iterates then inputting the new $\overline{M}$ at step 2a means we will get a new set of $\overline{M}$ after executing 2b. The iteration is stopped when $p(O|\overline{M_n}) = p(O|\overline{M_p})$ or is arbitrarily close enough and at this point the BW algorithm finishes.

Since the BW algorithm has been proven to always converge to the local optimum, the BW will output the local optimum. We also note that correct choice of R is important since $p(O|M)$ changes as M changes for a fixed O, however this disadvantage is common to all MLE methods.

3.3. Multivariate Gaussian Mixture Baum-Welch Calibration

To account for the variety of empirical distributions possible for various assets and capturing asymmetric properties arising from volatility (such as fat tails), we model each regime's distribution by a two component multivariate Gaussian mixture (GM), which is a mixture of two multinormal distributions.

Definition 2. *(Multinormal Distribution) Let* $\mathbf{X} = (X_1, X_2..., X_n)$ *be an n-dimensional random vector where each dimension is a random variable. Let* $\boldsymbol{u} = (u_1, u_2..., u_n)$ *represent an n dimensional vector of means,* $\boldsymbol{\Sigma}$ *represent an* $n \times n$ *covariance matrix. We say* $\mathbf{X}$ *follows a multinormal distribution if*

$$\mathbf{X} \sim \mathcal{N}_{\mathbf{n}}(\boldsymbol{u}, \boldsymbol{\Sigma}), \tag{42}$$

which may be alternatively written as

$$\begin{pmatrix} X_1 \\ X_{...} \\ X_n \end{pmatrix} \sim \mathcal{N}_n \left(\begin{pmatrix} u_1 \\ u_{...} \\ u_n \end{pmatrix}, \begin{pmatrix} \varphi_{11} & \varphi_{...} & \varphi_{1n} \\ \varphi_{...} & \varphi_{...} & \varphi_{...} \\ \varphi_{n1} & \varphi_{...} & \varphi_{nn} \end{pmatrix} \right). \tag{43}$$

The probability of $\mathbf{X}$ *is*

$$p(\mathbf{X}) = \frac{1}{2\pi\sqrt{det(\boldsymbol{\Sigma})}} exp\left(-\frac{1}{2}(\mathbf{X} - \boldsymbol{u})^T \boldsymbol{\Sigma}^{-1}(\mathbf{X} - \boldsymbol{u}) \right), \tag{44}$$

where $det(\boldsymbol{\Sigma})$ *denotes the determinant of* $\boldsymbol{\Sigma}$.

Definition 3. *(Multivariate Gaussian Mixture) A multivariate Gaussian mixture consists of a mixture of K multinormal distributions, spanning n-dimensions. It is defined by:*

$$\mathbf{X} \sim c_1 \mathcal{N}_{\mathbf{n}}(\mathbf{u}_1, \mathbf{\Sigma}_1) + ... + c_K \mathcal{N}_{\mathbf{n}}(\mathbf{u}_K, \mathbf{\Sigma}_K), \tag{45}$$

where c_k are weights and

$$\sum_{k=1}^{k=K} c_k = 1, c_k \geq 0. \tag{46}$$

The term $p_{gmm}(\mathbf{X})$ denotes the probability of a multivariate Gaussian mixture variable $\mathbf{X}$ and is defined as

$$p_{gmm}(\mathbf{X}) = \sum_{k=1}^{K} c_k p_k(\mathbf{X}), \tag{47}$$

where $p_k(\mathbf{X}) \sim \mathcal{N}_{\mathbf{n}}(\mathbf{u}_k, \mathbf{\Sigma}_k)$.

