arXiv:math/0001027v1 [math.DG] 5 Jan 2000

HYPERKÄHLER POTENTIALS VIA FINITE-DIMENSIONAL QUOTIENTS

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Abstract. It is known that nilpotent orbits in a complex simple Lie algebra admit hyperKähler metrics with a single function that is a global potential for each of the Kähler structures (a hyperKähler potential). In an earlier paper the authors showed that nilpotent orbits in classical Lie algebras can be constructed as finite-dimensional hyperKähler quotient of a flat vector space. This paper uses that quotient construction to compute hyperKähler potentials explicitly for orbits of elements with small Jordan blocks. It is seen that the Kähler potentials of Biquard and Gauduchon for $\mathsf{SL}(n,\mathbb{C})$-orbits of elements with $X^2 = 0$, are in fact hyperKähler potentials.

1. Introduction

Adjoint orbits in complex semi-simple Lie algebras are known to carry a compatible hyperKähler metric invariant under the compact group action (see [18, 17, 16, 2]). Nilpotent orbits are particularly interesting as they admit a hyperKähler structure which is closely related to twistor spaces and quaternion-Kähler geometries [20] and which comes equipped with a hyperKähler potential. If one only asks for a Kähler potential compatible with the hyperKähler structure, then several examples are known. Hitchin [8] gave an expression for a global Kähler potential for a hyperKähler structure on the regular semi-simple orbit of $\mathfrak{sl}(n,\mathbb{C})$ in terms of theta functions. Biquard and Gauduchon [3] determined a simple formula for the Kähler potential for the hyperKähler metric on semi-simple orbits of *symmetric type*. These orbits come in continuous families and by taking a limit Biquard and Gauduchon also obtain Kähler potentials for certain nilpotent orbits.

In [15, 13], Kähler and hyperKähler potentials were obtained for orbits of cohomogeneity one and two by considering the invariants preserved by the compact group action. The cohomogeneity of a complex

1991 *Mathematics Subject Classification.* (2000 version) Primary 53C26; Secondary 53D20, 14L35.

Key words and phrases. HyperKähler manifold, hyperKähler potential, hyperKähler quotient, classical Lie algebras, nilpotent orbit.

orbit $\mathcal{O} \subset \mathfrak{g}^{\mathbb{C}}$ is defined as the codimension of the generic orbits of the compact group G on $\mathcal{O}$. As the cohomogeneity increases, we move further away from homogeneous manifolds and the geometry of the orbits becomes more complicated.

But there are other ways of rating the level of complexity of nilpotent orbits. In the case when each simple component of $\mathfrak{g}^{\mathbb{C}}$ is classical (i.e., equals $\mathfrak{su}(n,\mathbb{C})$, $\mathfrak{so}(n,\mathbb{C})$, or $\mathfrak{sp}(n,\mathbb{C})$) it can be shown that nilpotent orbits in $\mathfrak{g}^{\mathbb{C}}$ arise as hyperKähler reductions of the flat hyperKähler spaces $\mathbb{H}^N$ (see [11]). This gives a more explicit description of the hyperKähler metric and the corresponding potential, as the latter comes simply from the radial function r^2 on $\mathbb{H}^N$. The space $\mathbb{H}^N$ in the construction arises from a *diagram* of unitary vector spaces; the longer the diagram, the more complicated the geometry of the orbit. But even orbits that arise from the simplest diagrams (i.e., those of length 2) may have arbitrary high cohomogeneity, which puts them beyond the scope of the "low cohomogeneity approach" mentioned above. In [10], we successfully applied this technique to construct the hyperKähler potential for the regular nilpotent orbit in $\mathfrak{sl}(3,\mathbb{C})$, which has cohomogeneity 4. The aim of this paper is to apply the same construction to calculate hyperKähler potentials for nilpotent orbits with diagrams of length two or three. This includes classical orbits of cohomogeneity one or two and also all orbits obtainable as limits of semi-simple orbits of symmetric type. In particular, we are able to prove (in the $\mathfrak{sl}(n,\mathbb{C})$ case) that the Kähler potentials obtained by Biquard and Gauduchon on nilpotent orbits are in fact hyperKähler potentials. This is not apparent from their work, particularly because we found in [13] that several of these orbits admit families of invariant hyperKähler metrics with Kähler potentials. We also determine the potential for orbits in $\mathfrak{so}(n,\mathbb{C})$ which have length three diagrams and Jordan type $(3, 2^{2k}, 1^{\ell})$. In the simplest cases there is a striking resemblance to the formulæ we have for the cohomogeneity two case, but for $k \geqslant 2$ matters complicate rapidly.

In the calculations we use finite covering maps between nilpotent orbits and the Beauville bundle construction. It is worth pointing out that these techniques combined with knowledge of the invariants of the compact group action can be used to find the potential in several other cases, for example for nilpotent orbits in the exceptional Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$ (see [14]).

Explicit knowledge of hyperKähler potentials is of interest in the study of real nilpotent orbits, cf. [5], and we expect to pursue this in future work.

The paper is organised as follows. Section 2 recalls the hyperKähler quotient construction of classical nilpotent orbits and gives some general results on hyperKähler potentials. In section 3 we derive formulæ for the potential for orbits with diagrams of length 2 and then, in section 4, apply the result to the low cohomogeneity case. Finally, in section 5 we work out the potential for the simplest orbits with diagrams of length 3.

Acknowledgements. We are grateful for financial support from the EPSRC of Great Britain and KBN in Poland.

2. Background and General Results

We begin by reviewing the general theory of the relationship between hyperKähler quotients, hyperKähler potentials and nilpotent orbits.

A Riemannian manifold (N, g) with complex structures I, J and K satisfying the quaternion identities $IJ = K = -JI$, etc., is *hyperKähler* if g is Hermitian with respect to each of the complex structures and the two-forms $\omega_I(X, Y) := g(X, IY)$, ω_J and ω_K are closed. Such a manifold is thus symplectic in three different ways. If one distinguishes the complex structure I, then N becomes a Kähler manifold with a holomorphic symplectic two-form $\omega_c := \omega_J + i\omega_K$.

An interesting general problem is to find hyperKähler structures compatible with a given complex structure I and a holomorphic symplectic form ω_c. One natural source of such manifolds is adjoint orbits $\mathcal{O}$ of a complex semi-simple Lie group $G^{\mathbb{C}}$. Such an orbit inherits a complex structure I as a submanifold of the complex vector space $\mathfrak{g}^{\mathbb{C}}$. The complex symplectic form on $\mathcal{O}$ is given at $X \in \mathcal{O}$ by

$$\omega_c^{\mathcal{O}}([A, X], [B, X]) = \langle X, [A, B] \rangle ,$$

where $\langle \cdot, \cdot \rangle$ is the *negative* of the Killing form on $\mathfrak{g}^{\mathbb{C}}$. If G is a compact real form of $G^{\mathbb{C}}$, then $\mathcal{O}$ admits a G-invariant hyperKähler structure compatible with I and $\omega_c^{\mathcal{O}}$ [18, 17, 16, 2].

