Symplectic quotients by a nonabelian group and by its maximal torus

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Introduction

This paper examines the relationship between the symplectic quotient $X/\!\!/G$ of a Hamiltonian G-manifold X, and the associated symplectic quotient $X/\!\!/T$, where $T \subset G$ is a maximal torus, in the case in which $X/\!\!/G$ is a compact manifold or orbifold.

The three main results are: a formula expressing the rational cohomology ring of $X/\!\!/G$ in terms of the rational cohomology ring of $X/\!\!/T$; an 'integration' formula, which expresses cohomology pairings on $X/\!\!/G$ in terms of cohomology pairings on $X/\!\!/T$; and an index formula, which expresses the indices of elliptic operators on $X/\!\!/G$ in terms of indices on $X/\!\!/T$.

The results of this paper are complemented by the results in a companion paper [15], in which different techniques are used to derive formulæ for cohomology pairings on symplectic quotients $X/\!\!/T$, where T is a torus, in terms of the T-fixed points of X. That paper also gives some applications of the formulæ proved here.

In order to state the main results of this paper, we introduce some notation. The symplectic quotient $X/\!\!/G$ is defined to be the topological quotient $\mu_G^{-1}(0)/G$, where $\mu_G : X \to \mathrm{Lie}(G)^*$ is a moment map for the G-action on X. A choice of maximal torus $T \subset G$ induces a natural projection $\mathrm{Lie}(G)^* \twoheadrightarrow \mathrm{Lie}(T)^*$, and composing with μ_G gives a moment map $\mu_T : X \to \mathrm{Lie}(T)^*$ for the T-action, with $X/\!\!/T := \mu_T^{-1}(0)/T$. In most of this paper we make some additional simplifying assumptions: we assume that both $\mu_G^{-1}(0)$ and $\mu_T^{-1}(0)$ are compact manifolds, on which the respective G- and T-actions are free. It follows that $X/\!\!/G$ and $X/\!\!/T$ are compact symplectic manifolds. In section 6 we show how to modify the main results when various of these assumptions are dropped.

For every *weight* α of T there is a characteristic class $e(\alpha) \in H^2(X/\!\!/T)$ naturally associated[1] to the principal T-bundle $\mu_T^{-1}(0) \to X/\!\!/T$ (for a precise definition see section 2).

Theorem A (Cohomology rings). *There is a natural ring isomorphism*

$$H^*(X/\!\!/G;\mathbb{Q}) \cong \frac{H^*(X/\!\!/T;\mathbb{Q})^W}{\mathrm{ann}(e)}.$$

Here W denotes the Weyl group of G, which acts naturally on $X/\!\!/T$; the class $e \in H^*(X/\!\!/T)^W$ is defined by $e := \prod_{\alpha\in\Delta} e(\alpha)$ for Δ the roots of G, and $\mathrm{ann}(e) \lhd H^*(X/\!\!/T;\mathbb{Q})^W$ is the ideal consisting of all W-invariant elements $c \in H^*(X/\!\!/T;\mathbb{Q})^W$ such that the product $c \smile e$ vanishes.

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[1] A weight of T is a homomorphism $\alpha : T \to S^1$. Denoting by $\mathbb{C}_{(\alpha)}$ the representation space on which T acts via this homomorphism, we define $L_\alpha \to X/\!\!/T$ to be the associated line bundle, that is, $L_\alpha := \mu_T^{-1}(0) \times_T \mathbb{C}_{(\alpha)}$, and $e(\alpha)$ to be the Euler class of L_α.

There is a natural notion of a **lift** of a cohomology class on $X/\!\!/G$ to a class on $X/\!\!/T$, compatible with the above isomorphism. The most concrete way of expressing this[2] involves the manifold $Y := \mu_G^{-1}(0)/T$. There is an obvious inclusion $i : Y \hookrightarrow X/\!\!/T$ and projection $\pi : Y \to X/\!\!/G$, and we say $\tilde{a} \in H^*(X/\!\!/T)$ is a **lift** of $a \in H^*(X/\!\!/G)$ if $\pi^* a = i^* \tilde{a}$.

Theorem B (Integration formula). *Given a cohomology class $a \in H^*(X/\!\!/G)$ with lift $\tilde{a} \in H^*(X/\!\!/T)$, then*

$$\int_{X/\!\!/G} a = \frac{1}{|W|} \int_{X/\!\!/T} \tilde{a} \smile e,$$

where $|W|$ is the order of the Weyl group of G, and e is the cohomology class defined in Theorem A.

This formula gives cohomology pairings on $X/\!\!/G$ because the lift of a class is compatible with cup product: given classes $a, b \in H^*(X/\!\!/G)$ with lifts $\tilde{a}, \tilde{b} \in H^*(X/\!\!/T)$, then $\tilde{a} \smile \tilde{b}$ is a lift of $a \smile b$.

The symplectic quotient $X/\!\!/G$ is a compact symplectic manifold, and it can be given an almost complex structure $J : T(X/\!\!/G) \to T(X/\!\!/G)$ which is compatible with its symplectic structure in a certain sense. Using the same prescription[3] as on a Kähler manifold, we can then define an elliptic operator

$$D := \overline{\partial} + \overline{\partial}^* : C^\infty\big(\Lambda^{\mathrm{even}} T^{0,1}\big) \to C^\infty\big(\Lambda^{\mathrm{odd}} T^{0,1}\big).$$

Furthermore, if $V \to X/\!\!/G$ is a complex vector bundle, then a choice of Hermitian connection on V lets us define an elliptic operator $D_V : C^\infty\big(V \otimes \Lambda^{\mathrm{even}} T^{0,1}\big) \to C^\infty\big(V \otimes \Lambda^{\mathrm{odd}} T^{0,1}\big)$. While the operator D_V depends on the various choices involved, its index does not, and we have

Theorem C (Index formula). *Suppose $V \to X/\!\!/G$ is a complex vector bundle, and $\tilde{V} \to X/\!\!/T$ is a lift of V. Then*

$$\mathrm{index}^{X/\!\!/G}\, D_V = \mathrm{index}^{X/\!\!/T}\, D_{\tilde{V} \otimes \Lambda^{even} E} - \mathrm{index}^{X/\!\!/T}\, D_{\tilde{V} \otimes \Lambda^{odd} E}$$

Here we can take $E \to X/\!\!/T$ to equal $\bigoplus_{\alpha \in \Delta^+} L_\alpha$ for any choice Δ^+ of positive roots of G, where $L_\alpha \to X/\!\!/T$ denotes the complex line bundle naturally associated to the weight α.

This formula is simpler than one would get by applying the Atiyah-Singer index theorem to the integration formula—its proof runs along similar lines, but brings in a result of Borel and Hirzebruch on the Todd genus of G/T.

The layout of this paper is as follows. Section 1 contains the main topological result, giving a detailed description of the topological relationship between $X/\!\!/G$ and $X/\!\!/T$. Section 2 uses this result, together with some cohomological facts, to prove theorem B, the integration formula. In section 3 the integration formula is combined with Poincaré duality to prove theorem A, and in section 4 the index formula is proved. Section 5 is a very short section

[2] A more natural way of expressing the fact that $\tilde{a}$ is a lift of a brings in the G-equivariant cohomology $H^*_G(X)$. There are natural maps from $H^*_G(X)$ to both $H^*(X/\!\!/G)$ and $H^*(X/\!\!/T)$, and $\tilde{a}$ is a lift of a if they are both images of the same class in $H^*_G(X)$.