If we model a stochastic process $\mathbf{X}$ by a Gaussian mixture for each regime then for a given regime j we have:

$$\mathbf{X} \sim c_{j1} \mathcal{N}_{\mathbf{n}}(\mathbf{u}_{j1}, \mathbf{\Sigma}_{j1}) + ... + c_{jK} \mathcal{N}_{\mathbf{n}}(\mathbf{u}_{jK}, \mathbf{\Sigma}_{jK}). \tag{48}$$

The probability of $\mathbf{X}$ for a given regime j, $p_{gmm}(\mathbf{X})_j$, is:

$$p_{gmm}(\mathbf{X})_j = \sum_{k=1}^{k=K} c_{jk} p_{jk}(\mathbf{X}). \tag{49}$$

where

- $p_{jk}(\mathbf{X}) \sim \mathcal{N}_{\mathbf{n}}(\mathbf{u}_{jk}, \mathbf{\Sigma}_{jk})$;

- c_{jk} are weights for each regime j and

$$\sum_{k=1}^{k=K} c_{jk} = 1, \forall j. \tag{50}$$

Note that the dimensions of multivariate distribution n are independent of the number of mixture components K.

For an n-asset portfolio $\mathbf{X(t)} = (X_1(t), X_2(t), ..., X_n(t))$, where $X_i(t)$ represents the stock price of asset i, with each asset following a Gaussian mixture, the portfolio returns would be modelled by:

$$d\mathbf{X}/\mathbf{X} \sim c_{j1} \mathcal{N}_{\mathbf{n}}(\mathbf{u}_{j1}, \mathbf{\Sigma}_{j1}) + c_{j2} \mathcal{N}_{\mathbf{n}}(\mathbf{u}_{j2}, \mathbf{\Sigma}_{j2}). \tag{51}$$

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For practial calibration purposes we set the multivariate observation *vector* $\mathbf{O_t}$ to annual log returns:

$$\mathbf{O_t} = log(\mathbf{X}(t + \Delta t)/\mathbf{X}(t)), \tag{52}$$

where $\Delta t = 1$ year.

Combining GM with HMM gives us a GM-HMM (Gaussian mixture HMM) model and the BW algorithm can be adapted to it: Gaussian mixture BW (GM-BW). For $\mathbf{O}_t$ our observation (vector) at time t we model $b_j(\mathbf{O})$ by GM:

$$b_j(\mathbf{O}) = p_{gmm}(\mathbf{O})_j. \tag{53}$$

The BW algorithm for calculating $\mathbf{A}, \pi_i$ remains the same; for $\mathbf{B}$ we have a GM. We would like to obtain the GM mixture coeffficents c_{jk}, mean vectors $\boldsymbol{u}_{jk}$ and covariance matrices $\boldsymbol{\Sigma}_{jk}$ whose estimates are $\overline{c}_{jk}$, $\overline{\boldsymbol{u}}_{jk}$ and $\overline{\boldsymbol{\Sigma}}_{jk}$ respectively. These can be incorporated within the BW algorithm as detailed by Rabiner [Rab89]:

$$\overline{\boldsymbol{u}}_{jk} = \frac{\sum_{t=1}^{T} \Gamma_t(j,k).\mathbf{O}_t}{\sum_{t=1}^{T} \Gamma_t(j,k)}, \tag{54}$$

$$\overline{\boldsymbol{\Sigma}}_{jk} = \frac{\sum_{t=1}^{T} \Gamma_t(j,k).(\mathbf{O}_t - \overline{\boldsymbol{u}}_{jk})(\mathbf{O}_t - \overline{\boldsymbol{u}}_{jk})^T}{\sum_{t=1}^{T} \Gamma_t(j,k)}, \tag{55}$$

$$\overline{c}_{jk} = \frac{\sum_{t=1}^{T} \Gamma_t(j,k)}{\sum_{t=1}^{T} \sum_{k=1}^{K} \Gamma_t(j,k)}, \tag{56}$$

$$\text{where } \Gamma_t(j,k) = \left[\frac{\kappa_t(j)\varrho_t(j)}{\sum_{j=1}^{N} \kappa_t(j)\varrho_t(j)} \right] \left[\frac{c_{jk} p_{jk}(\mathbf{O}_t)}{p_{gmm}(\mathbf{O}_t)_j} \right]. \tag{57}$$

Here $\Gamma_t(j,k)$ is the probability at time t of being in state j with the k mixture component accounting for $\mathbf{O}_t$. Using the same logic as in section 3 (for non mixture distributions) we can understand equations (54)-(56), for example $\overline{c}_{jk}$ is the expected number of times the HMM k-th component is in state j divided by the expected number of times in state j.

