NEW MODULI SPACES OF POINTED CURVES AND PENCILS OF FLAT CONNECTIONS

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Dedicated to William Fulton on the occasion of his 60th birthday

Abstract. It is well known that formal solutions to the Associativity Equations are the same as cyclic algebras over the homology operad $(H_*(\overline{M}_{0,n+1}))$ of the moduli spaces of n–pointed stable curves of genus zero. In this paper we establish a similar relationship between the pencils of formal flat connections (or solutions to the Commutativity Equations) and homology of a new series $\overline{L}_n$ of pointed stable curves of genus zero. Whereas $\overline{M}_{0,n+1}$ parametrizes trees of $\mathbf{P}^1$'s with pairwise distinct nonsingular marked points, $\overline{L}_n$ parametrizes strings of $\mathbf{P}^1$'s stabilized by marked points of two types. The union of all $\overline{L}_n$'s forms a semigroup rather than operad, and the role of operadic algebras is taken over by the representations of the appropriately twisted homology algebra of this union.

0. Introduction and plan of the paper

One of the remarkable basic results in the theory of the Associativity Equations (or Frobenius manifolds) is the fact that their formal solutions are the same as cyclic algebras over the homology operad $(H_*(\overline{M}_{0,n+1}))$ of the moduli spaces of n–pointed stable curves of genus zero. This connection was discovered by physicists, who observed that the data of both types come from models of topological string theories. Precise mathematical treatment was given in [KM] and [KMK].

In this paper we establish a similar relationship between the pencils of formal flat connections (or solutions to the Commutativity Equations: see 3.1–3.2 below) and homology of a new series $\overline{L}_n$ of pointed stable curves of genus zero. Whereas $\overline{M}_{0,n+1}$ parametrizes trees of $\mathbf{P}^1$'s with pairwise distinct nonsingular marked points, $\overline{L}_n$ parametrizes strings of $\mathbf{P}^1$'s, and all marked points with exception of two are allowed to coincide (see the precise definitions in 1.1 and 2.1). Moreover, the union of all $\overline{L}_n$'s forms a semigroup rather than operad, and the role of operadic algebras is taken over by the representations of the appropriately twisted homology algebra of this union: see precise definitions in 3.3.

This relationship was discovered on a physical level in [Lo1], [Lo2]. Here we give a mathematical treatment of some of the main issues raised in these papers.

This paper is structured as follows.

In §1 we introduce the notion of (A, B)–pointed curves whose combinatorial structure generalizes that of strings of projective lines described above. We then describe a construction of "adjoining a generic black point" which allows us to produce families of such curves and their moduli stacks inductively. This is a simple variation of one of the arguments due to F. Knudsen [1].

In §2 we define and study the spaces $\overline{L}_n$ for which we give two complementary constructions. The first one identifies $\overline{L}_n$ with one of the moduli spaces of pointed curves. The second one exhibits $\overline{L}_n$ as a well–known toric manifold associated with the polytope called permutohedron in [Ka2]. These constructions put $\overline{L}_n$ into two quite different contexts and suggest generalizations in different directions.

As moduli spaces, $\overline{L}_n$ become components of the extended modular operad which we define and briefly discuss in §4. We expect that there exists an appropriate extension of the Gromov–Witten invariants producing algebras over extended operads involving gravitational descendants.

As toric varieties, $(\overline{L}_n)$ form one of the several series related to the generalized flag spaces of classical groups: see [GeSe]. It would be interesting to generalize to other series our constructions.

In this paper we use the toric description in order to prove for $\overline{L}_n$'s an analog of Keel's theorem (Theorem 2.7.1) and its extension (Theorem 2.9), crucial for studying representations of the twisted homology algebra.

This twisted homology algebra H_*T and its relationship with pencils of formal flat connections are discussed in §3, which contains the main result of this paper: Theorem 3.3.1.

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§1. (A, B)–pointed curves

1.1. Definition. *Let A, B be two finite disjoint sets, S a scheme, $g \geq 0$. An (A, B)–pointed curve of genus g over S consists of the data*

$$(\pi : C \to S;\ x_i : S \to C,\ i \in A;\ x_j : S \to C,\ j \in B) \tag{1.1}$$

where

(i) π is a flat proper morphism whose geometric fibres C_s are reduced and connected curves, with at most ordinary double points as singularities, and $g = H^1(C_s, \mathcal{O}_{C_s})$.

(ii) $x_i, i \in A \cup B$, are sections of π not containing singular points of geometric fibres.

(iii) $x_i \cap x_j = \emptyset$ if $i \in A,\ j \in A \cup B,\ i \neq j$.

Such a curve (1.1) *is called stable, if the normalization of any irreducible component C' of a geometric fibre carries ≥ 3 pairwise different special points if C' is of genus* 0 *and ≥ 1 special points if C' is of genus* 1. *Special points are inverse images of singular points and of the structure sections x_i.*

1.2. Remarks. a) If we put in this definition $B = \emptyset$, we will get the usual notion of an A–pointed (pre)stable curve whose structure sections are not allowed to intersect pairwise. Now we divide the sections into two groups: "white" sections $x_i, i \in A$ are not allowed to intersect any other section, whereas "black" sections $x_j, j \in B$ cannot intersect white ones, but otherwise are free and can even pairwise coincide. (However, both types of sections are not allowed to intersect singularities of fibres).

If we take in this definition one–element set $B = \{*\}$, we will get a natural bijection between $(A, \{*\})$–pointed curves and $(A \cup \{*\}, \emptyset)$–pointed curves. If card $B \geq 2$, the two notions become essentially different.

b) The dual modular graph of a geometric fibre is defined in the same way as in the usual case (for the conventions we use see [Ma], III.2). Tails now can be of two types, and we may refer to them and their marks as "black" and "white" ones as well. Combinatorial type of a geometric fiber is, by definition, the isomorphism class of the respective modular graph with (A, B)–marking of its tails.

c) Let $T \to S$ be an arbitrary base change. It produces from any (A, B)–pointed (stable) curve (1.2) over S another (A, B)–pointed (stable) curve over T: $(C_T;\ x_{i,T})$.

1.3. A construction. In this subsection, we start with an (A, B)–pointed curve (1.1) and produce from it another (A, B')–pointed curve:

$$(\pi' : C' \to S';\ x'_i,\ i \in A \cup B'). \tag{1.2}$$

The base of the new curve will be $S' := C$. There will be one extra black mark, say, $*$, so that $B' = B \cup \{*\}$. The new curve and sections will be produced in two steps. At the first step we make the base change $C \to S$ as in 1.2 c), obtaining an (A, B)–pointed curve $X := C \times_S C$, with sections $x_{i,C}$. We then add the extra section $\Delta : C \to C \times_S C$ which is the relative diagonal, and mark it by $*$. We did not yet produce an (A, B')–pointed curve over $S' = C$, because the extra black section can (and generally will) intersect both singular points of the fibres and white sections as well.

At the second step of the construction, we remedy this by birationally modifying $C \times_S C \to C$ as in [Kn1], Definition 2.3. More precisely, we define $C' := \mathrm{Proj}\,\mathrm{Sym}\,\mathcal{K}$ as the relative projective spectrum of the symmetric algebra of the sheaf $\mathcal{K}$ on $X = C \times_S C$ defined as the cokernel of the map

$$\delta : \mathcal{O}_X \to \mathcal{J}_\Delta\check{} \oplus \mathcal{O}_X(\sum_{i \in A} x_{i,C}),\ \delta(t) = (t, t). \tag{1.3}$$

Here $\mathcal{J}_\Delta$ is the $\mathcal{O}_X$–ideal of Δ, and $\mathcal{J}_\Delta\check{}$ is its dual sheaf considered as a subsheaf of meromorphic functions, as in [Kn1], Lemma 2.2 and Appendix.

We claim now that we get an (A, B')–pointed curve, because Knudsen's treatment of his modification can be directly extended to our case. In fact, the modification we described is nontrivial only in a neighbourhood of those points, where Δ intersects either singular points of the fibres, or A–sections. The B–sections do not intersect these neighborhoods, if they are small enough, and do not influence the local analysis due to Knudsen ([Kn1], pp. 176–178).

1.3.1. Remark. We can try to modify this construction in order to be able to add an extra white point, instead of a black one. However, for $\mathrm{card}\, B \geq 2$, we will not be able then to avoid the local analysis of the situation by referring to [Kn1]. In fact, points where Δ intersects at least two B–sections simultaneously, will have to be treated anew.

§2. Spaces $\overline{L}_n$

2.1. Spaces $\overline{L}_n$. In this subsection we will inductively define for any $n \geq 1$ the $(\{0,\infty\},\{1,\dots,n\})$–pointed stable curve of genus zero

$$(\pi_n : C_n \to \overline{L}_n;\ x_0^{(n)}, x_\infty^{(n)};\ x_1^{(n)}, \dots, x_n^{(n)}). \tag{2.1}$$

Namely, put

$$C_1 := \mathbf{P}^1,\ \overline{L}_1 = \text{a point},$$

and choose for $x_0^{(1)}, x_\infty^{(1)}, x_1^{(1)}$ arbitrary pairwise distinct points.

If (2.1) is already constructed, we define the next family $(C_{n+1} \to \overline{L}_{n+1}, \dots)$ as the result of the application of the construction 1.3 to $C_n/\overline{L}_n$. In particular, we have a canonical isomorphism $C_n = \overline{L}_{n+1}$.

2.2. Theorem. *a) $\overline{L}_n$ is a smooth separated irreducible proper manifold of dimension $n-1$. It represents the functor which associates with every scheme T the set of the isomorphism classes of $(\{0,\infty\},\{1,\dots,n\})$–pointed stable curves of genus zero over T whose geometric fibers have combinatorial types described below.*

The symmetric group $\mathbf{S}_n$ renumbering the structure sections acts naturally and compatibly on $\overline{L}_n$ and the universal curve. In particular, we can define the spaces $\overline{L}_B, C_B$ for any finite set B, functorial with respect to the bijections of the sets.

b) Combinatorial types of geometric fibres of $C_n \to \overline{L}_n$ are in a natural bijection with ordered partitions

$$\{1,\dots,n\} = \sigma_1 \cup \dots \cup \sigma_l,\ 1 \leq l \leq n,\ \sigma_i \neq \emptyset. \tag{2.2}$$

Partition (2.2) corresponds to the linear graph with vertices $(v_1,\dots,v_l)$ of genus zero, edges joining $(v_i, v_{i+1}), 1 \leq i \leq l-1$, A–tail 0 at the vertex v_1, A–tail ∞ at the vertex v_l, and B–tails marked by the elements of σ_i at the vertex v_i.

We will call $l = l(\sigma)$ the length of the partition σ as in (2.2).

c) Denote by L_σ the set of all points of $\overline{L}_n$ corresponding to the curves of the combinatorial type σ, and by $\overline{L}_\sigma$ its Zariski closure. Then L_σ are locally closed subsets, and we have

$$\overline{L}_\sigma = \coprod_{\tau \leq \sigma} L_\tau \tag{2.3}$$

where $\tau \leq \sigma$ means that τ is obtained from σ by replacing each σ_i by an ordered partition of σ_i into non–empty subsets.

d) For every σ, there exists a natural isomorphism

$$L_{|\sigma_1|} \times \dots \times L_{|\sigma_l|} \to L_\sigma \tag{2.4}$$

such that the pointed curve induced by this isomorphism over $L_{|\sigma_1|}\times\cdots\times L_{|\sigma_l|}$ can be obtained by clutching the curves $C_{|\sigma_i|}/L_{|\sigma_i|}$ in an obvious linear order (∞–section of the i–th curve is identified with the 0–section of the $(i+1)$–th curve, see [Kn1], Theorem 3.4), and subsequent remarking of the B–sections.

In particular, L_σ is a smooth irreducible submanifold of codimension $l(\sigma)-1$.

The similar statements hold for the closed strata $\overline{L}_\sigma$.

Proof. Properness and smoothness follow by induction and Knudsen's local analysis which we already invoked.