[3] the operator $\overline{\partial}$ is the usual Cauchy-Riemann operator from $(0,i)$-forms to $(0,i+1)$-forms—this is well-defined on an almost-complex manifold, although $\overline{\partial} \circ \overline{\partial}$ does not necessarily vanish—see for example Griffiths and Harris [7, p. 80]; the almost complex structure combines with the symplectic structure to define a natural metric, which we use to define $\overline{\partial}^*$. (The complex structure can alternatively be viewed as defining a spin^c structure, with D defined as the spin^c-Dirac operator, taking even spinors to odd spinors.)

describing some formulæ for characteristic numbers of $X/\!\!/G$, such as the Euler characteristic and the signature. In section 6 various generalizations of the main results are described, including straightforward generalizations to the case when the two symplectic quotients are compact orbifolds, to the case in which $X/\!\!/T$ may have singularities or be noncompact, and finally, in a different direction, a generalization in which T is replaced by a subgroup of full rank. Finally, in section 7, the results of this paper are applied to the explicit example of the Grassmannian of k-dimensional planes in $\mathbb{C}^n$, which arises as a symplectic quotient by the group $U(k)$. The associated symplectic quotient by the maximal torus is the k-fold product $(\mathbb{CP}^{n-1})^k$. One result is a presentation of the cohomology of the Grassmannian which is different from the usual one.

Relation to other results

The results of this paper, together with the companion paper [15], have been applied by Jeffrey and Kirwan [11] to prove certain formulæ for cohomology pairings on moduli spaces of stable holomorphic bundles over a Riemann surface. These formulæ were first derived by Witten [21] using physical arguments. Indeed the main motivation for the results of this paper and its companion was to find a purely topological proof of Witten's formulæ.

This work was carried out in 1994, in Oxford, and at the Isaac Newton Institute in Cambridge. After completing this work, I was made aware of some related results of Ellingsrud and Strømme [5]. The present paper intersects with theirs in the case that X is a complex vector space; in that case they have a result closely related to theorem A. (Their general setting is the 'Geometric invariant theory' quotient [19] of a vector space over an arbitrary field, for which they calculate the Chow ring; their techniques are completely different from those used here.)

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Notation

Fixed throughout this paper are the following:

- X is a fixed smooth symplectic manifold (with symplectic form ω);
- $G \supset T$ is a connected compact Lie group and a fixed maximal torus, both acting on X, preserving ω;
- $\mathfrak{g} \supset \mathfrak{t}$ are their Lie algebras;
- $\mu_G : X \to \mathfrak{g}^*$ is a moment map for the G action on X, which we assume throughout to be proper, and to have 0 as a regular value (our sign convention is $d\langle \mu_G, \xi\rangle = \omega(V(\xi), \cdot),\ \forall \xi \in \mathfrak{g}$, where $V(\xi) \in \Gamma(TX)$ denotes the vector field generated by the infinitesimal action of ξ; see the companion paper [15] for more details);
- $\mu_T : X \to \mathfrak{t}^*$ is the corresponding moment map for the restriction of the action to T (given by composing μ_G with the natural projection $\mathfrak{g}^* \twoheadrightarrow \mathfrak{t}^*$);
- $X/\!\!/G, X/\!\!/T$ denote the symplectic quotients $\mu_G^{-1}(0)/G$ and $\mu_T^{-1}(0)/T$ respectively.

1. The main topological result

The results of this paper all follow from one topological result, which we prove in this section. This result is stated in terms of certain complex line bundles on $X/\!\!/T$:

Definition 1.1. Let α be a weight of T, that is, a one-dimensional representation, and let $\mathbb{C}_{(\alpha)}$ denote the corresponding representation space. Then we define the line bundle $L_\alpha \to X/\!\!/T$ to be the associated bundle

$$\begin{array}{rc} L_\alpha := & \mu_T^{-1}(0) \times_T \mathbb{C}_{(\alpha)} \\ & \big\downarrow \\ & X/\!\!/T \end{array}$$

We denote by Δ the set of roots of G, that is, Δ is the set of nonzero weights which occur in the action of T on $\mathfrak{g} \otimes \mathbb{C}$; we fix a choice $\Delta^+ \subset \Delta$ of positive roots, and denote by Δ^- the corresponding negative roots.

$X/\!\!/G$ and $X/\!\!/T$ are related by a fibering and an inclusion:

$$\begin{array}{ccc} \mu_G^{-1}(0)/T & \xhookrightarrow{\ i\ } & \mu_T^{-1}(0)/T = X/\!\!/T \\ \big\downarrow{\scriptstyle \pi} & & \\ X/\!\!/G = \mu_G^{-1}(0)/G. & & \end{array}$$

Note that $X/\!\!/G$ and $X/\!\!/T$ are symplectic manifolds, and hence possess compactible almost complex structures, unique up to homotopy [17, proposition 4.1].

Proposition 1.2.
1. *The vector bundle $\bigoplus_{\alpha\in\Delta^-} L_\alpha \to X/\!\!/T$ has a section s, which is transverse to the zero section, and such that the zeroset of s is the submanifold $\mu_G^{-1}(0)/T \subset X/\!\!/T$. It follows that the derivative of s identifies the normal bundle*

$$\nu(\mu_G^{-1}(0)/T) \cong \bigoplus_{\alpha\in\Delta^-} L_\alpha\big|_{\mu_G^{-1}(0)/T}\,.$$

2. *Letting* $\mathrm{vert}(\pi) \to \mu_G^{-1}(0)/T$ *denote the vector bundle of tangents to the fibres of π, we have*

$$\mathrm{vert}(\pi) \cong \bigoplus_{\alpha\in\Delta^+} L_\alpha\big|_{\mu_G^{-1}(0)/T}\,.$$

3. *There is a complex orientation[4] of $\mu_G^{-1}(0)/T$ such that the above isomorphisms are isomorphisms of complex-oriented spaces and vector bundles, with respect to the complex orientations of $X/\!\!/G$ and $X/\!\!/T$ induced by their symplectic forms.*

A complex orientation induces a real orientation in a standard manner, and for most of this paper we will only need that the above isomorphisms are isomorphisms of real-oriented spaces and vector bundles. We will need complex orientations for the results on indices of elliptic operators and characteristic numbers, in sections 4 and 5.

[4]an almost complex structure on a manifold or a vector bundle defines a complex orientation, and two almost complex structures which are homotopic (through almost complex structures) define the same complex orientation. For the definition of complex orientation (which also involves stabilization) see [20].

Proof. We prove the three statements in the proposition in order.
1. The inclusion. We have the commuting triangle of maps

$$\begin{array}{ccc} X & \xrightarrow{\mu_G} & \mathfrak{g}^* \\ & \underset{\mu_T}{\searrow} & \downarrow{\scriptstyle p} \\ & & \mathfrak{t}^* \end{array}$$

where the projection p is induced by the inclusion $T \hookrightarrow G$. Define $V \subset \mathfrak{g}^*$ by $V := p^{-1}(0)$. Then $\mu_T^{-1}(0) = \mu_G^{-1}(V)$, and the fact that $0 \in \mathfrak{t}^*$ is a regular value for μ_T is equivalent to the assertion that the subspace V is transverse to the map μ_G. Note that μ_T is a T-equivariant map and the coadjoint action of T on $\mathfrak{g}^*$ preserves the subspace V. Moreover, given our choice of positive and negative roots, we have $V \cong \bigoplus_{\alpha \in \Delta^-} \mathbb{C}_{(\alpha)}$.

The restriction of μ_G to $\mu_T^{-1}(0)$ defines a T-equivariant map $\tilde{s} : \mu_T^{-1}(0) \to V$, and taking the quotient by T, then $\tilde{s}$ defines a section s of the associated bundle $\mu_T^{-1}(0) \times_T V \to X/\!\!/T$. Since $0 \in \mathfrak{g}^*$ is a regular value of μ_G it follows that $0 \in V$ is a regular value of $\tilde{s}$, and hence s is transverse to the zero section.

