ON $[L]$-HOMOTOPY GROUPS

A. V. KARASEV

Abstract. The paper is devoted to investigation of some properties of $[L]$-homotopy groups. It is proved, in particular, that for any finite CW-complex L, satisfying double inequality $[S^n] < [L] \leq [S^{n+1}]$, $\pi_n^{[L]}(S^n) = \mathbb{Z}$. Here $[L]$ denotes extension type of complex L and $\pi_n^{[L]}(X)$ denotes n-th $[L]$-homotopy group of X.

1. Introduction

A new approach to dimension theory, based on notions of extension types of complexes and extension dimension leads to appearence of $[L]$-homotopy theory which, in turn, allows to introduce $[L]$-homotopy groups (see [1]). Perhaps the most natural problem related to $[L]$-homotopy groups is a problem of computation. It is necessary to point out that $[L]$-homotopy groups may differ from usual homotopy groups even for complexes.

More specifically the problem of computation can be stated as follows: describe $[L]$-homotopy groups of a space X in terms of usual homotopy groups of X and homotopy properties of complex L.

The first step on this way is apparently computation of n-th $[L]$-homotopy group of S^n for complex whose extension type lies between extension types of S^n and S^{n+1}.

In what follows we, in particular, perform this step.

2. Preliminaries

Follow [1], we introduce notions of *extension types of complexes, extension dimension,* $[L]$*-homotopy,* $[L]$*-homotopy groups* and other related notions.

We also state Dranishnikov's theorem, characterizing extension properties of complex [2].

All spaces are polish, all complexes are countable finitely-dominated CW complexes.

1991 *Mathematics Subject Classification.* Primary: 55Q05; Secondary: 54C20.
Key words and phrases. Extension dimension, $[L]$-homotopy.

For spaces X and L, the notation $L \in \mathrm{AE}(X)$ means, that every map $f : A \to L$, defined on a closed subspace A of X, admits an extension $\bar{f}$ over X.

Let L and K be complexes. We say (see [1]) that $L \le K$ if for each space X from $L \in \mathrm{AE}(X)$ follows $K \in \mathrm{AE}(X)$. Equivalence classes of complexes with respect to this relation are called *extension types*. By $[L]$ we denote extension type of L.

Definition 2.1. ([1]). The extension dimension of a space X is extension type $\mathrm{ed}(X)$ such that $\mathrm{ed}(X) = \min\{[L] : L \in \mathrm{AE}(X)\}$.

Observe, that if $[L] \le [S^n]$ and $\mathrm{ed}(X) \le [L]$, then $\dim X \le n$.

Now we can give the following

Definition 2.2. [1] We say that a space X is *an absolute (neighbourhood) extensor modulo* L (shortly X is $\mathrm{A(N)E}([L])$) and write $X \in \mathrm{A(N)E}([L])$ if $X \in \mathrm{A(N)E}(Y)$ for each space Y with $\mathrm{ed}(X) \le [L]$.

Definition of $[L]$-homotopy and $[L]$-homotopy equivalence [1] are essential for our consideration:

Definition 2.3. Two maps $f_0, f_1 : X \to Y$ are said to be $[L]$-homotopic (notation: $f_0 \overset{[L]}{\simeq} f_1$) if for any map $h : Z \to X \times [0,1]$, where Z is a space with $\mathrm{ed}(Z) \le [L]$, the composition $(f_0 \oplus f_1)h|_{h^{-1}(X\times\{0,1\})} : h^{-1}(X \times \{0,1\}) \to Y$ admits an extension $H : Z \to Y$.

Definition 2.4. A map $f : X \to Y$ is said to be $[L]$-homotopy equivalence if there is a map $g : Y \to X$ such that the compositions gf and fg are $[L]$-homotopic to id_X and id_Y respectively.

Let us observe (see [1]) that $\mathrm{ANE}([L])$-spaces have the following $[L]$-homotopy extension property.

Proposition 2.1. *Let* $[L]$ *be a finitely dominated complex and* X *be a Polish* $\mathrm{ANE}([L])$*-space. Suppose that* A *is closed in a space* B *with* $\mathrm{ed}(B) \le [L]$*. If maps* $f, g : A \to X$ *are* $[L]$*-homotopic and* f *admits an extension* $F : B \to X$ *then* g *also admits an extension* $G : B \to X$*, and it may be assumed that* F *is* $[L]$*-homotopic to* G*.*

To provide an important example of $[L]$-homotopy equivalence we need to introduce the class of approximately $[L]$-soft maps.

