THE HYPERKÄHLER GEOMETRY ASSOCIATED TO WOLF SPACES

PIOTR KOBAK AND ANDREW SWANN

1. Introduction

One of the glories of homogeneous geometry is Cartan's classification of the compact Riemannian symmetric spaces [5, 6]. Many manifolds that play a central rôle in geometry are symmetric and it is fascinating to look for patterns in the presentations G/H. One obvious family is provided by the sphere $S^n = SO(n+1)/SO(n)$, complex projective space $\mathbb{C}\mathrm{P}(n) = U(n+1)/(U(n)\,U(1))$, quaternionic projective space $\mathbb{H}\mathrm{P}(n) = Sp(n+1)/(Sp(n)\,Sp(1))$ and the Cayley projective plane $F_4/Spin(9)$. Another consists of the Hermitian symmetric spaces: these are of the form $G/(U(1)L)$ (see [4]). However, the most surprising is the family of quaternionic symmetric spaces $W(G) := G/(Sp(1)K)$, which has the feature that there is precisely one example for each compact simple simply-connected Lie group G. The manifolds in this last family have become known as Wolf spaces following [14]. Alekseevsky [1] proved that they are the only homogeneous positive quaternionic Kähler manifolds (cf. [2]).

Wolf showed that the quaternionic symmetric spaces may be constructed by choosing a highest root α for $\mathfrak{g}^{\mathbb{C}}$. The corresponding root vector E_α is a nilpotent element in $\mathfrak{g}^{\mathbb{C}}$. In [13] it was shown that there is a fibration of the nilpotent adjoint orbit $\mathcal{O}_{\min} = G^{\mathbb{C}} \cdot E_\alpha$ over the Wolf space $W(G)$.

Nilpotent orbits $\mathcal{O}$ in $\mathfrak{g}^{\mathbb{C}}$ have a rich and interesting geometry. Firstly, they are complex submanifolds of $\mathfrak{g}^{\mathbb{C}}$ with respect to the natural complex structure I. Secondly, the construction of Kirillov, Kostant and Souriau endows them with a $G^{\mathbb{C}}$-invariant complex symplectic form ω_c. It is natural to ask whether one can find a metric making the orbit hyperKähler, i.e., can one find a Riemannian metric g on $\mathcal{O}$, such that the real and imaginary parts of ω_c are Kähler forms with respect to complex structures J and K satisfying $IJ = K$. By identifying $\mathcal{O}$ with a moduli space of solutions to Nahm's equations, Kronheimer [12] showed that there is indeed such a hyperKähler metric on $\mathcal{O}$. This hyperKähler structure is invariant under the compact group G, and has the important additional property that it admits [13] a hyperKähler potential ρ: a function that is simultaneously a Kähler potential with respect to I, J and K. Using ρ, one can define an action of $\mathbb{H}^*$ on $\mathcal{O}$ such that the quotient is a quaternionic Kähler manifold. It is in this way that one may obtain the Wolf space $W(G)$ from $\mathcal{O}_{\min}$. In contrast to the semi-simple case [3], currently one does not know how many invariant hyperKähler metrics a given nilpotent orbit admits.

The aim of this paper is to study the hyperKähler geometry of $\mathcal{O}_{\min}$ in an elementary way. We look for all hyperKähler metrics on $\mathcal{O}_{\min}$ with a G-invariant Kähler potential and which are compatible with the complex symplectic structure. Note that we do not restrict our attention to metrics with hyperKähler potentials. We derive a simple formula for the a priori unknown complex structure J. The orbit $\mathcal{O}_{\min}$ is particularly straight-forward to study in this way, since G acts with orbits of codimension one. This means that the metrics we obtain are already known, they are covered by the classification [7], but it is interesting to see how these metrics can be constructed directly form their potentials. In agreement with the classification, the hyperKähler structure is found to be unique, unless $\mathfrak{g} = \mathfrak{su}(2)$, in which case one obtains a one-dimensional family of metrics, the Eguchi-Hanson metrics.