Finally, the identification of V in terms of negative roots gives $\mu_T^{-1}(0) \times_T V \cong \bigoplus_{\alpha \in \Delta^-} L_\alpha$.
2. The fibering. We write $Z := \mu_G^{-1}(0)$. Let π_G and π_T denote the projections

$$\begin{array}{ccc} \mu_G^{-1}(0) =: Z & \xrightarrow{\pi_T} & Z/T \\ & \underset{\pi_G}{\searrow} & \downarrow{\scriptstyle \pi} \\ & & Z/G \end{array}$$

(π was defined above the proposition).

Consider the foliation of Z given by the G-orbits: using a G-invariant metric to take orthgonal complements, and the infinitesimal action to identify tangents to the G-orbits, we have the G-equivariant identification

$$TZ \cong (Z \times \mathfrak{g}) \oplus \pi_G^* T(Z/G),$$

where G acts on $\mathfrak{g}$ by the adjoint action. Restricting to the action of T, we can refine this identification using the T-equivariant decomposition $\mathfrak{g} \cong \mathfrak{t} \oplus \mathfrak{v}$

$$TZ \cong (Z \times \mathfrak{t}) \oplus (Z \times \mathfrak{v}) \oplus \pi_G^* T(Z/G),$$

with $\mathfrak{v} \cong \bigoplus_{\alpha \in \Delta^+} \mathbb{C}_{(\alpha)}$. Identifying the $Z \times \mathfrak{t}$ factor as the tangents to the T-orbits, and taking the quotient by T,

$$T(Z/T) \cong (Z \times_T \mathfrak{v}) \oplus \pi^* T(Z/G).$$

Hence, identifying $\mathfrak{v}$ in terms of positive roots gives $\mathrm{vert}(\pi) \cong Z \times_T \mathfrak{v} \cong \bigoplus_{\alpha \in \Delta^+} L_\alpha\big|_{Z/T}$.
3. The orientation. We begin by summarizing the arguments in the final stage of the proof. We will first describe the symplectic form of $X/\!\!/T$, restricted to Z/T, in terms of a decomposition of the tangent bundle (equation (1.4) below). We then describe an almost complex structure $\tilde{J}_0$ on $X/\!\!/T$ which is compatible with the symplectic form on $X/\!\!/T$, and which has a simple description over Z/T. Finally, we show that $\tilde{J}_0$ is homotopic, through almost complex structures, to an almost complex structure $\tilde{J}_1$ on $X/\!\!/T$ with respect to which Z/T is an almost complex submanifold, and such that the almost complex structures on Z/T and on its normal bundle agree with the complex orientations described in the statement of the proposition.

3(i) The identification of the symplectic form. On $\mathfrak{g}\oplus\mathfrak{g}^*$ we define the G-invariant symplectic form η by using the duality pairing:

$$\eta(\xi,\alpha) := \langle \xi, \alpha \rangle\,, \quad \forall \xi \in \mathfrak{g}, \alpha \in \mathfrak{g}^*,$$

and demanding that η be skew-symmetric. Applying this definition fibrewise defines an invariant symplectic form on the vector bundle $Z\times(\mathfrak{g}\oplus\mathfrak{g}^*)\to Z$, which we will also denote by η. Then there exists a G-equivariant isomorphism of symplectic vector bundles over $Z=\mu_G^{-1}(0)$

$$TX|_Z \cong (Z\times\mathfrak{g})\oplus(Z\times\mathfrak{g}^*)\oplus\pi_G^*T(X/\!\!/G) \tag{1.3}$$

where the symplectic form on the left is the restriction of the symplectic form on X, and the symplectic form on the right is the direct sum of the symplectic form on $(Z\times\mathfrak{g})\oplus(Z\times\mathfrak{g}^*)$ given by η, and the pullback of the symplectic form on $X/\!\!/G$.

This isomorphism is defined as follows. A choice of connection on the principal G-bundle $Z\to X/\!\!/G$ defines a 'horizontal subbundle' $\mathcal{H}\subset TZ$, which is isomorphic to $\pi_G^*T(X/\!\!/G)$ (the isomorphism is induced by the derivative $d\pi_G$). Using the inclusion $TZ\hookrightarrow TX|_Z$ we can consider $\mathcal{H}$ to be a subbundle of $TX|_Z$, and we define $\mathcal{H}^\omega\subset TX|_Z$ to be the symplectic complement to $\mathcal{H}$, with respect to the restriction of the symplectic form ω on X. Standard calculations using the moment map then imply (1) the restriction of ω to $\mathcal{H}$ equals the pullback of the symplectic form on $X/\!\!/G$, (2) the subbundle $\mathcal{H}^\omega$ is a vector bundle complement to $\mathcal{H}$, containing $\mathrm{vert}(\pi_G)\cong(Z\times\mathfrak{g})$, and isomorphic to $(Z\times\mathfrak{g})\oplus(Z\times\mathfrak{g}^*)$ (with the isomorphism given by choosing an equivariant complement to $\mathrm{vert}(\pi_G)$ and identifying this complement with $Z\times\mathfrak{g}^*$ via $d\mu$), and (3) the restriction of ω to $\mathcal{H}^\omega\cong(Z\times\mathfrak{g})\oplus(Z\times\mathfrak{g}^*)$ equals the symplectic form η defined above.

The same arguments, applied to T and $\mu_T^{-1}(0)$ in place of G and $Z=\mu_G^{-1}(0)$, give an analogous isomorphism to that of equation (1.3) above. Combining these two isomorphisms, in a neighbourhood of Z, and arguing as in step 2 of this proof gives an isomorphism of symplectic vector bundles

$$T(X/\!\!/T)|_{Z/T} \cong (Z\times_T\mathfrak{v})\oplus(Z\times_T\mathfrak{v}^*)\oplus\pi^*T(X/\!\!/G) \tag{1.4}$$

such that the symplectic form on the left is the restriction of the symplectic form on $X/\!\!/T$, and on the right is the direct sum of the natural symplectic form on $(Z\times_T\mathfrak{v})\oplus(Z\times_T\mathfrak{v}^*)$ defined analogously to η, and the pullback of the symplectic form on $X/\!\!/G$.

3(ii) The almost complex structure $\tilde{J}_0$.

Fix (1) an almost complex structure on $X/\!\!/G$ which is compatible with the symplectic form, and (2) a T-invariant positive-definite inner product on $\mathfrak{v}$.

The inner product on $\mathfrak{v}$ gives a duality isomorphism $\mathfrak{v}\xrightarrow{\cong}\mathfrak{v}^*$, which is T-equivariant, and which thus descends to an isomorphism

$$\varphi : Z\times_T\mathfrak{v}\to Z\times_T\mathfrak{v}^*.$$

We now define J_0. On the subbundle $\pi^*T(X/\!\!/G)$ we define J_0 to equal the almost complex structure on $X/\!\!/G$ which we fixed above. On the subbundle $(Z\times_T\mathfrak{v})\oplus(Z\times_T\mathfrak{v}^*)$ we define J_0 to equal $\begin{pmatrix}0 & -\varphi\\ \varphi & 0\end{pmatrix}$. One easily checks that J_0 is compatible with the symplectic form on $T(X/\!\!/T)|_{Z/T}$, and it follows from standard results in symplectic geometry that there exists an almost complex structure $\tilde{J}_0$ on $X/\!\!/T$ which is compatible with the symplectic form, and whose restriction equals J_0 (see for example McDuff and Salamon [17, proposition 4.1]).

3(iii) The homotopy.