The Marsden-Weinstein quotient construction was adapted to hyperKähler manifolds in [9]. Suppose a Lie group H acts on a hyperKähler manifold N preserving g, I, J and K. Suppose also that there exist symplectic moment maps μ_I, μ_J and μ_K from N to $\mathfrak{h}^*$ for the action of H with respect to the symplectic forms ω_I, ω_J and ω_K. For I, this means that for each $V \in \mathfrak{h}$, the function $\mu_I^V := \langle \mu_I, V \rangle$ satisfies

$$d\mu_I^V = \xi_V \lrcorner\, \omega_I, \tag{2.1}$$

where ξ_V is the vector field generated by the action of V. We then define a *hyperKähler moment map* by

$$\mu \colon N \to \mathfrak{h}^* \otimes \operatorname{Im} \mathbb{H}, \qquad \mu = \mu_I i + \mu_J j + \mu_K k.$$

The *hyperKähler quotient* of N by H is defined to be

$$N /\!/\!/ H := \mu^{-1}(0)/H.$$

If H acts freely on N, then $N /\!/\!/ H$ is a hyperKähler manifold of dimension $\dim N - 4\dim H$. Even if the action of H is not free, there is a natural way to write $N /\!/\!/ H$ as a union of hyperKähler manifolds [6]. We will often distinguish the complex structure I and write $\mu = (\mu_{\mathbb{C}}, \mu_{\mathbb{R}})$, where $\mu_{\mathbb{C}} = \mu_J + i\mu_K$ and $\mu_{\mathbb{R}} = \mu_I$. The map $\mu_{\mathbb{C}}$ is then a complex symplectic moment map for the (infinitesimal) action of $H^{\mathbb{C}}$ on N.

For nilpotent orbits in the classical Lie algebras, a G-invariant hyperKähler metric may be constructed by finite-dimensional hyperKähler quotients [11]. The only other orbits for which such a construction is known are the semi-simple orbits in $\mathfrak{sl}(n,\mathbb{C})$ [19] together with finite quotients of a couple of orbits in exceptional algebras [12]. Let us briefly recall the construction for nilpotent orbits.

2.1. **Nilpotent Orbits for Special Linear Groups.** Given a nilpotent element $A \in \mathfrak{sl}(n,\mathbb{C})$ such that $A^{k-1} \neq 0$ and $A^k = 0$ one defines the associated *image flag* to be $\{0\} = V_0 \rightleftarrows V_1 \rightleftarrows V_2 \rightleftarrows \cdots \rightleftarrows V_k = \mathbb{C}^n$, where $V_i = \operatorname{Im} A^{k-i}$. We consider the complex vector space

$$W = \bigoplus_{i=0}^{k-1} \bigl(\operatorname{Hom}(V_i, V_{i+1}) \oplus \operatorname{Hom}(V_{i+1}, V_i)\bigr) \tag{2.2}$$

and represent elements $(\dots, \alpha_i, \beta_i, \dots)$ of W by diagrams

$$\{0\} = V_0 \underset{\beta_0}{\overset{\alpha_0}{\rightleftarrows}} V_1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} V_2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{k-1}}{\overset{\alpha_{k-1}}{\rightleftarrows}} V_k = \mathbb{C}^n.$$

Taking $\mathbb{C}^n$ to be equipped with a Hermitian two-form, induces Hermitian inner products on each V_i, $i = 0, 1, 2, \dots, k$, and we get a norm on W given by

$$r^2 = \|(\dots, \alpha_i, \beta_i, \dots)\|^2 = \sum_{i=1}^{k-1} \operatorname{Tr}(\alpha_i^*\alpha_i + \beta_i\beta_i^*). \tag{2.3}$$

The inner products enables us to make sense of Hermitian adjoints α_i^* and β_i^* and to endow the vector space W with a quaternionic structure by defining $j(\dots, \alpha_i, \beta_i, \dots) = (\dots, -\beta_i^*, \alpha_i^*, \dots)$.

The product $H = \mathsf{U}(V_1) \times \cdots \times \mathsf{U}(V_{k-1})$ of unitary groups acts in a natural way on W:

$$\begin{aligned}(a_1, \dots, a_{k-1})&(\dots, \alpha_i, \beta_i, \dots, \alpha_{k-1}, \beta_{k-1})\\ &= (\dots, a_{i+1}\alpha_i a_i^{-1}, a_i\beta_i a_{i+1}^{-1}, \dots, \alpha_{k-1}a_{k-1}^{-1}, a_{k-1}\beta_{k-1}).\end{aligned}$$

This action preserves the quaternionic structure on W, and the hyperKähler moment map $\mu = (\mu_{\mathbb{C}}, \mu_{\mathbb{R}})$ is given by

$$\begin{aligned} \mu_{\mathbb{C}} &= (\dots, \alpha_i\beta_i - \beta_{i+1}\alpha_{i+1}, \dots), \\ \mu_{\mathbb{R}} &= (\dots, \alpha_i\alpha_i^* - \beta_i^*\beta_i + \beta_{i+1}\beta_{i+1}^* - \alpha_{i+1}^*\alpha_{i+1}, \dots). \end{aligned} \tag{2.4}$$

The hyperKähler quotient $W /\!\!/\!\!/ H$ is homeomorphic to the closure $\overline{\mathcal{O}}$ of the nilpotent orbit $\mathcal{O} = \mathsf{SL}(n, \mathbb{C})A$, which is a singular algebraic variety. The identification is induced by the map $\psi\colon W \to \mathfrak{gl}(n, \mathbb{C})$ given by

$$\psi(\dots, \alpha_i, \beta_i, \dots) = \alpha_{k-1}\beta_{k-1}. \tag{2.5}$$

If $W_0 \subset W$ denotes the open set where each α_i is injective and each β_i is surjective, then $\psi\colon W_0 /\!\!/\!\!/ H \to \mathcal{O}$ is a diffeomorphism. In fact, ψ is the complex symplectic moment map for the action of $\mathsf{GL}(n, \mathbb{C})$ on $W_0 /\!\!/\!\!/ H$ and so the general theory of moment maps implies that $\psi^*\omega_c^{\mathcal{O}}$ agrees with the complex symplectic structure on $W_0 /\!\!/\!\!/ H$. Note that j on W acts on $\mathcal{O}$ by $\alpha_{k-1}\beta_{k-1} \mapsto -\beta_{k-1}^*\alpha_{k-1}^*$ which agrees with the real structure $X \mapsto -X^*$ on $\mathfrak{sl}(n, \mathbb{C})$ defining the Lie algebra of the compact group $\mathsf{SU}(n)$.

2.2. **Nilpotent Orbits in Orthogonal and Symplectic Algebras.** The above construction may be adapted to the remaining classical Lie algebras $\mathfrak{so}(n, \mathbb{C})$ and $\mathfrak{sp}(n, \mathbb{C})$. We start with a nilpotent element A in the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ with $A^k = 0$ and $A^{k-1} \neq 0$. Let δ be 0, if $\mathfrak{g}^{\mathbb{C}} = \mathfrak{so}(n, \mathbb{C})$, or 1, if $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sp}(n, \mathbb{C})$. We consider the image flag

$$\{0\} \rightleftarrows (V_1, \omega_1) \rightleftarrows (V_2, \omega_2) \rightleftarrows \cdots \rightleftarrows (V_k, \omega_k) = (\mathbb{C}^n, \omega_k), \tag{2.6}$$

where $\omega_i\colon V_i \times V_i \to \mathbb{C}$ are non-degenerate bilinear forms satisfying

$$\omega_i(X, Y) = (-1)^{k-i+\delta}\omega_i(Y, X).$$

(This implies that $\dim V_i$ is even if $k - i + \delta$ is odd). We denote by $\cdot^\dagger$ the adjoint with respect to the forms ω_i and define Lie groups

$$H_i = \{A \in \mathsf{U}(V_i) : A^\dagger A = \mathrm{Id}_{V_i}\}.$$

Then H_i is $\mathsf{Sp}(V_i)$, if $k - i + \delta$ is odd, or $\mathsf{O}(V_i)$, if $k - i + \delta$ is even.

Take $H = H_1 \times \cdots \times H_{k-1}$ and let W be the quaternionic vector space as in formula (2.2). The subspace $W^+ \subset W$ defined by the equations

$$\beta_i = \alpha_i^\dagger, \qquad i = 1, \dots, k-1,$$

is a quaternionic vector space. The equations (2.4) define a hyperKähler moment map for the action of H on W^+. Using the map ψ of (2.5), the hyperKähler quotient $W^+ /\!\!/\!\!/ H$ may be identified with the closure of the nilpotent orbit $H_k^{\mathbb{C}}A \subset \mathfrak{h}_k^{\mathbb{C}}$. Again, this identification is compatible with the complex-symplectic form $\omega_c^{\mathcal{O}}$ and the real structure.