The statement about the combinatorial types is proved by induction as well. In fact, if everything is already proved for C_n, then we must look at a geometric fibre $C_{n,s}$ of C_n and see what happens to it after the blow up described in 1.3. If Δ intersects a smooth point of $C_{n,s}$, not coinciding with $x_{0,s}, x_{\infty,s}$, nothing happens, except that we get a new black point on this fibre, and a new tail at the respective vertex of the dual graph. If Δ intersects an intersection point of two neighboring components of $C_{n,s}$, then after blowing up these two components become disjoint, and we get a new component intersecting both of them, with a new black point on it. The linear structure of the graph is preserved. Finally, if Δ intersects $C_{n,s}$ at $x_{0,s}$ or $x_{\infty,s}$, then after blowing up we will get a new end component, with $x_{0,s}$, resp. $x_{\infty,s}$ and the new black point on it. Thus the new combinatorial types will be linear and indexed by partitions of $(n+1)$. To check that all partitions are obtained in this way, it suffices to remark that Δ, being the relative diagonal, can intersect the fibre of a given type at any point.

In order to check the statement about the functor represented by $\overline{L}_n$ we apply the following inductive reasoning. For $n=1$ the statement is almost obvious. In fact, let $\pi: C\to S$ be a $(\{0,\infty\},\{1\})$–pointed stable curve of genus zero over T. From the stability it follows that all geometric fibres are projective lines. Since the three structure sections pairwise do not intersect, the family can be identified with $\mathbf{P}^1\times T$ endowed with three constant sections. This means that it is induced by the trivial morphism $T\to\overline{L}_1$.

Assume that the statement is true for n. In order to prove it for $n+1$, consider a $(\{0,\infty\},\{1,\dots,n+1\})$–pointed stable curve of genus zero $\pi: C\to T$. First of all, one can produce from it a $(\{0,\infty\},\{1,\dots,n\})$–pointed stable curve of genus zero $\pi: C'\to T$ obtained by forgetting x_{n+1} and subsequent stabilization. The respective map $C'\to C$ is given by the relative projective spectrum of the algebra $\sum_{k=0}^{\infty}\pi_*(\mathcal{K}^{\otimes k})$ where $\mathcal{K}:=\omega_{C/T}(x_0+x_1+\cdots+x_n+x_\infty)$. By induction, C' is induced by a morphism $p: T\to\overline{L}_n$. Addition of an extra black section to C' and subsequent stabilization boils down exactly to the construction 1.3 applied to C'/T which allows us to lift p to a unique morphism $q: T\to\overline{L}_{n+1}$.

Separatedness is checked by the standard deformation arguments.

The statement about renumbering follows from the description of the functor.

A similar adaptation of Knudsen's arguments allows us to prove the remaining statements, and we leave them to the reader.

Notice that below we will give another direct description of the spaces $\overline{L}_B$ and all the structure morphisms connecting them in terms of toric geometry. This will provide easy alternate proofs of their properties. Except for §4, we can restrict ourselves to this alternate description.

2.2.1. Remark. Dual graphs of the degenerate fibers of C_n over $\overline{L}_n$ come with a natural orientation from x_0 to x_∞. We could have allowed ourselves not to distinguish between the two white points, interchanging them by isomorphisms, but this would produce several upleasant consequences. First, our manifolds would become actual stacks, starting already with $\overline{L}_1$. Second, we would have lost the toric interpretation of these spaces. Third, and most important, we would meet an ambiguity in the definition of the multiplication between the homology spaces: see (3.5) below. With our choice, we can simply introduce the involution permuting x_0 and x_∞ as a part of the structure and look how it interacts with other parts.

2.3. Theorem. *$\overline{L}_n$ has no odd cohomology. Let*

$$p_n(q) := \sum_{i=0}^{n-1} \dim H^{2i}(\overline{L}_n)\, q^i \tag{2.5}$$

be the Poincaré polynomial of $\overline{L}_n$. Then we have

$$1 + \sum_{n=1}^{\infty} \frac{p_n(q)}{n!} y^n = \frac{q-1}{q - e^{(q-1)y}} \in \mathbf{Q}[q][[y]]. \tag{2.6}$$

Letting here $q \to 1$ we get $\dfrac{1}{1-y}$ so that $\chi(\overline{L}) = n!$.

Proof. Since $\overline{L}_n$ are defined over $\mathbf{Q}$, we can apply the classical Weil's technique of counting points over $\mathbf{F}_q$ (thus treating q not as a formal variable but as a power of prime). After the counting is done, we will see that card $\overline{L}_n(\mathbf{F}_q)$ is a polynomial in q with positive integer coefficients, so that we can right away identify it with p_n:

$$p_n(q) = \operatorname{card} \overline{L}_n(\mathbf{F}_q) \tag{2.7}$$

The latter number can be calculated by directly applying (2.3) to the one–element partition σ, so that we get

$$\frac{p_n(q)}{n!} = \sum_{l=1}^{n} \sum_{\substack{(s_1,\dots,s_l)\\ s_1+\dots+s_l=n \\ s_i\geq 1}} \frac{(q-1)^{s_1-1}}{s_1!} \cdots \frac{(q-1)^{s_l-1}}{s_l!}$$

$$\sum_{l=1}^{n}\left[\text{coeff. of } x^{n-l} \text{ in } \left(\frac{e^x-1}{x}\right)^l\right]\cdot(q-1)^{n-l}.$$

Inserting this in the left hand side of (2.6) and summing over n first, we obtain

$$\sum_{n=1}^{\infty}\frac{p_n(q)}{n!}y^n=\sum_{l=1}^{\infty}\sum_{n=l}^{\infty}\left[\text{coeff. of } x^n \text{ in } (e^x-1)^l\right]\cdot(q-1)^n$$

$$=\sum_{l=1}^{\infty}\frac{1}{(q-1)^l}\left(e^{(q-1)y}-1\right)^l$$

which gives (2.6).

2.3.1. Special cases. Here is a list of the Poincaré polynomials for small values of n:

$$p_1=1,\ p_2=q+1,\ p_3=q^2+4q+1,\ p_4=q^3+11q^2+11q+1,$$

$$p_5=q^4+26q^3+66q^2+26q+1,\ p_6=q^6+57q^5+302q^4+302q^2+57q+1.$$

The rank of $H^2(L_n)$ is 2^n-n-1. Individual coefficients of of $p_n(q)$ are well known in combinatorics. They are called Euler numbers:

$$a_{n,i}=\dim H^{2i}(\overline{L}_n)\,.$$

2.4. $\overline{L}_n$ and toric actions. Let ε be the trivial partition of B of length one. The "big cell" L_ε of $\overline{L}_B$ (see 2.2 c)) has a canonical structure of the torsor (principal homogeneous space) over the torus $T_B:=\mathbf{G}_m^B/\mathbf{G}_m$ (where the subgroup $\mathbf{G}_m$ is embedded diagonally). In fact, $\mathbf{P}^1\setminus\{x_0,x_\infty\}$ is a $\mathbf{G}_m$–torsor, and the respective action of $\mathbf{G}_m^B$ on L_ε, moving x_i, $i\in B$ via the i–th factor, produces an isomorphic marked curve exactly via the action of the diagonal.

Similarly, every stratum L_σ is a torsor over $T_\sigma:=\prod_i T_{\sigma_i}$ (see (2.4)), and there is a canonical surjective morphism $T_B\to T_\sigma$ so that L_B is a union of T_B–orbits. In order to show that L_B is a toric variety, it remains to show that these actions are compatible. This again can be done using the explicit construction of $\overline{L}_n$ and induction. For a change, we will provide a direct toric construction. We start with a more systematic treatment of the combinatorics involved.

2.4.1. Partitions of finite sets. For any finite set B, we call a partition σ of B *a totally ordered set of non–empty subsets of B whose union is B and whose pairwise intersectons are empty.* If a partition consists of N subsets, it is called N–partition. If its components are denoted $\sigma_1,\dots,\sigma_N$, or otherwise listed, this means that they are listed in their structure order. Another partition can be denoted τ,

$\sigma^{(1)}$ etc. Notice that no particular ordering of B is a part of the structure. This is why we replaced $\{1, \dots, n\}$ here by an unstructured set B.

Let σ be a partition of B, $i, j \in B$. We say that σ *separates* i *and* j if they belong to different components of σ. We then write $i\sigma j$ in order to indicate that the component containing i comes earlier that the one containing j in the structure order.

Let τ be an $N+1$–partition of B. If $N \geq 1$, it determines a well ordered family of N 2–partitions $\sigma^{(a)}$:

$$\sigma_1^{(a)} := \tau_1 \cup \dots \cup \tau_a,\ \sigma_2^{(a)} := \tau_{a+1} \cup \dots \cup \tau_N,\ a = 1, \dots, N\,. \tag{2.8}$$

In reverse direction, call a family of 2–partitions $(\sigma^{(i)})$ *good* if for any $i \neq j$ we have $\sigma^{(i)} \neq \sigma^{(j)}$ and either $\sigma_1^{(i)} \subset \sigma_1^{(j)}$, or $\sigma_1^{(j)} \subset \sigma_1^{(i)}$. Any good family is naturally well–ordered by the relation $\sigma_1^{(i)} \subset \sigma_1^{(j)}$, and we will consider this ordering as a part of the structure. If a good family of 2–partitions consists of N members, we will usually choose superscripts $1, \dots, N$ to number these partitions in such a way that $\sigma_1^{(i)} \subset \sigma_1^{(j)}$ for $i < j$.

Such a good family produces one $(N+1)$–partition τ:

$$\tau_1 := \sigma_1^{(1)},\ \tau_2 := \sigma_1^{(2)} \setminus \sigma_1^{(1)},\ \dots,\ \tau_N := \sigma_1^{(N)} \setminus \sigma_1^{(N-1)},\ \tau_{N+1} = \sigma_2^{(N)}. \tag{2.9}$$

This correspondence between good N–element families of 2–partitions and $(N+1)$–partitions is one–to–one, because clearly $\sigma_1^{(i)} = \tau_1 \cup \dots \cup \tau_i$ for $1 \leq i \leq N$.

Consider the case when $\tau^{(1)} = \sigma$ is a 2–partition, and $\tau^{(2)} = \tau$ is an N–partition, $N \geq 2$. Their union is good, iff there exists $a \leq N$ and a 2–partition $\alpha = (\tau_{a1}, \tau_{a2})$ of τ_a such that

$$\sigma = (\tau_1 \cup \dots \cup \tau_{a-1} \cup \tau_{a1}, \tau_{a2} \cup \tau_{a+1} \cup \dots \cup \tau_N). \tag{2.10}$$

In this case we denote

$$\sigma * \tau = \tau(\alpha) := (\tau_1, \dots, \tau_{a-1}, \tau_{a1}, \tau_{a2}, \tau_{a+1}, \dots, \tau_N). \tag{2.11}$$

2.4.2. Lemma. *Let τ be a partition of B of length ≥ 1, and σ a 2–partition. Then one of the three mutually exclusive cases occurs:*

(i) σ coincides with one of the partitions $\sigma^{(a)}$ in (2.8). In this case we will say that σ breaks τ between τ_a and τ_{a+1}.

(ii) σ coincides with one of the partitions (2.10). In this case we will say that σ breaks τ at τ_a.

(iii) None of the above. In this case we will say that σ does not break τ. This happens exactly when there is a neighboring pair (τ_b, τ_{b+1}) of elements of τ with the following property:

$$\tau_b \setminus \sigma_1 \neq \emptyset, \ \tau_{b+1} \cap \sigma_1 \neq \emptyset. \tag{2.12}$$

We will call (τ_b, τ_{b+1}) a bad pair (for σ).

Proof. Consider the sequence of sets

$$\sigma_1 \cap \tau_1, \sigma_1 \cap \tau_2, \dots, \sigma_1 \cap \tau_N.$$

Produce from it a sequence of numbers $0, 1, 2$ by the following rule: replace $\sigma_1 \cap \tau_b$ by 2, if it coincides with τ_b, by 0 if it is empty, and by 1 otherwise. Cases (i) and (ii) above together will furnish all sequences of the form $(2 \dots 20 \dots 0)$, $(2 \dots 210 \dots 0)$, $(10 \dots 0)$. Each remaining admissible sequence will contain at least one pair of neighbors from the list 01, 02, 11, 12. For the respective pair of sets, (2.12) will hold.