Fix a choice of positive roots $\Delta^+ \subset \Delta$. This choice of positive roots gives a T-invariant complex structure on $\mathfrak{v}$, which descends to a complex structure on $Z \times_T \mathfrak{v}$, which we denote by $i_{\mathfrak{v}}$. Similarly, the negative roots define a complex structure $i_{\mathfrak{v}^*}$ on $Z \times_T \mathfrak{v}^*$, and we have

$$i_{\mathfrak{v}^*} = \varphi \circ (-i_{\mathfrak{v}^*}) \circ \varphi^{-1}.$$

We now define J_1. On the subbundle $\pi^* T(X/\!\!/G)$ we define J_1 to equal the almost complex structure on $X/\!\!/G$ which we fixed above, and hence to agree with J_0. On the subbundle $(Z \times_T \mathfrak{v}) \oplus (Z \times_T \mathfrak{v}^*)$ we define J_1 to equal $\begin{pmatrix} i_{\mathfrak{v}} & 0 \\ 0 & i_{\mathfrak{v}^*} \end{pmatrix}$.

We now show that J_0 and J_1 are homotopic through almost complex structures. Consider the complex linear transformations $j_0 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $j_1 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathrm{End}(\mathbb{C} \oplus \mathbb{C})$. Since j_0 and j_1 are unitary matrices having the same eigenvalues, there is a unitary matrix g_1 such that $j_1 = g_1 j_0 g_1^{-1}$. Let g_t, for $t \in [0,1]$, be a path of unitary matrices with g_0 equal to the identity and g_1 the matrix we have just described. Then $j_t := g_t j_0 g_t^{-1}$ is a path of complex structures on the real vector space underlying $\mathbb{C} \oplus \mathbb{C}$.

Tensoring with $\mathfrak{v}$, and using the isomorphism provided by φ, we can thus define a path of almost complex structures J_t from J_0 to J_1 (keeping the almost complex structure on the subbundle $\pi^* T(X/\!\!/G)$ fixed throughout).

We can think of an almost complex structure over $X/\!\!/T$ as a section of a bundle with fibres $O(2n)/U(n)$, where $2n = \dim X/\!\!/T$ [17, proposition 2.46]. Thus $\tilde{J}_0$ is such a section, and J_t is a homotopy of sections restricted to the submanifold Z/T. By the homotopy extension property, we can extend J_t to a homotopy $\tilde{J}_t$ of almost complex structures on $X/\!\!/T$.

But $\tilde{J}_1$ has the property that Z/T is an almost complex submanifold, such that the complex structure on Z/T and on its normal bundle agree with the complex structures on the vector bundles in the proposition, hence completing the proof. □

2. The integration formula

In this section we prove the integration formula, theorem B. We begin by recalling some cohomological techniques needed in the proof.

Integration over the fibre

If $\pi : Y \to B$ is a fibre bundle with fibre F, such that Y, B and F are compact oriented manifolds, then **integration over the fibre** is a map

$$\pi_* : H^*(Y) \to H^{*-\dim F}(B)$$

satisfying

(Multiplication) $\qquad \pi_*(\pi^*(b) \smile a) = b \smile \pi_* a, \qquad \forall a \in H^*(Y), b \in H^*(B),$

(Restriction) If $i : S \hookrightarrow B$ is the inclusion a closed oriented submanifold, then the following square of maps commutes

$$\begin{array}{ccc} H^*(\pi^{-1}(S)) & \xleftarrow{\ i'^*\ } & H^*(Y) \\ \downarrow{\scriptstyle \pi'_*} & & \downarrow{\scriptstyle \pi_*} \\ H^{*-\dim F}(S) & \xleftarrow{\ i^*\ } & H^{*-\dim F}(B), \end{array}$$

where π' denotes the restriction of π to $\pi^{-1}(S)$, and i' denotes the inclusion of $\pi^{-1}(S)$ in Y.

(Composition) $$\int_B \pi_* a = \int_Y a, \qquad \forall a \in H^*(Y).$$

The Euler class of a vector bundle

The second fact we need involves the Euler class of a vector bundle. Suppose V is a real oriented vector bundle over a compact oriented manifold Y, and s is a section of V which is transverse to the zero section. Then the zeroset of s is a submanifold S of Y, and denoting by $i : S \hookrightarrow Y$ the inclusion, we have

$$\int_S i^* a = \int_Y a \smile e(V), \qquad \forall a \in H^*(Y)$$

where $e(V) \in H^{\mathrm{rk}\, V}(Y)$ is the Euler class of V. There is a natural identification $\nu S \cong V|_S$ of the normal bundle to S with the restriction of V, and hence the orientation of V induces an orientation on νS; we assume $S, \nu S$ and Y are oriented compatibly.

If instead V is a complex vector bundle, with its natural real orientation, then the Euler class of V (thought of as a real vector bundle) equals the top Chern class of V (thought of as a complex vector bundle). For proofs and more detailed explanations, see Bott and Tu [4, section 6].

Proof of the integration formula

Recall that we have maps $i : \mu_G^{-1}(0)/T \hookrightarrow X/\!\!/T$ and $\pi : \mu_G^{-1}(0)/T \to X/\!\!/G$ as defined in section 1, and we say that a class $\tilde{a} \in H^*(X/\!\!/T)$ is a **lift** of $a \in H^*(X/\!\!/G)$ if $\pi^*(a) = i^*(\tilde{a})$.

Definition 2.1. Given a weight α of T, then we define $e(\alpha)$ to be the Euler class

$$e(\alpha) := e(L_\alpha) \in H^*(X/\!\!/T).$$

(These are the cohomology classes which appear in Theorems A and B).

We are now ready to prove theorem B (as stated in the introduction). This integration formula is stated in terms of cohomology classes on $X/\!\!/G$ and $X/\!\!/T$. However, such classes often arise from equivariant cohomology classes on X via the 'Kirwan map'. After proving the integration formula, we will set up notation for equivariant cohomology and state a corollary in terms of equivariant classes.

Proof of Theorem B. Define the class $b \in H^*(X/\!\!/T)$ by $b := \prod_{\alpha \in \Delta^+} e(L_\alpha)$. Then $i^* b = e(\mathrm{vert}(\pi))$, by proposition 1.2. We can calculate the integral over the fibres $\pi_*(i^* b) \in H^0(X/\!\!/G)$ by restricting to a single fibre G/T, using the restriction property of the push-forward. By naturality of the Euler class, $e(\mathrm{vert}(\pi))|_{G/T} = e(\mathrm{vert}(\pi)|_{G/T}) = e(T(G/T))$, and so $\pi_*(i^* b) = \int_{G/T} e(T(G/T)) = \chi(G/T) = |W|$, using a standard identification of the Euler characteristic of G/T [3].

Then

$$\begin{aligned}
\int_{X/\!\!/G} a &= \frac{1}{|W|}\int_{\mu_G^{-1}(0)/T} \pi^* a \smile i^* b, \quad \text{applying the push-pull formula,}\\
&= \frac{1}{|W|}\int_{X/\!\!/T} i_* i^*(\tilde{a}\smile b), \quad \text{composition of pushforwards; and } \tilde{a} \text{ is a lift of } a,\\
&= \frac{1}{|W|}\int_{X/\!\!/T} \tilde{a}\smile b \smile \prod_{\alpha\in\Delta^-} e(L_\alpha), \quad \begin{array}{l}\text{properties of the Euler class, applied to the}\\ \text{vector bundle identified in proposition 1.2(1),}\end{array}\\
&= \frac{1}{|W|}\int_{X/\!\!/T} \tilde{a}\smile e, \quad \text{by definition of } e.
\end{aligned}$$

□

3. The relationship between the cohomology rings

In this section we prove theorem A, which relates the cohomology rings of the symplectic quotients $X/\!\!/G$ and $X/\!\!/T$. The proof involves some standard machinery in equivariant cohomology, and a crucial result concerning the 'Kirwan map', which we begin by reviewing. We also state a version of the integration formula in the language of equivariant cohomology.