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2. Definitions

On the simple complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$, let $\langle \cdot, \cdot \rangle$ be the *negative* of the Killing form and let σ be a real structure giving a compact real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$. An element X of $\mathfrak{g}^{\mathbb{C}}$ is said to be nilpotent if $(\mathrm{ad}_X)^k = 0$ for some integer k. Let $\mathcal{O}$ be the orbit of a nilpotent element X under the adjoint action of $G^{\mathbb{C}}$. At $X \in \mathcal{O}$, the vector field generated by A in $\mathfrak{g}^{\mathbb{C}}$ is $\xi_A = [A, X]$. Using the Jacobi identity it is easy to see that these vector fields satisfy $[\xi_A, \xi_B] = \xi_{-[A,B]}$, for $A, B \in \mathfrak{g}^{\mathbb{C}}$. The orbit $\mathcal{O}$ is a complex submanifold of the complex vector space $\mathfrak{g}^{\mathbb{C}}$ and so has a complex structure I given by $I\xi_A = i\xi_A = \xi_{iA}$.

On a hyperKähler manifold M with complex structures I, J and K and metric g, we define Kähler two-forms by $\omega_I(X, Y) = g(X, IY)$, etc., for tangent vectors X and Y. The condition that a function $\rho\colon M \to \mathbb{R}$ be a Kähler potential for I is

$$\omega_I = -i\partial_I\overline{\partial_I}\rho = -id\overline{\partial_I}\rho = -\tfrac{i}{2}d(d - iId)\rho = -\tfrac{1}{2}dId\rho. \tag{2.1}$$

On the orbit $\mathcal{O}$, the complex symplectic form of Kirillov, Kostant and Souriau is given by $\omega_c(\xi_A, \xi_B)_X = \langle X, [A, B] \rangle = -\langle \xi_A, B \rangle$.

We will be looking for hyperKähler structures with Kähler potential ρ and such that $\omega_c = \omega_J + i\omega_K$. This will be done by computing the Riemann metric g defined by ρ via (2.1) and then using this to determine an endomorphism J of $T_X\mathcal{O}$ via $\omega_J = g(\cdot, J\cdot)$. The constraints on ρ will come from the two conditions that g is positive definite and that $J^2 = -1$.

3. Highest Roots and Minimal Orbits

Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}^{\mathbb{C}}$. Fix a system of roots Δ with positive roots Δ_+. We write $\mathfrak{g}_\beta$ for the root space of $\beta \in \Delta$. Choose a Cartan basis $\{E_\beta, H_\beta, F_\beta : \beta \in \Delta_+\}$, which we may assume is compatible with the real structure σ, in the sense that $\sigma(E_\beta) = -F_\beta$ and $\sigma(H_\beta) = -H_\beta$. One important property of the Cartan basis is that for each β, $\mathrm{Span}_{\mathbb{C}}\{E_\beta, H_\beta, F_\beta\}$ is a subalgebra of $\mathfrak{g}^{\mathbb{C}}$ isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.

The Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ has Cartan basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.1}$$

The irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$ are the symmetric powers $S^k = S^k\mathbb{C}^2$ of the fundamental representation $S^1 = \mathbb{C}^2$. The representation S^k has dimension $k+1$ and E, H and F act as

$$\varphi_E = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & \ddots & \ddots & \\ & & & 0 & k \\ & & & & 0 \end{pmatrix}, \quad \varphi_H = \begin{pmatrix} k & & & & \\ & k-2 & & & \\ & & \ddots & & \\ & & & 2-k & \\ & & & & -k \end{pmatrix} \tag{3.2}$$

$$\text{and} \quad \varphi_F = \begin{pmatrix} 0 & & & & \\ k & 0 & & & \\ & \ddots & \ddots & & \\ & & 2 & 0 & \\ & & & 1 & 0 \end{pmatrix}$$

respectively. In particular, $(\varphi_E)^{k+1} = 0$ and $(\varphi_E)^k$ has rank one, with image the k-eigenspace of φ_H.

Let $\alpha \in \Delta_+$ be a highest root; this is characterised by the condition $[E_\alpha, E_\beta] = 0$ for all $\beta \in \Delta_+$. We define $\mathcal{O}_{\min}$ to be the adjoint orbit of E_α under the action of $G^{\mathbb{C}}$. Define $\mathfrak{sl}(2,\mathbb{C})_\alpha := \mathrm{Span}_{\mathbb{C}}\{E_\alpha, H_\alpha, F_\alpha\}$.