2.5. Fan F_B. In this subsection we will describe a fan F_B in the space $N_B \otimes \mathbf{R}$, where $N_B := \mathrm{Hom}\,(\mathbf{G}_m, T_B)$, $T_B := \mathbf{G}_m^B/\mathbf{G}_m$ as in the beginning of 2.4. Up to notation, we use [Fu] as the basic reference on fans and toric varieties.

Clearly, N_B can be canonically identified with $\mathbf{Z}^B/\mathbf{Z}$, the latter subgroup being embedded diagonally. Similarly, $N_B \otimes \mathbf{R} = \mathbf{R}^B/\mathbf{R}$. We will write the vectors of this space (resp. lattice) as functions $B \to \mathbf{R}$ (resp. $B \to \mathbf{Z}$) considered modulo constant functions. For a subset $\beta \subset B$, let χ_β be the function equal 1 on β and 0 elsewhere.

2.5.1. Definition. *The fan F_B consists of the following l–dimensional cones $C(\tau)$ labeled by $(l+1)$–partitions τ of B.*

If τ is the trivial 1–partition, $C(\tau) = \{0\}$.

If σ is a 2–partition, $C(\sigma)$ is generated by χ_{σ_1}, or, equivalently, $-\chi_{\sigma_2}$, modulo constants.

Generally, let τ be an $(l+1)$–partition, and $\sigma^{(i)}$, $i = 1, \dots, l$, the respective good family of 2–partitions (2.9). Then $C(\tau)$ as a cone is generated by all $C(\sigma^{(i)})$.

It is not quite obvious that F_B is well defined. We sketch the relevant arguments.

First, all cones $C(\tau)$ are strongly convex. In fact, according to [Fu], p. 14, it suffices to check that $C(\tau) \cap (-C(\tau)) = 0$. But $C(\tau)$ consists of classes of linear combinations with non–negative coefficients of functions

$$\chi_{\tau_1},\, \chi_{\tau_1} + \chi_{\tau_2},\, \dots,\, \chi_{\tau_1} + \dots + \chi_{\tau_l}$$

if τ has length $l+1$. Non–vanishing function of this type cannot be constant.

Second, the same argument shows that $C(\tau)$ is actually l–dimensional.

Third, since the cone $C(\tau)$ is simplicial, one sees that $(l-1)$–faces of $C(\tau)$ are exactly $C(\tau^{(i)})$ where $\tau^{(i)}$ is obtained from τ by uniting τ_i with τ_{i+1}, which is equivalent to omitting $C(\sigma^{(i)})$ from the list of generators. More generally, $C(\tau')$ is a face of $C(\tau)$ iff $\tau \le \tau'$ as in (2.3), that is, if τ is a refinement of τ'.

Fourth, let $C(\tau^{(i)})$, $i = 1, 2$, be two cones. We have to check that their intersection is a cone of the same type. An obvious candidate is $C(\tau)$ where τ is the crudest common refinement of $\tau^{(1)}$ and $\tau^{(2)}$. This is the correct answer.

In order to see this, let us a give a different description of F_B which will simultaneously show that the support of F_B is the whole space. Let $\chi : \mathbf{B} \to \mathbf{R}$ represent an element $\bar{\chi} \in N_B \otimes \mathbf{R}$. It defines a unique partition τ of B consisting of the level sets of χ ordered in such a way that the values of χ decrease. Clearly, τ depends only on $\bar{\chi}$, and χ modulo constants can be expressed as a linear combination of $\chi_{\tau_1} + \dots + \chi_{\tau_i}$, $1 \le i \le l$ with positive coefficients. In other words, χ belongs to the interior part of $C(\tau)$. On the boundary, some of the strict inequalities between the consecurive values of χ become equalities. This proves the last assertion.

We see now that F_B satisfies the definition of [Fu], p. 20, and so is a fan.

2.6. Toric varieties $\overline{\mathcal{L}}_B$. We now define $\overline{\mathcal{L}}_B$ (later to be identified with $\overline{L}_B$) as the toric variety associated with the fan F_B.

To check that it is smooth, it suffices to show that each $C(\tau)$ is generated by a part of a basis of N_B (see [Fu], p. 29). In fact, let us choose a total ordering of B such that if $i \in \tau_k$, $j \in \tau_l$ and $k < l$, then $i < j$. Let $B_k \subset B$ consist of the first k elements of B in this ordering. Then the classes of the characteristic functions of $B_1, B_2, \dots, B_{n-1}$, $n = \mathrm{card}\, B$, form a basis of N_B, and $\{\chi_{\sigma^{(i)}}\}$ is a part of it.

To check that $\overline{\mathcal{L}}_B$ is proper, we have to show that the support of F_B is the total space. We have already proved this.

As any toric variety, $\overline{\mathcal{L}}_B$ carries a family of subvarieties which are the closures of the orbits of T_B and which are in a natural bijection with the cones $C(\tau)$ in F_B. We denote them $\overline{\mathcal{L}}_\tau$. They are smooth. The respective orbit which is an open subset of $\overline{\mathcal{L}}_\tau$ is denoted $\mathcal{L}_\tau$.

2.6.1. Forgetful morphisms and a family of pointed curves over $\overline{\mathcal{L}}_B$. Assume that $B \subset B'$. Then we have the projection morphism $\mathbf{Z}^{B'} \to \mathbf{Z}^B$ which induces the morphism $f^{B',B} : N_{B'} \to N_B$. It satisfies the property stated in the last lines of [Fu], p. 22: for each cone $C(\tau') \in F_{B'}$, there exists a cone $C(\tau) \in F_B$ such that $f^{B',B}(C(\tau')) \subset C(\tau)$. In fact, τ is obtained from τ' by deleting elements of $B' \setminus B$ and then deleting the empty subsets of the resulting partition of B.

Therefore, we have a morphism $f_*^{B',B}:\overline{\mathcal{L}}_{B'}\to\overline{\mathcal{L}}_B$ ([Fu], p. 23) which we will call *forgetful one* (it forgets elements of $B'\setminus B$).

2.6.2. Proposition. *If $B'\setminus B$ consists of one element, then the forgetful morphism $\overline{\mathcal{L}}_{B'}\to\overline{\mathcal{L}}_B$ has a natural structure of a stable $(\{0,\infty\},B)$–pointed curve of genus zero.*

Proof. Let us first study the fibers of the forgetful morphism. Let τ be a partition of B of length $l+1$ and $\mathcal{L}_\tau$ the respective orbit in $\overline{\mathcal{L}}_B$. Its inverse image in $\overline{\mathcal{L}}_{B'}$ is contained in the union $\cup\overline{\mathcal{L}}_{\tau'}$ where τ' runs over partitions of B' obtained by adding the forgotten point either to one of the parts τ_i, or inserting it in between τ_i and τ_{i+1}, or else putting it at the very beginning or at the very end as a separate part.

The inverse image of any point $x\in\mathcal{L}_\tau$ is acted upon by the multiplicative group $\mathbf{G}_m=\mathrm{Ker}\,(T_{B'}\to T_B)$. This action breaks the fiber into a finite number of orbits which coincide with the intersections of this fiber with various $\mathcal{L}_{\tau'}$ described above. When τ' is obtained by adding the forgotten point to one of the parts, this intersection is a torsor over the kernel, otherwise it is a point. As a result, we get that the fiber is a chain of $\mathbf{P}^1$'s, whose components are labeled by the components of τ and singular points by the neighboring pairs of components.

The forgetful morphism is flat, because locally in toric coordinates it is described as adjoining a variable and localization.

In order to describe the two white sections of the forgetful morphism, consider two partitions $(B'\setminus B,B)$ and $(B,B'\setminus B)$ of B' and the respective closed strata. It is easily seen that the forgetful morphism restricted to these strata identifies them with $\overline{\mathcal{L}}_B$. We will call them x_0 and x_∞ respectively.

Finally, to define the j–th black section, $j\in B$, consider the morphism of lattices $s_j:N_B\to N_{B'}$ which extends a function χ on B to the function $s_j(\chi)$ on B' taking the value $\chi(j)$ at the forgotten point. This morphism satisfies the condition of [Fu], p. 22: each cone $C(\tau)$ from F_B lands in an appropriate cone $C(\tau')$ from $F_{B'}$. This must be quite clear from the description at the end of 2.5.1: τ' is obtained from τ by adding the forgotten point to the same part to which j belongs. Hence we have the induced morphisms $s_{j*}:\overline{\mathcal{L}}_B\to\overline{\mathcal{L}}_{B'}$ which obviously are sections. Moreover, they do not intersect x_0 and x_∞, and they are distributed among the components of the reducible fibers exactly as expected.

2.6.3. Theorem. *The morphism $\overline{\mathcal{L}}_B\to\overline{L}_B$ inducing the family described in the Proposition 2.6.2 is an isomorphism.*

This can be proved by induction on $\mathrm{card}\,B$ with the help of the more detailed analysis of the forgetful morphism, as above. We omit the details because they are not instructive.

An important corollary of this Theorem is the existence of a surjective birational morphism $\overline{M}_{0,n+2} \to \overline{L}_n$ corresponding to any choice of two different labels i, j in $(1, \dots, n+2)$. In terms of the of the respective functors, this morphism blows down all the components of a stable $(n+2)$–labeled curve except for those that belong to the single path from the component containing the i–th point to the one containing the j–th point.

In fact, M. Kapranov has shown the existence of such a morphism for $\overline{\mathcal{L}}_n$ in place of $\overline{L}_n$ (see [Ka2], p. 102). He used a different description of $\overline{\mathcal{L}}_n$ in terms of the defining polyhedron, which he identified with the so called permutohedron, the convex hull of the $\mathbf{S}_n$–orbit of $(1, 2, \dots, n)$. He has also proved that $\overline{\mathcal{L}}_n$ can be identified with the closure of the generic orbit of the torus in the space of complete flags in an n–dimensional vector space.

2.7. Combinatorial model of $H^*(\overline{\mathcal{L}}_B)$. We will denote by $[\overline{\mathcal{L}}_\sigma]_*$ (resp. $[\overline{\mathcal{L}}_\sigma]^*$) the homology (resp. the dual cohomology) class of $\overline{\mathcal{L}}_\sigma$.

The remaining parts of this section (and the Appendix) are dedicated to the study of linear and non–linear relations between these classes, in the spirit of [KM] and [KMK], but with the help of the standard toric techniques.

Consider a family of pairwise commuting independent variables l_σ numbered by 2–partitions of B and introduce the ring

$$H_B^* := \mathcal{R}_B / I_B \tag{2.13}$$

where $\mathcal{R}_B$ is freely generated by l_σ (over an arbitrary coefficient ring k), and the ideal I_B is generated by the following elements indexed by pairs $i, j \in B$:

$$r_{ij}^{(1)} := \sum_{\sigma\colon i\sigma j} l_\sigma - \sum_{\tau\colon j\tau i} l_\tau, \tag{2.14}$$

$$r^{(2)}(\sigma, \tau) := l_\sigma l_\tau \qquad \text{if } i\sigma j \text{ and } j\tau i \text{ for some } i, j. \tag{2.15}$$

2.7.1. Theorem. *a) There is a well defined ring isomorphism $\mathcal{R}_B/I_B \to A^*(\overline{\mathcal{L}}_B, k)$ such that $l_\sigma \bmod I_B \mapsto [\overline{\mathcal{L}}_\sigma]^*$. The Chow ring $A^*(\overline{\mathcal{L}}_B, k)$ and the cohomology ring $H^*(\overline{\mathcal{L}}_B, k)$ are canonically isomorphic.*

b) The boundary divisors (strata corresponding to 2–partitions) intersect transversally.

Proof. We must check that the ideal of relations between $2^n - 2$ dual classes of the boundary divisors $[\overline{\mathcal{L}}_\sigma]^*$ contains and is generated by the following relations:

$$R_{ij}^{(1)}: \qquad \sum_{\sigma\colon i\sigma j} [\overline{\mathcal{L}}_\sigma]^* - \sum_{\tau\colon j\tau i} [\overline{\mathcal{L}}_\tau]^* = 0. \tag{2.16}$$

If $i\sigma j$ and $j\tau i$, then

$$R^{(2)}(\sigma,\tau): \qquad [\overline{\mathcal{L}}_\sigma]^* \cdot [\overline{\mathcal{L}}_\tau]^* = 0. \tag{2.17}$$

We refer to the Proposition on p. 106 of [Fu] which gives a system of generators for this ideal for any smooth proper toric variety (Fulton additionally assumes projectivity which we did not check, but see [Da], Theorem 10.8 for the general proper case).

