SOLUTIONS TO CONGRUENCES USING SETS WITH THE PROPERTY OF BAIRE

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Abstract. Hausdorff's paradoxical decomposition of a sphere with countably many points removed (the main precursor of the Banach-Tarski paradox) actually produced a partition of this set into three pieces A, B, C such that A is congruent to B (i.e., there is an isometry of the set which sends A to B), B is congruent to C, and A is congruent to $B \cup C$. While refining the Banach-Tarski paradox, R. Robinson characterized the systems of congruences like this which could be realized by partitions of the sphere with rotations witnessing the congruences: the only nontrivial restriction is that the system should not require any set to be congruent to its complement. Later, Adams showed that this restriction can be removed if one allows arbitrary isometries of the sphere to witness the congruences.

The purpose of this paper is to characterize those systems of congruences which can be satisfied by partitions of the sphere or related spaces into sets with the property of Baire. A paper of Dougherty and Foreman gives a proof that the Banach-Tarski paradox can be achieved using such sets, and gives versions of this result using open sets and related results about partitions of spaces into congruent sets. The same method is used here; it turns out that only one additional restriction on a system of congruences is needed to make it solvable using subsets of the sphere with the property of Baire (or solvable with open sets if one allows meager exceptions to the congruences and the covering of the space) with free rotations witnessing the congruences. Actually, the result applies to any complete metric space acted on in a sufficiently free way by a free group of homeomorphisms. We also characterize the systems solvable on the sphere using sets with the property of Baire but allowing all isometries.

1. Introduction and definitions

The basic form of the Banach-Tarski paradox can be stated as follows: The two-dimensional sphere S^2 can be partitioned into finitely many pieces $A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_m$ with the property that the sets A_i can be rearranged by rigid motions (rotations) so as to cover the entire sphere, and so can the sets B_j. This contradicts standard intuitions concerning measure or area; the sets A_i and B_j cannot all be measurable with respect to the standard rotation-invariant probability measure on S^2.

This result of Banach and Tarski [2] was based on earlier work of Hausdorff [7, p. 469] who proved that the free product of cyclic groups $\mathbf{Z}_2$ and $\mathbf{Z}_3$ can be embedded in the rotation group SO_3 of S^2. Using this, Hausdorff showed that there is a countable set D such that $S^2 \setminus D$ can be partitioned into three sets A, B, C such that A is congruent to B (i.e., there is a rotation ρ such that $\rho(A) = B$), B is congruent to C, and C is congruent to $A \cup B$. This also is counterintuitive, and the sets A, B,

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and C cannot be measurable with respect to the standard isometry-invariant probability measure on S^2. (In fact, there is no rotation-invariant finitely additive probability measure on S^2 which assigns a measure to these sets.)

Later, R. Robinson [11] refined the Banach-Tarski construction, and characterized the systems of congruences for which one can partition S^2 (without a countable exceptional set) into pieces satisfying the congruences. In order to state Robinson's results precisely, we need some definitions.

Fix a positive integer r. A *congruence* is specified by two subsets L and R of $\{1, 2, \ldots, r\}$, and is written formally as $\bigcup_{k\in L} A_k \cong \bigcup_{k\in R} A_k$, where $A_1, A_2, \ldots, A_r$ are variables. The congruence is *proper* if both L and R are nonempty proper subsets of $\{1, \ldots, r\}$. Now suppose G is a group acting on a set X, and a system of congruences is given by pairs $L_i, R_i \subseteq \{1, \ldots, r\}$ for $i \leq m$; a *solution* to the system of congruences in X is a sequence of sets $A_k \subseteq X$ ($k \leq r$) which are pairwise disjoint and have union X, such that, for each $i \leq m$, there is $\sigma_i \in G$ such that $\sigma_i(\bigcup_{k\in L_i} A_k) = \bigcup_{k\in R_i} A_k$ (i.e., σ_i witnesses congruence number i). It is clear that only proper congruences are useful here, and that, if σ_i witnesses the congruence given by L_i and R_i, then σ_i also witnesses the complementary congruence given by L_i^c and R_i^c, where $S^c = \{1, \ldots, r\} \setminus S$. Also, congruence is transitive: if σ witnesses $A \cong B$ and τ witnesses $B \cong C$, then $\tau \circ \sigma$ witnesses $A \cong C$. A system of congruences is called *weak* if, among all congruences which can be deduced from the system by taking complements and applying transitivity, there is no congruence of the form $\bigcup_{k\in L} A_k \cong \bigcup_{k\in L^c} A_k$ which requires some set to be congruent to its complement.

