A GALOIS CORRESPONDENCE FOR II$_1$ FACTORS AND QUANTUM GROUPOIDS

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Abstract

We establish a Galois correspondence for finite quantum groupoid actions on II$_1$ factors and show that every finite index and finite depth subfactor is an intermediate subalgebra of a quantum groupoid crossed product. Moreover, any such a subfactor is completely and canonically determined by a quantum groupoid and its coideal $*$-subalgebra. This allows to express the bimodule category of a subfactor in terms of the representation category of a corresponding quantum groupoid and the principal graph as the Bratteli diagram of an inclusion of certain C^*-algebras related to it.

1 Introduction

This paper continues the research initiated in [13], where finite index II$_1$-subfactors of depth 2 were characterized in terms of weak C^*-Hopf algebra crossed products (the latter objects were introduced in [3]).

In what follows, we use the term "quantum groupoid" instead of "weak C^*-Hopf algebra" since we believe it is important to stress that this algebraic structure provides a natural non-commutative generalization of a usual finite groupoid. In particular, if it is commutative as an algebra (resp. cocommutative as a coalgebra), then it can be identified in a canonical way

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with the C^*-algebra of functions on a finite groupoid (resp. groupoid algebra). Quantum groupoids also generalize finite-dimensional Kac algebras ("finite quantum groups") [18], [12].

According to the characterization obtained in [13], if $N \subset M \subset M_1 \subset M_2 \subset \ldots$ is the Jones tower constructed from a finite index depth 2 inclusion $N \subset M$ of II_1 factors, then $B = M' \cap M_2$ has a canonical structure of a quantum groupoid acting outerly on M_1 such that $M = M_1^B$ and $M_2 = M_1 \rtimes B$, moreover $A = N' \cap M_1$ is a quantum groupoid dual to B.

In the present paper we extend the above result to show that quantum groupoids give a uniform description of arbitrary finite index and finite depth II_1-subfactors via a Galois correspondence. We also show how to express subfactor invariants such as bimodule categories and principal graphs in quantum groupoid terms.

After discussing basic definitions and constructions in Preliminaries (Section 2) we introduce and study coideal $*$-subalgebras of quantum groupoids (Section 3), that play an important role in the sequel.

Section 4 starts with a simple observation (see Proposition 4.1 and Corollary 4.2) that any finite depth subfactor $N \subset M$ ($[M : N] < \infty$) can be viewed as an intermediate for some depth 2 inclusion $N \subset \widetilde{M}$. Due to the above characterization result we have $\widetilde{M} \cong M \rtimes B$, which allows to describe $N \subset M$ via a Galois correspondence between intermediate von Neumann subalgebras of $N \subset \widetilde{M}$ and left coideal $*$-subalgebras of B (Theorem 4.3). Thus, every finite depth subfactor is completely determined by a pair (B, I), where I is a left coideal $*$-subalgebra of a quantum groupoid B, and can be realized as $N \subset N \rtimes I$, where $N \rtimes I$ is a von Neumann algebra generated by N and I inside $N \rtimes B$. Note that the Galois correspondence for quantum group actions on factors was established in [4], [6].

In Section 5 we discuss an equivalence between the tensor category of $(N - N)$-bimodules associated with $N \subset M$ and the co-representation category of B (Theorem 5.8). Given a quantum groupoid B acting on a II_1 factor N and a pair of its left coideal $*$-subalgebras H, K, we define in the spirit of [17] a category C_{H-K} of relative $(B, H - K)$ Hopf bimodules, whose objects are both B-comodules and $(H - K)$-bimodules such that the B-coaction commutes with the bimodule action and construct a functor from C_{H-K} to the category of $(N \rtimes H - N \rtimes K)$-bimodules preserving direct sums and compatible with operations of taking tensor products and adjoints. In the case when H and K are trivial, C_{H-K} is $\mathrm{Corep}(B)$, the co-representation category of B, and the above functor is an equivalence. We want to emphasize that a bimodule category of *any* finite depth subfactor $N \subset M$ (not

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only depth 2) is equivalent to $\operatorname{Corep}(B)$ for some B.

This functor also allows to express the principal graph of the inclusion $N \subset M$ in terms of B, as the Bratteli diagram of an inclusion of certain finite-dimensional C^*-algebras related to B (Proposition 5.9, Corollary 5.10).

Finally, in Appendix we explicitly write down the structure maps of a quantum groupoid associated with a finite depth subfactor.

2 Preliminaries

Throughout this paper we use Sweedler's notation for a comultiplication, writing $\Delta(b) = b_{(1)} \otimes b_{(2)}$.

A *weak Hopf C^*-algebra* [3] or *quantum groupoid B* is a finite dimensional C^*-algebra with the comultiplication $\Delta : B \to B \otimes B$, counit $\varepsilon : B \to \mathbb{C}$, and antipode $S : B \to B$ such that (B, Δ, ε) is a coalgebra and the following axioms hold for all $b, c, d \in B$:

(1) Δ is a (not necessarily unital) $*$-homomorphism :

$$\Delta(bc) = \Delta(b)\Delta(c), \quad \Delta(b^*) = \Delta(b)^*,$$

(2) The unit and counit satisfy the identities

$$
\begin{aligned}
\varepsilon(bc_{(1)})\varepsilon(c_{(2)}d) &= \varepsilon(bcd), \\
(\Delta(1) \otimes 1)(1 \otimes \Delta(1)) &= (\Delta \otimes \operatorname{id})\Delta(1),
\end{aligned}
$$

(3) S is an anti-algebra and anti-coalgebra map such that

$$
\begin{aligned}
m(\operatorname{id} \otimes S)\Delta(b) &= (\varepsilon \otimes \operatorname{id})(\Delta(1)(b \otimes 1)), \\
m(S \otimes \operatorname{id})\Delta(b) &= (\operatorname{id} \otimes \varepsilon)((1 \otimes b)\Delta(1)),
\end{aligned}
$$

where m denotes the multiplication.

The right hand sides of two last formulas define *target* and *source counital maps* :

$$\varepsilon_t(b) = (\varepsilon \otimes \mathrm{id})(\Delta(1)(b \otimes 1)), \quad \varepsilon_s(b) = (\mathrm{id} \otimes \varepsilon)((1 \otimes b)\Delta(1)),$$

and play an important role in this theory.

Let us remark that the axiom (2) of the definition of a quantum groupoid is equivalent to each of the following axioms expressed in terms of counital maps ([12], [13]) :

(2') $\qquad b\varepsilon^t(c) = \varepsilon(b_{(1)}c)b_{(2)}, \qquad b_{(1)} \otimes \varepsilon^t(b_{(2)}) = 1_{(1)}b \otimes 1_{(2)},$

(2") $\qquad \varepsilon^s(c)b = b_{(1)}\varepsilon(cb_{(2)}), \qquad \varepsilon^s(b_{(1)}) \otimes b_{(2)} = 1_{(1)} \otimes b1_{(2)},$

These axioms are convenient for concrete computations, as they show that the properties of the counital maps ε_t and ε_s are similar to those of a counit in an ordinary Hopf algebra.

The dual vector space B^* has a natural structure of a quantum groupoid given by dualizing the structure operations of B ([3], [12]) :

$$
\begin{aligned}
\langle \varphi\psi, b \rangle &= \langle \varphi \otimes \psi, \Delta(b) \rangle, \\
\langle \Delta(\varphi), b \otimes c \rangle &= \langle \varphi, bc \rangle, \\
\langle S(\varphi), b \rangle &= \langle \varphi, S(b) \rangle, \\
\langle \varphi^*, b \rangle &= \overline{\langle \varphi, S(b)^* \rangle},
\end{aligned}
$$

for all $b, c \in B$ and $\varphi, \psi \in B^*$. The unit of B^* is ε and the counit is 1.

The main difference between finite quantum groupoids and classical finite-dimensional Hopf C^*-algebras (Kac algebras) is that the images of the counital maps are, in general, non-trivial unital C^*-subalgebras of B, called *target* and *source counital subalgebras* :

$$
\begin{aligned}
B_t &= \{b \in B \mid \varepsilon_t(b) = b\} = \{b \in B \mid \Delta(b) = (b \otimes 1)\Delta(1) = \Delta(1)(b \otimes 1)\}, \\
B_s &= \{b \in B \mid \varepsilon_s(b) = b\} = \{b \in B \mid \Delta(b) = (1 \otimes b)\Delta(1) = \Delta(1)(1 \otimes b)\}.
\end{aligned}
$$

The counital subalgebras commute elementwise: $[B_t, B_s] = 0$, we also have $S \circ \varepsilon^s = \varepsilon^t \circ S$ and $S(B_t) = B_s$. We say that B is *connected* [11] if $B_t \cap Z(B) = \mathbb{C}$ (where $Z(B)$ denotes the center of B), i.e., if the inclusion $B_t \subset B$ is connected. B is connected iff $B_t^* \cap B_s^* = \mathbb{C}$ ([11], Proposition 3.11). We say that B is *biconnected* if both B and B^* are connected.

The antipode of a quantum groupoid is necessarily unique, invertible, and satisfies $(S \circ *)^2 = \mathrm{id}$. Furthermore, there exists a canonical positive

element H in the center of B_t such that S^2 is an inner automorphism implemented by $G = HS(H)^{-1}$, i.e., $S^2(b) = GbG^{-1}$ for all $b \in B$. The element G is group-like, i.e., $\Delta(G) = (G \otimes G)\Delta(1) = \Delta(1)(G \otimes G)$.

Quantum groupoids possess integrals in the following sense.

There exists a unique projection $p \in B$, called a *Haar projection*, such that for all $x \in B$:

$$xp = \varepsilon^t(x)p, \quad S(p) = p, \quad \varepsilon^t(p) = 1.$$

There exists a unique positive functional ϕ on B, called a *normalized Haar functional* (which is a trace iff B is a weak Kac algebra), such that

$$(\mathrm{id} \otimes \phi)\Delta = (\varepsilon^t \otimes \phi)\Delta, \quad \phi \circ S = S, \quad \phi \circ \varepsilon^t = \varepsilon.$$

The next proposition establishes a useful invariance property of the Haar functional.

Proposition 2.1 (cf. ([12], 2.3.5)) *The normalized Haar functional ϕ of B satisfies the following strong invariance property :*

$$x_{(1)}\phi(yx_{(2)}) = S(y_{(1)})\phi(y_{(2)}x), \qquad \phi(x_{(1)}y)x_{(2)} = \phi(xy_{(1)})S(y_{(2)}),$$

for all $x, y \in B$.

Proof. It follows from the axioms of quantum groupoid that

$$\varepsilon_t(S(c))b = \varepsilon(cb_{(1)})b_{(2)}$$

for all $b, c \in B$. Using this identity and the properties of ϕ one computes :

$$
\begin{aligned}
x_{(1)}\phi(yx_{(2)}) &= 1_{(1)}x_{(1)}\phi(y1_{(2)}x_{(2)}) \\
&= \varepsilon_s(y_{(1)})x_{(1)}\phi(y_{(2)}x_{(2)}) \\
&= S(y_{(1)})(y_{(2)}x)_{(1)}\phi((y_{(2)}x)_{(2)}) \\
&= S(y_{(1)})\varepsilon_t((y_{(2)}x)_{(1)})\phi((y_{(2)}x)_{(2)}) \\
&= S(y_{(1)})1_{(2)}\phi(\varepsilon(1_{(1)}(y_{(2)}x)_{(1)})(y_{(2)}x)_{(2)}) \\
&= S(y_{(1)})1_{(2)}\phi(\varepsilon_t(S(1_{(1)}))y_{(2)}x) \\
&= S(y_{(1)})S(1_{(1)})\phi(1_{(2)}y_{(2)}x) = S(y_{(1)})\phi(y_{(2)}x).
\end{aligned}
$$

The second identity is similar.