In our notation, these generators look as follows.

To get the complete system of linear relations, we must choose some elements m in the dual lattice of N_B spanning this lattice and form the sums $\sum_\sigma m(\chi_{\sigma_1})[\overline{\mathcal{L}}_\sigma]^*$, where σ runs over all 2–partitions. In our case, the dual lattice is spanned by the linear functionals $m_{ij}: \chi \mapsto \chi(i) - \chi(j)$ for all pairs $i,j \in B$. Writing the respective relation, we get (2.16).

The complete system of nonlinear relations is given by the monomials $l_{\sigma^{(1)}} \dots l_{\sigma^{(k)}}$ such that $(C(\sigma^{(1)}),\dots,C(\sigma^{(k)}))$ do not span a cone in F_B. This means that some pair $(C(\sigma^{(a)}),C(\sigma^{(b)}))$ already does not span a cone, because otherwise the respective 2–partitions would form a good family (cf. 2.4.1). And in view of Lemma 2.4.2 (iii), we can find $i,j \in B$ such that $i\sigma^{(a)}j$ and $j\sigma^{(b)}i$. Hence (2.16) and (2.17) together constitute a generating system of relations.

The remaining statements are true for all smooth complete toric varieties defined by simplicial fans.

2.8. Combinatorial structure of the cohomology ring. In the remaining part of this section we fix a finite set B and study H_B^* as an abstract ring.

For an $(N+1)$–partition τ define the respective *good monomial* $m(\tau)$ by the formula

$$m(\tau) = l_{\sigma^{(1)}} \dots l_{\sigma^{(N)}} \in \mathcal{R}_B.$$

If τ is the trivial 1–partition, we put $m(\tau) := 1$. In view of the Theorem 2.7.1, $m(\tau)$ represents the cohomology class of $\overline{\mathcal{L}}_\tau$.

Notice that if we have two good families of 2–partitions whose union is also good, then the product of the respective good monomials is a good monomial. This defines a partial operation $*$ on pairs of partitions

$$m(\tau^{(1)})\, m(\tau^{(2)}) = m(\tau^{(1)} * \tau^{(2)}).$$

2.8.1. Proposition. *Good monomials and I_B span $\mathcal{R}_B$. Therefore, images of good monomials span H_B^*.*

Proof. We make induction on the degree. In degrees zero and one the statement is clear because l_σ are good. If it is proved in degree N, it suffices to check that for any 2–partition σ and any nontrivial partition τ, $l_\sigma m(\tau)$ is a linear combination of good monomials modulo I_B. We will consider the three cases of Lemma 2.4.2 in turn.

(i) *σ breaks τ between τ_a and τ_{a+1}.*

This means that l_σ divides $m(\tau)$.

Choose $i \in \tau_a, j \in \tau_{a+1}$. In view of (2.14), we have

$$\left(\sum_{\rho:\, i\rho j} l_\rho - \sum_{\rho:\, j\rho i} l_\rho\right) m(\tau) \in I_B. \tag{2.18}$$

But if $j\rho i$, then $l_\rho m(\tau) \in I_B$ because of (2.15). Among the terms with $i\rho j$ there is one l_σ. For all other ρ's, l_ρ cannot divide $m(\tau)$ since other divisors put i and j in the same part of the respective partition. Therefore, $l_\rho m(\tau)$ either belongs to I_B, or is good. So finally (2.18) allows us to express $l_\sigma m(\tau)$ as a sum of good monomials and an element of I_B :

$$l_\sigma m(\tau) = -\sum_{\rho\neq\sigma,\, i\rho j} m(\rho * \tau) \bmod I_B$$

where the terms for which $\rho * \tau$ is not defined must be interpreted as zero. More precisely, there are two types of non–vanishing terms. One corresponds to all 2–partitions α of τ_a such that $i \in \tau_{a1}$ which we will write as $i\alpha$. Another corresponds to 2–partitions β of τ_{a+1} with j belonging to the second part, βj:

$$l_\sigma m(\tau) = -\sum_{\alpha:\, i\alpha} m(\tau(\alpha)) - \sum_{\beta:\, \beta j} m(\tau(\beta)) \bmod I_B. \tag{2.19}$$

Notice that there are several ways to write the right hand side, depending on the choice of i, j. Hence good monomials are not linearly independent modulo I_B.

(ii) *σ breaks τ at τ_a.*

According to the analysis above, this means that

$$l_\sigma m(\tau) = m(\sigma * \tau) = m(\tau(\alpha)) \tag{2.20}$$

for an appropriate partition α of τ_a.

(iii) *σ does not break τ.*

In this case, let (τ_b, τ_{b+1}) be a bad pair for σ. Then from (2.12) it follows that there exist $i, j \in B$ such that $i\sigma j$ and $j\sigma^{(a)} i$. Hence $l_\sigma m(\tau)$ is divisible by $r^{(2)}(\sigma, \sigma^{(a)})$ and

$$l_\sigma m(\tau) = 0 \bmod I_B.$$

2.8.2. Linear combinations of good monomials belonging to I_B. Let $\tau = (\tau_1, \dots, \tau_N)$ be a partition of B. Choose $a \le N$ such that $|\tau_a| \ge 2$, and two elements $i, j \in \tau_a,\ i \ne j$. For any ordered 2–partition $\alpha = (\tau_{a1}, \tau_{a2})$ of τ_a, denote by $\tau(\alpha)$ the induced $N+1$–partition of B as above:

$$(\tau_1, \dots, \tau_{a-1}, \tau_{a1}, \tau_{a2}, \tau_{a+1}, \dots, \tau_N).$$

Finally, put

$$r_{ij}^{(1)}(\tau, a) := \sum_{\alpha:\, i\alpha j} m(\tau(\alpha)) - \sum_{\alpha:\, j\alpha i} m(\tau(\alpha)). \tag{2.21}$$

Choosing for τ the trivial 1–partition, we get (2.14) so that these elements span the intersection of I_B with the space of good monomials of degree one.

Generally, all $r_{ij}^{(1)}(\tau, a)$ belong to I_B. In fact, keeping the notations above, consider

$$r_{ij}^{(1)} m(\tau) = \left(\sum_{\rho:\, i\rho j} l_\rho - \sum_{\rho:\, j\rho i} l_\rho \right) m(\tau) \in I_B. \tag{2.22}$$

Arguing as above, we see that the summand corresponding to ρ in (2.18) either belongs to I_B, or is a good monomial, and the latter happens exactly for those partitions ρ that are of the type $\tau(\alpha)$ with either $i\alpha j$, or $j\alpha i$. Hence (2.21) lies in I_B. This proves our claim.

2.9. Theorem. *Elements (2.21) span the intersection of I_B with the space generated by good monomials.*

Proof. Define the linear space H_{*B} generated by the symbols $\mu(\tau)$ for all partitions of B as above which satisfy analogs of the linear relations (2.21): for all (τ, τ_a, i, j) as above we have

$$\sum_{\alpha:\, i\alpha j} \mu(\tau(\alpha)) - \sum_{\alpha:\, j\alpha i} \mu(\tau(\alpha)) = 0. \tag{2.23}$$

2.9.1. Technical Lemma. *There exists an (obviously unique) structure of the H_B^*–module on H_{*B} with the following multiplication table.*

(i) If σ breaks τ between τ_a and τ_{a+1}, then for any choice of $i \in \tau_a$, $j \in \tau_{a+1}$

$$l_\sigma \mu(\tau) = -\sum_{\alpha:\, i\alpha} \mu(\tau(\alpha)) - \sum_{\beta:\, \beta j} \mu(\tau(\beta)). \tag{2.24}$$

(cf. (2.19)).

(ii) If σ breaks τ at τ_a, then

$$l_\sigma \mu(\tau) = \mu(\sigma * \tau). \tag{2.25}$$

(cf. (2.20)).

(iii) If σ does not break τ, then

$$l_\sigma \mu(\tau) = 0. \tag{2.26}$$

Our proof of the Technical Lemma consists in the direct verification that the prescriptions (2.24)–(2.23) are compatible with all relations that we have postulated. Unfortunately, such strategy requires the painstaking case–by–case treatment of a long list of combinatorially distinct situations, and we relegate it to the Appendix.

2.9.2. Deduction of Theorem 2.9 from the Technical Lemma. Since elements (2.21) belong to I_B, there exists a surjective linear map $s : H_{*B} \to H_B^*$, $\mu(\tau) \mapsto m(\tau)$. Now denote by $\mathbf{1}$ the element $\mu(\varepsilon)$ where ε is the 1–partition. Then $t : m(\sigma) \mapsto m(\sigma)\mathbf{1}$ is a linear map $H_B^* \to H_{*B}$. From (2.25) one easily deduces that $m(\tau)\mathbf{1} = \mu(\tau)$ so that s and t are mutually inverse. Therefore, (2.22) span the linear relations between the images of good monomials in H_B^*.

According to the Theorem 2.4.1, H_{*B}, together with its structure of H_B^*–module, is a combinatorial model of the homology module $H_*(\overline{\mathcal{L}}_B, k)$. The generators $\mu(\tau)$ correspond to $[\overline{\mathcal{L}}_\tau]_*$.

§3. Pencils of flat connections and the Commutativity Equations

3.1. Notation. Let M be a (super)manifold over a field k of characteristic zero in one of the standard categories (smooth, complex analytic, schemes, formal ...). We use the conventions spelled out in [Ma], I.1. In particular, differentials in the de Rham complex (Ω^*_M, d) and connections are odd. This determines our sign rules; parity of an object x is denoted $\widetilde{x}$.

Let $\mathcal{F}$ be a locally free sheaf (of sections of a vector bundle) on M, ∇_0 a connection on $\mathcal{F}$, that is an odd k–linear operator $\mathcal{F} \to \Omega^1_M \otimes \mathcal{F}$ satisfying the Leibniz identity

$$\nabla_0(\varphi f) = d\varphi \otimes f + (-1)^{\widetilde{\varphi}} \varphi \, \nabla_0 f, \ \varphi \in \mathcal{O}_M, \ f \in \mathcal{F}. \tag{3.1}$$

This operator extends to a unique operator on the Ω^*_M–module $\Omega^*_M \otimes \mathcal{F}$ denoted again ∇_0 and satisfying the same identity (3.1) for any $\varphi \in \Omega_M$. Any other connection differential ∇ restricted to $\mathcal{F}$ has the form $\nabla_0 + \mathcal{A}$ where $\mathcal{A} : \mathcal{F} \to \Omega^1_M \otimes \mathcal{F}$ is an odd $\mathcal{O}_M$–linear operator: $\mathcal{A}(\varphi f) = (-1)^{\widetilde{\varphi}} \varphi \mathcal{A}(f)$. Any connection naturally extends to the whole tensor algebra generated by $\mathcal{F}$, in particular, to $\mathcal{E}nd\,\mathcal{F}$.

The connection ∇_0 is called flat, iff $\nabla_0^2 = 0$. *A pencil of flat connections* is a line in the space of connections $\nabla_\lambda := \nabla_0 + \lambda \mathcal{A}$ such that $\nabla_\lambda^2 = 0$ (λ is an even parameter). In the smooth, analytic or formal category, ∇_0 is flat iff $\mathcal{F}$ locally admits a basis of flat sections $f, \nabla_0 f = 0$.

3.2. Proposition. $\nabla_0 + \lambda \mathcal{A}$ *is a pencil of flat connections iff the following two conditions are satisfied:*

(i) Everywhere locally on M, we have

$$\mathcal{A} = \nabla_0 \mathcal{B} \tag{3.2}$$

for some $\mathcal{B} \in \mathcal{E}nd\,\mathcal{F}$.

(ii) Such an operator $\mathcal{B}$ satisfies the quadratic differential equation

$$\nabla_0 \mathcal{B} \wedge \nabla_0 \mathcal{B} = 0. \tag{3.3}$$

Proof. Calculating the coefficient of λ in $\nabla_\lambda^2 = 0$ we get $\nabla_0 \mathcal{A} = 0$. But the complex $\Omega^*_M \otimes \mathcal{F}$ is the resolution of the sheaf of flat sections $\operatorname{Ker} \nabla_0 \subset \mathcal{F}$. This furnishes (i); (ii) means the vanishing of the coefficient at λ^2.