2.3. **HyperKähler Potentials.** A real-valued function $\rho\colon N \to \mathbb{R}$ on a hyperKähler manifold N is called a *hyperKähler potential* if ρ is simultaneously a Kähler potential for each of the Kähler structures (ω_I, I), (ω_J, J) and (ω_K, K). For I, this means that $\omega_I = i\overline{\partial_I}\partial_I\rho$, or equivalently

$$\omega_I = -\tfrac{1}{2} dId\rho.$$

In general, N will not admit a hyperKähler potential even locally. Indeed, the existence of ρ implies that if we set $\zeta = \frac{1}{2}\operatorname{grad}\rho$ then $\{\zeta, I\zeta, J\zeta, K\zeta\}$ generates an infinitesimal action of $\mathbb{H}^* \cong \mathbb{R} \times \mathsf{Sp}(1)$ such that

$$L_{I\zeta}g = 0, \quad L_{I\zeta}I = 0, \quad \text{and} \quad L_{I\zeta}J = 2K,$$

with similar expressions for the action of $J\zeta$ and $K\zeta$, obtained by permuting (I, J, K) cyclically (see [20, 4]).

We need to know how hyperKähler potentials behave with respect to hyperKähler quotients. An indirect proof of a slightly weaker form of the following result may be found in [20]. Beware that the hypotheses given in [4] are not quite strong enough.

Theorem 2.1. *Let (N, g, I, J, K) be a hyperKähler manifold admitting a hyperKähler potential ρ. Suppose a Lie group H acts freely and properly on N preserving g, I, J, K and ρ. Suppose also that there is a hyperKähler moment map μ for the action of H on N and that μ is equivariant with respect to the infinitesimal action of $\mathsf{Sp}(1)$ defined by ρ, meaning*

$$L_{I\zeta}\mu_I = 0, \quad L_{I\zeta}\mu_J = -2\mu_K, \quad \textit{etc.} \tag{2.7}$$

Then the function ρ induces a hyperKähler potential on the hyperKähler quotient $N/\!\!/\!\!/H$.

Proof. Let $i\colon \mu^{-1}(0) \hookrightarrow N$ be the inclusion and write $\pi\colon \mu^{-1}(0) \to Q := N/\!\!/\!\!/H$ for the projection. The hyperKähler structure on the quotient is defined by the relations $\pi^*\omega_I^Q = i^*\omega_I$, etc. In particular, at each $x \in \mu^{-1}(0)$ the tangent space to the fibre is spanned by the vector fields ξ_V, for $V \in \mathfrak{h}$ and $(T_x\mu^{-1}(0))^\perp = \{I\xi_V, J\xi_V, K\xi_V : V \in \mathfrak{h}\}$. Thus if $Y \in T_x\mu^{-1}(0)$ is orthogonal to each ξ_V, then IY, JY and KY lie in $T_x\mu^{-1}(0)$ too.

As ρ is invariant under the action of H, it descends to define a function $\rho_Q\colon Q \to \mathbb{R}$ satisfying $\pi^*\rho_Q = i^*\rho$. This implies $\pi^*d\rho_Q = i^*d\rho$.

Now $d\rho$ is metric dual to 2ζ, so ζ commutes with the action of H, and we claim that ζ is tangent to $\mu^{-1}(0)$.

The equivariance condition (2.7) gives,

$$2\mu_K^V = -L_{I\zeta}\mu_J^V = -I\zeta \lrcorner (\xi_V \lrcorner \omega_J) = \omega_K(\xi_V, \zeta),$$

using the J version of (2.1). But now

$$L_\zeta \mu_K^V = \zeta \lrcorner d\mu_K^V = \zeta \lrcorner (\xi_V \lrcorner \omega_K) = \omega_K(\xi_V, \zeta) = 2\mu_K^V.$$

Thus $L_\zeta\mu = 2\mu$ and ζ preserves $\mu^{-1}(0)$.

For $V \in \mathfrak{h}$, we have

$$g(\zeta, \xi_V) = \tfrac{1}{2}d\rho(\xi_V) = \tfrac{1}{2}L_{\xi_V}\rho = 0,$$

as ρ is H-invariant. So $I\zeta$ is also tangent to $\mu^{-1}(0)$. In particular, $i^*Id\rho = Ii^*d\rho$ and we have

$$\pi^*(-\tfrac{1}{2}dId\rho_Q) = i^*(-\tfrac{1}{2}dId\rho) = i^*\omega_I = \pi^*\omega_I^Q,$$

so ρ_Q is a Kähler potential for ω_I^Q. Similar computations apply for J and K and we have that ρ_Q is a hyperKähler potential on $Q = N /\!/\!/ H$. □

For the flat hyperKähler spaces W and W^+ introduced above, the hyperKähler potential is given by the function r^2 of equation (2.3). A hyperKähler potential on $\mathcal{O} = W_0 /\!/\!/ H \subset \mathfrak{sl}(n,\mathbb{C})$ or $\mathcal{O} = W_0^+ /\!/\!/ H \subset \mathfrak{so}(n,\mathbb{C})$ or $\mathfrak{sp}(n,\mathbb{C})$ is then given by the restriction of r^2 to the zero set of the hyperKähler moment map.

One can now ask whether this hyperKähler potential is any sense unique. In fact, one can answer such a question for nilpotent orbits in general. The following is an extension of an argument in [5].

Proposition 2.2. *Let G be a compact semi-simple Lie group and let σ be the corresponding real structure on $\mathfrak{g}^{\mathbb{C}}$. Let $\mathcal{O} \subset \mathfrak{g}^{\mathbb{C}}$ be a nilpotent orbit with the Kirillov-Kostant-Souriau complex symplectic structure $(I, \omega_c^{\mathcal{O}})$. Suppose (g, I, J, K) is a hyperKähler structure on $\mathcal{O}$ such that (a) $\omega_J + i\omega_K = \omega_c^{\mathcal{O}}$, (b) g is invariant under the compact group G and (c) the structure admits a hyperKähler potential such that for the induced $\mathbb{H}^*$-action $j \in \mathbb{H}^*$ acts as $\sigma|_{\mathcal{O}}$. Then the hyperKähler structure is unique.*

Proof. By averaging with the G-action we may assume that there is a G-invariant hyperKähler potential ρ on $\mathcal{O}$. Let $\zeta = \frac{1}{2}\operatorname{grad}\rho$, as above. Then $L_\zeta\omega_I = 2\omega_I$ and $L_\zeta\omega_c^{\mathcal{O}} = 2\omega_c^{\mathcal{O}}$, so

$$\omega_c^{\mathcal{O}} = \tfrac{1}{2}d(\zeta \lrcorner \omega_c^{\mathcal{O}}).$$

Note that as $\omega_c^{\mathcal{O}}$ is a $(2,0)$-form, $\zeta \lrcorner \omega_c^{\mathcal{O}}$ is of type $(1,0)$.

However, as $\mathcal{O}$ is nilpotent, the form $\omega_c^{\mathcal{O}}$ is exact in Dolbeault cohomology: $\omega_c^{\mathcal{O}} = d\theta$, with $\theta_X([X,A]) = \langle X, A\rangle$, which is holomorphic and $G^{\mathbb{C}}$-invariant. Therefore $\theta - \frac{1}{2}\zeta \lrcorner \omega_c^{\mathcal{O}}$ is closed. But $H^1(\mathcal{O},\mathbb{C}) = 0$, as for nilpotent orbits have finite fundamental groups. So $\theta - \frac{1}{2}\zeta \lrcorner \omega_c^{\mathcal{O}} = df$, for some function $f\colon \mathcal{O} \to \mathbb{C}$.