When B is connected, there exists a unique (non-degenerate) Markov trace ([5]) τ for the inclusion $B_t \subset B$ normalized by $\tau(1) = \dim B_t$. This trace is related to the Haar functional ϕ by $\phi(x) = \tau(HS(H)x)$ ([13], 5.7), where H is the canonical central positive element in B_t described earlier.

Corollary 2.2 *For all $x, y \in B$ we have*

$$
\begin{aligned}
x_{(1)}\tau(yx_{(2)}) &= S(y_{(1)})G\tau(y_{(2)}x), \\
\tau(x_{(1)}y)x_{(2)} &= \tau(xy_{(1)})G^{-1}S(y_{(2)}),
\end{aligned}
$$

where $G = HS(H)^{-1}$ is the canonical element implementing S^2.

Proof. From Proposition 2.1 we have

$$
x_{(1)}\tau(HS(H)yx_{(2)}) = S(y_{(1)})\tau(HS(H)y_{(2)}x),
$$

and replacing y by $HS(H)y$ we get the first identity, the second one is similar.

The following notions of action, crossed product, and fixed point subalgebra were introduced in [14]. A (left) *action* of a quantum groupoid B on a von Neumann algebra M is a linear map

$$
B \otimes M \ni b \otimes x \mapsto (b \triangleright x) \in M
$$

making M into a left B-module such that for all $b \in B$ the map $b \otimes x \mapsto (b \triangleright x)$ is weakly continuous and

(1) $b \triangleright xy = (b_{(1)} \triangleright x)(b_{(2)} \triangleright y)$,

(2) $(b \triangleright x)^* = S(b)^* \triangleright x^*$,

(3) $b \triangleright 1 = \varepsilon^t(b) \triangleright 1$, and $b \triangleright 1 = 0$ iff $\varepsilon^t(b) = 0$.

A *crossed product* algebra $M \rtimes B$ is constructed on the relative tensor product $M \otimes_{B_t} B$, where B is a left B_t-module via multiplication and M is a right B_t-module via $x \triangleleft z = S(z) \triangleright x = x(z \triangleright 1)$. Let $[x \otimes b]$ denote the class of $x \otimes b$ in $M \rtimes B$. A $*$-algebra structure on $M \rtimes B$ is defined by

$$
[x \otimes b][y \otimes c] = [x(b_{(1)} \triangleright y) \otimes b_{(2)}c], \quad [x \otimes b]^* = [(b_{(1)}^* \triangleright x^*) \otimes b_{(2)}^*]
$$

for all $x, y \in M$, $b, c \in B$. It is possible to show that this abstractly defined $*$-algebra $M \rtimes B$ is *-isomorphic to a weakly closed algebra of operators on some Hilbert space [14], i.e., $M \rtimes B$ is a von Neumann algebra.

The collection $M^B = \{x \in M \mid b \triangleright x = \varepsilon^t(b) \triangleright x, \ \forall b \in B\}$ is a von Neumann subalgebra of M, called a *fixed point subalgebra*. The relative commutant $M' \cap M \rtimes B$ always contains a *-subalgebra isomorphic to B_s [13]. The action of B is called minimal if $B_s = M' \cap M \rtimes B$.

It was shown in [13] that finite index depth 2 subfactors of II_1-factors can be characterized in terms of quantum groupoids. Namely, if $N \subset M$ is such a subfactor ($[M : N] = \lambda^{-1}$) and

$$N \subset M \subset M_1 \subset M_2 \subset \cdots$$

is the corresponding Jones tower, $M_1 = \langle M, e_1 \rangle$, $M_2 = \langle M_1, e_2 \rangle, \ldots$, where $e_1 \in N' \cap M_1$, $e_2 \in M' \cap M_2, \cdots$ are the Jones projections. The depth 2 condition means that $N' \cap M_2$ is the basic construction of the inclusion $N' \cap M \subset N' \cap M_1$. Let τ be the trace on M_2 normalized by $\tau(1) = 1$. There is a canonical non-degenerate duality form between $A = N' \cap M_1$ and $B = M' \cap M_2$ defined by

$$\langle a, b \rangle = \lambda^{-2} \tau(a e_2 e_1 H b),$$

for all $a \in A$ and $b \in B$, where H is a central element in $M' \cap M_1$ canonically defined by the property $\tau(Hz) = \text{Tr}(z)$, where $z \in M' \cap M_1$ and Tr is the trace of the regular representation of $M' \cap M_1$ on itself (in other words, H is the index [19] of $\tau|_{M' \cap M_1}$).

Using this duality, one defines the comultiplication, the counit and the antipode of B as follows :

$$\begin{aligned}
\langle a_1 \otimes a_2, \Delta(b) \rangle &= \langle a_1 a_2, b \rangle, \\
\varepsilon(b) &= \langle 1, b \rangle = \lambda^{-1} \tau(e_2 H b), \\
S(b) &= J(H b H^{-1})^* J,
\end{aligned}$$

for all $a, a_1, a_2 \in A$ and $b \in B$, where J is the canonical modular involution on $L^2(M_1)$ and $b \mapsto Jb^*J$ is a $*$-anti-automorphism of $B = M' \cap M_2$. The above expression for S follows from the explicit formula ([13], 4.5(i)).

With these operations and involution $b^\dagger = S(H)^{-1} b^* S(H)$, the $*$-algebra B becomes a biconnected quantum groupoid and A becomes its dual (see [13] for the proof).

The counital subalgebras of B are $B_s = M_1' \cap M_2$ and $B_t = M' \cap M_1$, moreover H is the canonical element of B_t. The map

$$\triangleright : B \otimes M_1 \to M_1 : b \otimes x \mapsto \lambda^{-1} E_{M_1}(b x e_2)$$

defines a left action of B on M_1, such that $M = M_1^B$ is the fixed point subalgebra for this action (here E_{M_1} denotes the τ-preserving conditional expectation on M_1) and

$$\theta : M_1 \rtimes B \to M_2 : [x \otimes b] \mapsto xS(H)^{1/2}bS(H)^{-1/2}$$

is an isomorphism of von Neumann algebras.

It is straightforward to show that the above left action of B on M_1 extends to an action $b \rhd \xi$ of B on $L^2(M_1) := L^2(M_1, \tau)$ such that

$$(b \rhd \xi, \eta) = (\xi, S(H)b^*S(H)^{-1} \rhd \eta),$$

where $b \in B, \xi, \eta \in L^2(M_1)$ (recall that the involution in $B = M_2 \cap M'$ is different from the one in M_2 - see above). This means that $L^2(M_1)$ equipped with a scalar product

$$(\xi, \eta)_{\widetilde{L^2(M_1)}} = (S(H) \rhd \xi, \eta)_{L^2(M_1)}$$

is a unitary left B-module.

One can also introduce a right action of B on $L^2(M_1)$ by setting $\xi \lhd b = S^{-1}(b) \rhd \xi$. This makes $L^2(M_1)$ a $(B_t - B_t)$- and $(B_s - B_s)$-bimodule (here and in what follows, the term bimodule means a unitary bimodule, see, e.g., [10]). Since $L^2(M_1)$ is also an $(M_1 - M_1)$-bimodule with respect to left and right multiplications, the properties of the left action of B on M_1 give:

$$b \rhd (a\xi) = (b_{(1)} \rhd a)(b_{(2)} \rhd \xi), \qquad b \rhd (\xi a) = (b_{(1)} \rhd \xi)(b_{(2)} \rhd a),$$

$$J(b \rhd \xi) = S(b)^* \rhd J\xi := J\xi \lhd b^*, \qquad (J\xi, J\eta)_{\widetilde{L^2(M_1)}} = (G \rhd \eta, \xi)_{\widetilde{L^2(M_1)}},$$

where $a \in M_1, \xi \in L^2(M_1)$ and $J : a \mapsto a^*$ is the canonical modular involution on $L^2(M_1)$.

3   Coideal $*$-subalgebras

Definition 3.1 A left (resp. right) *coideal* of a quantum groupoid B is a linear subspace $I \subset B$ such that $\Delta(I) \subset B \otimes I$ (resp. $\Delta(I) \subset I \otimes B$). A left (resp. right) *coideal $*$-subalgebra* is a unital C^*-subalgebra $I \subset B$ which is a left (resp. right) coideal.

Note that the target counital subalgebra B_t (resp. the source counital subalgebra B_s) is a left (resp. right) coideal $*$-subalgebra of B contained in every left (resp. right) coideal $*$-subalgebra $I \subset B$. It is easy to check that a linear subspace $I \subset B$ is a left (resp. right) coideal iff it is invariant under the right (resp. left) dual action of B^* iff its *annihilator* $I^0 = \{a \in B^* \mid \langle a, b \rangle = 0, \ \forall b \in I\} \subset B^*$ is a left (resp. right) ideal in B^* iff $S(I)$ is a right (resp. left) coideal. Note that if I is a left coideal $*$-subalgebra, then $u^* I u$ is a left coideal $*$-subalgebra for any unitary $u \in B_s$. In particular, there can be infinitely many non-equal conjugated coideal $*$-subalgebras.

For any quantum groupoid B, the set $\ell(B)$ of left coideal $*$-subalgebras is a lattice under the usual operations:

$$I_1 \wedge I_2 = I_1 \cap I_2, \qquad I_1 \vee I_2 = (I_1 \cup I_2)''$$

for all $I_1, I_2 \in \ell(B)$. The smallest element of $\ell(B)$ is B_t and the greatest element is B.

Proposition 3.2 *If $I \subset B$ is a left coideal $*$-subalgebra of B, then $\tilde{I} = G^{-1/2} S(I) G^{1/2}$ is a right coideal $*$-subalgebra of B. The map $I \mapsto \tilde{I}$ is an isomorphism of lattices.*

Proof. Clearly, $\tilde{I}$ is a subalgebra. Let $c \in \tilde{I}$, $c = G^{-1/2} S(b) G^{1/2}$ for some $b \in I$. Then, using the group-like propery of G, we have :

$$
\begin{aligned}
c^* &= G^{1/2} S^{-1}(b^*) G^{-1/2} = G^{-1/2} S(b^*) G^{1/2} \in \tilde{I}, \\
\Delta(c) &= G^{-1/2} S(b_{(2)}) G^{1/2} \otimes G^{-1/2} S(b_{(1)}) G^{1/2} \in \tilde{I} \otimes B,
\end{aligned}
$$

therefore $\tilde{I}$ is a $*$-invariant right coideal. It is easy to see that the map $I \mapsto \tilde{I}$ preserves the lattice structure.

We will show that $\ell(B)$ is the dual lattice of $\ell(B^*)$, i.e., $\ell(B) = \breve{\ell}(B^*)$. The following proposition describes an explicit isomorphism between these lattices.

Proposition 3.3 (cf. ([6], 4.6)) *Let $T \subset B$ be a selfadjoint subset and I be the minimal right coideal $*$-subalgebra of B containing T. Then $T' \cap B^* \subset B^* \rtimes B$ is a left coideal $*$-subalgebra of B^* and $T' \cap B^* = I' \cap B^*$.*

If we denote this coideal subalgebra by I^d, then the map $\delta : I \mapsto \tilde{I}^d$ defines a lattice anti-isomorphism between $\ell(B)$ and $\ell(B^*)$.