It is easy to see that, if a system of congruences has a solution in S^2 with rotations witnessing the congruences, then the system must be weak: any rotation has fixed points, and hence cannot witness that a set is congruent to its complement. Robinson showed that the converse is true: any weak system of congruences has a solution in S^2 with rotations witnessing the congruences. Dekker [3] gave the following abstract form of Robinson's result: If G is a free group on more than one generator which acts locally commutatively on a set X, then any weak system of congruences has a solution in X with elements of G witnessing the congruences. (An action of a group G is called *locally commutative* if any two elements of G with a common fixed point commute; clearly the rotation group on S^2 has this property. Variants or corollaries of Hausdorff's embedding of a free product of $\mathbf{Z}_2$ and $\mathbf{Z}_3$ into SO_3 show that free groups on any finite or countable number of generators can be embedded into SO_3.) Also, Adams showed that, if one allows arbitrary isometries of S^2 rather than just rotations to witness the congruences, then any system of proper congruences has a solution in S^2.

The preceding information is from Wagon [12], which is an excellent reference on the Banach-Tarski paradox and related work.

In general, a weak system of congruences need not have a solution using *measurable* subsets of S^2. For example, consider the system $A_2 \cong A_2 \cup A_3 \cup A_4$, $A_4 \cong A_1 \cup A_2 \cup A_4$, which Robinson used to get a minimal Banach-Tarski decomposition of S^2. If these congruences are satisfied by measurable subsets of S^2, then, since S^2 has finite measure, the first congruence forces A_3 and A_4 to have measure 0, and the second forces A_1 and A_2 to have measure 0, so the four sets together cannot cover S^2. (The same argument applies to Hausdorff's system of congruences, but this is not weak.) Another example is the system $A_1 \cong A_2 \cong A_3 \cong A_4 \cong A_5$, $A_1 \cup A_2 \cong A_1 \cup A_3 \cup A_4$; any measurable solution to the first part of this would have to give measure $1/5$ to each of the sets, making the final congruence impossible.

If one considers solutions using sets with the property of Baire instead of measurable sets, then the situation is quite different; the arguments of the preceding paragraph do not apply. It was shown in Dougherty and Foreman [6] that the Banach-Tarski paradox can be carried out using pieces with the property of Baire. In the present paper, the methods of Dougherty and Foreman [6] will be used to characterize those systems of congruences which have solutions in S^2 under rotations using sets with the property of Baire. The result applies more generally, to show that suitable systems of congruences have solutions using sets with the property of Baire in any Polish space (complete separable metric space) $\mathcal{X}$ with a nonabelian free group of homeomorphisms of $\mathcal{X}$ which acts locally commutatively on $\mathcal{X}$ and freely (without fixed points) on a comeager subset of $\mathcal{X}$. To specify which systems are 'suitable' requires further definitions.

We will call a system of congruences *nonredundant* if no congruence in the system can be deduced from the other congruences in the system by complementation and transitivity as above, and there is no identity congruence $A \cong A$ in the system.

Next, say that A is *subcongruent* to B ($A \preceq B$) if A is congruent to a subset of B. From a given system of congruences, one can deduce subcongruences by the following rules: if $A \subseteq B$, then $A \preceq B$; if $A \preceq B$ and $B \preceq C$, then $A \preceq C$; and, if $A \cong B$ is in the given system, then $A \preceq B$, $B \preceq A$, $A^c \preceq B^c$, and $B^c \preceq A^c$ (where A^c is the complement of A). We will call the system of congruences *consistent* if there do not exist sets $L, R \subseteq \{1, 2, \ldots, r\}$ with R a proper subset of L such that one can deduce $\bigcup_{k \in L} A_k \preceq \bigcup_{k \in R} A_k$ from the system. For example, the systems used by Hausdorff and Robinson as above are not consistent, but the other example system above is consistent (the only subcongruences deducible from it where the left side is a union of more sets than the right side are $A_1 \cup A_3 \cup A_4 \preceq A_1 \cup A_2$ and $A_3 \cup A_4 \cup A_5 \preceq A_2 \cup A_5$).