Now df is of type $(1,0)$ and holomorphic. It is also G-invariant, as ζ commutes with G. Therefore we may average f over the action of G to get a G-invariant holomorphic function $\tilde f$ satisfying $d\tilde f = \theta - \frac{1}{2}\zeta \lrcorner \omega_c^{\mathcal{O}}$. However, such a function is $G^{\mathbb{C}}$-invariant and $G^{\mathbb{C}}$ acts transitively on $\mathcal{O}$, so $\tilde f$ is constant and $\zeta \lrcorner \omega_c^{\mathcal{O}} = 2\theta$. Therefore, the $(1,0)$-part of ζ agrees with the $(1,0)$ part of the Euler vector field on $\mathcal{O}$. As both these vector fields preserve I, we have that ζ equals the Euler vector field.

We now have that the quotient of $\mathcal{O}$ by the $\mathbb{C}^*$-action generated by ζ and $I\zeta$ is the projectivised orbit $\mathbb{P}(\mathcal{O})$ with θ as its complex-contact structure and with real structure σ. By [21], $\mathbb{P}(\mathcal{O})$ is the twistor space of a unique quaternion-Kähler manifold M of positive scalar curvature and $\mathcal{O}$ is the associated hyperKähler manifold $\mathcal{U}(M)$. Thus the hyperKähler structure is uniquely determined. ☐

3. Nilpotent Orbits with Diagrams of Length Two

Assume that $\mathfrak{g}^{\mathbb{C}}$ is a classical complex simple Lie algebra and $\mathcal{O} \subset \mathfrak{g}^{\mathbb{C}}$ is an orbit of a rank k nilpotent matrix $X \in \mathcal{O} \subset \mathfrak{sl}(n,\mathbb{C})$ which satisfies $X^2 = 0$. Then X has Jordan type $(2^k, 1^{n-2k})$. Such orbits are precisely those that arise from diagrams of length two:

$$\{0\} \rightleftarrows \mathbb{C}^k \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathbb{C}^n.$$

It follows from §2.1 that there exist $\alpha\colon \mathbb{C}^2 \to \mathbb{C}^n$ and $\beta\colon \mathbb{C}^n \to \mathbb{C}^2$, such that $X = \alpha\beta$, with

$$\beta\alpha = 0 \quad \text{and} \quad \beta\beta^* = \alpha^*\alpha. \tag{3.1}$$

When $\mathfrak{g} = \mathfrak{su}(n)$ this is the full set of equations for $\mathcal{O}$. If $\mathfrak{g}$ is either $\mathfrak{o}(n)$ or $\mathfrak{sp}(n)$, then we have additionally

$$\beta = \alpha^\dagger. \tag{3.2}$$

In all cases $\operatorname{rank}\alpha = \operatorname{rank}\beta = \operatorname{rank} X = k$, so α is injective and β is surjective.

We shall use the above equations to calculate the hyperKähler potential ρ on $\mathcal{O}$. From Theorem 2.1 we know that ρ is the restriction of the radial function r^2. By (2.3) we have

$$\rho = \operatorname{Tr}(\alpha^*\alpha + \beta\beta^*) = 2\operatorname{Tr}\alpha^*\alpha = 2\operatorname{Tr}\Lambda, \tag{3.3}$$

where $\Lambda = \alpha^*\alpha = \beta\beta^*$. Since Λ is self-adjoint, there exists an orthonormal basis $\{e_1, \dots, e_k\}$ for $\mathbb{C}^k$ in which Λ is diagonal,

$$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_k).$$

Thus $\rho = 2(\lambda_1 + \dots + \lambda_k)$.

Note that

$$\langle \beta^* e_i, \beta^* e_j \rangle = \langle \beta\beta^* e_i, e_j \rangle = \langle \Lambda e_i, e_j \rangle = \lambda_i \delta_{ij}.$$

In particular, $\|\beta^* e_i\|^2 = \lambda_i$. But β^* is injective, so $\lambda_i > 0$ and $\{\beta^* e_1, \dots, \beta^* e_k\}$ is an orthogonal basis for $\operatorname{Im}\beta^*$.

Now consider the matrix X^*X. On $\operatorname{Im}\beta^*$, we have $X^*X = \Lambda^2$, since

$$X^*X\beta^* e_i = \beta^*\alpha^*\alpha\beta\beta^* e_i = \beta^*\Lambda^2 e_i = {\lambda_i}^2 \beta^* e_i.$$

On the other hand, $(\operatorname{Im}\beta^*)^\perp = \ker\beta$ and $X = \alpha\beta$, so X^*X vanishes on $(\operatorname{Im}\beta^*)^\perp$. As a result X^*X has eigenvalues ${\lambda_1}^2, \dots, {\lambda_k}^2$. Writing $\operatorname{Spec} X^*X = \{\mu_1, \dots, \mu_r\}$ with μ_i distinct and of multiplicity k_i we get

Theorem 3.1. *Let $\mathcal{O}$ be the adjoint orbit of a non-zero nilpotent matrix X in a complex classical Lie algebra, and assume that $X^2 = 0$. Then the hyperKähler potential for the canonical hyperKähler metric on $\mathcal{O}$ is given by the formula*

$$\rho(X) = 2 \sum_{\mu_i \in \operatorname{Spec}(X^*X)} k_i {\mu_i}^{1/2} \tag{3.4}$$

Remark 3.2. The above formula can be obtained from (3.3) by explicitly solving (3.1) and (3.2) for a given nilpotent element X. For example consider orbits in $\mathfrak{sl}(n, \mathbb{C})$. Then X is $\mathsf{U}(n)$-conjugate to

$$M = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \tag{3.5}$$

where $A = \operatorname{diag}(a_1, \dots, a_k)$ with a_i real and positive. To see this note that X^*X determines a set of orthonormal eigenvectors $e_1, \dots, e_k$ with positive eigenvalues $\mu_1, \dots, \mu_k$. Moreover, $\langle Xe_i, Xe_j \rangle = \mu_i \delta_{ij}$, so ${\mu_i}^{-1/2} X e_i$, $i = 1, \dots, k$ are also orthonormal. Since $X^2 = 0$ it follows that

$$0 = \left\langle X^2 e_i, X e_j \right\rangle = \langle X e_i, X^* X e_j \rangle = \mu_j \langle X e_i, e_j \rangle .$$

In effect the vectors

$$e_1, \dots e_k, {\mu_1}^{-1/2} X e_1, \dots, {\mu_k}^{-1/2} X e_k$$

form an orthonormal set. Complete this to an orthonormal basis in $\mathbb{C}^n$. In this basis X has the required form, with $a_i = {\mu_i}^{1/2}$.

Type	$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{so}(n,\mathbb{C})$	$\mathfrak{sp}(n,\mathbb{C})$
Cohomogeneity 1	(21^{n-2})	$(2^2 1^{n-4})$	(21^{2n-2})
Cohomogeneity 2	$(2^2 1^{n-4})$	(31^{n-3}), $(2^4 1^{n-8})$	$(2^2 1^{2n-4})$

TABLE 1. Nilpotent orbits of low cohomogeneity in classical Lie algebras

It follows that X is $\mathsf{SU}(n)$-conjugate to λM for some λ satisfying $\lambda\overline{\lambda} = 1$. The moment map equations (3.1) are now solved by

$$\alpha = \lambda \begin{pmatrix} A^{1/2} \\ 0 \end{pmatrix} \quad \text{and} \quad \beta = \overline{\lambda} \begin{pmatrix} 0 & A^{1/2} \end{pmatrix},$$

where $A^{1/2} = \operatorname{diag}(a_1^{1/2}, \dots a_k^{1/2})$. In particular $A = \alpha^*\alpha = \beta\beta^*$. We have $\operatorname{Spec}(XX^*) = \operatorname{Spec}(A^2) = \{{a_1}^2, \dots, {a_k}^2\}$ and, by (3.4)

$$\rho(X) = 2\sum_{i=1}^{k} |a_i|.$$

This agrees with the formula obtained in [3]. There Biquard & Gauduchon showed that this formula gives a Kähler potential for a hyper-Kähler structure on the nilpotent orbit. This was done by considering the orbit in $\mathfrak{sl}(n,\mathbb{C})$ as a limit of semi-simple orbits. However, we have now shown that the Biquard-Gauduchon Kähler potential is in fact a hyperKähler potential.