Some key facts in equivariant cohomology

The G-equivariant cohomology of the G-manifold X, which we denote by $H^*_G(X)$, is defined to be the ordinary cohomology of the **homotopy quotient**

$$X_G := (EG \times X)/G,$$

where EG is a universal space for G: that is, EG is contractible and has a free G-action. For various facts in equivariant cohomology, see [1, 6, 16]. We recall that, if $K \subset G$ is a subgroup, there is a natural restriction map $r^G_K : H^*_G(X) \to H^*_K(X)$.

A map of fundamental importance in symplectic geometry is the **Kirwan map**, which gives a surjective ring homomorphism from the equivariant cohomology of a symplectic manifold onto the ordinary cohomology of its symplectic quotient. Explicitly, we define the Kirwan map

$$\kappa_G : H^*_G(X) \to H^*(X/\!\!/G),$$

by taking the restriction to $\mu_G^{-1}(0)$, and composing this with the natural isomorphism $H^*_G(\mu_G^{-1}(0)) \xrightarrow{\cong} H^*(X/\!\!/G)$. This natural isomorphism is defined in rational cohomology whenever the G-action on $\mu_G^{-1}(0)$ is locally free, and we understand κ_G to only be defined when this is the case. We denote the analogous map for the maximal torus T by $\kappa_T : H^*_T(X) \to H^*(X/\!\!/T)$.

We observe that, for any equivariant class $a \in H^*_G(X)$, the class $\kappa_T \circ r^G_T(a)$ is a lift of $\kappa_G(a)$ (see comments in the proof of theorem A for more elucidation on this point). We can thus restate the integration formula, theorem B, in a form which is more natural in many applications:

Corollary 3.1 (Integration formula in terms of equivariant cohomology). *For all $a \in H^*_G(X)$,*

$$\int_{X/\!\!/G} \kappa_G(a) = \frac{1}{|W|}\int_{X/\!\!/T} \kappa_T \circ r^G_T(a) \smile e,$$

where

$$e = \prod_{\alpha\in\Delta} \kappa_T(e_T(\mathbb{C}_{(\alpha)})),$$

and e_T denotes the T-equivariant Euler class.

The proof of theorem A

Observe that the Weyl group W of G acts on $X/\!\!/T$: since the normalizer $N(T)$ of T preserves $\mu_T^{-1}(0)$, the action of $N(T)$ on $\mu_T^{-1}(0)$ descends to an action of $W = N(T)/T$ on the quotient $X/\!\!/T$.

Proof of theorem A. Consider the fibre bundle $\mu_G^{-1}(0)/T \xrightarrow{\pi} X/\!\!/G$. This has fibre G/T, and the Weyl group W acts on the fibres, covering the trivial action on the base (this is the restriction of the W-action on $X/\!\!/T$). By a result of Borel [2, section 27], the pullback π^* gives an isomorphism between the rational cohomology of the base $X/\!\!/G$ and the W-invariant cohomology of the total space $\mu_G^{-1}(0)/T$. This means there is a natural ring homomorphism

$$\varphi : H^*(X/\!\!/T)^W \to H^*(X/\!\!/G) \tag{3.2}$$

given by restriction to $\mu_G^{-1}(0)/T$ followed by the above identification, which we will now show to be onto.

Applying the Borel-Hirzebruch result to the homotopy quotients X_G and X_T, one can also recover the known fact that the restriction r_T^G gives an isomorphism with the W-invariants $r_T^G : H_G^*(X) \xrightarrow{\cong} H_T^*(X)^W$. By naturality of the maps involved, we have

$$\kappa_G = \varphi \circ \kappa_T \circ r_T^G : H_G^*(X) \to H^*(X/\!\!/G),$$

and since κ_G is onto, it follows that φ is onto.

To prove the theorem, we thus need to show that $\ker\varphi = \mathrm{ann}(e)$. Let $a \in H^*(X/\!\!/T)^W$. Then

$$\begin{aligned}
\varphi(a) = 0 \iff & \forall c \in H^*(X/\!\!/T)^W, \quad \int_{X/\!\!/G} \varphi(a)\smile\varphi(c) = 0, && \text{Poincaré duality on } X/\!\!/G; \text{ surjectivity of } \varphi,\\
\iff & \forall c \in H^*(X/\!\!/T)^W, \quad \int_{X/\!\!/T} a\smile c\smile e = 0, && \text{by the integration formula,}\\
\iff & \forall d \in H^*(X/\!\!/T), \quad \int_{X/\!\!/T} (a\smile e)\smile d = 0, && \text{since we can average } d \text{ by } W \text{ (see below),}\\
\iff & a\smile e = 0, \quad \text{by Poincaré duality on } X/\!\!/T.
\end{aligned}$$

In the second-last step, we note that W acts by symplectomorphisms on $X/\!\!/T$, hence preserves orientation and integrals, and since a and e are W-invariant, we can average d by W to obtain a W-invariant class without changing the integral. □

4. The index formula

We now prove the index formula, theorem C, by applying the Atiyah-Singer index theorem to the main topological result. For more details on the spinc^c Dirac operators D and D_V (described in the introduction) see for example [14, appendix D].

In the proof we will use K-theory, but only in a rudimentary way, and we recall a few facts and set up some notation. Given a compact space Y, then $K(Y)$ is a commutative ring, whose elements are represented by formal sums (with integer coefficients) of complex vector bundles over Y. Given a vector bundle $V \to Y$, we write $[V] \in K(Y)$ for the equivalence class it represents. Addition and multiplication in $K(Y)$ are induced by the direct sum and tensor product of vector bundles respectively, and these operations are extended to formal sums of vector bundles by the usual laws of a commutative ring.

We will use the Chern character and the Todd class of complex vector bundles. It is a standard fact that these characteristic classes only depend on the equivalence class $[V] \in K(Y)$ of a complex vector bundle $V \to Y$, and that these characteristic classes can be extended to every element of $K(Y)$ by setting

$$\mathrm{Td}([V]-[W]) = \frac{\mathrm{Td}(V)}{\mathrm{Td}(W)}, \qquad \mathrm{ch}([V]-[W]) = \mathrm{ch}(V) - \mathrm{ch}(W),$$

for all vector bundles $V, W \to Y$ (the Todd class $\mathrm{Td}(V)$ is a cohomology class of mixed degree, but it has degree-0 part equal to $1 \in H^0(Y)$, and it follows that $\mathrm{Td}(V)$ has a multiplicative inverse in the cohomology ring). Finally, if a vector bundle V is given as a sum of line bundles $V = \bigoplus_{1\le i\le k} L_i$ then

$$\mathrm{Td}(V) = \prod_{1\le i\le k} \frac{c_1(L_i)}{1-\exp(-c_1(L_i))}, \qquad \mathrm{ch}(V) = \sum_{1\le i\le k} \exp(c_1(L_i)).$$

We use the maps $i : \mu_G^{-1}(0)/T \hookrightarrow X/\!\!/T$ and $\pi : \mu_G^{-1}(0)/T \to X/\!\!/G$ as defined in section 1, and we extend the definition of the 'lift' of a cohomology class to both vector bundles and K-theory in the obvious way. Thus we say a class $\tilde{a} \in K(X/\!\!/T)$ is a **lift** of $a \in K(X/\!\!/G)$ if $\pi^* a = i^* \tilde{a}$.

Proof of theorem C. Fix almost complex structures on $X/\!\!/G$ and $X/\!\!/T$, compatible with their respctive symplectic forms. Throughout this proof we will let $T(X/\!\!/G)$ and $T(X/\!\!/T)$ denote the tangent bundles, thought of as complex vector bundles given by these almost complex structures.