Proof. Obviously, $T' \cap B^*$ is a $*$-subalgebra of B^*. In order to prove that it is a left coideal, we need to show that it remains invariant under the right dual action of B, i.e., that $(x \triangleleft a)$ belongs to T' for all $a \in B$, $x \in T'$. The latter means that $[x_{(1)} \otimes (t \triangleleft x_{(2)})] = [x \otimes t]$ for all $t \in T$. Applying $a \in B$ to the above identity on the left (i.e., using the right dual action of B on $B^* \bowtie B = B^* \rtimes B$), we get

$$[(x \triangleleft a)_{(1)} \otimes (t \triangleleft (x \triangleleft a)_{(2)})] = [(x_{(1)} \triangleleft a) \otimes (t \triangleleft x_{(2)})] = [(x \triangleleft a) \otimes t]$$

therefore, $(x \triangleleft a) \in T' \cap B^*$ for all $a \in B$. Note that I is generated, as an algebra, by elements of the form $(y \triangleright t)$, with $t \in T$ and $y \in B^*$. We need to show that $I' \cap B^* \subset T' \cap B^*$, i.e., that any x from $T' \cap B^*$ commutes with $(y \triangleright t)$. The latter follows from considering the left dual action of B^* on $B^* \rtimes B$:

$$[(y \triangleright t)_{(1)} \triangleright x \otimes (y \triangleright t)_{(2)}] = [(t_{(1)} \triangleright x) \otimes (y \triangleright t_{(2)})] = [x \otimes (y \triangleright t)].$$

The opposite inclusion is obvious.

Since $I_1' \cap I_2' = (I_1 \vee I_2)'$ for all $I_1, I_2 \in \ell(B)$, the map $\delta : I \mapsto \tilde{I}^d$ is a homomorphism of lattices. Its inverse is given by the composition of the maps $\delta(I) \mapsto \delta(I)' \cap B \subset B^* \bowtie B$ and $I \mapsto \tilde{I}$, since we have

$$\delta(I)' \cap B = (\tilde{I}' \cap B^*)' \cap B = (\tilde{I} \vee (B^*)') \cap B = \tilde{I} \vee ((B^*)' \cap B) = \tilde{I},$$

for all $I \in \ell(B)$. Therefore, δ is an isomorphism.

Definition 3.4 A left coideal $*$-subalgebra $I \subset B$ is said to be *connected* if $Z(I) \cap B_s = \mathbb{C}$.

To justify this definition, note that if $I = B$, then this is precisely the definition of B being connected, and if $I = B_t$, then this definition is equivalent to B^* being connected ([11], 3.10, 3.11).

Let $I \subset B$ be a connected left coideal $*$-subalgebra of B, then there is a uniquely determined positive element $x_I \in I$ such that $\varepsilon(b) = \tau(x_I b)$, for all $b \in I$.

Proposition 3.5 *For any system* $\{f_{rs}^\alpha\}$ *of matrix units in* $I = \Sigma_\alpha M_{n_\alpha}(C)$ *the value of the comultiplication on* x_I *is*

$$\Delta(x_I) = \sum_{\alpha rs} \frac{1}{\tau(f_{ss}^\alpha)} GS^{-1}(f_{sr}^\alpha) \otimes f_{rs}^\alpha = \sum_{\alpha rs} \frac{1}{\tau(f_{ss}^\alpha)} S(f_{sr}^\alpha)G \otimes f_{rs}^\alpha.$$

We also have $S(x_I) = x_I G^{-1}$.

Proof. From Corollary 2.2 we get

$$
\begin{aligned}
b &= b_{(1)}\varepsilon(b_{(2)}) = b_{(1)}\tau(x_I b_{(2)}) \\
&= S(H^{-1}x_{I(1)})\tau(Hx_{I(2)}b).
\end{aligned}
$$

If we write $\Delta(x_I) = \Sigma_{\alpha rs}\, g^\alpha_{sr} \otimes f^\alpha_{sr}$ with $g^\alpha_{sr} \in B$, then applying the above identity to $b = f^\beta_{kl}H^{-1}$ on has $f^\beta_{kl}H^{-1} = \tau(f^\beta_{ll})S(H^{-1}g^\beta_{lk})$, therefore $g^\beta_{lk} = \frac{1}{\tau(f^\beta_{ll})}GS^{-1}(f^\beta_{kl})$. Comparing $\Delta(S(x_I))$ and $\Delta(x_I)$, we get the last identity.

Definition 3.6 Let $e_I \in I$ be the support of the restriction of ε on I, i.e., $e_I = e$, where e is the minimal projection having property $\varepsilon(ebe) = \varepsilon(b)$ for all $b \in I$ (note that ε is faithful on $e_I I e_I$). We will call e_I a *distinguished projection* of I.

Note that $x_I e_I = e_I x_I = x_I$ and e_I is the minimal projection with this property, i.e., e_I is the support of x_I. Also, it is easy to see that $I_1 \subset I_2$ implies $e_{I_2} \leq e_{I_1}$.

Proposition 3.7 *Let $I \subset B$ be a left coideal $*$-subalgebra. Then the distinguished projection e_I satisfies the following Haar property :*

$$
be_I = \varepsilon_t(b)e_I, \qquad \text{for all } b \in I.
$$

Proof. Since $\varepsilon(xy) = \varepsilon(x\varepsilon_t(y))$ for all $x, y \in B$, we get

$$
\varepsilon((\varepsilon_t(b) - b)^*(\varepsilon_t(b) - b)) = 0, \qquad \forall\, b \in B,
$$

which implies $\varepsilon(e_I(\varepsilon_t(b) - b)^*(\varepsilon_t(b) - b)e_I) = 0, \qquad \forall\, b \in I$.
    Therefore, $(\varepsilon_t(b) - b)e_I = 0$, since $\varepsilon|_I(e_I \cdot e_I)$ is faithful on $e_I I e_I$.

Remark 3.8 (i) For right coideal $*$-subalgebras one can prove a similar identity $e_I b = e_I \varepsilon_s(b)$.

(ii) For $I = B$ Proposition 3.7 is the Haar theorem for quantum groupoids ([12], 2.2.5), ([3], 4.5), e_B is the Haar projection, and x_B is a scalar multiple of e_B (since e_B is minimal in B ([3], 4.6)).

(iii) For $I = B_t$ one has $x_{B_t} = H$ and $e_{B_t} = 1$.

Corollary 3.9 *The map $E_I(y) = y_{(1)}\tau(x_I y_{(2)})$, $y \in B$ is the τ-preserving conditional expectation from B to I.*

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Proof. Using the relation from Proposition 2.2, the formula of Proposition 3.5, and that $S(G) = G^{-1}$ we get

$$
\begin{aligned}
\tau(bE_I(y)) &= \tau(y_{(1)}b)\tau(x_I y_{(2)}) \\
&= \tau(yb_{(1)})\tau(x_I G^{-1} S(b_{(2)})) \\
&= \tau(yb_{(1)})\tau(x_I b_{(2)}) = \tau(yb_{(1)})\varepsilon(b_{(2)}) = \tau(yb),
\end{aligned}
$$

therefore E_I is the τ-preserving conditional expectation from B to I.

Remark 3.10 From the explicit form of the isomorphism $\theta : M_1 \rtimes B \to M_2$ one can see that the map $[x \otimes b] \mapsto [x \otimes E_{M_1}(S(H)^{1/2}bS(H)^{-1/2})]$ from $M_1 \rtimes B$ onto M_1 is the image of $E_{M_1} : M_2 \to M_1$ under θ^{-1}, and thus it is a trace preserving projection in $M_1 \rtimes B$ onto M_1. Note that the map $E_{M_1} : B \to B_t$ is uniquely defined by the relation $\tau(zb) = \tau(zE_{M_1}(b))$ ($\forall z \in B_t, b \in B$), and the same relation determines the τ-preserving conditional expectation E_{B_t} from Corollary 3.9. So, these two maps coincide, and the formula for $E_{B_t}(x) = x_{(1)} \otimes \tau(Hx_{(2)})$ shows that $E_{M_1}(S(H)^{1/2}bS(H)^{-1/2}) = E_{M_1}(b)$.

As a result, the above inverse image of $E_{M_1} : M_2 \to M_1$ is the map $[x \otimes b] \mapsto [x \otimes E_{B_t}(b)]$. Together with Corollary 3.9 this gives the following expression for $\tau_{M_1 \rtimes B}$:

$$
\begin{aligned}
\tau_{M_1 \rtimes B}([x \otimes b]) &= \tau_{M_1}(x(E_{B_t}(b) \rhd 1)) = \tau_{M_1}(x(b_{(1)} \rhd 1))\tau(Hb_{(2)}) \\
&= \tau_{M_1}(x(S(H1_{(1)})G \rhd 1))\tau(1_{(2)}b) = \tau_{M_1}(x(H \rhd 1)\tau(b) \\
&= \tau_{M_1}(S(H) \rhd x)\tau(b) = \tau_{M_1}(x)\tau(Hb),
\end{aligned}
$$

where $x \in M_1, b \in B$. Since $\tau_{M_1}(H \rhd 1) = \tau(H) = 1$, we have

$$
\tau_{M_1 \rtimes B}([x \otimes 1]) = \tau_{M_1}(x) \qquad \text{and} \qquad \tau_{M_1 \rtimes B}([1 \otimes b]) = \tau(b).
$$

Then one can write down the GNS inner product on $M_1 \rtimes B$ as

$$
([x \otimes b], [y \otimes c])_{M_1 \rtimes B} = (x, y)_{\widetilde{L^2(M_1)}}(b, c)_B,
$$

where $(\cdot, \cdot)_B$ is the GNS-scalar product on B with respect to the Markov trace.

4 A Galois Correspondence

Let $N \subset M$ be a finite depth inclusion of II_1 factors with finite index $\lambda^{-1} = [M : N]$ and

$$
N \subset M \subset M_1 \subset M_2 \subset \cdots
$$

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be the corresponding Jones tower, $M_i = \langle M_{i-1}, e_i \rangle$, where $e_i \in M'_{i-2} \cap M_i$, $i = 1, 2, \ldots$ are Jones projections (we denote $M_{-1} = N$ and $M_0 = M$). Let n be the depth ([5], 4.6.4) of $N \subset M$, i.e.,

$$n = \min\{k \in \mathbb{Z}^+ \mid \dim Z(N' \cap M_{k-2}) = \dim Z(N' \cap M_k)\}.$$

The case $n = 2$ is completely understood in [13], where it was shown that the symmetries of depth 2 subfactors are described by quantum groupoids (see Preliminaries). For the case of general depth $n \geq 2$ we have the following result.

Proposition 4.1 *For all $k \geq 0$ the inclusion $N \subset M_k$ has depth $d + 1$, where d is the smallest positive integer $\geq \frac{n-1}{k+1}$. In particular, $N \subset M_i$ has depth 2 for all $i \geq n - 2$.*

Proof. Note that $\dim Z(N' \cap M_i) = \dim Z(N' \cap M_{i+2})$ for all $i \geq n - 2$. By [16], the tower of basic construction for $N \subset M_k$ is

$$N \subset M_k \subset M_{2k+1} \subset M_{3k+2} \subset \cdots,$$

therefore, the depth of this inclusion is equal to $d+1$, where d is the smallest positive integer such that $d(k + 1) - 1 \geq n - 2$.

Corollary 4.2 *Any finite depth subfactor $N \subset M$ is an intermediate subfactor of some depth 2 inclusion.*

Proof. Consider $N \subset M \subset M_k$, $k \geq n - 2$.