4. HyperKähler Potentials for Low Cohomogeneity Orbits

In the simplest case $\mathcal{O}$ is a minimal nilpotent orbit in a classical Lie algebra. Such orbit arises from a length two diagram. Its Jordan type is given in Table 1. Minimal orbits are cohomogeneity one so any two elements $X, X' \in \mathcal{O}$ are conjugate if and only if $\|X\| = \|X'\|$. It follows that for all $X \in \mathcal{O}$ the matrix X^*X has only one non-zero eigenvalue, say λ, with multiplicity κ. Then, by (3.4) $\rho = 2\kappa\lambda^{1/2}$, so $\rho^2 = 4\kappa^2\lambda$. But $\operatorname{Tr} X^*X = \kappa\lambda$, so

$$\rho^2 = 4\kappa \operatorname{Tr} X^*X, \qquad \text{where} \quad \kappa = \begin{cases} 1 & \text{for } \mathfrak{sl}(n,\mathbb{C}),\ \mathfrak{sp}(n,\mathbb{C}), \\ 2 & \text{for } \mathfrak{so}(n,\mathbb{C}). \end{cases} \tag{4.1}$$

One finds the multiplicity κ simply by calculating X^*X where X is the block matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$ with $A = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ for $\mathfrak{sl}(n,\mathbb{C})$ and $\mathfrak{sp}(n,\mathbb{C})$, and $A = \left(\begin{smallmatrix} 0 & 1 & 0 \\ -1 & 0 & i \\ 0 & -i & 0 \end{smallmatrix}\right)$ for $\mathfrak{so}(n,\mathbb{C})$.

In fact the potential on a minimal nilpotent orbit in any complex simple Lie algebra is equal to $\|X\| = \sqrt{\operatorname{Tr} X^*X}$, up to a constant multiplier, see for example [15].

It is known that, with one exception, the next-to-minimal orbits in complex semi-simple Lie algebras are precisely the cohomogeneity-two orbits [7]. The exception is the next-to-minimal nilpotent orbit in $\mathfrak{sl}(3,\mathbb{C})$ which has cohomogeneity 4. This case was dealt with in [10] while in [13] hyperKähler potentials for cohomogeneity-two nilpotent orbits were calculated: the latter were expressed in terms of two invariants $\eta_1(X) := -K(X,\sigma X)$ and $\eta_2(X) := \eta_1([X,\sigma X])$, where K denotes the Killing form. In our situation it will be more convenient to use the following two invariants (which in fact are multiples of η_1 and η_2):

$$\begin{aligned} c_1(X) &= \operatorname{Tr} XX^*, \\ c_2(X) &= \operatorname{Tr} YY^*, \qquad \text{where } Y = [X, X^*]. \end{aligned}$$

Theorem 4.1. *Let $\mathcal{O}$ be a cohomogeneity-two nilpotent orbit in a classical Lie algebra. Then the hyperKähler potential for $\mathcal{O}$ is given by the formula*

$$\rho^2 = 4\kappa c_1 + 4\kappa\sqrt{2{c_1}^2 - \kappa c_2}$$

where $\kappa = 1$ for $\mathfrak{sl}(n,\mathbb{C})$ and $\mathfrak{sp}(n,\mathbb{C})$, and $\kappa = 2$ for $\mathfrak{so}(n,\mathbb{C})$.

In the proof we shall consider the three classes of orbits which have length two diagrams, and postpone the length three case to §5.

Proof. We use the notation of Remark 3.2. Since $\mathcal{O}$ is a cohomogeneity-two orbit, X^*X has at most two different eigenvalues. By considering a matrix defined in (3.5), with a_1, a_2 arbitrary, and $a_3 = \cdots = a_k = 0$ one finds that for a generic element X in nilpotent orbits $\mathcal{O}_{(2,1^{n-k})} \subset \mathfrak{sl}(n,\mathbb{C})$, and $\mathcal{O}_{(2,1^{2n-k})} \subset \mathfrak{sp}(n,\mathbb{C})$ we have $\operatorname{Spec}(X^*X) = \{\mu_1, \mu_2\}$ where the eigenvalues μ_1, μ_2 have multiplicities $\kappa = k_1 = k_2 = 1$. An element X of $\mathcal{O}_{(2^4,1^{n-8})} \subset \mathfrak{so}(n,\mathbb{C})$ has, by Lemma 4.2 below, eigenvalues with even multiplicities. But X^*X has rank 4 so again $\operatorname{Spec}(X^*X) = \{\mu_1, \mu_2\}$, this time with multiplicities $\kappa = k_1 = k_2 = 2$. This can be verified by a direct calculation: a typical matrix in this orbit is conjugate to the matrix obtained by taking X as in (3.5) with $a_1 = -a_k, a_2 = -a_{k-1}$ arbitrary, and $a_3 = \cdots = a_{k-2} = 0$; note that this is possible if we take the quadratic form which defines $\mathfrak{so}(n,\mathbb{C})$ to be $\frac{1}{2}(x_1x_n + x_2x_{n-1} + \dots + x_nx_1)$, cf. §5.2.

From (3.4) we have $\rho = 2\kappa({\mu_1}^{1/2} + {\mu_2}^{1/2})$. The invariants c_i are not difficult to compute in terms of μ_1 and μ_2:

$$c_1 = \operatorname{Tr} XX^* = \kappa(\mu_1 + \mu_2),$$

and, since $X^2 = 0$, we have

$$\begin{aligned} c_2 &= \operatorname{Tr}([X,X^*][X,X^*]^*) = \operatorname{Tr}(XX^* - X^*X)^2 = 2\operatorname{Tr}(X^*X)^2 \\ &= 2\kappa({\mu_1}^2 + {\mu_2}^2). \end{aligned}$$

Thus

$$\begin{gathered} \rho = 2\kappa({\mu_1}^{1/2} + {\mu_2}^{1/2}), \\ c_1 = \kappa(\mu_1 + \mu_2) \quad \text{and} \quad c_2 = 2\kappa({\mu_1}^2 + {\mu_2}^2) \end{gathered}$$

which leads to the required formula for length two orbits.

There is only one cohomogeneity 2 orbit with diagram of length greater than two, for proof in this case see §5.1. □

The above proof used the following lemma:

Lemma 4.2. *If $X \in \mathfrak{so}(n,\mathbb{C})$ then the non-zero eigenvalues for X^*X have even multiplicities.*

Proof. We consider $\mathbb{C}^n$ with the standard quadratic and Hermitian forms, so that $\mathfrak{so}(n,\mathbb{C})$ consists of skew-symmetric matrices, and $X^* = \overline{X}^{\mathsf{T}}$. Let J denote the $\mathbb{R}$-linear automorphism of $\mathbb{C}^n$, defined by the formula

$$Jv = X^*\overline{v}.$$

Suppose λ is a non-zero eigenvalue of X^*X and that v is a corresponding eigenvector. Now $X^{\mathsf{T}} = -X$, so $X^* = -\overline{X}$, and we get

$$X^*XJv = X^*XX^*\overline{v} = X^*\overline{X^*Xv} = \lambda X^*\overline{v} = \lambda Jv,$$

since the eigenvalues of X^*X are real. Thus Jv is also a λ-eigenvector of X^*X.