3.2.1. Remarks. a) Write $\mathcal{B}$ as a matrix in a basis of ∇_0–flat sections of $\mathcal{F}$, whose entries are local functions on M. Then (3.3) becomes

$$d\mathcal{B} \wedge d\mathcal{B} = 0. \tag{3.4}$$

These equations written in local coordinates (t^i) on M were called "t–part of the t–t^* equations" by S. Cecotti and C. Vafa. A. Losev in [Lo1] suggested to call them "the Commutativity Equations".

b) If $\nabla_0\varphi_0 = 0$, then

$$(\nabla_0 + \lambda\nabla_0\mathcal{B})(e^{-\lambda\mathcal{B}}\varphi_0) = 0.$$

3.2.2. Pencils of flat connections related to Frobenius manifolds. Any solution to the Associativity Equations produces a pencil of flat connections.

To explain this we will use the geometric language due to B. Dubrovin (and the notation of [Ma], I.1.5). Consider a Frobenius manifold $(M, g, \circ)$ where

$$\circ : \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{T}_M \to \mathcal{T}_M$$

is a (super)commutative associative multiplication on the tangent sheaf satisfying the potentiality condition, and g is an invariant flat metric (no positivity condition is assumed, only symmetry and non–degeneracy). Denote by ∇_0 the Levi–Civita connection of g. Finally, denote by $\mathcal{A}$ the operator obtained from the Frobenius multiplication in $\mathcal{T}_M$ ([Ma], I.1.4). In other words, consider the pencil of connections on $\mathcal{F} = \mathcal{T}_M$ whose covariant derivatives are

$$(\nabla_0 + \lambda\mathcal{A})_X(Y) := \nabla_{0,X}(Y) + \lambda\, X \circ Y\,.$$

This pencil is flat (see [Ma], Theorem I.1.5, p. 20). In fact, $\mathcal{B}$ written in a basis of ∇_0–flat coordinates and the respective flat vector fields is simply the matrix of the second derivatives of a local potential Φ (with one subscript raised). This is the first structure connection of M.

This pencil admits an infinite dimensional deformation: one should take the canonical extension of the potential to the large phase space and consider the coordinates with gravitational descendants as parameters of the deformation.

Another family of flat connections, this time on the *cotangent* sheaf of a Frobenius manifold M admitting an Euler vector field E (see [Ma], pp. 23–24), is defined as follows. Denote the scalar product on vector fields $\check{g}_\lambda(X,Y) := g((E-\lambda)^{-1}\circ X, Y)$. The inverse form induces a pencil of flat metrics on the cotangent sheaf, whose Levi–Civita connections however do not form a pencil of flat connections in our sense (see [Du1], Appendix D, and [Du3] for a general discussion of such setup). This is the second structure connection of M.

3.2.3. Flat coordinates and gravitational descendants. One can show that 1–forms on M flat with respect to the dualized first structure connection are closed and therefore locally exact. Their integrals are called deformed flat coordinates. In [Du2], Example 2.3 and Theorem 2.2, B. Dubrovin gives explicit formal series in λ (z in his notation) for suitably normalized deformed flat coordinates. Coefficients of these series involve some correlators with gravitational descendants, namely those for which the non–trivial operators τ_p are applied only at one point. In [KM2] and [Ma], VI.7.2, p.278, it was shown that two–point correlators of this kind determine a linear operator in the large phase space which transforms the modified correlators with descendants into non–modified ones (in any genus). This is important because apriori only modified correlators are defined for an arbitrary Cohomological Field Theory in the sense of [KM1], which is not necessarily quantum cohomology of a manifold.

3.2.4. Pencils of flat connections in a global setting. Pencils of flat connections appear also in the context of Simpson's non–abelian Hodge theory. Briefly, consider a smooth projective manifold M over $\mathbf{C}$. One can define two moduli spaces, Mod_1 and Mod_2. The first one classifies flat connections (on variable vector bundles $\mathcal{F}$ with vanishing rational Chern classes) with semisimple Zariski closure of the monodromy group. The second one classifies semistable Higgs pairs $(\mathcal{F}, \mathcal{A})$ where $\mathcal{A}$ is an operator as in 3.1, satisfying only the condition $\mathcal{A} \wedge \mathcal{A} = 0$. (In fact, one should only consider smooth points of the respective moduli spaces). N. Hitchin, C. Simpson, Fujiki et al. established that Mod_1 and Mod_2 are canonically isomorphic as C^∞–manifolds, but their complex structures I, J are different, and together with $K = IJ$ produce a hypercomplex manifold.

P. Deligne has shown that the respective twistor space is precisely the moduli space of the pencils of flat connections on M (where the Higgs complex structure corresponds to the point $\lambda = \infty$ in our notation).

For details, see [Si].

3.3. Formal solutions to the Commutativity Equations and the homology of $\overline{L}_n$. In [KM1] and [KMK] it was shown that formal solutions to the Associativity Equations are cyclic algebras over the cyclic genus zero homology modular operad $(H_*(\overline{M}_{0,n+1}))$ (see also [Ma], III.4). The main goal of this section is to show the similar role of the homology of the spaces $\overline{L}_n$ in the theory of the Commutativity Equations. This was discovered and discussed on a physical level in [Lo1], [Lo2]. Here we supply precise mathematical statements with proofs.

Unlike the case of the Associativity Equations, we will have to deal here with modules over an algebra (depending explicitly on the base space) rather than with algebras over an operad. The main ingredient of the construction is the direct sum of the homology spaces of all $\overline{L}_n$ endowed with the multiplication coming from the boundary morphisms. We work with the combinatorial models of these spaces defined in 2.9.1.

We start with some preparations. Let $V = \oplus_{n=1}^{\infty} V_n$ be a graded associative k–algebra (without identity) in the category of vector k–superspaces over a field k. We will call it *an* $\mathbf{S}$*–algebra*, if for each n, an action of the symmetric group $\mathbf{S}_n$ on V_n is given such that the multiplication map $V_m \otimes V_n \to V_{m+n}$ is compatible with the action of $\mathbf{S}_m \times \mathbf{S}_n$ embedded in an obvious way into $\mathbf{S}_{m+n}$.

If V is an $\mathbf{S}$–algebra, then the sum of subspaces J_n spanned by $(1-s)v$, $s \in \mathbf{S}_n$, $v \in V_n$, is a double–sided ideal in V. Hence the sum of the coinvariant spaces $V_{\mathbf{S}_n} = V_n/J_n$ is a graded ring which we denote $V_{\mathbf{S}}$.

If V, W are two $\mathbf{S}$–algebras, then the diagonal part of their tensor product $\oplus_{n=1}^{\infty} V_n \otimes W_n$ is an $\mathbf{S}$–algebra as well.

Let T be a vector superspace (below always assumed finite–dimensional). Its tensor algebra (without the rank zero part) is an $\mathbf{S}$–algebra.

As a less trivial example, consider $H_* := \oplus_{n=1}^{\infty} H_{*n}$ where we write H_{*n} for $H_{*\{1,\dots,n\}}$. The multiplication law is given by what becomes the boundary morphisms in the geometric setting: if $\tau^{(1)}$ (resp. $\tau^{(2)}$) is a partition of $\{1,\dots,m\}$ (resp. of $\{1,\dots,n\}$), then

$$\mu(\tau^{(1)})\mu(\tau^{(2)}) = \mu(\tau^{(1)} \cup \tau^{(2)}) \tag{3.5}$$

where the concatenated partition of $\{1,\dots,m,\, m+1,\dots,m+n\}$ is defined in an obvious way, shifting all the components of $\tau^{(2)}$ by m.

Our main protagonist is the algebra of coinvariants of the diagonal tensor product of these examples:

$$H_*T := \left(\oplus_{n=1}^{\infty} H_{*n} \otimes T^{\otimes n}\right)_{\mathbf{S}}. \tag{3.6}$$

We now fix T and another vector superspace F and assume that the ground field k has characteristic zero.

3.3.1. Theorem. *There is a natural bijection between the set of representations of H_*T in F and the set of pencils of flat connections on the trivial bundle with fiber F on the formal completion of T at the origin.*

This bijection will be precisely defined and discussed below: see Proposition 3.6.1. Before passing to this definition and the proof of the Theorem, we will give a down–to–earth coordinate–dependent description of the representations of H_*T.

3.4. Matrix correlators. Fix T and choose its parity homogeneous basis $(\Delta_a \,|\, a \in I)$ where I is a finite set of indices.

For any $n \geq 1$, the space $H_{*n} \otimes T^{\otimes n}$ is spanned by the elements

$$\mu(\tau^{(n)}) \otimes \Delta_{a_1} \otimes \dots \otimes \Delta_{a_n} \tag{3.7}$$

where $\tau^{(n)}$ runs over all partitions of $\{1,\dots,n\}$ whereas $(a_1,\dots,a_n)$ runs over all maps $\{1,\dots,n\}\mapsto I:\, i\to a_i$.

In view of the Theorem 2.9, all linear relations between these elements are spanned by the following ones: choose $(a_1,\dots,a_n)$ and $(\tau^{(n)},\tau_r^{(n)},\, i\neq j\in\tau_r^{(n)})$, then

$$\sum_{\alpha:\, i\alpha j}\mu(\tau^{(n)}(\alpha))\otimes\Delta_{a_1}\otimes\dots\otimes\Delta_{a_n}-\sum_{\alpha:\, j\alpha i}\mu(\tau^{(n)}(\alpha))\otimes\Delta_{a_1}\otimes\dots\otimes\Delta_{a_n}=0 \quad (3.8)$$

where the summation is taken over all 2–partitions α of $\tau_r^{(n)}$ separating i and j.

The action of a permutation $i\mapsto s(i)$ on (3.7) is defined by

$$s\left(\mu(\tau^{(n)})\otimes\Delta_{a_1}\otimes\dots\otimes\Delta_{a_n}\right)=\varepsilon(s,(a_i))\,\mu(s(\tau^{(n)}))\otimes\Delta_{a_{s(1)}}\otimes\dots\otimes\Delta_{a_{s(n)}}. \quad (3.9)$$

Here $\varepsilon(s,(a_i))=\pm1$ is the sign of the permutation induced by s on the subfamily of odd Δ_{a_i}'s, and $s(\tau^{(n)})$ is defined as follows:

$$s(i)\in s(\tau^{(n)})_r \quad \text{iff} \quad i\in\tau_r^{(n)}. \quad (3.10)$$

Finally, the multiplication rule between the generators in the diagonal tensor product is given by:

$$\mu(\tau^{(m)})\otimes\Delta_{a_1}\otimes\dots\otimes\Delta_{a_m}\cdot\mu(\tau^{(n)})\otimes\Delta_{b_1}\otimes\dots\otimes\Delta_{b_n}$$

$$=\mu(\tau^{(m)}\cup\tau^{(n)})\otimes\Delta_{a_1}\otimes\dots\otimes\Delta_{a_m}\otimes\Delta_{b_1}\otimes\dots\otimes\Delta_{b_n}. \quad (3.11)$$

Any linear representation $K:\, H_*T\to\operatorname{End}F$ can be described as a linear representation of the diagonal tensor product satisfying additional symmetry restrictions. To spell it out explicitly, we define *the matrix correlators* of K as the following family of endomorphisms of F:

$$\tau^{(n)}\langle\Delta_{a_1}\dots\Delta_{a_n}\rangle:=K(\mu(\tau^{(n)})\otimes\Delta_{a_1}\otimes\dots\otimes\Delta_{a_n}). \quad (3.12)$$

3.4.1. Claim. *Matrix correlators of any representation satisfy the following relations:*

(i) $\mathbf{S}_n$*–symmetry:*

$$s^{-1}(\tau^{(n)})\langle\Delta_{a_1}\dots\Delta_{a_n}\rangle=\varepsilon(s,(a_i))\,\tau^{(n)}\langle\Delta_{a_{s(1)}}\dots\Delta_{a_{s(n)}}\rangle\,. \quad (3.13)$$

(ii) Factorization:

$$(\tau^{(m)} \cup \tau^{(n)})\langle \Delta_{a_1} \dots \Delta_{a_m} \Delta_{b_1} \dots \Delta_{b_n} \rangle = \tau^{(m)} \langle \Delta_{a_1} \dots \Delta_{a_m} \rangle \cdot \tau^{(n)} \langle \Delta_{b_1} \dots \Delta_{b_n} \rangle \,. \tag{3.14}$$

(iii) Linear relations:

$$\sum_{\alpha :\, i\alpha j} \tau^{(n)}(\alpha)\langle \Delta_{a_1} \dots \Delta_{a_n} \rangle - \sum_{\alpha :\, j\alpha i} \tau^{(n)}(\alpha)\langle \Delta_{a_1} \dots \Delta_{a_n} \rangle = 0 \tag{3.15}$$

Conversely, any family of elements of $\mathrm{End}\, F$ *defined for all* n, $(a_1, \dots, a_n)$, $\tau^{(n)}$ *and satisfying (3.13)–(3.15) consists of matrix correlators of a well defined representation* $K :\, H_*T \to \mathrm{End}\, F$.