The last result means that $N \subset M$ can be realized as an intermediate subfactor of a crossed product inclusion $N \subset N{\rtimes}B$ for some quantum groupoid B :

$$N \subset M \subset N{\rtimes}B.$$

Recall that in the case of a usual C^*-Hopf algebra (i.e., Kac algebra) action there is a Galois correspondence between intermediate von Neumann subalgebras of $N \subset N{\rtimes}B$ and left coideal $*$-subalgebras of B [6],[4]. Thus, it is natural to ask about a quantum groupoid analogue of this correspondence.

Clearly, the set $\ell(M_1 \subset M_2)$ of intermediate von Neumann subalgebras of $M_1 \subset M_2$ forms a lattice under the operations

$$K_1 \wedge K_2 = K_1 \cap K_2, \qquad K_1 \vee K_2 = (K_1 \cup K_2)''$$

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for all $K_1, K_2 \in \ell(M_1 \subset M_2)$. The smallest element of this lattice is M_1 and the greatest element is M_2.

Given a left (resp. right) action of B on a von Neumann algebra N, we will denote (by an abuse of notation)

$$N \rtimes I = \text{span}\{[x \otimes b] \mid x \in N, \, b \in I\} \subset N \rtimes B.$$

The next theorem establishes a Galois correspondence between intermediate von Neumann subalgebras of depth 2 inclusions of II_1 factors and coideal $*$-subalgebras of a quantum groupoid, i.e., a lattice isomorphism between $\ell(M_1 \subset M_2)$ and $\ell(B)$.

Theorem 4.3 *Let $N \subset M \subset M_1 \subset M_2 \subset \cdots$ be the tower constructed from a depth 2 subfactor $N \subset M$, $B = M' \cap M_2$ be the corresponding quantum groupoid, and θ be the isomorphism between $M_1 \rtimes B$ and M_2 ([13], 6.3). Then the following formulas*

$$\phi \;:\; \ell(M_1 \subset M_2) \to \ell(B) : K \mapsto \theta^{-1}(M' \cap K) \subset B$$
$$\psi \;:\; \ell(B) \to \ell(M_1 \subset M_2) : I \mapsto \theta(M_1 \rtimes I) \subset M_2.$$

define isomorphisms between $\ell(M_1 \subset M_2)$ and $\ell(B)$ inverse to each other.

Proof. First, we need to check that ϕ and ψ are indeed maps between the specified lattices. It follows immediately from the definition of the crossed product that $M_1 \rtimes I$ is a von Neumann subalgebra of $M_1 \rtimes B$, therefore $\theta(M_1 \rtimes I)$ is a von Neumann subalgebra of $M_2 = \theta(M_1 \rtimes B)$, so ψ is a map to $\ell(M_1 \subset M_2)$. To show that ϕ maps to $\ell(B)$, it is enough to show that the annihilator $(M' \cap K)^0 \subset B^*$ of $M' \cap K \subset B$ is a left ideal in B^*.

For all $x \in A$, $y \in (M' \cap K)^0$, and $b \in M' \cap K$ we have

$$
\begin{aligned}
\langle\, xy, \, b \,\rangle \;&=\; \lambda^{-2}\tau(xy e_2 e_1 Hb) = \lambda^{-2}\tau(y e_2 e_1 Hbx) \\
&=\; \lambda^{-3}\tau(y e_2 e_1 E_{M'}(e_1 Hbx)) = \langle\, y, \, \lambda^{-1} E_{M'}(e_1 Hbx) \,\rangle,
\end{aligned}
$$

and it remains to show that $E_{M'}(e_1 Hbx) \in M' \cap K$. By ([5], 4.2.7), the square

$$
\begin{array}{ccc}
K & \subset & M_2 \\
\cup & & \cup \\
M' \cap K & \subset & M' \cap M_2
\end{array}
$$

is commuting, therefore, $E_{M'}(K) \subset M' \cap K$. Since $e_1 Hbx \in K$, we have $xy \in (M' \cap K)^0$, i.e., $(M' \cap K)^0$ is a left ideal and $\phi(K) = \theta^{-1}(M' \cap K)$ is a left coideal $*$-subalgebra.

Clearly, ϕ and ψ preserve $\wedge$ and $\vee$, moreover $\phi(M_1) = B_t$, $\phi(M_2) = B$ and $\psi(B_t) = M_1$, $\psi(B) = M_2$, therefore they are morphisms of lattices.

To see that they are inverses for each other, we first observe that the condition $\psi \circ \phi = \mathrm{id}$ is equivalent to $M_1(M' \cap K) = K$, and the latter follows from applying the conditional expectation E_K to $M_1(M' \cap M_2) = M_1 B = M_2$. The condition $\phi \circ \psi = \mathrm{id}$ translates into $\theta(I) = M' \cap \theta(M_1 \rtimes I)$. If $b \in I$, $x \in M = M_1^B$, then

$$
\begin{aligned}
\theta(b)x &= \theta([1 \otimes b][x \otimes 1]) = \theta([(b_{(1)} \triangleright x) \otimes b_{(2)}]) \\
&= \theta([x(1_{(2)} \triangleright 1) \otimes \varepsilon(1_{(1)}b_{(1)})b_{(2)}]) = \theta([x \otimes b]) = x\theta(b),
\end{aligned}
$$

i.e., $\theta(I)$ commutes with M. Conversely, if $x \in M' \cap \theta(M_1 \rtimes I) \subset B$, then $x = \theta(y)$ for some $y \in (M_1 \rtimes I) \cap B = I$, therefore $x \in \theta(I)$.

The following proposition describes the center of $K = M_1 \rtimes I$ and the first relative commutant in terms of I.

Proposition 4.4 *In the above situation,*

(i) $Z(K) = Z(M_1 \rtimes I) = Z(I) \cap B_s$,

(ii) $M_1' \cap K = M_1' \cap M_1 \rtimes I = I \cap B_s$.

Proof. Recall that $B_s = M_1' \cap M_2$. If $x \in Z(I) \cap B_s \subset K$, then x commutes with both I and M_1 and, therefore, with $K = M_1 \rtimes I$, i.e., $x \in Z(K)$. Conversely, if $x \in Z(K)$ then $x \in K' \cap K \subset M_1' \cap M_2 = B_s$ and $x \in M' \cap K$, so $x \in Z(M' \cap K) = Z(I)$ and (i) follows. To prove (ii), note that since $B_s = M_1' \cap M_2 \subset M_1'$ and $I = M' \cap K \subset K$, we have $M_1' \cap K \subset (M_1' \cap M_2) \cap (M' \cap K) = B_s \cap I \subset M_1' \cap K$.

Corollary 4.5 (i) $K = M \rtimes I$ *is a factor iff* $Z(I) \cap B_s = \mathbb{C}$.

(ii) *The inclusion* $M_1 \subset K = M_1 \rtimes I$ *is irreducible iff* $B_s \cap I = \mathbb{C}$.

Corollary 4.6 *There is no subfactor* $N \subset M$ *of depth* n *and index* $\sqrt[k]{p}$, *where* p *is prime and* $k \geq n - 1$ *(unless* $n = 2$ *and* $k = 1$, *in which case* $M \cong N \rtimes \mathbb{Z}/p\mathbb{Z}$).

Proof. Suppose that such a subfactor $N \subset M$ exists, then $[M_{k-1} : N] = p$ and $\mathrm{depth}(N \subset M_{k-1}) = 2$ by Proposition 4.1. Therefore, $(N \subset M_{k-1}) \cong (N \subset N \rtimes \mathbb{Z}/p\mathbb{Z})$ (see [13], Corollary 4.19) and Theorem 4.3 implies the

existence of a subgroup of $\mathbb{Z}/p\mathbb{Z}$ corresponding to the intermediate subfactor $N \subset M$. But $\mathbb{Z}/p\mathbb{Z}$ does not have any non-trivial subgroups, therefore $M = M_{k-1}$, i.e., $k = 1$ and $n = 2$.

Note that intermediate von Neumann subalgebras of $M \subset M_1$ can be characterized in terms of projections in $M' \cap M_2$ having certain properties [1]. Namely, every projection $q \in M' \cap M_2$ such that

(IS 1) $qe_2 = e_2$,

(IS 2) $E_{M_1}(q)$ is a scalar,

(IS 3) $\lambda^{-1}E_{M_1}(qe_1e_2)$ is a multiple of a projection,

implements a conditional expectation from M_1 to an intermediate subalgebra $Q = \{q\}' \cap M_1$. If Q is a factor, then $[M_1 : Q] = \tau(q)^{-1}$ ([1], Theorem 3.2). This result is true for all finite index subfactors (regardless of depth). The goal of next two propositions is to relate such projections and coideal $*$-subalgebras in the case of finite depth inclusions (which are intermediate subfactors of depth 2 inclusions by Corollary 4.2).

Proposition 4.7 *If $I \subset B$ is a connected left coideal $*$-subalgebra, then there is a constant $\lambda_I > 0$ such that $p_I = \lambda_I^{-1}H^{-1/2}x_IH^{-1/2}$ is a projection in I.*

Proof. From the formula for $\Delta(x_I)$ (Proposition 3.5) we get

$$\varepsilon_s(H^{-1}x_I) = m(S \otimes \mathrm{id})\Delta(H^{-1}x_I) = \sum_{\alpha rs} \frac{1}{\tau(f_{ss}^\alpha)} f_{sr}^\alpha H^{-1} f_{rs}^\alpha \in Z(I) \cap B_s.$$

Since I is connected, we conclude that $\varepsilon_s(H^{-1}x_I) = \lambda_I 1$ for some constant λ_I. Using this result and Proposition 3.5 one can check by a direct computation that $\Delta(H^{-1}x_I)^2 = \lambda_I\Delta(H^{-1}x_I)$, from where it follows that $\lambda_I^{-1}H^{-1}x_I$ is an idempotent and $p_I = \lambda_I^{-1}H^{-1/2}x_IH^{-1/2}$ is a projection.

Proposition 4.8 *Let $N \subset M$ be a depth 2 inclusion, B be the quantum groupoid constructed on $M' \cap M_2$, and $\theta : M_1 \bowtie B \to M_2$ be the isomorphism of II_1 factors (see Preliminaries). Then for any connected left coideal $*$-subalgebra $I \subset B$ the projection $q_I = \theta(p_I)$ satisfies properties (IS 1)–(IS 3).*

Proof. The relation $\varepsilon_t(x_I H^{-1}) = \lambda_I 1$ follows easily from definitions, from where

$$\varepsilon_t(H^{1/2} p_I H^{-1/2}) = \lambda_I \varepsilon_t(x_I H^{-1}) = 1,$$

hence $H^{1/2} p_I H^{-1/2} e_B = e_B$. Applying θ to the last equality, and observing that $e_2 = \theta(H^{-1/2} e_B H^{-1/2})$, we get $\theta(p_I) e_2 = e_2$ which is (IS 1).