Note that $J^2v = X^*\overline{X^*\overline{v}} = -X^*Xv = -\lambda v$. It follows that v and Jv are linearly independent. We conclude that λ-eigenvectors with $\lambda \neq 0$ come in pairs v, Jv which span J-invariant two-dimensional λ-eigenspaces. □

5. Orbits with Diagrams of Length Three

The hyperKähler potential calculations for orbits that correspond to diagrams of length three can be quite involved, and the result is known only in few special cases. One of the early results is the calculation of the hyperKähler potential for the generic orbit $\mathcal{O}_{(3)} \subset \mathfrak{sl}(3,\mathbb{C})$, given in [10]. The formula

$$\rho(X) = 2\sqrt{(a^{2/3}+c^{2/3})^3+b^2}, \qquad \text{where} \qquad X = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$$

was derived from moment map equation (2.4) for $\mathcal{O}_{(3)}$. This seems to be the most efficient formula; the attempts to write the potential for this orbit in another language, for example in terms of Lie algebra invariants, yield much more complicated results. Note, however, that the regular orbit in $\mathfrak{sl}(3,\mathbb{C})$ is a three-to-one quotient of the minimal orbit in $\mathfrak{g}_2^{\mathbb{C}}$, the potential in question is proportional to the invariant $\sqrt{c_1}$ on $\mathfrak{g}_2^{\mathbb{C}}$.

In this section we shall consider nilpotent orbits in $\mathfrak{so}(n,\mathbb{C})$ which have a single Jordan block of size three. For nilpotent orbits in $\mathfrak{so}(n,\mathbb{C})$ the Jordan blocks of even size come in pairs, so these orbits have Jordan type $(3, 2^{2k}, 1^{n-4k-3})$ and the corresponding diagram is

$$\{0\} \rightleftarrows \mathbb{C} \rightleftarrows \mathbb{C}^{2k+2} \rightleftarrows \mathbb{C}^n$$

We may assume that the orthogonal structures ω_1 on $\mathbb{C}$ and ω_3 on $\mathbb{C}^n$, cf. formula (2.6), are the standard quadratic forms. In particular $\mathfrak{so}(n,\mathbb{C})$ consists of skew-symmetric matrices.

By §2.2, the orbit $\mathcal{O}_{(3,2^{2k})} \subset \mathfrak{so}(n,\mathbb{C})$ is a hyperKähler quotient

$$\mathbb{H}^{(2k+2)(n+1)} /\!\!/\!\!/ (\mathsf{Sp}(k,\mathbb{C}) \times \mathbb{Z}_2)$$

and $\mathcal{O}_{(2^{2k+2})} \subset \mathfrak{so}(n+1,\mathbb{C})$ is $\mathbb{H}^{(2k+2)(n+1)} /\!\!/\!\!/ \mathsf{Sp}(k,\mathbb{C})$. This indicates that there is a $\mathbb{Z}_2$-quotient map $\mathcal{O}_{(2^{2k+2})} \to \mathcal{O}_{(3,2^{2k})}$. Moreover, the hyperKähler potentials on $\mathcal{O}_{(2^{2k+2})}$ and on $\mathcal{O}_{(3,2^{2k})}$ are restrictions of the radial function r^2 on $\mathbb{H}^{(2k+2)(n+1)}$, so they are preserved by the quotient map.

Now $\mathcal{O}_{(2^{2k+2})}$ is given by a diagram of length two, so one can use Theorem 3.1 to calculate the potential for $\mathcal{O}_{(2^{2k+2})}$, and hence for $\mathcal{O}_{(3,2^{2k})}$. By making the inverse to the two-to-one quotient map explicit one gets an algorithmic method of calculating the hyperKähler potential on $\mathcal{O}_{(3,2^{2k})}$. This is shown in the following technical lemma.

Lemma 5.1. *Let $X \in \mathcal{O}_{(3,2^{2k})}$ and denote by $x \in \mathbb{C}^n$ the (unique up to sign) vector such that $X^2 = xx^{\mathsf{T}}$. Then the hyperKähler potential ρ*

on $\mathcal{O}_{(3,2^{2k})}$ *is given by the formula*

$$\rho(X) = 2 \sum_{\mu_i \in \mathrm{Spec}(X'X'^*)} k_i {\mu_i}^{1/2} \qquad \text{where} \qquad X' = \begin{pmatrix} X & x \\ -x^{\mathsf{T}} & 0 \end{pmatrix}.$$

Proof. We begin by writing down the diagram for $\mathcal{O}_{(3,2^{2k})}$:

$$\{0\} \rightleftarrows V_1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} V_2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} V_3, \qquad \text{with} \quad V_1 = \mathbb{C}, \quad V_2 = \mathbb{C}^{2k+2}, \quad V_3 = \mathbb{C}^n,$$

and the corresponding moment map equations

$$\beta_1\alpha_1 = 0, \tag{5.1}$$

$$\alpha_1\beta_1 = \beta_2\alpha_2, \tag{5.2}$$

$$\beta_1{\beta_1}^* = {\alpha_1}^*\alpha_1, \tag{5.3}$$

$$\alpha_1{\alpha_1}^* + \beta_2{\beta_2}^* = {\beta_1}^*\beta_1 + {\alpha_2}^*\alpha_2. \tag{5.4}$$

We also have

$$\beta_i = {\alpha_i}^\dagger, \quad \text{and} \quad X = \alpha_2\beta_2.$$

Consider now the diagram

$$\{0\} \rightleftarrows V_2 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} V_1 \oplus V_3.$$

The moment map equations are

$$\beta\alpha = 0, \tag{5.5}$$

$$\beta\beta^* = \alpha^*\alpha \tag{5.6}$$

and it is easy to see that the map

$$(\alpha_1, \alpha_2) \mapsto \alpha = \alpha_1^\dagger \oplus \alpha_2$$

transforms the solutions of (5.1)–(5.4) into solutions of (5.5)–(5.6). To verify this simply write α and β in block-matrix form:

$$\alpha = \begin{pmatrix} \alpha_2 \\ \beta_1 \end{pmatrix}, \qquad \beta = \alpha^\dagger = \begin{pmatrix} \beta_2 & -\alpha_1 \end{pmatrix}.$$

Then it is clear that (5.2) is equivalent to (5.5) and (5.3) to (5.6). The remaining two equations (5.1) and (5.3) are $\mathsf{O}(1,\mathbb{C}) = \mathbb{Z}_2$ moment map equations and are trivially satisfied.

Note that if $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ solves (5.1)–(5.4) then so does $(-\alpha_1, -\beta_1, \alpha_2, \beta_2)$. This corresponds to

$$\alpha = \begin{pmatrix} \alpha_2 \\ -\beta_1 \end{pmatrix}, \qquad \beta = \alpha^\dagger = \begin{pmatrix} \beta_2 & \alpha_1 \end{pmatrix}.$$

A solution $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ represents an element $X = \beta_2\alpha_2 \in \mathcal{O}_{(2^{2k+2})}$ while the lifts $X'_\pm$ are given by

$$X'_\pm = \alpha\beta = \begin{pmatrix} \alpha_2\beta & \mp\alpha_2\alpha_1 \\ \pm\beta_1\beta_2 & 0 \end{pmatrix}$$

Define $x = \alpha_2\alpha_1(1)$. With our conventions (ω_1 and ω_3 are the identity matrices) the dagger operator acts on maps $\mathbb{C} \to \mathbb{C}^n$ as the transpose, so $\beta_1\beta_2 = (\alpha_2\alpha_1)^\dagger = x^\mathsf{T}$. Also,

$$\begin{aligned} X^2 &= (\alpha_2\beta_2)^2 = \alpha_2\beta_2\alpha_2\beta_2 \\ &= \alpha_2\alpha_1\beta_1\beta_2 = xx^\mathsf{T} \end{aligned}$$

where the penultimate equality follows from (5.2). This shows that X' is of the required form. Finally, note that

$$\begin{aligned} r'^2 &= \operatorname{Tr}\alpha\alpha^* + \operatorname{Tr}\beta^*\beta \\ &= \operatorname{Tr}\alpha_2\alpha_2^* + \operatorname{Tr}\beta_2^*\beta_2 + \operatorname{Tr}\alpha_1\alpha_1^* + \operatorname{Tr}\beta_1^*\beta_1 = r^2 \end{aligned}$$

which shows directly, that the two-to-one map respects the hyperKähler potentials. □

We shall apply the above lemma to determine the hyperKähler potential on $\mathcal{O}_{(3,2^{2k})}$ in a few simple cases. The first completes the proof of Theorem 4.1.