Proposition 3.1. (i) *Under the action of $\mathfrak{sl}(2,\mathbb{C})_\alpha$ the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ decomposes as*

$$\mathfrak{g}^{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C})_\alpha \oplus \mathfrak{k}^{\mathbb{C}} \oplus (V \otimes S^1),$$

where $\mathfrak{k}^{\mathbb{C}}$ is the centraliser of $\mathfrak{sl}(2,\mathbb{C})$, V is a $\mathfrak{k}^{\mathbb{C}}$-module.

(ii) *The action of the compact group G on the nilpotent orbit $\mathcal{O}_{min}$ has cohomogeneity one.*

Proof. (i) Consider the action of $\operatorname{ad} E_\alpha$ on $\mathfrak{g}^{\mathbb{C}}$. For $\beta \in \Delta_+$, we have $[E_\alpha, F_\beta] \in \mathfrak{g}_{\alpha-\beta}$. If $\beta \neq \alpha$, then we have two cases: (a) if $\alpha - \beta$ is not a root then $\mathfrak{g}_{\alpha-\beta} = \{0\}$ and $[E_\alpha, F_\beta] = 0$; (b) if $\alpha - \beta$ is a root, then the condition that α is a highest root implies $\alpha - \beta \in \Delta_+$, since otherwise $\alpha - \beta = -\gamma$ for some $\gamma \in \Delta_+$ and then $[E_\alpha, E_\gamma]$ is non-zero, which for a highest root α is impossible. We therefore have that $(\operatorname{ad} E_\alpha)^2$ is zero on the complement of $\mathfrak{sl}(2,\mathbb{C})_\alpha$ and the decomposition follows.

(ii) At E_α the tangent space to $\mathcal{O}_{\min}$ is

$$\operatorname{ad}_{E_\alpha} \mathfrak{g}^{\mathbb{C}} = \operatorname{Span}_{\mathbb{C}} \{E_\alpha, H_\alpha\} + \operatorname{Span}_{\mathbb{C}} \{E_{\alpha-\beta} : \beta \in \Delta_+\}.$$

The real Lie algebra $\mathfrak{g}$ is the real span of $\{E_\beta - F_\beta, iH_\beta, i(E_\beta + F_\beta)\}$. Thus the tangent space $\operatorname{ad}_{E_\alpha} \mathfrak{g}$ to the G-orbit is

$$\operatorname{Span}_{\mathbb{R}} \{iE_\alpha, H_\alpha, iH_\alpha\} + \operatorname{Span}_{\mathbb{R}} \{E_{\alpha-\beta}, iE_{\alpha-\beta} : \beta \in \Delta_+\}$$

and we see that it has codimension one in $T_{E_\alpha}\mathcal{O}_{\min}$, the complement being $\mathbb{R}E_\alpha$. As G is compact, this implies G acts with cohomogeneity one. □

As in [8], it is possible to use this result to show that $\mathcal{O}_{\min}$ is the minimal with respect to the partial order on nilpotent orbits given by inclusions of closures. This explains the name $\mathcal{O}_{\min}$, but will not be needed in the subsequent discussion.

4. Kähler Potentials in Cohomogeneity One

Let $\rho\colon \mathcal{O}_{\min} \to \mathbb{R}$ be a smooth function invariant under the action of the compact group G. The group G acts with cohomogeneity one, and the function $\eta(X) = \|X\|^2 = \langle X, \sigma X\rangle$ is G-invariant and distinguishes orbits of G. We may therefore assume that ρ is just a function of η, i.e., $\rho = \rho(\eta)$.

We wish to consider ρ as a Kähler potential for the complex manifold $(\mathcal{O}_{\min}, I)$. The corresponding Kähler form is given by (2.1):

$$\omega_I = -\tfrac{1}{2} d(\rho' I d\eta) = -\tfrac{1}{2}\rho' dId\eta - \tfrac{1}{2}\rho'' d\eta \wedge Id\eta, \tag{4.1}$$

where $\rho' = d\rho/d\eta$, etc.