To establish the second property, note that $E_{M_1}(\theta(p_I)) = \theta(E_{M_1}(p_I)) = \theta(E_{B_t}(p_I))$, so it is enough to show that $E_{B_t}(p_I)$ is a scalar. Using Proposition 3.9 ($x_{B_t} = H$) and Corollary 2.2 we have :

$$E_{B_t}(p_I) \;=\; p_{I(1)} \tau(H p_{I(2)}) = \lambda_I^{-1} S(1_{(1)}) \tau(1_{(2)} x_I) = \lambda_I^{-1} 1.$$

For the third property we must verify that $z_I = E_{M_1}(q_I e_1 e_2)$ is a multiple of a projection. Using the duality between $B^* = N' \cap M_1$ and $B = M' \cap M_2$ and the formula for E_I from Proposition 3.9 we have, for all $b \in B$:

$$
\begin{aligned}
\langle z_I, b \rangle &= \lambda^{-3} \tau(E_{M_1}(q_I e_1 e_2) e_2 e_1 H b) \\
&= \tau(q_I H b) = \lambda_I^{-1} \tau(x_I G^{1/2} b G^{-1/2}), \\
&= \lambda_I^{-1} \tau(x_I E_I(G^{1/2} b G^{-1/2})) \\
&= \lambda_I^{-1} \tau(x_I (G^{1/2} b G^{-1/2})_{(1)}) \tau(x_I (G^{1/2} b G^{-1/2})_{(2)}) \\
&= \lambda_I \langle z_I, b_{(1)} \rangle \langle z_I, b_{(2)} \rangle = \lambda_I \langle z_I^2, b \rangle,
\end{aligned}
$$

which implies that $z_I^2 = \lambda_I^{-1} z_I$. Finally, in order to show that z_I is selfadjoint, we first compute $S(q_I)$, using properties of the antipode and Proposition 3.5 :

$$
\begin{aligned}
S(q_I) &= \lambda_I^{-1} S(S(H)^{1/2} H^{-1/2} x_I H^{-1/2} S(H)^{-1/2}) \\
&= \lambda_I^{-1} S(H)^{-1/2} H^{-1/2} x_I H^{-1/2} S(H)^{1/2} = S(H)^{-1} q_I S(H),
\end{aligned}
$$

from where, using the definition of S we have

$$
\begin{aligned}
z_I^* &= E_{M_1}(q_I e_1 e_2)^* = E_{M_1}(e_2 e_1 S(H^{-1} q_I H))^* \\
&= E_{M_1}(e_2 e_1 q_I)^* = E_{M_1}(q_I e_1 e_2) = z_I.
\end{aligned}
$$

Remark 4.9 Let $Q_I = \{q_I\}' \cap M_1$. Then $[M_1 : Q_I] = \tau(q_I) = \lambda_I^{-1}$ (cf. [1]).

Proposition 4.10 *If I is a left coideal $*$-subalgebra of B and $\delta(I)$ is a left coideal subalgebra of B^* constructed in Proposition 3.3, then the triple*

$$\delta(I) \subset B^* \subset B^* \bowtie I$$

is a basic construction.

Proof. It follows from Proposition 4.8 that q_I implements the conditional expectation from $N' \cap M_1 = B^*$ to $\{q_I\}' \cap (N' \cap M_1) = \{q_I\}' \cap B^* = \{p_I\}' \cap B^* = \tilde{I}' \cap B^* = \delta(I)$ (observe that $\tilde{I}$ is the right coideal $*$-subalgebra of B^* generated by p_I, cf. Propositions 3.2 and 4.7). Since B^* and p_I generate $B^* \bowtie I$ we conclude that $\delta(I) \subset B^* \subset B^* \bowtie I$ is a basic construction. $\blacksquare$

5 Bimodule categories and principal graphs

In this section we establish an equivalence between the tensor category of $N - N$ bimodules of a finite index and finite depth subfactor $N \subset M$ and the co-representation category of a quantum groupoid B canonically associated with it as in Theorem 4.3. The principal graph of $N \subset M$ can be described in terms of relative Hopf modules over B. Alternatively, we show that it can also be obtained as a certain Bratteli diagram.

Our methods follow those of [8], where the special case of the invariants associated with the group-subgroup subfactors was considered.

A left (resp., right) B-comodule V (with the structure map denoted by $v \mapsto v^{(1)} \otimes v^{(2)}$, $v \in V$) is said to be *unitary*, if

$$
\begin{aligned}
(v_2^{(1)})^*(v_1, v_2^{(2)}) &= S(v_1^{(1)})G(v_1^{(2)}, v_2) \\
(\text{resp.,}\ (v_1^{(2)})(v_1^{(1)}, v_2) &= G^{-1}S((v_1^{(1)})^*)(v_1, v_2^{(1)})),
\end{aligned}
$$

where $v_1, v_2 \in V$, and G is the canonical group-like element of B. The notion of a unitary comodule in the Hopf $*$-algebra case can be found, e.g., in ([7], 1.3.2).

Given left coideal $*$-subalgebras H and K of B, we consider a category C_{H-K} of left relative $(B, H - K)$ Hopf bimodules (cf. [17]), whose objects are Hilbert spaces which are both $H - K$-bimodules and left unitary B-comodules such that the bimodule action commutes with the coaction of B, i.e., for any object V of C_{H-K} and $v \in V$ one has

$$(h \triangleright v \triangleleft k)^{(1)} \otimes (h \triangleright v \triangleleft k)^{(2)} = h_{(1)}v^{(1)}k_{(1)} \otimes (h_{(2)} \triangleright v^{(2)} \triangleleft k_{(2)}),$$

where $v \mapsto v^{(1)} \otimes v^{(2)}$ denotes the coaction of B on v, $h \in H$, $k \in K$, and morphisms are intertwining maps.

Similarly one can define a category of right relative $(B, H - K)$ Hopf bimodules.

Remark 5.1 Note that any left B-comodule V is automatically a $B_t - B_t$-bimodule via $z_1 \cdot v \cdot z_2 = \varepsilon(z_1 v^{(1)} z_2) v^{(2)}$, $v \in V, z_1, z_2 \in B_t$. For any object of C_{H-K}, this $B_t - B_t$-bimodule structure is a restriction of the given $H - K$-bimodule structure; it is easily seen by applying $(\varepsilon \otimes id)$ to both sides of the relation of commutation between the $H - K$-bimodule action and the coaction of B, and taking $h, k \in B_t$. Therefore, in the case when $H = B_t$ (resp. $K = B_t$) we can speak about right (resp. left) relative Hopf modules, which are a special case of weak Doi-Hopf modules [2].

Let us also mention obvious relations $\varepsilon_s(v^{(1)}) \otimes v^{(2)} = 1_{(1)} \otimes (v \triangleleft 1_{(2)})$ and $\varepsilon_t(v^{(1)}) \triangleright v^{(2)} = v$.

Proposition 5.2 *If B is a group Hopf C^*-algebra and H, K are subgroups, then there is a bijection between simple objects of C_{H-K} and double cosets of $H \backslash B / K$.*

Proof. If V is an object of C_{H-K}, then every simple subcomodule of V is 1-dimensional. Let $U = Cu$ ($u \mapsto g \otimes u$, $g \in B$) be one of these comodules, then all other simple subcomodules of V are of the form $h \triangleright U \triangleleft k$, where $h \in H$, $k \in K$, and

$$V = \oplus_{h,k} (h \triangleright U \triangleleft k) = \text{span}\{HgK\}.$$

Vice versa, span$\{HgK\}$ with natural $H - K$ bimodule and B-comodule structures is a simple object of C_{H-K}.

Example 5.3 If H, V, K are left coideal $*$-subalgebras of B, $H \subset V, K \subset V$, then V is an object of C_{H-K} with the structure maps given by $h \triangleright v \triangleleft k = hvk$ and Δ, where $v^{(1)} \otimes v^{(2)} = \Delta(v)$, $v \in V$, $h \in H$, $k \in K$. The scalar product is defined by the restriction on V of the Markov trace of B (this B-comodule is unitary due to Corollary 2.2).

Similarly, right coideal $*$-subalgebras of B give examples of right relative $(B, H - K)$ Hopf bimodules.

Given an object V of C_{H-K}, the conjugate Hilbert space $\overline{V}$ is an object of C_{K-H} with the bimodule action

$$k \triangleright \overline{v} \triangleleft h = \overline{h^* \triangleright v \triangleleft k^*} \quad (\forall h \in H,\, k \in K)$$

(here $\overline{v}$ denotes the vector $v \in V$ considered as an element of $\overline{V}$) and the coaction $\overline{v} \mapsto \overline{v}^{(1)} \otimes \overline{v}^{(2)} = (v^{(1)})^* \otimes \overline{v}^{(2)}$. The relation of commutation between the actions and the coaction and the unitarity of $\overline{V}$ are straightforward.

Define V^*, the dual object of V, to be $\overline{V}$ with the above structures. One can directly check that $V^{**} \cong V$ for any object V. Let us remark, that in Example 5.3 the dual object can be obtained by putting $\overline{v} = v^*$ for all $v \in V$.

Definition 5.4 Let L be another coideal $*$-subalgebra of B. For any objects $V \in C_{H-L}$ and $W \in C_{L-K}$, we define an object $V \otimes_L W$ from C_{H-K} as a tensor product of bimodules V and W [10] equipped with a comodule structure

$$(v \otimes_L w)^{(1)} \otimes (v \otimes_L w)^{(2)} := v^{(1)} w^{(1)} \otimes (v^{(2)} \otimes_L w^{(2)}).$$

Let us verify that we have indeed an object from C_{H-K}. First, the above coproduct is clearly coassociative and compatible with counit (see the properties of ε and Remark 5.1):

$$
\begin{aligned}
\varepsilon(v^{(1)} w^{(1)})(v^{(2)} \otimes_L w^{(2)}) &= \varepsilon(v^{(1)} 1_{(1)}) v^{(2)} \otimes_L \varepsilon(1_{(2)} w^{(1)}) w^{(2)} \\
&= (v \triangleleft S(1_{(1)})) \otimes_L (1_{(2)} \triangleright w) = v \otimes_L w.
\end{aligned}
$$

Second, the commutation relation between the $H - K$-bimodule and B-comodule structures can be proved as follows:

$$
\begin{aligned}
(h \triangleright (v \otimes_L w) \triangleleft k)^{(1)} \otimes (h \triangleright (v \otimes_L w) \triangleleft k)^{(2)} &= \\
&= ((h \triangleright v) \otimes_L (w \triangleleft k))^{(1)} \otimes ((h \triangleright v) \otimes_L (w \triangleleft k))^{(2)} \\
&= (h \triangleright v)^{(1)} (w \triangleleft k)^{(1)} \otimes (h \triangleright v)^{(2)} \otimes_L (w \triangleleft k)^{(2)} \\
&= h_{(1)} v^{(1)} w^{(1)} k_{(1)} \otimes (h_{(2)} \triangleright v^{(2)}) \otimes_L (w^{(2)} \triangleleft k_{(2)}) \\
&= h_{(1)} (v \otimes_L w)^{(1)} k_{(1)} \otimes (h_{(2)} \triangleright (v \otimes_L w)^{(2)} \triangleleft k_{(2)}).
\end{aligned}
$$

Finally, let us show that $V \otimes_L W$ is unitary. To this end, recall the following expression of the scalar product in this bimodule [10]:

$$
\begin{aligned}
(v_1 \otimes_L w_1, v_2 \otimes_L w_2)_{V \otimes_L W} &= (v_1 \triangleleft \langle w_1, w_2 \rangle_L, v_2)_V = \\
&= (\langle v_1, v_2 \rangle_L \triangleright w_1, w_2)_W = \tau(\langle v_1, v_2 \rangle_L \langle w_1, w_2 \rangle_L),
\end{aligned}
$$

where $v_1, v_2 \in V, w_1, w_2 \in W$ and the elements $\langle v_1, v_2 \rangle_L, \langle w_1, w_2 \rangle_L \in L$ (L-valued scalar products on V and W respectively) are defined in a unique way by the relations

$$(v_1 \triangleleft l, v_2)_V = \tau(l\langle v_1, v_2 \rangle_L), \qquad (l \triangleright w_1, w_2)_W = \tau(l\langle w_1, w_2 \rangle_L) \quad (\forall l \in L).$$