5.1. $\mathcal{O}_{(3,1^{n-3})}$ **in** $\mathfrak{so}(n,\mathbb{C})$**.** As in Lemma 5.1 define

$$X' = \begin{pmatrix} X & x \\ -x^\mathsf{T} & 0 \end{pmatrix}.$$

Then X' lies in the minimal nilpotent orbit $\mathcal{N}_{2^2,1^{n-3}} \subset \mathfrak{so}(n+1,\mathbb{C})$, so the potential is given by (4.1). We have

$$X'^* = \begin{pmatrix} X^* & -\overline{x} \\ x^* & 0 \end{pmatrix},$$

and

$$\rho^2 = 4\kappa \operatorname{Tr} X'X'^* = 4\kappa(\operatorname{Tr} XX^* + 2\,\|x\|^2). \tag{5.7}$$

Putting $Y = [X, X^*]$ we get

$$\begin{aligned} c_2 &:= \operatorname{Tr} YY^* \\ &= \operatorname{Tr}(XX^* - X^*X)(XX^* - X^*X) \\ &= 2\operatorname{Tr}(XX^*)^2 - 2\operatorname{Tr} X^2X^{*2} \\ &= 2\operatorname{Tr}(XX^*)^2 - 2\,\|x\|^4 \end{aligned}$$

since $X = xx^\mathsf{T}$.

We know that $\operatorname{rank} X = 2$ so X^*X has at most two non-zero eigenvalues. It follows from Lemma 4.2 that it has a unique non-zero double eigenvalue, which we denote by λ. Then, in a suitable basis,

$$XX^* = \operatorname{diag}(\lambda, \lambda, 0, \dots, 0),$$

so $c_2 = 4\lambda^2 - 2\left\|x\right\|^4 = {c_1}^2 - 2\left\|x\right\|^4$, since $c_1 = \operatorname{Tr} XX^* = 2\lambda$. This implies that $\left\|x\right\|^2 = \sqrt{(c_1^2 - c_2)/2}$. Thus

$$\rho^2 = 4\kappa(c_1 + 2\left\|x\right\|^2) = 4\kappa c_1 + 4\kappa\sqrt{2c_1^2 - 2c_2}$$

which ends the proof of Theorem 4.1.

5.2. $\mathcal{O}_{(3,2^2,1^{n-7})}$ **in** $\mathfrak{so}(n,\mathbb{C})$**.** For this orbit $X'X'^*$ has two double eigenvalues, $\operatorname{Spec}(X'X'^*) = \{\lambda_1, \lambda_2\}$, so the computation of (5.7) yields $\lambda_1 + \lambda_2$ and not ρ^2:

$$2(\lambda_1 + \lambda_2) = \operatorname{Tr} X'X'^* = \operatorname{Tr} XX^* + 2\left\|x\right\|^2 = c_1 + 2\left\|x\right\|^2 \qquad (5.8)$$

Moreover, by Theorem 3.1,

$$\rho^2 = 4(\lambda_1 + \lambda_2 + 2\sqrt{\lambda_1\lambda_2})$$

so one needs to calculate the product of eigenvalues. This can be done by calculating $\operatorname{Tr}(X'X'^*)^2$ but then it is necessary to determine invariants like $\left\|X\overline{x}\right\|^2$. The most straightforward approach is to take a generic nilpotent element X, augment it to get X', and find the eigenvalues of X'.

To simplify the calculations we can use the action of the compact group $\mathsf{SO}(n)$ on $\mathcal{O}$ to put X in a *canonical form.* This is achieved by using the Beauville bundle [1]. We shall briefly outline this approach here; it is explained in more detail in [13, Section 4].

Consider $e \in \mathcal{O} \subset \mathfrak{g}^{\mathbb{C}}$ and choose $f, h \in \mathfrak{g}^{\mathbb{C}}$ so that e, f, h is an $\mathfrak{sl}(2,\mathbb{C})$-triple. Then use the ad_h-eigenspaces $\mathfrak{g}^{\mathbb{C}}(i)$ to define the algebras

$$\mathfrak{p} = \bigoplus_{i\geqslant 0} \mathfrak{g}^{\mathbb{C}}(i), \qquad \mathfrak{n} = \bigoplus_{i\geqslant 2} \mathfrak{g}^{\mathbb{C}}(i).$$

It turns out that $\mathfrak{p}$ is a parabolic algebra and it does not depend on the choice of f, h. This gives what is sometimes referred to as the *canonical* fibration $\mathcal{O} \to \mathcal{F}$ where $\mathcal{F} = G^{\mathbb{C}}/P$ is a flag manifold with P the normaliser of $\mathfrak{p}$. Moreover, $\mathcal{O}$ is an open dense subset of the *Beauville bundle*

$$N(\mathcal{O}) = G^{\mathbb{C}} \times_P \mathfrak{n},$$

the canonical fibration being the restriction to $\mathcal{O}$ of the Beauville bundle fibration.

Choose a flag $v \in \mathcal{F}$. Since $\mathcal{F}$ is G-homogeneous any element $e \in \mathcal{O}$ can be moved by the action of the compact group G into the Beauville bundle fibre $N(\mathcal{O})_v$. It is enough to calculate the hyperKähler potential ρ for nilpotent elements $e \in \mathcal{O} \cap N(\mathcal{O})_v$.

We now calculate the hyperKähler potential on $\mathcal{O}_{(3,2^2,1^{n-7})}$. First, assume that the quadratic form on $\mathbb{C}^n$ is given by the anti-diagonal matrix $(S)_{ij}$ with $S_{ij} = \delta_{i,n+1-j}$. Then $\mathfrak{so}(n,\mathbb{C})$ consists of matrices that are skew-symmetric about the anti-diagonal. The advantage of this choice for the quadratic form is that nilpotent matrices in $\mathfrak{so}(n,\mathbb{C})$ are $\mathsf{SO}(n,\mathbb{C})$-conjugate to matrices consisting of Jordan blocks. In our situation we can arrange for the size three block to be in the middle with the size two blocks placed symmetrically about the anti-diagonal:

$$e = \begin{pmatrix} J_2 & & 0 \\ & J_3 & \\ 0 & & -J_2 \end{pmatrix} \quad \text{with} \quad J_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \quad J_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

(For simplicity we write everything for $\mathcal{O}_{(3,2^2)}$, the formulæ are identical in other cases.)

Moreover, we can choose the maximal torus to consist of the diagonal matrices $\mathrm{diag}(a_1, a_2, \dots, -a_2, -a_1)$. Then we have an $\mathfrak{sl}(2,\mathbb{C})$-triple e, f, h with e as above and $h = \mathrm{diag}(1, 1, 2, 0-2, -1, -1)$. To make matters simpler we use the Weyl group to rearrange the diagonal matrix h, so take $h' = \mathrm{diag}(2, 1, 1, 0, -1, -1, -2)$. It is enough to work out the $\mathrm{ad}_{h'}$ eigenspaces that have eigenvalues $\geqslant 2$ to see that a typical element of the Beauville bundle fibre has the following form

$$Y = \begin{pmatrix} 0 & \begin{smallmatrix} a & b & 0 & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & -v & -b \\ 0 & 0 & 0 & -a \end{smallmatrix} \\ 0 & 0 \end{pmatrix}.$$

(To be precise the $(1,6)$ and $(6,1)$ entries in the matrix have weight 3 and thus belong to the Beauville bundle fibre, but one can assume they vanish by using the action of the stabiliser $\mathsf{SO}(2)\mathsf{SO}(2)\mathsf{Sp}(1)$.)