Define $E := \bigoplus_{\alpha\in\Delta^+} L_\alpha$, as in the statement of theorem C. Then we have $E^* \cong \bigoplus_{\alpha\in\Delta^-} L_\alpha$. The main topological result, proposition 1.2, implies that $[T(X/\!\!/T)] - [E \oplus E^*] \in K(X/\!\!/T)$ is a lift of $[T(X/\!\!/G)] \in K(X/\!\!/G)$. Since taking characteristic classes commutes with pullback, it follows that

$$\frac{\mathrm{Td}\, T(X/\!\!/T)}{\mathrm{Td}(E) \smile \mathrm{Td}(E^*)}$$

is a lift of $\mathrm{Td}\, T(X/\!\!/G)$.

Define the class $b \in H^*(X/\!\!/T)$ by $b := \mathrm{Td}(E)$. Then $i^* b = \mathrm{Td}(\mathrm{vert}(\pi))$. Now π is the projection of a fibre bundle, with fibres G/T, which arises as the global quotient of a principal G bundle by the maximal torus T. A result of Borel and Hirzebruch asserts that in this case $\pi_* \mathrm{Td}(\mathrm{vert}(\pi)) = 1$ [3, sections 7.4 and 22.3].

Applying the Atiyah-Singer index theorem, and arguing as in the proof of the integration formula,

$$
\begin{aligned}
\operatorname{index}^{X/\!\!/G} D_V &= \int_{X/\!\!/G} \operatorname{ch}(V) \smile \operatorname{Td} T(X/\!\!/G) \\
&= \int_{X/\!\!/T} \operatorname{ch}(\tilde{V}) \smile \frac{\operatorname{Td} T(X/\!\!/T)}{\operatorname{Td}(E) \smile \operatorname{Td}(E^*)} \smile b \smile \prod_{\alpha \in \Delta^-} e(\alpha) \\
&= \int_{X/\!\!/T} \operatorname{ch}(\tilde{V}) \smile \frac{\operatorname{Td} T(X/\!\!/T)}{\operatorname{Td}(E^*)} \smile \prod_{\alpha \in \Delta^-} e(\alpha) \\
&= \int_{X/\!\!/T} \operatorname{ch}(\tilde{V}) \smile \operatorname{Td} T(X/\!\!/T) \smile \prod_{\alpha \in \Delta^-} (1 - \exp(-e(\alpha))) \\
&= \int_{X/\!\!/T} \operatorname{ch}(\tilde{V}) \smile \operatorname{Td} T(X/\!\!/T) \smile \prod_{\alpha \in \Delta^+} (1 - \exp(e(\alpha)))
\end{aligned}
$$

But $[\Lambda^{\text{even}} E] - [\Lambda^{\text{odd}} E] = \sum_{i=0}^{\operatorname{rk} E} (-1)^i [\Lambda^i E] = \prod_{\alpha \in \Delta^+} ([\underline{\mathbb{C}}] - [L_\alpha])$ hence, applying the Chern character, $\operatorname{ch}(\Lambda^{\text{even}} E) - \operatorname{ch}(\Lambda^{\text{odd}} E) = \prod_{\alpha \in \Delta^+} (1 - \exp(e(\alpha)))$.

Combining these formulæ, using additive and multiplicative properties of the Chern character, gives

$$
\begin{aligned}
\operatorname{index}^{X/\!\!/G} D_V &= \int_{X/\!\!/T} \operatorname{ch}(\tilde{V}) \smile \operatorname{Td} T(X/\!\!/T) \smile \big(\operatorname{ch}(\Lambda^{\text{even}} E) - \operatorname{ch}(\Lambda^{\text{odd}} E) \big) \\
&= \int_{X/\!\!/T} \big(\operatorname{ch}(\tilde{V} \otimes \Lambda^{\text{even}} E) - (\operatorname{ch}(\tilde{V} \otimes \Lambda^{\text{odd}} E) \big) \smile \operatorname{Td} T(X/\!\!/T) \\
&= \operatorname{index}^{X/\!\!/T} D_{\tilde{V} \otimes \Lambda^{\text{even}} E} - \operatorname{index}^{X/\!\!/T} D_{\tilde{V} \otimes \Lambda^{\text{odd}} E}.
\end{aligned}
$$

Note finally that the formula we have derived is stated in terms of a choice of positive roots, but the proof does not depend on any properties of that choice, and hence the result holds for any choice of positive roots. □

5. Characteristic numbers

Using the K-theoretic arguments from section 4, it is a simple matter to derive formulæ which express various characteristic numbers of $X/\!\!/G$ in terms of characteristic numbers of $X/\!\!/T$. Recall that the tangent bundles $T(X/\!\!/G)$ and $T(X/\!\!/T)$ can be considered as complex vector bundles in an essentially unique way, by taking almost complex structures compatible with their symplectic forms.

In the proof of the index formula, we used the fact that $[T(X/\!\!/T)] - [E \oplus E^*] \in K(X/\!\!/T)$ is a lift of $[T(X/\!\!/G)] \in K(X/\!\!/G)$. It follows that

$$
\frac{c(T(X/\!\!/T))}{c(E) \smile c(E^*)}
$$

is a lift of the total Chern class $c(T(X/\!\!/G))$. Hence, applying the integration formula, we get the following formula for the Euler characteristic of $X/\!\!/G$

$$
\chi(X/\!\!/G) = \frac{1}{|W|} \int_{X/\!\!/T} c(T(X/\!\!/T)) \smile \prod_{\alpha \in \Delta} \frac{e(\alpha)}{1 + e(\alpha)}. \tag{5.1}
$$

(We are using the fact that the top Chern class equals the Euler class.)

Similarly, taking the L-class,

$$\text{signature}(X/\!\!/G) = \frac{1}{|W|} \int_{X/\!\!/T} L(T(X/\!\!/T)) \smile \prod_{\alpha \in \Delta} \tanh e(\alpha). \tag{5.2}$$

In general, for any 'multiplicative characteristic class' m (see Hirzebruch [8, section 1], or Milnor and Stasheff [18, section 19]), we have

$$\int_{X/\!\!/G} m(T(X/\!\!/G)) = \frac{1}{|W|} \int_{X/\!\!/T} \frac{m(T(X/\!\!/T))}{m(E) \smile m(E^*)} \smile \prod_{\alpha \in \Delta} e(\alpha). \tag{5.3}$$

6. Generalizations

In the previous sections of this paper we have assumed that both $\mu_G^{-1}(0)$ and $\mu_T^{-1}(0)$ are compact manifolds on which the respective G- and T-actions are free. In this section we show how to remove some of these assumptions. We will keep the assumption that μ_G is a proper map, having 0 as a regular value. From this it follows that $\mu_G^{-1}(0)$ is a compact manifold, on which the G-action is locally free, and hence that $X/\!\!/G$ is a compact symplectic orbifold.

The case in which μ_T is proper and has 0 as a regular value

If μ_T is proper and has 0 as a regular value, then $X/\!\!/T$ is a compact symplectic orbifold. The arguments in sections 1–3, in which we proved the integration formula and the formula relating the cohomology rings, go through with straightforward modifications, which we now describe.

In the main topological result, proposition 1.2, the line bundles L_α, as well as the normal bundle and the fibering must all be replaced by their orbifold equivalents. (In the companion paper to this one [15], appendix A summarizes the main topological and cohomological properties of orbifolds, orbifold vector bundles, and orbifold fibre bundles, including describing how integration over the fibre goes over to that case.)