Definition 2.5. [1] A map $f : X \to Y$ is said to be approximately $[L]$-soft, if for each space Z with $\mathrm{ed}(Z) \le [L]$, for each closed subset $A \subset Z$, for an open cover $\mathcal{U} \in \mathrm{cov}(Y)$, and for any two maps $g : A \to X$ and $h : Z \to Y$ such that $fg = h|_A$ there is a map $k : Z \to X$ satisfying condition $k|_A = g$ and the composition fk is $\mathcal{U}$-close to h.

Proposition 2.2. [1] *Let $f : X \to Y$ be a map between* ANE($[L]$)*-compacta and* $\mathrm{ed}(Y) \leq [L]$*. If f is approximately $[L]$-soft then f is a $[L]$-homotopy equivalence.*

In order to define $[L]$-homotopy groups it is necessary to consider an *n-th $[L]$-sphere* $S^n_{[L]}$ [1], namely, an $[L]$-dimensional ANE($[L]$) - compactum admitting an approximately $[L]$-soft map onto S^n. It can be shown that all possible choices of an $[L]$-sphere $S^n_{[L]}$ are $[L]$-homotopy equivalent. This remark, coupled with the following proposition, allows us to consider for every finite complex L, every $n \geq 1$ and for any space X, the set $\pi^{[L]}_n(X) = [S^n_{[L]}, X]_{[L]}$ endowed with natural group structure (see [1] for details).

Theorem 2.3. [1] *Let L be a finitely dominated complex and X be a finite polyhedron or a compact Hilbert cube manifold. Then there exist a $[L]$-universal* ANE($[L]$) *compactum $\mu^{[L]}_X$ with* $\mathrm{ed}(\mu^{[L]}_X) = [L]$ *and an $[L]$-invertible and approximately $[L]$-soft map $f^{[L]}_X : \mu^{[L]}_X \to X$.*

The following theorem is essential for our consideration.

Theorem 2.4. *Let L be simply-connected CW-complex, X be finite-dimensional compactum. Then $L \in \mathrm{AE}(X)$ iff $\mathrm{c}-\dim_{H_i(L)} X \leq i$ for any i.*

From the proof of Theorem 2.4 one can conclude that the following theorem also holds:

Theorem 2.5. *Let L be a CW-complex (not necessary simply-connected). Then for any finite-dimensional compactum X from $L \in \mathrm{AE}(X)$ follows that $\mathrm{c}-\dim_{H_i(L)} X \leq i$ for any i.*

3. Cohomological properties of L

In this section we will investigate some cohomological properties of complexes L satisfying condition $[L] \leq S^n$ for some n. To establish these properties let us first formulate the following

Proposition 3.1. [4] *Let (X, A) be a topological pair, such that $H_q(X, A)$ is finitely generated for any q. Then free submodules of $H^q(X, A)$ and $H_q(X, A)$ are isomorphic and torsion submodules of $H^q(X, A)$ and $H_{q-1}(X, A)$ are isomorphic.*

Now we use Theorem 2.5 to obtain the following lemma.

Lemma 3.2. *Let L be finite CW complex such that $[L] \leq [S^{n+1}]$ and n is minimal with this property. Then for any $q \leq n$ $H_q(L)$ is torsion group.*

Proof. Suppose that there exists $q \le n$ such that $H^q(L) = \mathbb{Z} \oplus G$. To get a contradiction let us show that $[L] \le [S^q]$. Consider X such that $L \in \mathrm{AE}(X)$. Observe, that X is finite-dimensional since $[L] \le [S^{n+1}]$ by our assumption.

Denote $H = H_q(L)$. By Theorem 2.5 we have $\mathrm{c}-\dim_H X \le q$. Hence, for any closed subset $A \subseteq X$ we have $H^{q+1}(X, A; H) = \{0\}$. From the other hand, univeral coefficients formula implies that
$H^{q+1}(X, A) \approx H^{q+1}(X, A) \otimes H \oplus \mathrm{Tor}(H^{q+2}(X, A), H)$.