Then the needed relation follows from

$$
\begin{aligned}
&((v_2 \otimes_L w_2)^{(1)})^*(v_1 \otimes_L w_1, (v_2 \otimes_L w_2)^{(2)})_{V\otimes_L W} = \\
&= (w_2^{(1)})^*(v_2^{(1)})^*(v_1 \otimes_L w_1, v_2^{(2)} \otimes_L w_2^{(2)})_{V\otimes_L W} \\
&= (w_2^{(1)})^*(v_2^{(1)})^*(v_1 \triangleleft \langle w_1, w_2^{(2)} \rangle_L, v_2^{(2)})_V \\
&= (w_2^{(1)})^* S(\langle w_1, w_2^{(2)} \rangle_L^{(1)}) S(v_1^{(1)}) G(v_1^{(2)} \triangleleft \langle w_1, w_2^{(2)} \rangle_L^{(2)}, v_2)_V \\
&= (w_2^{(1)})^* S(\langle w_1, w_2^{(2)} \rangle_L^{(1)}) \tau(\langle w_1, w_2^{(2)} \rangle_L^{(2)} \langle v_1^{(2)}, v_2 \rangle_L) S(v_1^{(1)}) G \\
&= (w_2^{(1)})^* (\langle v_1^{(2)}, v_2 \rangle_L^{(1)})^* G^{-1} \tau(\langle w_1, w_2^{(2)} \rangle_L \langle v_1^{(2)}, v_2 \rangle_L^{(2)}) S(v_1^{(1)}) G \\
&= (w_2^{(1)})^* (\langle v_1^{(2)}, v_2 \rangle_L^{(1)})^* G^{-1} (\langle v_1^{(2)}, v_2 \rangle_L \triangleright w_1, w_2)_W^{(2)} S(v_1^{(1)}) G \\
&= S(w_1^{(1)}) (\langle v_1^{(2)}, v_2 \rangle_L \triangleright w_1^{(2)}, w_2)_W S(v_1^{(1)}) G \\
&= S(w_1^{(1)}) S(v_1^{(1)}) G(v_1^{(2)} \otimes_L w_1^{(2)}, v_2 \otimes_L w_2)_{V\otimes_L W} \\
&= S((v_1 \otimes_L w_1)^{(1)}) G((v_1 \otimes_L w_1)^{(2)}, v_2 \otimes_L w_2)_{V\otimes_L W},
\end{aligned}
$$

where we used the unitarity of V and W and Corollary 2.2.

Lemma 5.5 (cf. [8]) *The operation of tensor product is*
(i) associative, i.e., $V \otimes_L (W \otimes_P U) \cong (V \otimes_L W) \otimes_P U$,
(ii) compatible with duality, i.e., $(V \otimes_L W)^* \cong W^* \otimes_L V^*$,
(iii) distributive, i.e., $(V \oplus V') \otimes_L W = (V \otimes_L W) \oplus (V' \otimes_L W)$.

Proof. Easy excercise left to the reader.

The tensor product of morphisms $T \in \mathrm{Hom}(V, V')$ and $S \in \mathrm{Hom}(W, W')$ is defined as usual :

$$(T \otimes_L S)(v \otimes_L w) = T(v) \otimes_L S(w).$$

From now on let us suppose that B is biconnected and acts outerly on the left on a II_1 factor N and that the extension of this action on $L^2(N)$ satisfies the relations mentioned in the end of Preliminaries.

Given an object V of C_{H-K}, we construct an $N \rtimes H - N \rtimes K$-bimodule $\widehat{V}$ as follows. We put

$$\widehat{V} = \mathrm{span}\{\Delta(1) \triangleright (\xi \otimes v) \mid \xi \otimes v \in L^2(N) \otimes V\}$$

and denote $[\xi \otimes v] = \Delta(1) \triangleright (\xi \otimes v)$. It is straightforward to show that $\widehat{V}$ is characterized by the property $[\xi \otimes (z \triangleright v)] = [(\xi \triangleleft z) \otimes v] = [\xi(z \triangleright 1) \otimes v] \; \forall z \in B_t$, i.e., that $\widehat{V} = L^2(N) \otimes_{B_t} V$.

Let us consider $\widehat{V}$ as a Hilbert space with the scalar product

$$([\xi \otimes v], [\eta \otimes w])_{\widehat{V}} = (\xi, \eta)_{\widetilde{L^2(N)}} (v, w)_V.$$

and define the actions of N, H, K on $\widehat{V}$ by

$$a[\xi \otimes v] = [a\xi \otimes v], \qquad [\xi \otimes v]a = [\xi(v^{(1)} \triangleright a) \otimes v^{(2)}],$$

$$h[\xi \otimes v] = [(h_{(1)} \triangleright \xi) \otimes (h_{(2)} \triangleright v)], \qquad [\xi \otimes v]k = [\xi \otimes (v \triangleleft k)],$$

for all $a \in N$, $h \in H$, $k \in K$. One can check that these actions are well-defined and that

$$ha[\xi \otimes v] = (h_{(1)} \triangleright a)h_{(2)}[\xi \otimes v], \qquad [\xi \otimes v]ka = [\xi \otimes v](k_{(1)} \triangleright a)k_{(2)},$$

$$(ha[\xi \otimes v])ka' = ha([\xi \otimes v]ka'), \qquad a' \in N,$$

i.e., that the above formulas define the structure of an $(N \rtimes H) - (N \rtimes K)$ bimodule on $\widehat{V}$ in the algebraic sense. Let us show that this bimodule is unitary. We only need to check that

$$([\xi \otimes v]a, [\eta \otimes w])_{\widehat{V}} = ([\xi \otimes v], [\eta \otimes w]a^*)_{\widehat{V}}$$

(all other relations of unitarity are trivial). The following computation uses the above definitions, the properties of the action of B on $L^2(N)$ and the unitarity of V:

$$
\begin{aligned}
([\xi \otimes v]a, [\eta \otimes v])_{\widehat{V}} &= ([\xi(v^{(1)} \triangleright a) \otimes v^{(2)}], [\eta \otimes v])_{\widehat{V}} \\
&= (\xi(v^{(1)} \triangleright a), \eta)_{\widetilde{L^2(N)}} (v^{(2)}, w)_V \\
&= (\xi(S(H)v^{(1)} \triangleright a)), \eta)_{L^2(N)} (v^{(2)}, w)_V \\
&= (\xi, \eta(HS^{-1}((v^{(1)})^*) \triangleright a^*))_{L^2(N)} (v^{(2)}, w)_V \\
&= (\xi, \eta(GS^{-1}((v^{(1)})^*) \triangleright a^*))_{\widetilde{L^2(N)}} (v^{(2)}, w)_V \\
&= (\xi, \eta(w^{(1)} \triangleright a^*))_{\widetilde{L^2(N)}} (v, w^{(2)})_V \\
&= ([\xi \otimes v], [\eta \otimes w]a^*)_{\widehat{V}}.
\end{aligned}
$$

For any morphism $T \in \mathrm{Hom}(V, W)$, define a morphism $\widehat{T} \in \mathrm{Hom}(\widehat{V}, \widehat{W})$ by

$$\widehat{T}([\xi \otimes v]) = [\xi \otimes T(v)].$$

Example 5.6 For $V \in C_{H-K}$ from Example 5.3, we have $\widehat{V} = N{\rtimes}V$ as $N{\rtimes}H - N{\rtimes}K$-bimodules. Indeed, the algebraic operations and the scalar products are the same (see the definitions above and Remark 3.10).

Theorem 5.7 *The above assignments $V \mapsto \widehat{V}$ and $T \mapsto \widehat{T}$ define a functor from C_{H-K} to the category of $N{\rtimes}H - N{\rtimes}K$ bimodules. This functor preserves direct sums and is compatible with operations of taking tensor products and adjoints in the sense that if W is an object of C_{K-L}, then*

$$\widehat{V \otimes_K W} \cong \widehat{V} \otimes_{N{\rtimes}K} \widehat{W}, \quad and \quad \widehat{V^*} \cong (\widehat{V})^*.$$

Proof. 1. Directly from definitions we have $\widehat{V \oplus W} = \widehat{V} \oplus \widehat{W}$ for any objects V and W of C_{H-L}.

2. In order to show that $\widehat{V \otimes_L W} \cong \widehat{V} \otimes_{N{\rtimes}L} \widehat{W}$, let us define a map

$$\alpha : \widehat{V \otimes_L W} \to \widehat{V} \otimes_{N{\rtimes}L} \widehat{W} : [\xi \otimes (v \otimes_L w)] \mapsto [\xi \otimes v] \otimes_{N{\rtimes}L} [1 \otimes w],$$

for all $\xi \in L^2(M)$, $v \in V$, $w \in W$.

(a) α is well-defined, since, for all $\xi \in L^2(N), z \in B_t, v \in V, w \in W$

$$
\begin{aligned}
\alpha([(\xi \cdot z) \otimes (v \otimes_L w)]) &= [\xi \otimes \varepsilon(zv^{(1)})v^{(2)}] \otimes_{N{\rtimes}L} [1 \otimes w] \\
&= [\xi \otimes \varepsilon(zv^{(1)}1_{(1)})(v^{(2)} \cdot 1_{(2)})] \otimes_{N{\rtimes}L} [1 \otimes w] \\
&= [\xi \otimes \varepsilon(zv^{(1)}1_{(1)})v^{(2)}] \otimes_{N{\rtimes}L} [(1_{(2)} \cdot 1) \otimes w] \\
&= [\xi \otimes \varepsilon(zv^{(1)}1_{(1)})v^{(2)}] \otimes_{N{\rtimes}L} [1 \otimes \varepsilon(1_{(2)}w^{(1)})w^{(2)}] \\
&= [\xi \otimes \varepsilon(zv^{(1)}w^{(1)})v^{(2)}] \otimes_{N{\rtimes}L} [1 \otimes w^{(2)}] \\
&= \alpha([\xi \otimes \varepsilon(zv^{(1)}w^{(1)})(v^{(2)} \otimes_L w^{(2)})]) \\
&= \alpha([\xi \otimes z \cdot (v \otimes_L w)]).
\end{aligned}
$$

(b) α preserves the bimodule structure. Indeed, for all $h \in H, k \in K, a, a' \in N$ we have :

$$
\begin{aligned}
\alpha(ah[\xi \otimes (v \otimes_L w)]) &= \alpha([a(h_{(1)} \rhd \xi) \otimes ((h_{(2)} \rhd v) \otimes_L w)]) \\
&= [a(h_{(1)} \rhd \xi) \otimes (h_{(2)} \rhd v)] \otimes_{N{\rtimes}L} [1 \otimes w] \\
&= ah[\xi \otimes v] \otimes_{N{\rtimes}L} [1 \otimes w] \\
&= ah\alpha([\xi \otimes (v \otimes_L w)]), \\
\alpha([\xi \otimes (v \otimes_L w)]a'k) &= \alpha([\xi(v^{(1)}w^{(1)} \cdot a') \otimes (v^{(2)} \otimes_L (w^{(2)} \lhd k))]) \\
&= [\xi(v^{(1)}w^{(1)} \cdot a') \otimes v^{(2)}] \otimes_{N{\rtimes}L} [1 \otimes (w^{(2)} \lhd k)] \\
&= [\xi \otimes v] \otimes_{N{\rtimes}L} [(w^{(1)} \cdot a') \otimes (w^{(2)} \lhd k)] \\
&= [\xi \otimes v] \otimes_{N{\rtimes}L} ([1 \otimes w]a'k) \\
&= \alpha([\xi \otimes (v \otimes_L w)])a'k.
\end{aligned}
$$

(c) Observe that the map

$$\beta : \widehat{V} \otimes_{N \rtimes L} \widehat{W} \to V \,\widehat{\otimes_L}\, W : [\xi \otimes v] \otimes_{N \rtimes L} [\eta \otimes w] \mapsto [\xi(v^{(1)} \cdot \eta) \otimes (v^{(2)} \otimes_L w)],$$

where $\xi, \eta \in L^2(N), v \in V, w \in W$, is the inverse of α.