The main result of this paper is that, if a given system of m congruences is weak and consistent, and if $\mathcal{X}$ is a Polish space on which a free group G of homeomorphisms with m generators acts locally commutatively everywhere and freely on a comeager set, then the system of congruences has a solution on $\mathcal{X}$ using sets with the property of Baire; furthermore, if the system is nonredundant, then one can use a specified list of m free generators of G to serve as the witnesses for the congruences. (This latter condition holds for the Robinson-Dekker construction, without any extra assumption.) The conditions of weakness and consistency are necessary for the case of S^2 with a free group of rotations, at least if we require the sets to be nonmeager; without this requirement, a system has a solution in S^2 using free rotations if and only if it has a subsystem (obtained by deleting zero or more of the sets $A_1, \ldots, A_r$ from all congruences) which is weak and consistent. Also, requiring that a redundant congruence be witnessed by a free rotation can make a system unsolvable on S^2.

As in Dougherty and Foreman [6], the results here concerning sets with the property of Baire are obtained by combining known results about arbitrary sets with new results about open sets. In most cases, one cannot expect to get actual solutions to systems of congruences using open sets; in particular, a connected space cannot be nontrivially partitioned into open sets at all. We will therefore allow meager exceptional sets when trying to satisfy congruences using open sets. This leads to the following definitions: Suppose G is a group of homeomorphisms of a space $\mathcal{X}$. Two sets $A, B \subseteq \mathcal{X}$ will be called *quasi-disjoint* if their intersection is meager. (Of course, quasi-disjoint open sets in a Polish space are actually disjoint.) Sets A and B are *quasi-congruent*, as witnessed by $\sigma \in G$, if $\sigma(A)$ differs from B by a meager set. A *quasi-solution* to a system of congruences $\bigcup_{k \in L_i} A_k \cong \bigcup_{k \in R_i} A_k$ is a sequence of sets $A_k \subseteq \mathcal{X}$ ($k \leq r$) which are pairwise quasi-disjoint and whose union is a comeager subset of $\mathcal{X}$, such that, for each $i \leq m$, there is $\sigma_i \in G$ which witnesses that $\bigcup_{k \in L_i} A_k$ is quasi-congruent to $\bigcup_{k \in R_i} A_k$.

The remainder of this paper is as follows. In section 2, it will be shown that, if G is a suitable free group of homeomorphisms of a Polish space $\mathcal{X}$, then any weak consistent system of congruences has a quasi-solution in $\mathcal{X}$ using nonempty open sets (with specified free generators of G witnessing the congruences, if the system is nonredundant). This result is then combined with the results of Robinson and Dekker to produce solutions (not just quasi-solutions) to any weak consistent system using sets with the property of Baire. Section 3 shows the necessity of weakness, consistency, and nonredundancy. Section 4 gives the proof that, if one allows arbitrary isometries of S^2 as witnesses, then any consistent system of congruences has a quasi-solution using nonempty open sets, and a solution using nonmeager sets with the property of Baire.

In a later paper [5], we consider the problem of finding open sets which actually satisfy congruences rather than quasi-congruences (but still are only required to cover a dense subset of the space, rather than all of it).

We will use the symbol $\circ$ or simple juxtaposition to denote a group operation, interchangeably. All group actions will be written on the left. For standard basic facts about free groups, such as the unique expression of any element as a reduced word in the generators and the fact that any nonidentity element has infinite order, see any text on combinatorial group theory, such as Magnus, Karrass, and Solitar [9]. More advanced facts will be referred to specifically as needed.

2. Positive results

Theorem 2.1. *Suppose $\mathcal{X}$ is a Polish space and G is a countable group of homeomorphisms of $\mathcal{X}$ which acts freely on a comeager subset of $\mathcal{X}$, and which has a subgroup which is free on m generators ($m \geq 1$). Suppose that a system of m congruences is specified by pairs (L_i, R_i) ($1 \leq i \leq m$) of subsets of $\{1, 2, \ldots, r\}$; also suppose that this system is weak and consistent. Then there is a sequence of nonempty open sets $A_k \subseteq \mathcal{X}$ ($k \leq r$) which is a quasi-solution to the system. Furthermore, if the system is nonredundant, and elements f_i ($1 \leq i \leq m$) of G are free generators for a free subgroup of G, then there is a sequence of nonempty open sets A_k as above such that, for each $i \leq m$, f_i witnesses that $\bigcup_{k\in L_i} A_k$ is quasi-congruent to $\bigcup_{k\in R_i} A_k$.*

Proof. First note that, if one congruence in a system is deducible from the other congruences, then one can delete that one congruence to get a smaller system, and any quasi-solution to the smaller system will be a quasi-solution to the original system. (The same holds for solutions.) By iterating this, one can reduce the original system to a nonredundant system with the same quasi-solutions. Therefore, it will suffice to prove only the second part of the theorem. We may assume that G is the free group generated by the elements f_i.