Hence, $H^{q+1}(X, A) \otimes H = \{0\}$. Observe, however, that by our assumtion we have $H^{q+1}(X, A) \otimes H = H^{q+1} \otimes (\mathbb{Z} \oplus G) = H^{q+1}(X, A) \oplus (H^{q+1}(X, A) \otimes G)$. Therefore, $H^{q+1}(X, A) = 0$.

From the last fact we conclude that $\mathrm{c}-\dim X \le q$ and therefore since X is finite-dimensional, $\dim X \le q$ which iplies $S^q \in \mathrm{AE}(X)$. □

From this lemma and Proposition 3.1 we obtain

Corollary 3.3. *In the same assumptions $H^q(L)$ is torsion group for any $q \le n$.*

The following fact is essential for constraction of compacts with some specific properties which we are going to construct further.

Lemma 3.4. *Let L be as in previous lemma. For any m there exists $p \ge m$ such that $H^q(L; \mathbb{Z}_p) = \{0\}$ for any $q \le n$.*

Proof. From Corollary 3.3 we can conclude that $H^q(L) = \bigoplus\limits_{i=1}^{l_k} \mathbb{Z}_{m_{qi}}$ for any $q \le n$. Additionally, let $\mathrm{Tor}\, H^{n+1}(L) = \bigoplus\limits_{i=1}^{l_{n+1}} \mathbb{Z}_{m_{(n+1)i}}$

For any m consider $p \ge m$ such that (p, m_{ki}) for every $k = 1 \dots n+1$ and $i = 1 \dots l_k$. Universal coefficients formula implies that $H^q(L; \mathbb{Z}_p) = \{0\}$ for every $k \le n$. □

Finally let us proof the following

Lemma 3.5. *Let X be a metrizable compactum, A be a closed subset of X. Consider a map $f : A \to S^n$. If there exists extension $\bar{f} : X \to S^n$ then for any k we have $\delta^*_{X,A}(f^*(\zeta)) = 0$ in group $H^{n+1}(X, A; \mathbb{Z}_k)$, where ζ is generator in $H^n(S^n, \mathbb{Z}_k)$.*

Proof. Let $\bar{f}$ be an extension of f. Commutativity of the following diagram implies assertion of lemma:

$$\begin{array}{ccc} H^n(A;\mathbb{Z}_k) & \xrightarrow{\delta^*_{X,A}} & H^{n+1}(X,A;\mathbb{Z}_k) \\ \Big\uparrow \bar{f}^*=f^* & & \Big\uparrow \bar{f}^* \\ H^n(S^n;\mathbb{Z}_k) & \xrightarrow{\delta^*_{S^n,S^n}} & H^{n+1}(S^n,S^n;\mathbb{Z}_k)=\{0\} \end{array}$$

□

4. Some properties of [L]-homotopy groups

In this section we will investigate some properties of $[L]$-homotopy groups.

From this point and up to the end of the text we consider finite complex L such that $[S^n] < [L] \leq [S^{n+1}]$ for some fixed n.

Remark 4.1. Let us observe that for such complexes $S^n_{[L]}$ is $[L]$-homotopic equivalent to S^n (see Proposition 2.2). Therefore for any X $\pi^{[L]}_n(X)$ is isomorphic to $G = \pi_n(S^n)/N([L])$ where $N([L])$ denotes the relation of $[L]$-homotopic equivalence between elements of $\pi_n(S^n)$.

From this observation one can easely obtain the following fact.

Proposition 4.1. *For $\pi^{[L]}_n(S^n)$ there are three variants: $\pi^{[L]}_n(S^n) = \mathbb{Z}$, $\pi^{[L]}_n(S^n) = \mathbb{Z}_m$ for some integer m or this group is trivial.*

Let us characterize the hypothetical equality $\pi^{[L]}_n(S^n) = \mathbb{Z}_m$ in terms of extensions of maps.