In fact, we obtain (3.13) by applying K to (3.9) written for $s^{-1}(\tau^{(n)})$ in place of $\tau^{(n)}$, because K, coming from H_*T, vanishes on the image of $1 - s$. Moreover, (3.14) means the compatibility with the multiplication of the generators. Finally, (3.15) is a necessary and sufficient condition for the extendability of the system of matrix correlators to a linear map K.

Notice that we can replace here $\mathrm{End}\, F$ by an arbitrary associative superalgebra over k.

3.5. Top matrix correlators. Define *top matrix correlators of* K as the subfamily of correlators corresponding to the identical partitions $\varepsilon^{(n)}$ of $\{1, \dots, n\}$:

$$\langle \Delta_{a_1} \dots \Delta_{a_n} \rangle := \varepsilon^{(n)} \langle \Delta_{a_1} \dots \Delta_{a_n} \rangle \,.$$

3.5.1. Proposition. *Top matrix correlators satisfy the following relations:*

$$\langle \Delta_{a_1} \dots \Delta_{a_n} \rangle = \varepsilon(s, (a_i))\, \langle \Delta_{a_{s(1)}} \dots \Delta_{a_{s(n)}} \rangle \tag{3.16}$$

and

$$\sum_{\sigma :\, i\sigma j} \varepsilon(\sigma, (a_k)) \langle \prod_{k \in \sigma_1} \Delta_{a_k} \rangle \cdot \langle \prod_{k \in \sigma_2} \Delta_{a_k} \rangle - \sum_{\sigma :\, j\sigma i} \varepsilon(\sigma, (a_k)) \langle \prod_{k \in \sigma_1} \Delta_{a_k} \rangle \cdot \langle \prod_{k \in \sigma_2} \Delta_{a_k} \rangle = 0 \,. \tag{3.17}$$

Here σ *runs over 2–partitions of* $\{1, \dots, n\}$. *We choose additionally an arbitrary ordering of both parts* σ_1, σ_2 *determining the ordering of* Δ*'s in the angular brackets, and compensate this choice by the* ± 1*–factor* $\varepsilon(\sigma, (a_k))$.

Conversely, any family of elements $\langle \Delta_{a_1} \dots \Delta_{a_n} \rangle \in \mathrm{End}\, F$ *defined for all* n *and* $(a_1, \dots, a_n)$ *and satisfying (3.16), (3.17) is the family of top matrix correlators of a well defined representation* $K :\, H_*T \to \mathrm{End}\, F$.

Proof. Clearly, (3.16) is a particular case of (3.13). To get (3.17), we apply (3.15) to the identical partition $\tau^{(n)} = \varepsilon^{(n)}$ and then replace each term by the double product of top correlators using (3.14).

Conversely, assume that we are given $\langle\Delta_{a_1}\dots\Delta_{a_n}\rangle$ satisfying (3.16) and (3.17). There is a unique way to extend this system to a family of elements $\tau^{(n)}\langle\Delta_{a_1}\dots\Delta_{a_n}\rangle$ defined for all N–partitions $\tau^{(n)}$ and satisfying the factorization property (3.14) and at least a part of the symmetry relations (3.13):

$$\tau^{(n)}\langle\Delta_{a_1}\dots\Delta_{a_n}\rangle := \varepsilon(\tau^{(n)},(a_k))\prod_{r=1}^{r=N}\langle\prod_{k\in\tau_r^{(n)}}\Delta_{a_k}\rangle\,. \qquad (3.18)$$

Here, as in (3.17), we choose arbitrary orderings of each $\tau_r^{(n)}$ and compensate this by the appropriate sign so that the result does not depend on the choices made. All the relations (3.13) become automatically satisfied with this definition. In fact, the left hand side of (3.13) puts into $s^{-1}(\tau^{(n)})_r$ those i for which $s(i)\in\tau_r^{(n)}$ (see (3.10)) so that the expression of both sides of (3.13) through the top correlators consists of the same groups taken in the same order. The coincidence of the signs is left to the reader.

It remains to check that (3.18) satisfy the linear relations (3.15). Recall now that to write a concrete relation (3.15) down we choose $\tau^{(n)}$, r, $i,j\in\tau_r^{(n)}$ and $(a_1,\dots,a_n)$ and then sum over 2–partitions α of $\tau_r^{(n)}$. Hence replacing each term of the left hand side of (3.15) by the prescriptions (3.18) we get

$$\prod_{p=1}^{r-1}\langle\prod_{k\in\tau_p^{(n)}}\Delta_{a_k}\rangle\cdot\left(\sum_{\alpha:\,i\alpha j}\pm\langle\prod_{k\in\alpha_1}\Delta_{a_k}\rangle\cdot\langle\prod_{k\in\alpha_2}\Delta_{a_k}\rangle-\sum_{\alpha:\,j\alpha i}\pm\langle\prod_{k\in\alpha_1}\Delta_{a_k}\rangle\cdot\langle\prod_{k\in\alpha_2}\Delta_{a_k}\rangle\right)$$

$$\cdot\prod_{q=r+1}^{N}\langle\prod_{k\in\tau_q^{(n)}}\Delta_{a_k}\rangle\,.$$

This expression vanishes because its middle term is an instance of (3.17).

3.6. Precise statement and proof of the Theorem 3.3.1. Assume that we are given a representation $K:\,H_*T\to\mathrm{End}\,F$. We will produce from it a formal solution of the Commutativity Equations using only its top correlators. Let (x^a) be the basis of formal coordinates on T dual to (Δ_a). Put

$$\mathcal{B}=\sum_{n=1}^{\infty}\sum_{(a_1,\dots,a_n)}\frac{x^{a_n}\dots x^{a_1}}{n!}\,\langle\Delta_{a_1}\dots\Delta_{a_n}\rangle\in k[[x]]\otimes\mathrm{End}\,F. \qquad (3.19)$$

3.6.1. Proposition. *a) We have*

$$d\mathcal{B} \wedge d\mathcal{B} = 0. \tag{3.20}$$

b) Conversely, let $\Delta(a_1, \dots, a_n) \in \operatorname{End} F$ be a family of linear operators defined for all $n \geq 1$ and all maps $\{1, \dots, n\} \to I : i \mapsto a_i$. Assume that the parity of $\Delta(a_1, \dots, a_n)$ coincides with the sum of the parities of Δ_{a_i} and that for any $s \in \mathbf{S}_n$

$$\Delta(a_{s(1)}, \dots, a_{s(n)}) = \varepsilon(s, (a_i))\, \Delta(a_1, \dots, a_n)\,.$$

Finally, assume that the formal series

$$\mathcal{B} = \sum_{n=1}^{\infty} \sum_{(a_1, \dots, a_n)} \frac{x^{a_n} \dots x^{a_1}}{n!}\, \Delta(a_1, \dots, a_n) \in k[[x]] \otimes \operatorname{End} F \tag{3.21}$$

satisfies the equations (3.20). Then there exists a well defined representation $K : H_*T \to \operatorname{End} F$ such that $\Delta(a_1, \dots, a_n)$ are the top correlators $\langle \Delta_{a_1} \dots \Delta_{a_n} \rangle$ of this representation.

Notice that any even element of $k[[x]] \otimes \operatorname{End} F$ without constant term can be uniquely written in the form (3.21).

Proof. Clearly, the equations $d\mathcal{B} \wedge d\mathcal{B} = 0$ written for the series (3.21) are equivalent to a family of bilinear relations between the symmetric matrix–valued tensors $\Delta(a_1 \dots a_n)$. In view of the Proposition 3.5.1, it remains to check only that this family of relations is equivalent to the family (3.17). This is a straightforward exercise.

§4. Stacks $\overline{L}_{g;A,B}$ and the extended modular operad

4.1. Introduction. The basic topological operad $(\overline{M}_{0;n+1}, n \geq 2)$ of Quantum Cohomology lacks the $n = 1$ term which is usually formally defined as a point. We argued elsewhere (cf. [MaZ], sec. 7 and [Ma], VI.7.6) that it would be very desirable to find a non–trivial DM–stack which could play the role of $\overline{M}_{0;2}$. There are several tests that such an object should pass:

a) It must be a semigroup (because for any operad $\mathcal{P}$, the operadic multiplication makes a semigroup of $\mathcal{P}(1)$).

b) It must be a part of an extended genus zero operad, say, $(\widetilde{L}_{0;n+1}, n \geq 1)$ geometrically related to $(\overline{M}_{0;n+1}, n \geq 2)$ in such a way that the theory of Gromov–Witten invariants with gravitational descendants could be formulated in this new context. In particular, it must geometrically explain two–point correlators with gravitational descendants.

c) In turn, the extended genus zero operad must be a part of an extended modular operad containing moduli spaces of arbitrary genus, in such a way that algebras over classical modular operads produce extended algebras.

In this section we will try to show that the space

$$\widetilde{L}_{0;2} := \coprod_{n\geq 1} \overline{L}_n \tag{4.1}$$

passes at least a part of these tests. (Another candidate which might be interesting is $\lim\mathrm{proj}\,\overline{L}_n$ with respect to the forgetful morphisms).

4.2. Semigroup structure. It is defined as the union of boundary (clutching) morphisms

$$b := (b_{n_1,n_2}) : \ \widetilde{L}_{0;2} \times \widetilde{L}_{0;2} \to \widetilde{L}_{0;2} \tag{4.2}$$

where

$$b_{n_1,n_2} : \ \overline{L}_{n_1} \times \overline{L}_{n_2} \to \overline{L}_{n_1+n_2}$$

glues x_∞ of the first curve to x_0 of the second curve and renumbers the black points of the second curve keeping their order (cf. [MaZ], section 7). This is the structure that induced our multiplication on H_* in 3.3 above.

4.3. Extended operads. In (4.2), only white points $\{x_0, x_\infty\}$ are used to define the operadic composition whereas the black ones serve only to stabilize the strings of $\mathbf{P}^1$'s which otherwise would be unstable. This is a key observation for our attempt to define an extended operad.

A natural idea would be to proceed as follows. Denote by $\overline{M}_{g;A,B}$ the stack of stable (A,B)–pointed curves of genus g (see Definition 1.1). Check that it is a

DM–stack. Put $\widetilde{M}_{g;m+1} := \coprod_{n\geq 0} \overline{M}_{g;m+1,n}$ and define the operadic compositions via boundary maps, using only white points as above. (We sometimes write here and below n instead of $\{1,\dots,n\}$).

It seems however that this object is too big for our purposes and that it must be replaced by a smaller stack which we will define inductively, by using the Construction 1.3 which we will call here simply the adjoining of a generic black point. The components of this stack will be defined inductively.

If $g\geq 2$, $m\geq 0$, we start with $\overline{M}_{g;m} = \overline{M}_{g;m,\emptyset}$ and add n generic black points, one in turn. Denote the resulting stack by $\overline{L}_{g;m,n}$.

For $g=1$, one should add one more sequence of stacks, corresponding to $m=0$. Since we want to restrict ourselves to Deligne–Mumford stacks, we start at $\overline{M}_{1;0,1}$ identified with $\overline{M}_{1;1}$ (see 1.2 a)), and add black points to get the sequence $\overline{L}_{1;0,n}$, $n\geq 1$. These spaces are needed to serve as targets for the clutching morphisms gluing x_0 to x_∞ on the same curve of genus zero: cf. below.