We will follow the method of Dougherty and Foreman [6], but with a few differences. One difference is that we will concentrate on the points to be excluded from the sets A_k, and not construct the sets A_k themselves until the excluded sets are complete. (The reason for this is that it is easier to work with congruences between intersections than with congruences between unions; we can actually make intersections congruent, rather than quasi-congruent.) We will construct sets B_k ($1 \leq k \leq r$) with the following properties: $\bigcap_{k=1}^r B_k = \varnothing$; the sets $\bigcap_{k'\neq k} B_{k'}$ for $k \leq r$ are all nonempty, and their union is dense in $\mathcal{X}$; and, for each $i \leq m$, $f_i(\bigcap_{k\in L_i} B_k) = \bigcap_{k\in R_i} B_k$. Once we have these sets, we can define A_k to be $\bigcap_{k'\neq k} B_{k'}$; then the sets A_k will be as desired. (The intersection of any two sets A_k will be $\bigcap_{k'=1}^r B_{k'} = \varnothing$. For any $L \subseteq \{1, 2, \ldots, r\}$, the set $\bigcap_{k\in L} B_k$ includes $A_{k'}$ for $k' \notin L$ and is disjoint from $A_{k'}$ for $k' \in L$, so $\bigcup_{k\in L} A_k$ differs from the complement of $\bigcap_{k\in L} B_k$ by a meager set; hence, a congruence between $\bigcap_{k\in L} B_k$ and $\bigcap_{k\in R} B_k$ yields a quasi-congruence between $\bigcup_{k\in L} A_k$ and $\bigcup_{k\in R} A_k$.)

The open sets B_k will be built in stages: we will construct open sets $B_k^0 \subseteq B_k^1 \subseteq B_k^2 \subseteq \ldots$ for $k \leq r$ and then let $B_k = \bigcup_{n=0}^\infty B_k^n$. The sets B_k^n will satisfy the following properties, to be maintained as induction hypotheses:

(2) $\bigcap_{k=1}^r B_k^n = \varnothing$.
(3) For each $i \leq m$, $f_i(\bigcap_{k\in L_i} B_k^n) = \bigcap_{k\in R_i} B_k^n$ and $f_i(\bigcap_{k\in L_i^c} B_k^n) = \bigcap_{k\in R_i^c} B_k^n$.
(4) For any $x \in \mathcal{X}$, the set of $y \in \mathcal{X}$ which are connected to x by a chain of active links is finite.

(There is no property (1); this numbering is used for compatibility with Dougherty and Foreman [6].) Of course, we must define the terms used in (4):

Definition. Two points x and x' are *linked*, or there is a *link* from x to x', if $x' = f_i(x)$ or $x = f_i(x')$ for some $i \le m$. Points x and x' are *connected by a chain of links* if there are points $x_0, x_1, \dots, x_J$ with $x_0 = x$ and $x_J = x'$ such that there is a link from x_{j-1} to x_j for each $j \le J$. A link from x to x' is *active* (for the sets B_k^n) if there is a point in one or more of the sets B_k^n which is connected to x or to x' by a chain of at most 2^r links.

Note that adding one new point to a set B_k^{n+1} activates only a finite number of new links, although the finite number is very large.

Let $B_k^0 = \varnothing$ for all k; clearly this makes (2)–(4) true for $n = 0$. Let $\langle Z_n : n = 0, 1, 2, \dots \rangle$ be a listing of the nonempty sets in some base for the topology of $\mathcal{X}$; we may assume that $\mathcal{X}$ itself occurs at least r times in the list. We must show how to get from B_k^n to B_k^{n+1}, preserving properties (2)–(4), so that, for a given nonempty open set $Z = Z_n$, one of the sets $\bigcap_{k' \ne \bar{k}} B_{k'}^{n+1}$ for $\bar{k} \le r$ will meet Z. The t'th time that $Z = \mathcal{X}$ ($t \le r$), we will set $\bar{k} = t$ in order to ensure that $\bigcap_{k' \ne t} B_{k'}$ will be nonempty. Once this is accomplished for all n, the resulting sets B_k will have the desired properties.