Proposition 4.2. *If $\pi^{[L]}_n(S^n) = \mathbb{Z}_m$ then for any X such that $\mathrm{ed}(X) \leq [L]$, for any closed subset A of X and for any map $f : A \to S^n$, there exists extension $\bar{h} : X \to S^m$ of composition $h = z_m f$, where $z_m : S^n \to S^n$ is a map having degree m.*

Proof. Suppose, that $\pi^{[L]}_n(S^n) = \mathbb{Z}_m$. Then from Remark 4.1 and since $[z_m] = m[\mathrm{id}_{S^n}] = [*]$ (where $[f]$ denotes homotopic class of f) we conclude that $z_m : S^n \to S^n$ is $[L]$-homotopic to constant map. Let us show that $h = z_m f : A \to S^n$ is also $[L]$-homotopic to constant map. This fact will prove our statement. Indeed, by our assumption $\mathrm{ed}(X) \leq [L]$ and $S^n \in ANE$ and therefore we can apply Proposition 2.1.

Consider Z such that $\mathrm{ed}(Z) \leq [L]$ and a map $H : Z \to A \times I$, where $I = [0,1]$. Pick a point $s \in S^n$. Let $f_0 = z_m f$, $f_1 \equiv s$ – constant map considered as $f_i : A \times \{i\} \to S^n$, $i = 0,1$.

Define $F : A \times I \to S^n \times I$ as follows: $F(a,t) = (f(a),t)$ for each $a \in A$ and $t \in I$. Let $f'_0 \equiv z_m$ and $f'_1 \equiv s$ considered as $f'_i : S^n \times \{i\} \to S^n$, $i = 0,1$.

Consider a composition $G = FH : Z \to S^n \times I$. By our assumption f'_0 is $[L]$-homotopic to f'_1. Therefore a map $g : G^{-1}(S^n \times \{0\} \bigcup S^n \times \{1\}) \to S^n$, defined as $g|_{G^{-1}(S^n\times\{i\})} = f'_i G$ for $i = 0, 1$, can be extended over Z. From the other hand we have $G^{-1}(S^n \times \{i\}) \equiv H^{-1}(A \times \{i\})$ and $g|_{G^{-1}(S^n\times\{i\})} = f'_i fH = f_i$ for $i = 0, 1$. This remark completes the proof. □

Now consider a special case of complex having a form $S^n < L = K_s \vee K \leq S^{n+1}$, where K_s is a complex obtained by attaching to S^n a $(n+1)$-dimensional cell using a map of degree s.

Proposition 4.3. *Let $[\alpha] \in \pi_n(X)$ be an element of order s. Then α is $[L]$-homotopy to constant map.*

Proof. Observe that simillar to proof of Proposition 4.2 it is enough to show that for every Z with $\mathrm{ed}(Z) \leq [L]$, for every closed subspace A of Z and for any map $f : Z \to S^n$ a composition $\alpha f : A \to X$ can be extended over Z.

Let $g : S^n \to K_s^{(n)}$ be an embedding (by $M^{(n)}$ we denote n-dimensional skeleton of complex M) and $r : L \to K_s$ be a retraction.

Since $\mathrm{ed}(Z) \leq [L]$, a composition gf has an extension $F : Z \to L$. Let $F' = rF$ and α' be a map α considered as a map $\alpha' : K_s^{(n)} \to X$. Observe that $\alpha' F'$ is a necessary extension of αf. □

5. Computation of $\pi_n^{[L]}(S^n)$

In this section we will prove that $\pi_n^{[L]}(S^n) = \mathbb{Z}$.

Suppose the oppsite, i.e. $\pi_n^{[L]}(S^n) = \mathbb{Z}_m$ (we use Proposition 4.1; the same arguments can be used to prove that $\pi_n^{[L]}(S^n)$ is non-trivial).

To get a contradiction we need to obtain a compact with special extension properties. We will use a construction of [3]

Let us recall the following definition.

Definition 5.1. [3] Inverse sequence $S = \{X_i, p_i^{i+1} : i \in \omega\}$ consisting of metrizable compacta is said to be L-resolvable if for any i, $A \subseteq X_i$ - closed subspace of X_i and any map $f : A \to L$ there exists $k \leq i$ such that composition $fp_i^k : (p_i^k)^{-1}A \to L$ can be extended over X_k.

The following lemma (see [3]) expresses an important property of $[L]$-resolvable inverse sequences.