Finally, for $g=0$ we obtain our series of spaces $\overline{L}_n = \overline{L}_{0;2,n}$, $n\geq 1$ and moreover $\overline{L}_{0;m,n}$, for all $m\geq 3, n\geq 0$.

4.3.1. Combinatorial types of fibers. Let us remind that combinatorial types of classical (semi)stable curves with (only white) points labeled by a finite set A are isomorphism classes of graphs, whose vertices are labeled by "genera" $g\geq 0$ and tails are bijectively labeled by elements of A. Stability means that vertices of genus 0 bound ≥ 3 flags, and vertices of genus 1 bound ≥ 1 flags. Graphs can have edges with only one vertex, that is, simple loops. See [Ma], III.2 for more details.

Starting with such a graph Γ, or rather with its geometric realization, we can obtain an infinite series of graphs, which will turn out to be exactly combinatorial types of (semi)stable (A,B)–pointed curves that are fibers of the families described above. Namely, subdivide edges and tails of Γ by a finite set of new vertices of genus zero (on each edge or tail, this set may be empty). If a tail was subdivided, move the respective label (from A) to the newly emerged tail. Distribute the black tails labeled by elements of B arbitrarily among the old and the new vertices. Call the resulting graph *a stringy stable combinatorial type* if it becomes stable after repainting black tails into white ones. Clearly, new vertices depict strings of $\mathbf{P}^1$'s stabilized by black points and eventually two special points on the end components.

4.3.2. Theorem. *a) $\overline{L}_{g;m,n}$ is the Deligne–Mumford stack classifying (m,n)–pointed curves of genus g of stringy stable combinatorial types. It is proper and smooth.*

b) Therefore, one can define boundary morphisms gluing two white points of two different curves

$$\overline{L}_{g_1;m_1+1,n_1}\times\overline{L}_{g_2;m_2+1,n_2}\to\overline{L}_{g_1+g_2;m_1+m_2+1,n_1+n_2}$$

and gluing two white points of the same curve:

$$\overline{L}_{g;m+1,n} \to \overline{L}_{g+1,m-1;n}$$

such that the locally finite DM–stacks

$$\widetilde{L}_{g,m+1} := \coprod_{n\geq 0} \overline{L}_{g;m+1,n}$$

will form components of a modular operad.

The statement a) can be proved in the same way as the respective statement 2.2 a).

It remains to see whether one can develop an extension of the Gromov–Witten invariants, preferably with descendants, to this context. The Remark 3.2.3 seems promising in this respect.

Appendix. Proof of the Technical Lemma

We break the proof into several steps whose content is indicated in the title of the corresponding subsection. An advice for the reader who might care to check the details: the most daunting task is to convince oneself that none of the alternatives has been inadvertently omitted.

A.1. The right hand side of (2.24) does not depend on the choice of i, j.

We must check that a different choice leads to the same answer modulo relations (2.23). We can pass from one choice to another by consecutively replacing only one element of the pair. Consider, say, the passage from (i,j) to (i',j). Form the difference of the right hand sides of (2.24) written for (i',j) and for (i,j).

In this difference, the terms corresponding to the partitions β will cancel. The remaining terms will correspond to the partitions α of τ_a which separate i and i'. Their difference will vanish in H_{*B} because of (2.23).

A.2. Multiplications by l_σ are compatible with linear relations (2.23) between $\mu(\tau)$.

Choose and fix one linear relation (2.23), that is, a quadruple $(\tau,\tau_a,i,j\in\tau_a)$, $i\neq j$. Choose also a 2–partition σ. We want to check that after multiplying the left hand side of (2.23) by l_σ according to the prescriptions (2.23)–(2.26) we will get zero modulo all relations of the type (2.23). There are several basic cases to consider.

(i) σ breaks τ at τ_b, $b\neq a$. Then put $\tau'=\sigma*\tau$. After multiplication we will get again (2.23) written for τ' and one of its components τ_a.

(ii) σ breaks τ at τ_a. Let (τ_{a1},τ_{a2}) be the induced partition; it is now fixed. We must analyze $l_\sigma\mu(\tau(\alpha))$ for variable 2–partitions α of τ_a with $i\alpha j$ or $j\alpha i$.

Those α which do not break (τ_{a1},τ_{a2}) will contribute zero because of (2.26).

Those α which break (τ_{a1},τ_{a2}) will produce a 3–partition of τ_a, say $(\tau_{a11},\tau_{a12},\tau_{a2})$ or else $(\tau_{a1},\tau_{a21},\tau_{a22})$. Finally, there will be one α which is induced by σ that is, coincides with (τ_{a1},τ_{a2}). We must show that the sum total of the respective terms vanishes. However, the pattern of cancellation will depend on the positions of i and j. In order to present the argument more concisely, we will first introduce the numerotation of all possible positions *with respect to a variable α* as follows. Partitions which break (τ_{a1},τ_{a2}) with $i\alpha j$:

$$\mathrm{(I)}:\ i\in\tau_{a11},\,j\in\tau_{a12}\qquad \mathrm{(II)}:\ i\in\tau_{a11},\,j\in\tau_{a2}$$

$$\mathrm{(III)}:\ i\in\tau_{a1},\,j\in\tau_{a22}\qquad \mathrm{(IV)}:\ i\in\tau_{a21},\,j\in\tau_{a22}$$

Partitions which break (τ_{a1}, τ_{a2}) and satisfy $j\alpha i$ will be denoted similarly, but with prime. Say, (III)$'$ means (III) with positions of i and j reversed.

Now we will explain the patterns of cancellation depending on the positions of i, j with respect to σ. Recall that this latter data is fixed and determined by the choices we made at the beginning of this subsection.

If $i, j \in \tau_{a1}$, the only non–vanishing terms are of the types (I) and (I)$'$. Their sum over all α will vanish because of (2.23). Similarly, if $i, j \in \tau_{a2}$, (IV) and (IV)$'$ will cancel, and everything else will vanish.

Finally, assume that $i \in \tau_{a1}$, $j \in \tau_{a2}$. that is, σ separates i, j. Then we may have non–vanishing terms of the types (II) and (III) and in addition the terms coming from (the partition of τ_a induced by) σ which must be treated using the formula (2.24), applied however to $(\tau_1, \dots, \tau_{a-1}, \tau_{a1}, \tau_{a2}, \tau_{a+1}, \dots)$ in place of τ. Half of these latter terms (with $i \in \tau_{a11}$) will cancel (II), whereas the other half (with $j \in \tau_{a22}$) will cancel (III).

The case $j \in \tau_{a1}$, $i \in \tau_{a2}$ is treated similarly.

(iii) σ breaks τ between τ_b and τ_{b+1}. In this case σ breaks any $\tau(\alpha)$ in (2.23) between two neighbors as well. A contemplation will convince the reader that only the cases $b = a - 1$ and $b = a$ may present non–obvious cancellations. Let us treat the first one; the second one is simpler.

For $\alpha = (\tau_{a1}, \tau_{a2})$ we will calculate each term $l_\sigma \mu(\tau(\alpha))$ using a formula of the type (2.24), first choosing some $k \in \tau_{a-1}$, $l \in \tau_{a1}$ (in place of i, j of (2.24): these letters are already bound). The choice of k does not matter, but we will choose $l = i$ if $i\alpha j$, and $l = j$ if $j\alpha i$. We get then for $i\alpha j$:

$$l_\sigma \mu(\tau(\alpha)) = l_\sigma \mu(\dots \tau_{a-1} \tau_{a1} \tau_{a2} \dots)$$

$$= - \sum_{\beta:\, k \in \tau_{a-1,1}} \mu(\dots \tau_{a-1,1} \tau_{a-1,2} \tau_{a1} \tau_{a2} \dots) - \sum_{\gamma:\, i \in \tau_{a12}} \mu(\dots \tau_{a-1} \tau_{a11} \tau_{a12} \tau_{a2} \dots) \tag{A.1}$$

where β runs over 2–partitions of τ_{a-1} and γ runs over 2–partitions of τ_{a1}. Write now a similar expression for $j\alpha i$ (with the choice $l = j$). The second sum in this expression will term–by–term cancel the second sum in (A.1), because our choices force $i \in \tau_{a12}$, $j \in \tau_{a2}$ in both cases.

If we sum first over α, we will see that the first two sums cancel modulo relations (2.23) because our choices imply $i \in \tau_{a1}$, $j \in \tau_{a2}$ in the first sum of (A.1) and the reverse relation in the first sum written for $j\alpha i$.

(iv) σ does not break τ. In this case we choose a bad pair (τ_b, τ_{b+1}) for σ and τ (see Lemma 2.4.2(iii)). One easily sees that if $a \neq b$, $b + 1$, then it remains a bad

pair for σ and $\tau(\alpha)$ for any α in (2.23). Therefore, l_σ annihilates all terms of (2.23) in view of (2.26).

We will show that in the exceptional cases we still can find a bad pair for σ and $\tau(\alpha)$, but it will depend on $\alpha = (\tau_{a1}, \tau_{a2})$, which does not change the remaining argument.

Assume that $b = a$ that is, $\tau_a \setminus \sigma_1 \neq \emptyset, \tau_{a+1} \cap \sigma_1 \neq \emptyset$ (see (2.12)). Then (τ_{a2}, τ_{a+1}) is a bad pair for σ and $\tau(\alpha)$, unless $\tau_{a2} \subset \sigma_1$, in which case σ_1 cannot contain τ_{a1} so that (τ_{a1}, τ_{a2}) form a bad pair.

Similarly, if $b = a - 1$, then τ_{a-1}, τ_{a1} will be a bad pair unless $\tau_{a1} \cap \sigma_1 = \emptyset$, in which case (τ_{a1}, τ_{a2}) will be a bad pair.

By this time we have checked that multiplications by l_σ are well defined linear operators on the space H_{*B}. We will now prove that they pairwise commute and therefore define an action of $\mathcal{R}_B$ upon H_{*B}.

A.3. Multiplications by l_σ pairwise commute.

We start with fixing τ, $\sigma^{(1)}$ and $\sigma^{(2)}$. We want to check that

$$l_{\sigma^{(1)}}(l_{\sigma^{(2)}}\mu(\tau)) = l_{\sigma^{(2)}}(l_{\sigma^{(1)}}\mu(\tau)).$$

We may and will assume that $\sigma^{(1)} \neq \sigma^{(2)}$. The following alternatives can occur for $\sigma^{(1)}$ and $\sigma^{(2)}$ separately:

(i) $\sigma^{(1)}$ *breaks* τ *at* τ_a.
(ii) $\sigma^{(1)}$ *breaks* τ *between* τ_a *and* τ_{a+1}.
(iii) $\sigma^{(1)}$ *does not break* τ.

(i)′ $\sigma^{(2)}$ *breaks* τ *at* τ_b.
(ii)′ $\sigma^{(2)}$ *breaks* τ *between* τ_b *and* τ_{b+1}.
(iii)′ $\sigma^{(2)}$ *does not break* τ.

We will have to consider the combined alternatives (i)(i)′, (i)(ii)′, … , (iii)(iii)′ in turn. The symmetry of $\sigma^{(1)}$ and $\sigma^{(2)}$ allows us to discard a few of them.

Subcase (i)(i)′

We will first assume that $a \neq b$, say $a < b$. Denote by α (resp. β) the partition induced by $\sigma^{(1)}$ (resp. $\sigma^{(2)}$) on τ_a (resp. τ_b). Then

$$l_{\sigma^{(1)}}(l_{\sigma^{(2)}}\mu(\tau)) = l_{\sigma^{(2)}}(l_{\sigma^{(1)}}\mu(\tau)) = \mu(\tau(\alpha)(\beta)) = \mu(\tau(\beta)(\alpha)).$$

Now assume that $a = b$. If α breaks β, we will again have the desired equality, because $\alpha * \beta == \beta * \alpha$. If α does not break β, the both sides will vanish.

After having treated this subcase, we add one more remark which will be used below, in the subsection A.5. Namely, α does not break β exactly when $\sigma^{(1)}$ does not break $\sigma^{(2)}$. Therefore, if $l_{\sigma^{(1)}}l_{\sigma^{(2)}}$ is one of the quadratic generators of I_B, then consecutive multiplication by the respective elements annihilates $\mu(\tau)$.

Subcase (i)(ii)$'$

If $a\neq b$, $b+1$, the modifications induced in τ by the two multiplications are made in mutually disjoint places and therefore commute as above. Consider now the case $a=b$, the case $a=b+1$ being similar.