So suppose we are given B_k^n ($k \le r$) and $Z = Z_n$. The first step is to find a point $x_0 \in Z$ to be put into all but one of the sets B_k^{n+1}. In fact, we will find x_0 in Z', where Z' is Z unless $Z = \mathcal{X}$ and this is one of the first r occurrences of $\mathcal{X}$ in the list of open sets; in the latter case, if this is the t'th occurrence of $\mathcal{X}$, then let Z' be the interior of the complement of B_t^n. (To see that this set is nonempty, look at a G-orbit on which G acts freely and which does not meet the boundary of B_t^n; such orbits form a comeager subset of $\mathcal{X}$. By (4), some point in this orbit is not in any of the sets B_k^n, and hence must be in the interior of the complement of B_t^n.) So x_0 will be in Z in any case.

Let D be the complement of a (G-invariant) comeager set on which G acts freely, and let D' be the union of the images under the elements of G of the boundaries of the sets B_k^n; then $D \cup D'$ is meager. Let x_0 be any point in $Z' \setminus (D \cup D')$. By (2), we can find $\bar{k} \le r$ such that $x_0 \notin B_{\bar{k}}^n$. In the case where $Z = \mathcal{X}$ for the t'th time ($t \le r$), we have $x_0 \notin B_t^n$ by the definition of Z', so we may set $\bar{k} = t$. We will ensure that $x_0 \in B_k^{n+1}$ for all $k \ne \bar{k}$; this will take care of the current density requirement (or the current nonemptiness requirement).

As in Dougherty and Foreman [6], we will construct sets $\hat{B}_k$ by adding finitely many points to the sets B_k^n; $\hat{B}_k$ will be defined to be $B_k^n \cup \{g(x_0) : g \in T_k\}$ for some $T_k \subseteq G$. However, we will describe the construction a little differently; instead of giving inductive clauses to define the sets T_k, we will define a set $M_g \subseteq \{1, 2, \dots, r\}$ for each $g \in G$ and then let $T_k = \{g \in G : k \in M_g\}$.

We define M_g recursively, based on the reduced form of $g \in G$ in terms of the generators f_i. If g is the identity of G, then let $M_g = \{k \le r : k \ne \bar{k}\}$. Otherwise, we can write g uniquely as $f_i \circ g'$ or $f_i^{-1} \circ g'$ where g' has a shorter reduced form than g does, and hence $M_{g'}$ is already defined. Let $M_{g'}^+ = M_{g'} \cup \{k : g'(x_0) \in B_k^n\}$. If $M_{g'} = \varnothing$, let $M_g = \varnothing$. If $M_{g'} \ne \varnothing$ and $g = f_i \circ g'$, then define M_g as follows: if $L_i \subseteq M_{g'}^+$, let $M_g = R_i$; if $L_i^c \subseteq M_{g'}^+$, let $M_g = R_i^c$; otherwise, let $M_g = \varnothing$. (If both L_i and L_i^c are subsets of $M_{g'}^+$, make some arbitrary definition such as $M_g = \{1, 2, \dots, r\}$; we will see in the next paragraph that this case cannot occur.) If $M_{g'} \ne \varnothing$ and $g = f_i^{-1} \circ g'$, then define M_g in the same way, but with L_i and R_i interchanged.

First, we show by induction on $g \in G$ that $M_g^+ \ne \{1, 2, \dots, r\}$. If g is the identity, then $\bar{k} \notin M_g^+$ by the definition of x_0. Otherwise, we have $g = f_i \circ g'$ or $g = f_i^{-1} \circ g'$ for some simpler g'. If $M_g = \varnothing$, then $M_g^+ = \{k : g(x_0) \in B_k^n\} \ne \{1, 2, \dots, r\}$ by (2). If $M_g \ne \varnothing$, $g = f_i \circ g'$, and $L_i \subseteq M_{g'}^+$, then $L_i^c \not\subseteq M_{g'}^+$ by the induction hypothesis, so $L_i^c \not\subseteq \{k : g'(x_0) \in B_k^n\}$, so $R_i^c \not\subseteq \{k : g(x_0) \in B_k^n\}$ by (3), so $M_g^+ = R_i \cup \{k : g(x_0) \in B_k^n\} \neq \{1, 2, \dots, r\}$. The remaining cases are handled the same way.