The classes $e(\alpha)$ are well-defined rational cohomology classes, and theorem A extends to this case unchanged (rational Poincaré duality holds for compact oriented orbifolds). Theorem B must be modified to take into account the existence of global finite stabilizers, and becomes

Theorem B′ (Integration formula). *If μ_G and μ_T are proper maps, both having 0 as a regular value, then for any class $a \in H^*(X/\!\!/G)$ with lift $\tilde{a}$,*

$$\int_{X/\!\!/G} a = \frac{1}{|W|} \cdot \frac{o_T(\mu_T^{-1}(0))}{o_G(\mu_G^{-1}(0))} \int_{X/\!\!/T} \tilde{a} \smile e,$$

where $o_G(Y)$ denotes the order of the maximal subgroup of G which fixes every point in Y, and $|W|$ and e are as defined in theorem B.

The case in which μ_T is proper, but does not have 0 as a regular value

The integration formula, theorem B′, can be generalized to the case in which 0 is not a regular value for μ_T in two different ways. One way involves perturbing the value at which we take the symplectic quotient by T, which we now describe. We will then describe the other alternative, which makes use of compactly-supported cohomology: that alternative can also handle the case in which μ_T is not compact.

A tubular neighbourhood of $\mu_G^{-1}(0)/T$ is an orbifold, since we have assumed that 0 is a regular value for μ_G. By assumption 0 is not a regular value for μ_T, but by transversality there exist regular values arbitrarily close to 0. Let $\epsilon \in \mathfrak{t}^*$ be a regular value, and consider the family of symplectic quotients $X/\!\!/T(p) := \mu_T^{-1}(p)/T$, as p moves between 0 and ϵ. If ϵ is sufficiently close to 0, then we can find diffeomorphisms between a neighbourhood of $\mu_G^{-1}(0)/T$ and open sets in the quotients $X/\!\!/T(p)$. To do this, we note that the (orbifold) vector bundle $\bigoplus_{\alpha\in\Delta^-} L_\alpha \to X/\!\!/T(0)$ with section s, defined in proposition 1.2, is naturally defined over all quotients $X/\!\!/T(p)$. For ϵ sufficiently small, the section s will remain transverse to the zero-section, and hence the zeros of s on the symplectic quotients $X/\!\!/T(p)$ will be diffeomorphic to one another, along with tubular neighbourhoods of these zerosets. Thus, for ϵ sufficiently small, we have an injection $i' : \mu_G^{-1}(0)/T \hookrightarrow X/\!\!/T(\epsilon)$, and the main topological result, proposition 1.2 applies with the map i' in place of the map i. A sufficient condition on ϵ is that there exist path joining ϵ and 0, consisting entirely of regular values (except of course for 0). Note also that the notion of a 'lift' of a cohomology class from $X/\!\!/G$ to $X/\!\!/T(\epsilon)$ is well-defined in this case.

Theorem B″ (Integration formula). *Suppose μ_G and μ_T are proper maps, and 0 is a regular value for μ_G. Then for any regular value $\epsilon \in \mathfrak{t}^*$ sufficiently close to $0 \in \mathfrak{t}^*$, and any class $a \in H^*(X/\!\!/G)$, with lift $\tilde{a} \in H^*(X/\!\!/T(\epsilon))$,*

$$\int_{X/\!\!/G} a = \frac{1}{|W|} \cdot \frac{o_T(\mu_T^{-1}(0))}{o_G(\mu_G^{-1}(0))} \int_{X/\!\!/T(\epsilon)} \tilde{a}\smile e,$$

where $o_G(Y)$ denotes the order of the maximal subgroup of G which fixes every point in Y, and $|W|$ and e are as defined in theorem B.

The case in which μ_T is not proper

If μ_T is not proper but $0 \in \mathfrak{t}^*$ is a regular value, then $X/\!\!/T$ is a noncompact orbifold, with (orbifold) line bundles L_α, and with $\mu_G^{-1}(0)/T$ as a compact suborbifold.

The section s of the bundle $\bigoplus_{\alpha\in\Delta^-} L_\alpha \to X/\!\!/T$ has compactly-supported zeroset, and hence the pair $(\bigoplus_{\alpha\in\Delta^-} L_\alpha, s)$ possesses a **relative Euler class**[5]

$$e^- := e(\textstyle\bigoplus_{\alpha\in\Delta^-} L_\alpha, s) \in H_c^*(X/\!\!/T),$$

lying in the cohomology with compact support of $X/\!\!/T$. Setting $e^+ := e(\bigoplus_{\alpha\in\Delta^+} L_\alpha)$ (the regular Euler class), then the product $e := e^+\smile e^-$ lies in $H_c^*(X/\!\!/T)$, hence for any class $\tilde{a} \in H^*(X/\!\!/T)$, the product $a\smile e$ has compact support and thus has a well-defined integral over $X/\!\!/T$.

With this interpretation of the class e, the integration formula of theorem B′ holds as stated. Moreover, if 0 is not a regular value of μ_T, we can remove the non-orbifold points and apply the above reasoning.

[5]Let $E \to Y$ be an oriented vector bundle over a noncompact manifold Y, and let s be a section whose zeroset is compact. The bundle E possesses a Thom class Φ, and we define the relative Euler class by $e(E,s) := s^*\Phi \in H_c^*(Y)$. This cohomology class is an invariant of the homotopy class of s, through homotopies for which the zeroset remains compact at all times. Given a section in this homotopy class which is transverse to the zero section of E, then $e(E,s)$ represents the Poincaré dual of the compact submanifold given by the zeroset (Poincaré duality on a noncompact manifold is an isomorphism between homology and compactly-supported cohomology, see for example [4, propositions 6.24 and 6.25]). These statements go over to orbifolds, with rational cohomology.

Replacing T by a full-rank subgroup

Suppose $H \subset G$ is a connected closed subgroup which contains a maximal torus T. Many of the results of this paper generalize in a straightforward way to give relationships between $X/\!\!/G$ and $X/\!\!/H$.

Denoting by $\mathfrak{h}$ the Lie algebra of H, then moment map μ_H for the H-action is defined analogously to μ_T, by composing μ_G with the natural projection $\mathfrak{g}^* \twoheadrightarrow \mathfrak{h}^*$. We assume for simplicity that μ_H is proper and has 0 as a regular value, and that the G-action is free on $\mu_G^{-1}(0)$, and the H-action is free on $\mu_H^{-1}(0)$ (there are obvious generalizations when these conditions are not met, as described above).

Under the adjoint action of H, the Lie algebra $\mathfrak{g}$ decomposes into subrepresentations

$$\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{e}.$$

This decomposition is compatible with the T-action, and $\mathfrak{t} \subset \mathfrak{h}$, hence a choice of positive roots for T gives a complex structure to the H-representation $\mathfrak{e}$. We denote by $\mathcal{E} \to X/\!\!/H$ the associated vector bundle $\mu_H^{-1}(0) \times_H \mathfrak{e} \to X/\!\!/H$.

With these definitions, the main topological result generalizes in the obvious way, with the bundle $\bigoplus_{\alpha\in\Delta^+} L_\alpha$ replaced by $\mathcal{E}$, and $\bigoplus_{\alpha\in\Delta^-} L_\alpha$ replaced by the dual bundle $\mathcal{E}^*$. Thus, with the obvious definition of a lift of a cohomology class or vector bundle, we have

Theorem B$_H$ (Integration formula). *Given a cohomology class $a \in H^*(X/\!\!/G)$ with lift $\tilde{a} \in H^*(X/\!\!/H)$, then*

$$\int_{X/\!\!/G} a = \frac{|W(H)|}{|W(G)|} \int_{X/\!\!/H} \tilde{a} \smile e(\mathcal{E} \oplus \mathcal{E}^*),$$

where $W(H)$ and $W(G)$ are the Weyl groups of H and G respectively.