Denote by (τ_{a1},τ_{a2}) the partition induced by σ on τ_a. Then we have

$$l_{\sigma^{(1)}}\mu(\tau)=\mu(\dots\tau_{a-1}\tau_{a1}\tau_{a2}\tau_{a+1}\dots)=\mu(\tau').$$

Clearly, $\sigma^{(2)}$ breaks τ' between τ_{a2} and τ_{a+1} so that, after choosing $i\in\tau_{a2}, j\in\tau_{a+1}$ we have

$$l_{\sigma^{(2)}}(l_{\sigma^{(1)}}\mu(\tau))=-\sum_{\alpha:\, i\alpha}\mu(\tau'(\alpha))-\sum_{\beta:\,\beta j}\mu(\tau'(\beta)) \qquad (A.2)$$

where α runs over 2–partitions of τ_{a2} and β runs over 2–partitions of τ_{a+1}.

On the other hand, with the same choice of i, j we have

$$l_{\sigma^{(2)}}\mu(\tau)=-\sum_{\gamma:\, i\gamma}\mu(\tau(\gamma))-\sum_{\beta:\,\beta j}\mu(\tau(\beta)) \qquad (A.3)$$

where γ runs over 2–partitions of τ_a and β runs over 2–partitions of τ_{a+1}. After multiplication of (A.3) by $l_{\sigma^{(1)}}$, the second sum in (A.3) will become the second sum of (A.2). In the first sum, only partitions δ breaking (τ_{a1},τ_{a2}) will survive, and they will produce exactly the first sum in (A.2).

Subcase (i)(iii)$'$

Here $\sigma^{(1)}$ breaks τ at τ_a, and there exists a bad pair (τ_b,τ_{b+1}) for $\sigma^{(2)}$ and τ. Since $l_{\sigma^{(2)}}\mu(\tau)=0$, it remains to check that $l_{\sigma^{(2)}}(l_{\sigma^{(1)}}\mu(\tau))=0$. But $l_{\sigma^{(1)}}\mu(\tau)=\mu(\tau')$ as in the previous subcase. So it remains to find a bad pair for $\sigma^{(2)}$ and τ'.

If $a\neq b,b+1$, then (τ_b,τ_{b+1}) will will be such a bad pair

If $a=b$, denote by (τ_{a1},τ_{a2}) the partition of τ_a induced by $\sigma^{(1)}$. If τ_{a2} is not contained in $\sigma^{(2)}$, (τ_{a2},τ_{a+1}) will form a bad pair. Otherwise this role will pass to (τ_{a1},τ_{a2}).

Finally, consider the case when $a=b+1$. In the previous notation, if $\sigma_1^{(2)}\cap\tau_{a1}\neq\emptyset$, then (τ_{a-1},τ_{a1}) is the bad pair we are looking for, otherwise we should take (τ_{a1},τ_{a2}).

Subcase (ii)(ii)$'$

Here $\sigma^{(1)}$ (resp. $\sigma^{(2)}$) breaks τ between a and $a+1$ (resp. between b and $b+1$), and $a \neq b$.

If $a \neq b-1,\ b+1$, the modifications indiced in τ by $\sigma^{(1)}$ and $\sigma^{(2)}$ do not interact and the respective multiplications commute.

By symmetry, it remains to consider the case $a = b-1$. Choose $i \in \tau_a$, $j \in \tau_{a+1}$. Summing first over partitions $\alpha = (\tau_{a1}, \tau_{a2})$ and $\beta = (\tau_{a+1,1}, \tau_{a+1,2})$ we have

$$l_{\sigma^{(1)}}\mu(\tau) = -\sum_{\alpha\colon i\alpha} \mu(\dots \tau_{a1}\tau_{a2}\dots) - \sum_{\beta\colon \beta j} \mu(\dots \tau_{a+1,1}\tau_{a+1,2}\dots).$$

Now, $\sigma^{(2)}$ will break the terms of the first (resp. second) sum between τ_{a+1} and τ_{a+2} (resp. between $\tau_{a+1,2}$ and τ_{a+2}). In order to multiply by $l_{\sigma^{(2)}}$ each term of these sums we choose the same $j \in \tau_{a+1}$ and some $l \in \tau_{a+2}$. Below we sum additionally over 2–partitions $\beta = (\tau_{a+1,1}, \tau_{a+1,2})$ and $\gamma = (\tau_{a+2,1}, \tau_{a+2,2})$ in the first two sums. In the second two sums the respective notation is $\beta' = (\tau_{a+1,2,1}, \tau_{a+1,2,2})$:

$$\begin{aligned} &l_{\sigma^{(2)}}(l_{\sigma^{(1)}}\mu(\tau)) = \\ &= \sum_{\substack{\alpha\colon i\alpha \\ \beta\colon j\beta}} \mu(\dots \tau_{a1}\tau_{a2}\tau_{a+1,1}\tau_{a+1,2}\dots) + \sum_{\substack{\alpha\colon i\alpha \\ \gamma\colon \gamma l}} \mu(\dots \tau_{a1}\tau_{a2}\tau_{a+1}\tau_{a+2,1}\tau_{a+2,2}\dots) \\ &+ \sum_{\substack{\beta\colon \beta j \\ \beta'\colon j\beta'}} \mu(\dots \tau_{a+1,1}\tau_{a+1,2,1}\tau_{a+1,2,2}\tau_{a+2}\dots) + \sum_{\substack{\beta\colon \beta j \\ \gamma\colon \gamma l}} \mu(\dots \tau_{a+1,1}\tau_{a+1,2}\tau_{a+2,1}\tau_{a+2,2}\dots) \end{aligned} \tag{A.4}$$

On the other hand, with the same notation we have:

$$l_{\sigma^{(2)}}\mu(\tau) = -\sum_{\beta\colon j\beta} \mu(\dots \tau_{a+1,1}\tau_{a+1,2}\dots) - \sum_{\gamma\colon \gamma l} \mu(\dots \tau_{a+2,1}\tau_{a+2,2}\dots)$$

and

$$\begin{aligned} &l_{\sigma^{(1)}}(l_{\sigma^{(2)}}\mu(\tau)) = \\ &= \sum_{\substack{\alpha\colon i\alpha \\ \beta\colon j\beta}} \mu(\dots \tau_{a1}\tau_{a2}\tau_{a+1,1}\tau_{a+1,2}\dots) + \sum_{\substack{\beta\colon j\beta \\ \beta''\colon \beta'' j}} \mu(\dots \tau_{a+1,1,1}\tau_{a+1,1,2}\tau_{a+1,2}\dots) \\ &+ \sum_{\substack{\alpha\colon i\alpha \\ \gamma\colon \gamma l}} \mu(\dots \tau_{a1}\tau_{a2}\tau_{a+1}\tau_{a+2,1}\tau_{a+2,2}\dots) + \sum_{\substack{\beta\colon \beta j \\ \gamma\colon \gamma l}} \mu(\dots \tau_{a+1,1}\tau_{a+1,2}\tau_{a+2,1}\tau_{a+2,2}\dots) \end{aligned} \tag{A.5}$$

where $\beta'' = (\tau_{a+1,1,1}, \tau_{a+1,1,2})$. Three of the four sums in (A.4) and (A.5) obviously coincide. The third sum in (A.4) coincides with the second sum in (A.5) because both are taken over 3–partitions of τ_{a+1} with j in the middle part.

Subcase (ii)(iii)$'$

Here $\sigma^{(1)}$ breaks τ between a and $a+1$, $\sigma^{(2)}$ does not break τ. We must check that $l_{\sigma^{(2)}}(l_{\sigma^{(1)}}\mu(\tau)) = 0$, by finding a bad pair for $\sigma^{(2)}$ and each term in the right hand side of

$$l_{\sigma^{(1)}}\mu(\tau) = -\sum_{\alpha:\, i\alpha} \mu(\dots\tau_{a1}\tau_{a2}\dots) - \sum_{\beta:\, \beta j} \mu(\dots\tau_{a+1,1}\tau_{a+1,2}\dots).$$

Denote by (τ_b, τ_{b+1}) a bad pair for $\sigma^{(2)}$ and τ. As in the subcase (i)(iii)$'$, it will remain the bad pair unless $b \in \{a-1, a, a+1\}$, and will change somewhat in the exceptional cases.

More preciasely, if $b = a-1$, then for the terms of the second sum (τ_{a-1}, τ_a) will be bad. For the first sum, if $\sigma_1^{(2)} \cap \tau_{a1} \neq \emptyset$, the bad pair will be (τ_{a-1}, τ_{a1}). Otherwise it will be (τ_{a1}, τ_{a2}).

If $b = a$, then for the terms of the first sum (τ_{a2}, τ_{a+1}) will be bad. For the second sum, if $\sigma_1^{(2)} \cap \tau_{a+1,1} \neq \emptyset$, the bad pair will be $(\tau_a, \tau_{a+1,1})$. Otherwise it will be $(\tau_{a+1,1}, \tau_{a+1,2})$.

Finally, if $b = a+1$, then for the terms of the first sum (τ_{a+1}, τ_{a+2}) will be bad. For the second sum, it will be $(\tau_{a+1,2}, \tau_{a+2})$.

In the last remaining case (iii)(iii)$'$ both multiplications produce zero.

To complete the proof of the Technical Lemma, it now remains to check that the elements (2.14), (2.15) generating I_B annihilate H_{*B}.

A.4. Elements $r_{ij}^{(1)}$ annihilate H_{*B}.

Fix i, j and a partition τ. If τ does not separate i and j, we have $i, j \in \tau_a$ for some a, and then

$$\begin{aligned} r_{ij}^{(1)}\mu(\tau) &= \left(\sum_{\sigma:\, i\sigma j} l_\sigma - \sum_{\sigma:\, j\sigma i} l_\sigma\right)\mu(\tau) \\ &= \sum_{\alpha:\, i\alpha j} \mu(\tau(\alpha)) - \sum_{\alpha:\, j\alpha i} \mu(\tau(\alpha)) \qquad (A.6) \end{aligned}$$

where α runs over partitions of τ_a. This expression vanishes because of (2.23).

Assume now that τ separates i and j, say, $i \in \tau_a, j \in \tau_b$, $a < b$. In this case $l_\sigma\mu(\tau) = 0$ for all σ with $j\sigma i$. The remaining terms of (A.6) vanish unless σ breaks

τ at some τ_c, $a \le c \le b$, or else between τ_c and τ_{c+1} for $a \le c \le b-1$. In the latter cases each term corresponding to one σ can be replaced by a sum of terms corresponding to the 2–partitions α_c of τ_c with the help of (2.24) and (2.25).

Let us choose $k_c \in \tau_c$ for all $a \le c \le b$ so that $k_a = i$, $k_b = j$ and spell out the resulting expression:

$$\sum_{\sigma:\, i\sigma j} l_\sigma \mu(\tau) = {\sum_{c=a}^{c=b}}' \left(\sum_{\alpha_c:\, k_c\alpha_c} + \sum_{\alpha_c:\, \alpha_c k_c} \right) \mu(\tau(\alpha_c))$$

$$- \sum_{c=a}^{c=b-1} \left(\sum_{\alpha_c:\, k_c\alpha_c} + \sum_{\alpha_{c+1}:\, \alpha_{c+1}k_{c+1}} \right) \mu(\tau(\alpha_{c+1}))\,.$$

Here prime at the first sum indicates that the terms with $\alpha_a i$ and $j\alpha_b$ should be skipped.

All the terms of this expression cancel.

A.5. Elements $r^{(2)}(\sigma^{(1)},\sigma^{(2)})$ annihilate H_{*B}.

These elements correspond to the pairs $(\sigma^{(1)}, \sigma^{(2)})$ that do not break each other. If at least one of them, say $\sigma^{(1)}$, does not break τ either, then $l_{\sigma^{(1)}}\mu(\tau) = 0$ so that $r^{(2)}(\sigma^{(1)},\sigma^{(2)})\mu(\tau) = 0$. If both $\sigma^{(1)}$, $\sigma^{(2)}$ break τ, a contemplation will convince the reader that they must break τ at one and the same component τ_a. This is the subcase (i)(i)$'$ of A.3, and we made the relevant comment there.

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