The aim is now to apply Lemma 5.1 but we need to go back to the standard basis, where the quadratic form is diagonal. To diagonalise the quadratic form S consider the matrix Q written in a block form:

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1}_3 & 0 & -i\mathbf{1}_3 \\ 0 & \sqrt{2} & 0 \\ \mathbf{1}_3 & 0 & i\mathbf{1}_3 \end{pmatrix},$$

where $\mathbf{1}_3$ is the 3×3 identity matrix. Then $Q^{\mathsf{T}} S Q = 1$, so $X = Q^{-1} Y Q = Q^* Y Q$ is skew-symmetric. Lemma 5.1, applied to X, gives $x = \frac{1}{\sqrt{2}}(ai, 0, \dots, 0, ai)^{\mathsf{T}}$, and a direct calculation yields

$$\lambda_1 \lambda_2 = 2|a|^2|v|^2.$$

Let us introduce a new invariant

$$c_{21} = c_1(X^2) = \operatorname{Tr} XXX^*X^* = \left\|X^2\right\|^2.$$

A simple calculation shows that

$$c_1{}^2 - c_2 - 2c_{21} = 8|a|^2|v|^2 = 4\lambda_1\lambda_2.$$

By combining this with (5.8) we get the following formula:

Proposition 5.2. *The hyperKähler potential ρ for the canonical hyperKähler structure on the nilpotent orbit $\mathcal{O}_{(3,2^2)} \subset \mathfrak{so}(n,\mathbb{C})$ is given by the formula*

$$\rho^2 = 8c_1 + 16\sqrt{c_{21}} + 16\sqrt{c_1{}^2 - c_2 - 2c_{21}}.$$

Note the similarity of this formula to that in Theorem 4.1: for length two diagrams $c_{21} = 0$ while in the cohomogeneity two situation (the orbit $\mathcal{O}_{(3,1^{n-3})}$ in $\mathfrak{so}(n,\mathbb{C})$) the invariant c_{21} is a combination of c_1 and c_2.

5.3. **$\mathcal{O}_{(3,2^4,1^{n-11})}$ in $\mathfrak{so}(n,\mathbb{C})$.** Finally, we shall only indicate here how the matters tend to complicate if one tries to proceed in the same manner and calculate the hyperKähler potential for $\mathcal{O}_{(3,2^4,1^{n-11})}$. We start with the same strategy as in the previous section (again, it is enough to analyse the case of $\mathcal{O}_{(3,2^4)}$).

Here we take the semi-simple element

$$h' = \operatorname{diag}(2,1,1,1,1,0,-1,-1,-1,-1,-2).$$

Taking into account the action of the stabiliser $\mathsf{SO}(2)\mathsf{SO}(2)\mathsf{Sp}(2)$, a typical element of the fibre of the Beauville bundle can be written as

$$Y = \begin{pmatrix} 0 & \begin{smallmatrix} a & b & 0 & 0 & 0 & 0 \\ 0 & v_1 & -w_2 & w_3 & 0 & 0 \\ 0 & v_2 & w_1 & 0 & -w_3 & 0 \\ 0 & v_3 & 0 & -w_1 & w_2 & 0 \\ 0 & 0 & -v_3 & -v_2 & -v_1 & -b \\ 0 & 0 & 0 & 0 & 0 & -a \end{smallmatrix} \\ 0 & 0 \end{pmatrix}$$

As before, set $X = Q^*YQ$ and then $x = \frac{1}{\sqrt{2}}(ai,0,\dots,0,ai)^{\mathsf{T}}$. Calculations now become complex enough and the authors used MAPLE.

The result can be described as follows. Denote $v = (v_1,v_2,v_3)^{\mathsf{T}}$ and $w = (w_1,w_2,w_3)^{\mathsf{T}}$. Also, write $\zeta = v^{\mathsf{T}}w = \sum v_iw_i$. Then the hyperKähler potential ρ for the canonical hyperKähler metric on $\mathcal{O}_{(3,2^4,1^{n-11})}$ is given by the formula

$$\rho = 2(\lambda_1{}^{1/2} + \lambda_2{}^{1/2} + \lambda_3{}^{1/2})$$

where λ_i are the roots of the cubic $z^3 - pz^2 + qz - r$ with

$$p = 2|a|^2 + |b|^2 + |v|^2 + |w|^2 = c_1 + |a|^2$$
$$q = |\zeta|^2 + |b|^2|w|^2 + 2|a|^2(|v|^2 + |w|^2)$$
$$r = |a|^2|\zeta|^2.$$

References

[1] A. Beauville, *Fano contact manifolds and nilpotent orbits*, Comment. Math. Helv. **73** (1998), 566–583.

[2] O. Biquard, *Sur les équations de Nahm et la structure de Poisson des algèbres de Lie semi-simples complexes*, Math. Ann. **304** (1996), 253–276.

[3] O. Biquard and P. Gauduchon, *La métrique hyperkählérienne des orbites coadjointes de type symétrique d'un groupe de Lie complexe semi-simple*, C. R. Acad. Sci. Paris Sér. I Math. **323** (1996), no. 12, 1259–1264.

[4] C. P. Boyer, K. Galicki, and B. M. Mann, *Quaternionic reduction and Einstein manifolds*, Comm. Anal. Geom. **1** (1993), no. 2, 229–279.

[5] R. Brylinski, *Instantons and Kähler geometry of nilpotent orbits*, Representation theories and algebraic geometry (Montreal, PQ, 1997), Kluwer Acad. Publ., Dordrecht, 1998, pp. 85–125.

[6] A. S. Dancer and A. F. Swann, *The geometry of singular quaternionic Kähler quotients*, International J. Math. **8** (1997), 595–610.

[7] ———, *HyperKähler metrics of cohomogeneity one*, J. Geom. and Phys. **21** (1997), 218–230.

[8] N. Hitchin, *Integrable systems in Riemannian geometry*, J. Differential Geom. **S4** (1998), 21–81.

[9] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček, *HyperKähler metrics and supersymmetry*, Comm. Math. Phys. **108** (1987), 535–589.

[10] P. Z. Kobak and A. F. Swann, *Quaternionic geometry of a nilpotent variety*, Math. Ann. **297** (1993), 747–764.

[11] ———, *Classical nilpotent orbits as hyperKähler quotients*, International J. Math. **7** (1996), 193–210.

[12] ———, *Exceptional hyperKähler reductions*, Twistor Newsletter **44** (1998), 23–26.

[13] ———, *HyperKähler potentials in cohomogeneity two*, preprint 98/33, Department of Mathematical Sciences, University of Bath, December 1998.

[14] ———, *Computations in G_2 using Maple*, in preparation, see `http://www.imada.sdu.dk/~swann/g2/index.html`, 1999.

[15] ———, *The hyperKähler geometry associated to Wolf spaces*, preprint 99/14, Department of Mathematical Sciences, University of Bath, July 1999.

[16] A. G. Kovalev, *Nahm's equations and complex adjoint orbits*, Quart. J. Math. Oxford Ser. (2) **47** (1996), no. 185, 41–58.

[17] P. B. Kronheimer, *A hyper-Kählerian structure on coadjoint orbits of a semisimple complex group*, J. London Math. Soc. (2) **42** (1990), 193–208.

[18] ———, *Instantons and the geometry of the nilpotent variety*, J. Differential Geom. **32** (1990), 473–490.

[19] H. Nakajima, *Instantons on* ALE *spaces, quiver varietes and Kac-Moody algebras*, Duke Math. J. **76** (1994), 365–416.

[20] A. F. Swann, *HyperKähler and quaternionic Kähler geometry*, Math. Ann. **289** (1991), 421–450.

[21] ______, *Homogeneous twistor spaces and nilpotent orbits*, Math. Ann. **313** (1999), 161–188.

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