Lemma 4.1. *The Kähler form defined by $\rho(\eta)$ is*

$$\omega_I(\xi_A, \xi_B) = 2 \operatorname{Im} \left(\rho' \langle \xi_A, \sigma\xi_B\rangle + \rho'' \langle \xi_A, \sigma X\rangle \langle \sigma\xi_B, X\rangle\right). \tag{4.2}$$

Proof. The exterior derivative of η is

$$d\eta(\xi_A)_X = \langle [A, X], \sigma X\rangle + \langle X, \sigma[A, X]\rangle = 2 \operatorname{Re} \langle \xi_A, \sigma X\rangle \tag{4.3}$$

so $Id\eta(\xi_A)_X = 2 \operatorname{Im} \langle \xi_A, \sigma X\rangle$ and hence

$$(d\eta \wedge Id\eta)(\xi_A, \xi_B) = -4 \operatorname{Im} \left(\langle \xi_A, \sigma X\rangle \langle \sigma\xi_B, X\rangle\right).$$

Using the Jacobi identity we find that the exterior derivative of $Id\eta$ is given by

$$\begin{aligned} dId\eta(\xi_A,\xi_B)_X &= \xi_A(Id\eta(\xi_B)) - \xi_B(Id\eta(\xi_A)) - Id\eta([\xi_A,\xi_B]) \\ &= 2\operatorname{Im}\langle \xi_B, \sigma\xi_A\rangle + 2\operatorname{Im}\langle [B,[A,X]], \sigma X\rangle \\ &\qquad - 2\operatorname{Im}\langle \xi_A, \sigma\xi_B\rangle - 2\operatorname{Im}\langle [A,[B,X]], \sigma X\rangle \\ &\qquad + 2\operatorname{Im}\langle [[A,B],X], \sigma X\rangle \\ &= -4\operatorname{Im}\langle \xi_A, \sigma\xi_B\rangle \end{aligned}$$

Putting these expressions into (4.1) gives the result. □

Using the relation $g(\xi_A,\xi_B) = \omega_I(I\xi_A,\xi_B)$, we can now obtain the induced metric on $\mathcal{O}_{\min}$. In general, this metric will be indefinite; the signature may be determined by considering $\operatorname{Span}_{\mathbb{R}}\{X,\sigma X\}$ and its orthogonal complement with respect to the Killing form.

Proposition 4.2. *The pseudo-Kähler metric defined by* $\rho(\eta)$ *is*

$$g(\xi_A,\xi_B) = 2\operatorname{Re}\left(\rho'\langle \xi_A,\sigma\xi_B\rangle + \rho''\langle \xi_A,\sigma X\rangle\langle \sigma\xi_B, X\rangle\right). \tag{4.4}$$

This is positive definite if and only if $\rho' > \max\{0, -\eta\rho''\}$. □

5. HyperKähler Metrics

Given a function $\rho(\eta)$ on $\mathcal{O}_{\min}$ we have obtained a metric g. Let us assume that g is non-degenerate. Using the definition of ω_c and its splitting into real imaginary parts, we get endomorphisms J and K of $T_X\mathcal{O}_{\min}$ via

$$g(\xi_A,\xi_B) = \omega_J(J\xi_A,\xi_B) = -\operatorname{Re}\langle J\xi_A, B\rangle,$$

etc. This implies that

$$J_X\xi_A = -2\rho'[X,\sigma\xi_A] - 2\rho''\langle \sigma\xi_A, X\rangle [X,\sigma X]. \tag{5.1}$$

and $K = IJ$. Note that (5.1) implies $JI = -K$.

Suppose $J^2 = -1$ and that g is positive definite. Then we have I, J and K satisfying the quaternion identities, and with ω_I, ω_J and ω_K closed two-forms. By a result of Hitchin [10], this implies that I, J and K are integrable and that g is a hyperKähler metric.

Proposition 5.1. *The nilpotent orbit of* $\mathfrak{sl}(2,\mathbb{C})$ *has a one-parameter family of hyperKähler metrics with* $SU(2)$*-invariant Kähler potential and compatible with the Kostant-Kirillov-Souriau complex symplectic form* ω_c.