Theorem C$_H$ (Index formula). *Suppose $V \to X/\!\!/G$ is a complex vector bundle, and $\tilde{V} \to X/\!\!/H$ is a lift of V. Then*

$$\operatorname{index}^{X/\!\!/G} D_V = \operatorname{index}^{X/\!\!/H} D_{\tilde{V}\otimes\Lambda^{even}\mathcal{E}} - \operatorname{index}^{X/\!\!/T} D_{\tilde{V}\otimes\Lambda^{odd}\mathcal{E}}$$

Theorem A generalizes in a special case. Suppose $W(H)$ is a *normal* subgroup of $W(G)$. Then the quotient group $W(G)/W(H)$ can be thought of as the relative Weyl group for H in G, and $X/\!\!/H$ carries an action of this relative Weyl group. In this case we have

Theorem A$_H$ (Cohomology rings). *There is a natural ring isomorphism*

$$H^*(X/\!\!/G;\mathbb{Q}) \cong \frac{H^*(X/\!\!/H;\mathbb{Q})^W}{\operatorname{ann}\big(e(\mathcal{E}\oplus\mathcal{E}^*)\big)},$$

where $W := W(G)/W(H)$ is the relative Weyl group.

7. Example: The complex Grassmannian

This section contains a worked example: the complex Grassmannian of k-planes in $\mathbb{C}^n$, which we denote $G(k,n)$. We first describe the results of applying theorems A and B, and we then describe the derivation of these results in more detail.

The Grassmannian can be described as the symplectic quotient of the set of complex matrices with n rows and k columns by the unitary group

$$G(k,n) \cong \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) /\!\!/ U(k),$$

where $g \in U(k)$ acts on a matrix $A \in \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ by $A \circ g^{-1}$.

The associated symplectic quotient by the maximal torus $T \subset U(k)$ turns out to be the k-fold product $(\mathbb{CP}^{n-1})^k$. Its cohomology ring is generated by degree-2 elements $\{u_1, \dots, u_k\}$, where u_i is the positive generator of the cohomology ring of the i-th copy of $\mathbb{CP}^{n-1}$. The Weyl group of $U(k)$ is the symmetric group on k elements S^k, which acts by permuting the factors in $(\mathbb{CP}^{n-1})^k$. The roots α of $U(k)$ can be enumerated by pairs of positive integers (i,j) with $1 \le i, j \le k$ and $i \ne j$, and the cohomology class corresponding to the root (i,j) is the class $e(\alpha) = u_j - u_i$. Hence, theorem A states

Proposition 7.1. *The cohomology ring of the Grassmannian $G(k,n)$ is given by*

$$H^*(G(k,n); \mathbb{Q}) \cong \frac{\mathbb{Q}[u_1, \dots, u_k]^{S_k}}{\langle u_1^n, \dots, u_k^n \rangle : \prod_{i \ne j}(u_i - u_j)},$$

where the expression $I : e$ in the denominator denotes the ideal quotient of the ideal I by the element e, that is, the ideal consisting of all elements b such that $b \cdot e \in I$.

The Grassmannian possesses a tautological vector bundle $V \to G(k,n)$ of rank k, and the cohomology ring is generated by the Chern classes of the dual bundle V^*. In the above description, $c_i(V^*)$ is represented by the i-th symmetric polynomial in the u_j. The above description of the cohomology ring is quite different from the usual description (which involves the Segre classes of V), and I have been able to find neither a general algebraic proof of the equivalence of the two descriptions, nor any reference to the above description in the literature.

Theorem B gives

Proposition 7.2.

$$\int_{G(k,n)} c_1(V^*)^{m_1} \smile \dots \smile c_k(V^*)^{m_k} = \frac{1}{k!} \operatorname{coeff}_{u_1^{n-1} \dots u_k^{n-1}} \left(\sigma_1^{m_1} \dots \sigma_k^{m_k} \cdot \prod_{i \ne j} (u_i - u_j) \right)$$

where σ_i is the i-th elementary symmetric polynomial of the u_j, and $\operatorname{coeff}_m(p)$ denotes the coefficient of the monomial m in the polynomial p.

The construction of $G(k,n)$

The symplectic structure on $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ is the standard one for a complex vector space with coordinates, namely, if $a_{ij} = x_{ij} + \sqrt{-1} y_{ij}$, for $1 \le i \le n, 1 \le j \le k$, then

$$\omega := \sum_{i,j} dx_{ij} \wedge dy_{ij}.$$

The moment map takes values in the dual of the Lie algebra of $U(k)$, which can be identified with the space of Hermitian $k \times k$ matrices. It is a straightforward calculation using the definition of the moment map to show that a moment map is given by

$$\mu_{U(k)}(A) = A^* A - \mathbb{1},$$

where $A^* := \overline{A}^{\mathrm{tr}}$ (precisely, given a skew-Hermitian matrix $\xi \in \mathrm{Lie}(U(k))$, then the pairing $\langle \mu_{U(k)}(A), \xi \rangle$ is given by $\frac{i}{2}\mathrm{Tr}\big((A^*A - \mathbb{1})\xi^*\big)$).

The k column vectors of the matrix $A \in \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ define vectors $v_1, \ldots, v_k \in \mathbb{C}^n$, and the (i,j)-entry of A^*A is the Hermitian inner product (v_j, v_i). Hence $\mu_{U(k)}^{-1}(0)$ consists of the unitary k-frames in $\mathbb{C}^n$, and taking the quotient by $U(k)$ gives the Grassmannian $G(k,n)$.

The maximal torus $T \subset U(k)$ of diagonal matrices has associated moment map given by the diagonal entries of the matrix $A^*A - \mathbb{1}$, and so a k-tuple $(v_1, \ldots, v_k)$ lies in $\mu_T^{-1}(0)$ precisely when each v_i has length 1. The torus T equals the product $(S^1)^k$, the factors of which rotate the vectors independently, hence identifying the quotient $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)/\!\!/T$ with $(\mathbb{CP}^{n-1})^k$. To calculate the classes $e(\alpha)$, consider the conjugation action of the diagonal matrices on the skew-symmetric matrices whose diagonal entries vanish (this is the complement of $\mathfrak{t}$ in $\mathrm{Lie}(U(k))$). The matrix with diagonal entries $(\lambda_1, \ldots, \lambda_k)$ acts by on the (i,j)-entry by $\lambda_i \lambda_j^{-1}$; the complex line bundle constructed with this weight has Euler class $u_j - u_i$.

We now describe the tautological bundle $E \to G(k,n)$ in terms of the symplectic quotient construction, so that we can identify the Chern classes of its dual on the T-symplectic quotient $(\mathbb{CP}^{n-1})^k$. A point of $G(k,n)$ is a $U(k)$-orbit of nondegenerate homomorphisms $A : \mathbb{C}^k \to \mathbb{C}^n$, and the corresponding fibre of E is $\mathrm{im}(A) \subset \mathbb{C}^n$. Two points A and Ag^{-1} in the same $U(k)$-orbit give different identifications of $\mathbb{C}^k$ with the subspace $\mathrm{im}(A)$, and, taking this into account gives

$$E \cong \mu^{-1}(0) \times_{U(k)} \mathbb{C}^k_{(\mathrm{def.})},$$

where $\mathbb{C}^k_{(\mathrm{def.})}$ denotes the defining representation of $U(k)$. Thus E^* is constructed from the dual of the defining representation, and when we restrict this dual representation to the maximal torus, it decomposes into k one-dimensional representations, which have associated line bundles on the $(\mathbb{CP}^{n-1})^k$ with Euler classes $u_1, \ldots, u_k$. The identification of the Chern classes as elementary symmetric polynomials then follows.

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