It is worth noting that a well known problem in maximum likelihood estimation of GM is that observations with low variances give extremely high likelihoods, in which case the likelihood function does not converge [MT07]. To overcome this problem in the univariate case Messina and Toscani [MT07] implement Ridolfi's and Idier's [RI02] penalised maximum likelihood function, which limits the likelihood value of observations. This is beneficial in [MT07] because the observation time scales are of the order of days and therefore the variance of samples may approach zero. For our applications we calibrate the GM-HMM to annual return data, therefore the samples are unlikely to approach variances anywhere near zero.

3.4. Advantages of Baum-Welch Calibration

The BW algorithm has significant advantages over the Hamilton filter. Firstly, the Hamilton filter requires observation data to be taken from the invariant distribution in order to estimate the parameters (see equation (12)). To obtain observations from the invariant distribution implies the number of state transitions approaches a large limit, so is not suited to Markov chains that have run for a short time. Furthermore, the time to reach the invariant distribution increases with the number of regimes R and the number of Gaussian mixtures K.

Psaradakis and Sola [PS98] investigated the finite sample properties of the Hamilton filter for financial data. They concluded that samples of at least 400 observations are required for a simple two state regime switching model where each state's observation is modelled by a normal distribution.

Secondly, the Hamilton filter has no method of estimating the initial state probabilities whereas the BW is able to take account of and estimate initial state probabilities. This has a number of important consequences:

1. BW does not require observations from the invariant distribution and so can be calibrated to data of any observation length.

2. the Hamilton filter cannot fully define the entire HMM model since the initial state probabilities are one of the key HMM parameters in the definition (see HMM definition in section 2).

3. we cannot determine the probability of observation sequences $p(O|M)$, since we require the initial state probabilities. This can be understood from the forward algorithm.

4. we cannot determine the most likely state sequence that accounts for a given observation sequence and HMM, which can be obtained by the Viterbi algorithm. The Viterbi algorithm tells us the most likely state sequence for a given observation sequence and HMM parameters M (see Forney [FJ73] for more information).

5. without the initial state probabilities, we cannot simulate state sequences since the initial state radically alters the state sequence and its influence on the state sequence increases as the sequence size decreases. Consequently we cannot validate a model's feasibility by simulation.

Note that BW estimates initial state probabilities independently of the transition probabilities, whereas in the Hamilton filter η_i is a function of estimated transition probabilities. Hence BW is able to independently estimate more HMM parameters than the Hamilton filter.

Thirdly, to our knowledge the Hamilton filter cannot be applied to multivariate distributions, nor more complicated univariate distributions than Gaussians. Particularly for financial applications, we use multivariate data to model portfolios and multivariate stochastic volatility is becoming an increasingly important research area (see Bauwens et al. [BLR06] for a survey on multivariate GARCH). Hamilton has proposed a calibration method for univariate mixture distributions, the Quasi-Bayesian MLE approach [Ham91], yet this requires some prior knowledge regarding the reliability of observations. The GM-BW calibrates a multivariate Gaussian mixture to multivariate data, thereby capable of modelling most empirically observed distributions.

Fourthly, the GM-BW can calibrate time varying correlations. It is known that correlations amongst random variables tend to be unstable with time; for example Buckley et al. [BSS07] give evidence of covariances varying with time and model them as regime dependent. The GM-BW algorithm gives the covariance matrix for each regime and each regime is postulated to be linked to an economic state. Therefore, we can model and extract information on changing correlations with changing economic conditions. For instance, some stocks are considered to be strongly correlated with the economic cycle (known as cyclical stocks) e.g. British Airways, whereas other stocks are considered independent of the economy (known as defensive stocks) e.g. Tesco.

Finally, the BW algorithm is a complete HMM estimation method whereas the Hamilton filter is not. Hamilton's method provides no method or guidance as to the optimisation algorithm to apply for finding the parameters from the non-linear filter, yet the solutions can be significantly influenced by the non-convex optimisation method applied. The BW algorithm includes an estimation method for the full HMM and a numerical optimisation scheme. Additionally, the BW method is guaranteed to find the local optimum.