(d) α is an isometry of Hilbert spaces. Indeed, by the above definitions,

$$\|[\xi \otimes (v \otimes_L w)]\|^2_{V\widehat{\otimes_L}W} = (S(H) \triangleright \xi, \xi)_{L^2(N)}(v \triangleleft \langle w, w \rangle_L, v)_V,$$

where the L-valued scalar product $\langle w, w \rangle_L$ on W is defined in a unique way by the relation $(l \triangleright w, w)_W = \tau(l\langle w, w \rangle_L)$ $(\forall l \in L)$.

On the other hand, the definition of the tensor product of bimodules (see, for example, [10]) gives:

$$\|[\xi \otimes v] \otimes_{N \rtimes L} [1 \otimes w]\|^2_{\widehat{V}\otimes_{N \rtimes L}\widehat{W}} = ([\xi \otimes v] \triangleleft \langle [1 \otimes w], [1 \otimes w] \rangle_{N \rtimes L}, [\xi \otimes v])_{\widehat{V}},$$

where the element $\langle [1 \otimes w], [1 \otimes w] \rangle_{N \rtimes L}$ is defined in a unique way by

$$([n \otimes l] \triangleright [1 \otimes w], [1 \otimes w])_{\widehat{W}} = \tau_{N \rtimes L}([n \otimes l]\langle [1 \otimes w], [1 \otimes w] \rangle_{N \rtimes L}).$$

Since the left-hand side of this equality can be rewritten as:

$$([n(l_{(1)} \triangleright 1) \otimes (l_{(2)} \triangleright w)], [1 \otimes w])_{\widehat{W}} =$$
$$= ([n \otimes (\varepsilon_t(l_{(1)})l_{(2)}) \triangleright w)], [1 \otimes w])_{\widehat{W}}\tau_N(S(H) \triangleright n)(l \triangleright w, w)_W,$$

we can see that $\langle [1 \otimes w], [1 \otimes w] \rangle_{N \rtimes L} = [1 \otimes \langle w, w \rangle_L]$. Together with the formula for the scalar product on $\widehat{V}$ this gives

$$\|[\xi \otimes (v \otimes_L w)]\|^2_{V\widehat{\otimes_L}W} = \|[\xi \otimes v] \otimes_{N \rtimes L} [1 \otimes w]\|^2_{\widehat{V}\otimes_{N \rtimes L}\widehat{W}}.$$

3. In order to show that $\widehat{V^*} \cong (\widehat{V})^*$, let us define a map

$$\gamma : \widehat{V^*} \to (\widehat{V})^* : [\xi \otimes \overline{v}] \mapsto \overline{[(v^{(1)} \triangleright J\xi) \otimes v^{(2)}]}.$$

(a) γ is well-defined, since, for all $\xi \in L^2(N), z \in B_t, v \in V$

$$\gamma([(S(z) \triangleright \xi) \otimes \overline{v}]) = \overline{[(v^{(1)} \triangleright J(S(z) \triangleright \xi)) \otimes v^{(2)}]} =$$
$$= \overline{[(v^{(1)}z^* \triangleright \xi) \otimes v^{(2)}]} = \gamma([\xi \otimes (z \triangleright \overline{v})]).$$

(b) It is straightforward to show that γ preserves the bimodule structure and that the map

$$\overline{[\xi \otimes v]} \mapsto [((v^{(1)})^* \triangleright J\xi) \otimes \overline{v}^{(2)}]$$

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from $(\widehat{V})^*$ to $\widehat{V^*}$ is the inverse of γ.

(c) γ is an isometry of Hilbert spaces. Indeed, using the above definitions, the relation $(J\xi, J\eta)_{\widetilde{L^2(M_1)}} = (G \triangleright \eta, \xi)_{\widetilde{L^2(M_1)}}$ and the unitarity of V^*, we have:

$$
\begin{aligned}
||\gamma([\xi \otimes \overline{v}])||^2_{\widehat{(V)}^*} &= ||\overline{[(v^{(1)} \triangleright J\xi) \otimes v^{(2)}]}||^2_{\widehat{(V)}^*} \\
&= ||v^{(1)} \triangleright J\xi||^2_{\widetilde{L^2(N)}} ||\overline{v}^{(2)}||^2_{V^*} \\
&= ||G^{1/2} S^{-1}(\overline{v}^{(1)}) \triangleright \xi||^2_{\widetilde{L^2(N)}} ||\overline{v}^{(2)}||^2_{V^*} \\
&= (\xi, (\overline{v}^{(1)})^{(2)} S^{-1}((\overline{v}^{(1)})^{(1)}) \triangleright \xi)_{\widetilde{L^2(N)}} (\overline{v}, \overline{v}^{(2)})_{V^*} \\
&= (\xi, S(\varepsilon_t(\overline{v}^{(1)})) \triangleright \xi)_{\widetilde{L^2(N)}} (\overline{v}, \overline{v}^{(2)})_{V^*} \\
&= ||\xi||_{\widetilde{L^2(N)}} (\overline{v}, \varepsilon_t(\overline{v}^{(1)}) \triangleright \overline{v}^{(2)})_{V^*} = ||[\xi \otimes \overline{v}]||^2_{\widehat{V^*}}.
\end{aligned}
$$

According to Remark 5.1, $C_{B_t - B_t}$ is nothing but the category of B-comodules, $\mathrm{Corep}(B)$. The next theorem shows that this category is equivalent to $Bimod_{N-N}(N \subset M)$, the category of $N - N$ bimodules of a subfactor $N \subset M$. Recall that the latter is the tensor category generated by simple subobjects of $_N(M_n)_N$, $n \geq 1$.

Theorem 5.8 *Let $N \subset M$ be a finite depth subfactor ($[M : N] < \infty$), k be a number such that $N \subset M_k$ has depth ≤ 2, and let B be a canonical quantum groupoid such that $(N \subset M_k) \cong (N \subset N \rtimes B)$. Then $Bimod_{N-N}(N \subset M)$ and $Rep(B^*)$ are equivalent as tensor categories.*

Proof. First, we observe that

$$Bimod_{N-N}(N \subset M) = Bimod_{N-N}(N \subset M_l)$$

for any $l \geq 0$. Indeed, since both categories are semisimple, it is enough to check that they have the same set of simple objects. Clearly, all objects of $Bimod_{N-N}(N \subset M_l)$ are objects of $Bimod_{N-N}(N \subset M)$. Conversely, since irreducible $N - N$ subbimodules of $_N L^2(M_i)_N$ are contained in the decomposition of $_N L^2(M_{i+1})_N$ for all $i \geq 0$, we see that objects of $Bimod_{N-N}(N \subset M)$ belong to $Bimod_{N-N}(N \subset M_l)$.

Hence, by Proposition 4.1, the problem can be reduced to the case when $N \subset M$ has depth 2 ($M = N \rtimes B$), i.e., it will suffice to prove that $Bimod_{N-N}(N \subset N \rtimes B)$ is equivalent to $\mathrm{Corep}(B)$. The previous theorem

gives a functor from $\mathrm{Corep}(B) = \mathrm{Rep}(B^*)$ to $Bimod_{N-N}(N \subset N{\rtimes}B)$. To prove that this functor is, in fact, an equivalence, we need to check that it yields a bijection between classes of simple objects of these categories.

Observe that B itself is an object of $\mathrm{Corep}(B)$ via $\Delta : B \to B \otimes B$ and $\widehat{B} = {}_N L^2(M)_N$. Since the inclusion $N \subset M$ has depth 2, the simple objects of $Bimod_{N-N}(N \subset M)$ are precisely irreducible subbimodules of ${}_N L^2(M)_N$. We have $\widehat{B} = {}_N L^2(M)_N = \oplus_i {}_N(p_i L^2(M))_N$, where $\{p_i\}$ is a family of mutually orthogonal minimal projections in $N' \cap M_1$ so that every bimodule $p_i L^2(M)$ is irreducible. On the other hand, B is cosemisimple, hence $B = \oplus_i V_i$, where each V_i is an irreducible subcomodule. Note that $N' \cap M_1 = B^* = \sum p_i B^*$ and every $p_i B$ is a simple submodule of B ($=$ simple subcomodule of B^*). Thus, we see that there is a bijection between simple objects of $\mathrm{Corep}(B)$ and $Bimod_{N-N}(N \subset M)$, so that the categories are equivalent.

The principal and dual principal graphs of a subfactor $N \subset M$ are defined as follows [10], [5], [8]. Let $X = {}_N L^2(M)_M$ and consider the following sequence of $N-N$ and $N-M$ bimodules :

$$ {}_N L^2(N)_N, \ X, \ X \otimes_M X^*, \ X \otimes_M X^* \otimes_N X, \ldots $$

obtained by right tensoring with X^* and X. The vertex set of the principal graph is indexed by the classes of simple bimodules appearing as summands in the the above sequence. We connect vertices corresponding to bimodules ${}_N Y_N$ and ${}_N Z_M$ by l edges if ${}_N Y_N$ is contained in the decomposition of ${}_N Z_N$, the restriction of ${}_N Z_M$, with multiplicity l.

The dual principal graph can be constructed in a similar way from the following $M-M$ and $M-N$ bimodules :

$$ {}_M L^2(M)_M, \ X^*, \ X^* \otimes_N X, \ X^* \otimes_N X \otimes_M X^*, \ldots $$

We apply Theorem 5.7 to express the principal and dual graphs of a finite depth subfactor $N \subset M$ in terms of the quantum groupoid associated with it.

Let B and K be a quantum groupoid and its left coideal $*$-subalgebra such that B acts on N and $(N \subset M) \cong (N \subset N{\rtimes}K)$. Then $\widehat{K} = {}_N L^2(M)_M$, where we view K as a relative (B, K) Hopf module as in Example 5.3

By Theorem 5.7 we can identify irreducible $N-N$ (resp. $N-M$) bimodules with simple B-comodules (resp. relative right (B, K) Hopf modules).

Consider a bipartite graph with vertex set given by the union of (classes of) simple B-comodules and simple relative right (B, K) Hopf modules and the number of edges between the vertices U and V representing B-comodule and relative right (B, K) Hopf module respectively being equal to the multiplicity of U in the decomposition of V (when the latter is viewed as a B-comodule) :

<div align="center">

simple B-comodules

$\cdots\,|\,\cdots\,|\,\cdots$

simple relative right (B, K) Hopf modules

</div>

The principal graph of $N \subset M$ is the connected part of the above graph containing the trivial B-comodule.

Similarly, the dual principal graph can be obtained from the following diagram

<div align="center">

simple relative $(B, K - K)$ Hopf bimodules

$\cdots\,|\,\cdots\,|\,\cdots$

simple relative left (B, K) Hopf modules

</div>

as the connected component containing the relative Hopf $(B, K - K)$ bimodule K (it corresponds to the bimodule $_M L^2(M)_M$.