Lemma 5.1. *Suppose that L is a countable complex and that X is a compactum such that $X = limS$ where $S = (X_i, \lambda_i), q_i^{i+1}$ is a L-resolvable inverse system of compact polyhedra X_i with triangulations λ_i such that* $\mathrm{mesh}\{\lambda_i\} \to 0$. *Then $L \in \mathrm{AE}(X)$*

Let us recall that in [3] inverse sequence $S = \{(X_i, \tau_i), p_i^{i+1}\}$ was constructed such that X_i is compact polyhedron with fixed triangulation τ_i, $X_0 = S^{n+1}$, $\operatorname{mesh} \tau_i \to 0$, S is $[L]$-resolvable and for any $x \in X_i$ we have $(p_i^{i+1})^{-1}x \simeq L$ or $*$.

It is easy to see that using the same construction one can obtain inverse sequence $S = \{(X_i, \tau_i), p_i^{i+1}\}$ having the same properties with exeption of $X_0 = D^{n+1}$ where D^{n+1} is $n+1$-dimensional disk.

Let $X = \lim S$. Observe, that $\mathrm{ed}(X) \leq [L]$. Let $p_0 : X \to D^{n+1}$ be a limit projection.

Pick $p \geq m+1$ which Lemma 3.4 provides us with. By Vietoris-Begle theorem (see [4]) and our choice of p, for every i and every $X_i' \subseteq X_i$ a homomorphism $(p_i^{i+1})^* : H^k(X_i'; \mathbb{Z}_p) \to H^k((p_i^{i+1})^{-1}X_i'; \mathbb{Z}_p)$ is isomorphism for $k \leq n$ and monomorphism for $k = n+1$.

Therefore for each $D' \subseteq X_0 = D^{n+1}$ homomorphism $p_0^* : H^k(D'; \mathbb{Z}_p) \to H^k((p_0)^{-1}D'; \mathbb{Z}_p)$ is isomorphism for $k \leq n$ and monomorphism for $k = n+1$. In particular, $H^n(X; \mathbb{Z}_p) = \{0\}$ since $X_0 = D^{n+1}$ has trivial cohomology groups.

Let $A = (p_0)^{-1}S^n$ and $\zeta \in H^n(S^n; \mathbb{Z}_p) \approx \mathbb{Z}_p$ be a generator.

Since $p_0^* : H^n(S^n; \mathbb{Z}_p) \to H^n(A; \mathbb{Z}_p)$ is isomorphism, $p_0^*(\zeta)$ is generator in $H^n(A, \mathbb{Z}_p) \approx \mathbb{Z}_p$. In particular, $p_0^*(\zeta)$ is element of order p.

From exact sequence of pair (X, A)

$$\ldots \to H^n(X; \mathbb{Z}_p) = \{0\} \xrightarrow{i_{X,A}} H^n(A; \mathbb{Z}_p) \xrightarrow{\delta^*_{X,A}} H^{n+1}(X, A; \mathbb{Z}_p) \to \ldots$$

we conclude that $\delta^*_{X,A}$ is monomorphism and hence $\delta^*_{X,A}(p_0^*(\zeta)) \in H^{n+1}(X, A; \mathbb{Z}_p)$ is element of order p.

Consider now a composition $h = z_m p_0$. By our assumption this map can be extended over X (see Proposition 4.2). This fact coupled with Lemma 3.5 implies that $\delta^*_{X,A}(h^*(\zeta)) = 0$ in $H^{n+1}(X, A; \mathbb{Z}_p)$. But $\delta^*_{X,A}(h^*(\zeta)) = m\delta^*_{X,A}(p_0^*(\zeta))$. We arrive to a contradiction which shows that

Theorem 5.2. *Let L be a complex such that $[S^n] < [L] \leq [S^{n+1}]$. Then $\pi_n^{[L]}(S^n) \approx \mathbb{Z}$.*

The author is greatfull to A. C. Chigogidze for usefull discussions.

References

[1] A. Chigogidze, *Infinite dimensional topology and shape theory*, to appear in: "Handbook of Geometric Topology" edited by R. Daverman and R. B. Sher, North Holland, Amsterdam, 1999.

[2] A. N. Dranishnikov, *Extension of mappings into CW-complexes*, Math. USSR Sbornik **74** (1993), 47-56.

[3] A. N. Dranishnikov and D. Repovš, *Cohomological dimension with respect to perfect groups*, Topology Appl. **74** (1996), 123-140.
[4] E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

Department of Mathematics and Statistics, University of Saskatchewan, McLean Hall, 106 Wiggins Road, Saskatoon, SK, S7N 5E6, Canada
E-mail address: karasev@math.usask.ca