4. Numerical Experiment: Baum-Welch GM-HMM Calibration Results

In this section we calibrate a 2-state regime switching model with 2 Gaussian components to S&P 500 data from 1976-1996. We fitted the model:

$$dX/X \sim c_{j1}\mathcal{N}_1(u_{j1}, \varphi_{j1}) + c_{j2}\mathcal{N}_1(u_{j2}, \varphi_{j2}), j \in \{1, 2\}. \tag{58}$$

We set our observation to annual log returns:

$$O_t = log(X(t + \Delta t)/X(t)), \tag{59}$$

where Δt=1 year and X(t) is the stock price.

Due to GM-BW's wide usage in engineering it has already been implemented by numerous authors. We chose K. Murphy's Matlab implementation [Mur08] because it is considered one of the most standard and cited GM-BW programs. It also offers many useful features that are unavailable on other implementations e.g. the Viterbi algorithm for obtaining state sequences.

The GM-BW algorithm finds the best GM-HMM parameters that maximise the likelihood of the observations, however, this involves searching a nonconvex search space and BW only finds the local optimum. Theoretically, the globally optimal parameters can be determined by initialising the GM-BW algorithm over every possible starting point, then the globally optimal parameters are those that give the highest likelihood. However, the GM-BW algorithm finds the locally optimal $\overline{M} = (\overline{\mathbf{A}}, \overline{\mathbf{B}}, \overline{\pi})$ (where $\overline{\mathbf{B}}$ is parameterised by $\overline{c}_{jk}, \overline{\varphi}_{jk}$ and $\overline{u}_{jk}$), so that the calibration problem has a nonconvex solution search space over thirteen dimensions.

Due to the high dimensionality of the parameter estimation problem for either the univariate or multivariate case, determining the optimal parameters by initialising through different starting points is impractical. Instead we concentrated our effort on finding good initial parameter estimates for M. This was to significantly increase the probability of finding the best GM-BW solutions, particularly as initialisation strongly influences the GM-BW optimisation [LDLK04].

$\overline{\pi}$ *Initialisation*

To initialise $\overline{\pi}$, we assign a probability of 0.5 to state one if the first observed return is positive, with the probability increasing in value the more positive it is.

$\overline{\mathbf{A}}$ *Initialisation*

Given the HMM structure was chosen due to its ability to capture long term and fundamental properties, we can initialise $\overline{\mathbf{A}}$ based on the long term and fundamental properties we expect it to possess:

$$\overline{\mathbf{A}} = \begin{pmatrix} 0.6 & 0.4 \\ 0.7 & 0.3 \end{pmatrix}. \tag{60}$$

Each state represents an economic regime so $\overline{\mathbf{A}}$ can be interpreted as follows. The economy in the long term follows an upward drift, so we would expect it is more likely the HMM remains in state one rather than goto state two, given it is already in state one -hence we assign probability 0.6. Similarly, if the HMM is in state two we would

expect it is far more likely to return to state one than state two to capture the cyclical behaviour and in the long term the economy follows an upward trend, hence we assign probabilities 0.7 and 0.3.

GM Initialisation ($\overline{\mathbf{B}}$)

The GM distribution fitting strongly affects the GM-BW algorithm optimisation, hence it must be satisfactorily initialised for GM-BW to provide acceptable results. However it is well known that GM distribution fitting in general (without any regime switching) is a non-trivial problem:

- there are a large set of parameters to estimate.

- there exists the issue of uniqueness, that is for a given non-parametric distribution there does not always exist a unique set of GM parameter values.

- the flexibility of GM distributions to model virtually any unimodal or bimodal distribution means that it incorporates rather than rejects any noise in the data into the distribution. Therefore GM fitting is highly sensitive to noise.

- parameter estimation is further complicated with regime switching and the fact we cannot identify with certainty the (hidden) state associated with each observation.

Rather than randomly initialise the GM parameters (as is done in Murphy's program) we approximately divided data into each regime by classifying positive returns as belonging to state one, otherwise they belong to state two. We then fit a GM for each "regime's" data using a GM fitting program used by Lund University (stixbox).