Using the antiisomorphism $K \mapsto \delta(K)$ between the lattices of left coideal subalgebras of B and B^* from Proposition 3.3, it is possible to express the principal graph of $N \subset N \rtimes K$ as a certain Bratteli diagram.

Proposition 5.9 *If K is a coideal $*$-subalgebra of B then the principal graph of the subfactor $N \subset N \rtimes K$ is given by the connected component of the Bratteli diagram of the inclusion $\delta(K) \subset B^*$ containing the trivial representation of B^*.*

Proof. First, let us show that there is a bijective correspondence between right relative (B, K) Hopf modules and $(B^* \rtimes K)$-modules. Indeed, every right (B, K) Hopf module V carries a right action of K. If we define a right action of B^* by

$$v \triangleleft x = \langle\, v^{(1)}, x\, \rangle v^{(2)}, \quad v \in V, x \in B^*,$$

then we have

$$
\begin{aligned}
(v \triangleleft k) \triangleleft x &= \langle\, v^{(1)} k_{(1)}, x\, \rangle (v^{(2)} \triangleleft k_{(2)}) \\
&= \langle\, v^{(1)}, (k_{(1)} \triangleright x)\, \rangle (v^{(2)} \triangleleft k_{(2)}) \\
&= (v \triangleleft (k_{(1)} \triangleright x)) \triangleleft k_{(2)},
\end{aligned}
$$

<div align="center">27</div>

for all $x \in B^*$ and $k \in K$ which shows that kx and $(k_{(1)} \rhd x)k_{(2)}$ act on V exactly in the same way, therefore V is a right $(B^* \rtimes K)$-module.

Conversely, given an action of $(B^* \rtimes K)$ on V, we automatically have a B-comodule structure such that

$$
\begin{aligned}
\langle v^{(1)}k_{(1)}, x \rangle (v^{(2)} \rhd k_{(2)}) &= \langle v^{(1)}, x_{(1)} \rangle \langle k_{(1)}, x_{(2)} \rangle (v^{(2)} \lhd k_{(2)}) \\
&= (v \lhd (k_{(1)} \rhd x)) \lhd k_{(2)} = (v \lhd k) \lhd x \\
&= \langle x, (v \lhd k)^{(1)} \rangle (v \lhd k)^{(2)},
\end{aligned}
$$

which shows that $v^{(1)}k_{(1)} \otimes (v^{(2)} \lhd k_{(2)}) = (v \lhd k)^{(1)} \otimes (v \lhd k)^{(2)}$, i.e., that V is a right relative (B, K)-module.

Thus, we see that the principal graph is given by the connected component the Bratteli diagram of the inclusion $B^* \subset B^* \rtimes K$ containing the trivial representation of B^*. Recall that $B^* \rtimes K$ is the basic construction for the inclusion $\delta(K) \subset B^*$, therefore the Bratteli diagrams of the above two inclusions are the same.

Corollary 5.10 *If $N \subset N \rtimes B$ is a depth 2 inclusion corresponding to the quantum groupoid B, then its principal graph is given by the Bratteli diagram of the inclusion $B_t^* \subset B^*$.*

Proof. In this case $K = B$ and inclusion $B_t^* \subset B^*$ is connected, so that $\delta(K) = B_t^*$ (note that B^* is biconnected).

Let us mention two properties of the set X_n of finite index values of subfactors with depth $\leq n$.

Remark 5.11 (a) We can use Corollary 5.10 to give a short proof of the fact that for any given n the set X_n is a discrete subset of $\{4\cos^2 \frac{\pi}{n} \mid n \geq 3\} \cup [4, +\infty)$ [9].

It follows from Corollary 4.2 that the index of any depth $\leq n$ subfactor is the n-th root of the index of a depth ≤ 2 subfactor, therefore we have $X_n = \{\sqrt[n]{x} \mid x \in X_2\}$, therefore it suffices to prove that X_2 is discrete.

Let B be a biconnected quantum groupoid and Λ be the inclusion matrix of $B_t \subset B$. We will show that all the entries of $\Lambda\Lambda^t$ are strictly positive. Indeed, let $\pi_1, \ldots \pi_N$ (resp. $\rho_1, \ldots \rho_M$) be all the classes of irreducible representations of B_t (resp. B), and assume that ρ_1 is the trivial representation of B on B_t (i.e., $\rho_1(b)z = \varepsilon_t(bz)$ for all $b \in B$, $z \in B_t$ ([3], 2.4, [12], 2.2). Then Λ_{ij}, the ij-th entry of Λ, is equal to the multiplicity of π_i in $\rho_j|_{B_t}$.

Since $\rho_1|_{B_t}$ is faithful, we have $\Lambda_{i1} > 0$ for all $i = 1 \ldots M$, therefore

$$(\Lambda\Lambda^t)_{ik} = \Sigma_j \, \Lambda_{ij}\Lambda_{kj} \geq \Lambda_{i1}\Lambda_{k1} > 0.$$

Thus, it follows from Corollary 5.10 that every element of X_2 is the norm of a matrix with strictly positive entries. But for any given m the number of such matrices with norm $\leq m$ is clearly finite, hence $X_2 \cap (0, \, m]$ is finite for every m, i.e., X_2 is discrete.

(b) X_n is also a multiplicative subsemigroup of $\mathbb{R}^+$. Indeed, if $N \subset M$ and $P \subset Q$ are two subfactors of depth $\leq n$ then $(N \otimes P) \subset (M \otimes Q)$ has depth $\leq n$ and $[(M \otimes Q) : (N \otimes P)] = [M : N][Q : P]$.

Appendix : The structure of a quantum groupoid associated with a finite depth subfactor

We will write down explicit formulas that define a quantum groupoid canonically associated with a finite depth subfactor $N \subset M$ ($[M : N] = \lambda^{-1}$). It follows from Proposition 4.1 that the subfactor $N \subset M_k$ is of depth 2 for k large enough. According to [16], the Jones tower for the latter inclusion is

$$N \subset M_k \subset M_{2k+1} \subset M_{3k+2} \subset \cdots$$

Therefore, there is a non-degenerate duality between algebras $A = N' \cap M_{2k+1}$ and $B = M_k' \cap M_{3k+2}$ making them quantum groupoids dual to each other [13]. The corresponding bilinear form (cf. Preliminaries) is given by

$$\langle\, a, \, b\,\rangle = \lambda^{-2(k+1)}\tau(af_2f_1Hb), \qquad a \in A, b \in B,$$

where $H = \text{Index}\,\tau|_{M_k' \cap M_{2k+1}}$ and

$$
\begin{aligned}
f_1 &= \lambda^{k(k+1)/2}(e_{k+1}e_k \ldots e_1)(e_{k+2}\ldots e_2)\ldots(e_{2k+1}\ldots e_{k+1}), \\
f_2 &= \lambda^{k(k+1)/2}(e_{2k+2}e_{2k+1}\ldots e_{k+2})(e_{2k+3}\ldots e_{k+3})\ldots(e_{3k+2}\ldots e_{2k+2}),
\end{aligned}
$$

are the Jones projections of the k-step basic construction [16] such that f_1 (resp. f_2) implements the conditional expectation from M_k (resp. M_{2k+1}) to N (resp. M_k).

The target and source counital subalgebras of B are $B_t = M_k' \cap M_{2k+1}$ and $B_s = M_{2k+1}' \cap M_{3k+2}$. Note that B is generated by B_s, B_t, and e_{2k+2} as an algebra. Indeed, $M_k' \cap M_{3k+2} = \langle\, M_k' \cap M_{2k+1}, e_{2k+2}, \ldots e_{3k+2}\,\rangle$ because of the finite depth condition.

The antipode of B is given by

$$S(b) = j(HbH^{-1}), \quad b \in B,$$

where $j(b) = J_{2k+2} b^* J_{2k+2}$ is a canonical $*$-anti-isomorphism of $B = M'_k \cap M_{3k+2}$ (here J_{2k+2} is the modular involution on $L^2(M_{2k+1})$).

It is convenient to describe the comultiplication in terms of separability elements. By definition, a *separability element* ([15], 10.2) of a finite-dimensional C^*-algebra D is a projection $P_D \in D^{op} \otimes D$ uniquely determined by the properties

$$(x_1 \otimes 1)P_D(x_2 \otimes 1) = (x_2 \otimes 1)P_D(x_1 \otimes 1), \quad \text{and} \quad m(P_D) = 1,$$

for all $x_1, x_2 \in D$, where m denotes the multiplication of D.

Let $I = M'_k \cap M_{2k+2}$, then we have

$$\begin{aligned}
\Delta(yz) &= (z \otimes y) \cdot (S \otimes \mathrm{id}) P_{B_t}, \quad y \in B_s, \ z \in B_t, \\
\Delta(e_{2k+2}) &= (S \otimes \mathrm{id}) P_I.
\end{aligned}$$

Indeed, the first formula holds true in every weak Hopf algebra (see, e.g., [12]). To establish the second formula for all $a, c \in A$ we compute (using the notation $P_I = P_I^{(1)} \otimes P_I^{(2)}$) :

$$\begin{aligned}
\langle a, \, H^{-1} S(P_I^{(1)}) \rangle \langle c, \, P_I^{(2)} \rangle &= \\
&= \lambda^{-4(k+1)} \, \tau(a f_2 f_1 S(P_I^{(1)} H)) \, \tau(c f_2 f_1 P_I^{(2)}) \\
&= \lambda^{-4(k+1)} \, \tau(H P_I^{(1)} f_1 f_2 a) \, \tau(c f_2 f_1 P_I^{(2)}) \\
&= \lambda^{-4(k+1)+1} \, \tau(\lambda^{-1} H P_I^{(1)} E_{M'_k \cap M_{2k+2}}(f_1 f_2 a)) \, \tau(c f_2 f_1 P_I^{(2)}) \\
&= \lambda^{-4(k+1)+1} \, \tau(c f_2 f_1 E_{M'_k \cap M_{2k+2}}(f_1 f_2 a)) \\
&= \lambda^{-3(k+1)+1} \, \tau(c f_2 f_1 E_{M_{2k+2}}(f_1 E_{M'_k}(f_1 f_2 a))) \\
&= \lambda^{-2(k+1)} \, \tau(c f_2 f_1 e_{2k+2} a) = \langle ac, \, e_{2k+2} \rangle,
\end{aligned}$$

where we used that $\mathrm{Index}\, \tau|_{M'_k \cap M_{2k+2}} = \lambda^{-1} H$ and that $E_{M_{2k+2}}(f_2) = \lambda^k e_{2k+2}$. Thus, $\Delta(H^{-1} e_{2k+2}) = (H^{-1} \otimes 1) \cdot (S \otimes \mathrm{id}) P_I$ and, therefore, $\Delta(e_{2k+2}) = (S \otimes \mathrm{id}) P_I$.

Finally, the counit is given by

$$\varepsilon(b) = \lambda^{-(k+1)} \tau(f_2 H b), \quad b \in B.$$

Note that $I = M_k' \cap M_{2k+2}$ is a left coideal $*$-subalgebra in B and that

$$(N \subset M) \cong (M_{2k+1} \subset M_{2k+2}) \cong (M_k \subset M_k{\rtimes}I).$$

An example of a quantum groupoid of dimension 13, associated to the subfactor with index $4\cos^2\frac{\pi}{5}$, was considered in ([13], 7.3). One can also describe the quantum groupoids corresponding to the whole sequence of subfactors with principal graphs A_n [9].

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