Proof. The algebra $\mathfrak{sl}(2,\mathbb{C})$ has only one nilpotent orbit $\mathcal{O} = \mathcal{O}_{\mathrm{min}}$ and this has real dimension 4. Using the action of $SU(2)$ we may assume that $X = tE$, where $t > 0$ and E is given by (3.1). Then $T_X\mathcal{O}$ is spanned by H and E. We have $J_XH = -4\rho' t\, E$ and $J_XE = 2t(\rho'+\eta\rho'')H$, which implies $J^2 = -\operatorname{Id}$ if and only if $8t^2({\rho'}^2+\eta\rho'\rho'') = 1$. Now $\eta(E) = 4$, so we get the following ordinary differential equation for ρ:

$$2(\eta{\rho'}^2 + \eta^2\rho'\rho'') = 1.$$

The left-hand side of this equation is $(\eta^2{\rho'}^2)'$, so $\rho' = \sqrt{\eta + c}/\eta$, for some real constant c. For this to be defined for all positive η, we need $c \geqslant 0$. Now $\rho'' = -(\eta + 2c)/(2\eta^2\sqrt{\eta + c})$, so the metric is

$$\begin{aligned} g(\xi_A, \xi_B) = \frac{1}{\eta^2\sqrt{\eta + c}} \operatorname{Re}\bigl(2\eta(\eta + c)\, \langle \xi_A, \sigma\xi_B\rangle \\ - (\eta + 2c)\, \langle \xi_A, \sigma X\rangle\, \langle \sigma\xi_B, X\rangle\bigr), \end{aligned} \tag{5.2}$$

which is positive definite. □

This hyperKähler metric is of course well-known. We put it in standard form as follows. Using (4.3), we find $(\partial/\partial\eta) = E/(8t)$ at $X = tE$. An $SU(2)$-invariant basis of $T_X\mathcal{O}$ is now given by $\{\partial/\partial\eta, \xi_{s_1}, \xi_{s_2}, \xi_{s_3}\}$, where

$$s_1 = \tfrac{1}{2}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right),\ s_2 = \tfrac{1}{2}\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right),\ s_3 = \tfrac{1}{2}\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right).$$

This basis is orthogonal with respect to (5.2) and in terms of the dual basis of one-forms is $\{d\eta, \sigma_1, \sigma_2, \sigma_3\}$, g is

$$\frac{1}{4\eta^2\rho'}d\eta^2 + \eta\rho'\left(\sigma_1^2 + \sigma_2^2\right) + \frac{1}{\rho'}\sigma_3^2.$$

Substituting $\eta = (r/2)^4 - c$, we get

$$g = W^{-1}dr^2 + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2 + W\sigma_3^2),$$

with $W = 1 - 16c/r^4$, which are the Eguchi-Hanson metrics [9].

Theorem 5.2. *For $\mathfrak{g}^{\mathbb{C}} \neq \mathfrak{sl}(2,\mathbb{C})$, the minimal nilpotent orbit $\mathcal{O}_{min}$ admits a unique hyperKähler metric with G-invariant Kähler potential compatible with the complex symplectic form ω_c.*

Proof. Let α be a highest root. Using the action of G, we may assume that $X = tE_\alpha$, for some $t > 0$. On $\xi_A \in \mathfrak{sl}(2,\mathbb{C})_\alpha$, the condition $J^2 = -\operatorname{Id}$ gives $8t^2({\rho'}^2 + \eta\rho'\rho'') = 1$, as in Proposition 5.1. Putting

$\lambda^2 = \eta(E_\alpha)$, we have $t^2 = \eta(X)/\lambda^2$ and hence $\rho' = \sqrt{\lambda^2\eta + c}/2\eta$. Now for ξ_A Killing-orthogonal to $\mathfrak{sl}(2,\mathbb{C})$, we have

$$J\xi_A = -2\rho'[X, \sigma\xi_A] = -2t\rho'[E_\alpha, \sigma\xi_A]$$

and hence

$$J^2\xi_A = -(4\eta\rho'^2/\lambda^2)\,\mathrm{ad}_{E_\alpha}\mathrm{ad}_{F_\alpha}\xi_A = -\left(1 + \frac{c}{\lambda^2\eta}\right)\mathrm{ad}_{E_\alpha}\mathrm{ad}_{F_\alpha}\xi_A.$$