4.2. Results and Discussion

We present the results of the Baum-Welch GM-HMM calibration to S&P 500 data from 1976-1996:

Table 1: Initial State Probabilities (π_i)

State (i)	Probability
1	1×10^{-6}
2	$1 - 1 \times 10^{-6}$

$$\overline{\mathbf{A}} = \begin{pmatrix} 0.78 & 0.22 \\ 0.82 & 0.18 \end{pmatrix}$$

Table 2: Mixture Means u_{jk} (%/year)

Gaussian Component (k)	State (j)	
	1	2
$\mathcal{N}_1$	13.0	-4.8
$\mathcal{N}_2$	28.0	1.4
Overall	14.8	-4.7

Table 3: Mixture Standard Deviations $\sqrt{\varphi_{jk}}$ (%/year)

Gaussian Component (k)	State (j)	
	1	2
$\mathcal{N}_1$	4.5	5.6
$\mathcal{N}_2$	28.0	110.0
Overall	11.6	13.3

Table 4: Weighting Matrix (c_{jk})

Gaussian Component (k)	State (j)	
	1	2
$\mathcal{N}_1$	0.88	0.99
$\mathcal{N}_2$	0.12	0.01

From table 2 we can infer that the BW algorithm has attributed state two as the down state since its overall mean is negative, unlike state one. Using the Markowitz's variance measure of risk [Mar52] it is encouraging that we can conclude that the risk level in state two is higher than in state one (see table 3), since a declining economy (state two) is a riskier economic state. Additionally, an increase in variance (and therefore volatility) with lower returns is consistent with the leverage effect. The initial state probabilities $\bar{\pi}$ strongly suggest the HMM starts in state two and this is consistent with the data as the first year's return (see table 5) is relatively low (1.2%).

The transition matrix $\bar{\mathbf{A}}$ is similar to the transition matrix theoretically postulated (for an economy); thus it is consistent with our theoretical expectations. The matrix $\bar{\mathbf{A}}$ tells us that given the model is in state one it is likely to stay in that state most of the time, only transitioning to state two for 22% of the time. Additionally, in state two the model is most likely to return to state one (probability 0.78), thus captures the cyclical nature of the economy.

To validate the quality of the GM-HMM calibration in terms of state sequence generation, we ran the Viterbi algorithm. Note that calibration under the Hamilton filter does not provide or enable state sequence estimation. The Viterbi algorithm gave the following state sequence result for in sample data 1976-96 in table 5:

Table 5: In Sample Regime Sequence Results

Year	Regime	Empirical Annual Return (%)
1976	2	1.2
1977	2	-13.4
1978	1	11.3
1979	1	13.4
1980	1	12.6
1981	2	-7.3
1982	1	18.8
1983	1	11.7
1984	1	9.5
1985	1	16.5
1986	1	25.8
1987	2	-6.5
1988	1	14.6
1989	1	10.1
1990	1	4.4
1991	1	17.3
1992	1	7.1
1993	1	9.3
1994	2	-2.4
1995	1	30.2
1996	1	21.2

The regime sequence concurs with the empirical observations; negative or low returns were categorised into state two, whereas other returns were classed as state one. The results also illustrate the behaviour of the transition matrix $\overline{\mathbf{A}}$; generally remaining in state one (up state) whilst occasionally entering state two, in which case it quickly reverts back to state one. Hence the GM-HMM is able to capture the key characteristics of the economy, namely an upward trend and cyclicity.

The Viterbi algorithm was also applied to out of sample data from 1997-2007. Again the GM-HMM was able to satisfactorily classify the years for each state, as shown in table 6:

Table 6: Out of Sample Regime Sequence Results

Year	Regime	Empirical Annual Return (%)
1997	1	22.1
1998	1	26.6
1999	1	8.6
2000	2	-2.1
2001	2	-18.9
2002	1	-27.8
2003	1	27.9
2004	1	4.3
2005	1	8.0
2006	1	11.6
2007	2	-2.6

5. Conclusions

This paper has shown the advantages of Baum-Welch calibration over the standard Hamilton filter method. Not only does the Baum-Welch method offer a complete calibration procedure but also is able to estimate the full set of HMM parameters, unlike the Hamilton filter. We have also validated the usage of the Baum-Welch method through numerical experiments on S&P 500 data in and out of sample.

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