As η is not constant, the condition $J^2 = -\mathrm{Id}$ implies $c = 0$ and we have a unique hyperKähler metric. □

The proof enables us to write down J explicitly for $\mathcal{O}_{\min}$ in $\mathfrak{g}^{\mathbb{C}} \neq \mathfrak{sl}(2,\mathbb{C})$:

$$J_X\xi_A = -\frac{\lambda}{2\eta^{3/2}}\left(2\eta[X,\sigma\xi_A] - \langle\sigma\xi_A, X\rangle\,[X,\sigma X]\right).$$

The number λ^2 is a constant depending only on the Lie algebra $\mathfrak{g}^{\mathbb{C}}$, with values $2n$ ($\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{sp}(n-1,\mathbb{C})$, $\mathfrak{so}(n+2,\mathbb{C})$), 8 ($G_2$), 18 ($F_4$), 24 ($E_6$), 36 ($E_7$), 70 ($E_8$).

Remark 5.3. Theorem 5.2 only assumes that ρ is a Kähler potential. However, the uniqueness result implies that this potential is in fact hyperKähler (cf. [13]). This corresponds to Proposition 5.1, where ρ is a hyperKähler potential only when $c = 0$.

Finally, let us observe that the form of the potential determines the nilpotent orbit.

Proposition 5.4. *If a nilpotent orbit $\mathcal{O}$ has a Kähler potential ρ that is only a function of $\eta = \|X\|^2$ and which defines a hyperKähler structure compatible with ω_c, then $\mathcal{O}$ is a minimal nilpotent orbit.*

Proof. Choose $X \in \mathcal{O}$, such that $\mathrm{Span}_{\mathbb{C}}\{X, \sigma X, [X,\sigma X]\}$ is a subalgebra isomorphic to $\mathfrak{sl}(2,\mathbb{C})$; this is always possible by a result of Borel (cf. [11]). Let $X = tE$, for $t > 0$, and write $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{m}$. The proofs of Proposition 5.1 and Theorem 5.2 imply that $\rho' = \lambda\eta^{-1/2}/2$ and $J^2\xi_A = -\mathrm{ad}_E\mathrm{ad}_F\xi_A$ on $\mathfrak{m}$. Let S^k, $k > 0$, be an irreducible $\mathfrak{sl}(2,\mathbb{C})$-summand of $\mathfrak{m}$. Then ad_E and ad_F act via the matrices φ_E and φ_F of (3.2), so $\mathrm{ad}_E\mathrm{ad}_F$ acts as a diagonal matrix with entries k, $2(k-1)$, $3(k-2)$, … , $(k-1)2$, k and 0. As ξ_A is in the image of ad_E, in order to have $J^2\xi_A = -\xi_A$, we need all the non-zero eigenvalues of $\mathrm{ad}_E\mathrm{ad}_F$ to be 1. This forces $k = 1$.

Let $\mathfrak{g}(i)$ be the i-eigenspace of ad_H on $\mathfrak{g}^{\mathbb{C}}$. Then $\mathfrak{p} = \bigoplus_{i\geqslant 0}\mathfrak{g}(i)$ is a parabolic subalgebra, so we may choose a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$ lying in $\mathfrak{p}$ and a root system such that the positive root spaces are also in $\mathfrak{p}$. The discussion above shows that ad_E is zero on all these positive root spaces, and so E is a highest root vector. Therefore $\mathcal{O} = \mathcal{O}_{\min}$. □

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(Kobak) Insytut matematyki, Uniwersytet Jagielloński, ul. Reymonta 4, 30-059 Kraków, Poland

E-mail address: `kobak@im.uj.edu.pl`

(Swann) Department of Mathematical Sciences, University of Bath, Claverton Down, Bath BA2 7AY, England

E-mail address: `A.F.Swann@maths.bath.ac.uk`

Current address: From 1/9/99: Department of Mathematics and Computer Science, SDU, Odense University, Campusvej 55, DK-5230 Odense M, Denmark

E-mail address: `swann@imada.sdu.dk`