We can now check that, for any g and g' in G and $i \le r$, if $g = f_i \circ g'$, then $L_i \subseteq M_{g'}^+$ iff $R_i \subseteq M_g^+$, and $L_i^c \subseteq M_{g'}^+$ iff $R_i^c \subseteq M_g^+$. First, suppose the reduced form of g' does not have f_i^{-1} as its leftmost term; then M_g is defined from $M_{g'}$ as above. If $M_{g'} = \varnothing$ and hence $M_g = \varnothing$, then these two equivalences follow directly from (3); so suppose $M_{g'} \neq \varnothing$. Now the two left-to-right implications are immediate. For the first right-to-left implication, if $L_i \not\subseteq M_{g'}^+$, then $R_i \cap M_g = \varnothing$ by definition of M_g, while $R_i \not\subseteq \{k : g(x_0) \in B_k^n\}$ because otherwise (3) would give $L_i \subseteq \{k : g'(x_0) \in B_k^n\} \subseteq M_{g'}^+$, so $R_i \not\subseteq M_g^+$. The other implication is proved in the same way. This completes the case where g' does not have f_i^{-1} as its leftmost term. If g' does have f_i^{-1} as its leftmost term, then g does not have f_i as its leftmost term, so we can write $g' = f_i^{-1} \circ g$ and proceed as above.

We are now ready to prove (2)–(4) for the sets $\hat{B}_k$. The definitions of T_k and $\hat{B}_k$ (and the fact that G acts freely on the orbit of x_0) easily imply that $\{k : g(x_0) \in \hat{B}_k\} = M_g^+$ for all $g \in G$, while $\{k : x \in \hat{B}_k\} = \{k : x \in B_k^n\}$ if x is not in the G-orbit of x_0. Therefore, properties (2) and (3) for $\hat{B}_k$ follow from the same properties for B_k^n and the above facts about M_g^+.

To prove property (4) for the sets $\hat{B}_k$, we will need the following claims, which are the part of this proof where all of the restrictions on the system of congruences are needed.

Define a labeled directed graph $\mathcal{G}$ from the system of congruences as follows. The vertices of $\mathcal{G}$ are the nonempty proper subsets of $\{1, 2, \dots, r\}$. If S is such a subset, and $L_i \subseteq S$, then $\mathcal{G}$ has an edge from S to R_i labeled f_i. If $L_i^c \subseteq S$, then $\mathcal{G}$ has an edge from S to R_i^c labeled f_i. Similarly, if $R_i \subseteq S$, then $\mathcal{G}$ has an edge from S to L_i labeled f_i^{-1}; if $R_i^c \subseteq S$, then $\mathcal{G}$ has an edge from S to L_i^c labeled f_i^{-1}.

The digraph $\mathcal{G}$ has cycles of length 2 connecting pairs (L_i, R_i) or (L_i^c, R_i^c); each such cycle consists of an f_i-edge and an f_i^{-1}-edge. Call the edges in these 2-cycles (the edges which come from actual congruences rather than subcongruences) *good* edges, and call all other edges (e.g., an f_i-edge from a proper superset of L_i to R_i) *bad* edges.

Claim 1. No cycle in $\mathcal{G}$ contains a bad edge.

Proof. Suppose the edges $e_1, e_2, \dots, e_J$ form a cycle. Let $N_0, N_1, \dots, N_J$ be the vertices of this cycle (with $N_J = N_0$), so that e_j is an edge from N_{j-1} to N_j. For each j, e_j has a label, which is either f_{i_j} or $f_{i_j}^{-1}$ for some i_j. We will abuse notation by writing $M \preceq N$ for $M, N \subseteq \{1, 2, \dots, r\}$ to mean that the subcongruence $\bigcup_{k \in M} A_k \preceq \bigcup_{k \in N} A_k$ is deducible from the given system of congruences by the usual rules.

For each j such that $0 < j \le J$, define a set N_j^- as follows. First, suppose e_j is labeled f_{i_j}. Then either $L_{i_j} \subseteq N_{j-1}$ and $N_j = R_{i_j}$, or $L_{i_j}^c \subseteq N_{j-1}$ and $N_j = R_{i_j}^c$; let N_j^- be L_{i_j} in the former case and $L_{i_j}^c$ in the latter. Similarly, if e_j is labeled $f_{i_j}^{-1}$, let N_j^- be R_{i_j} or $R_{i_j}^c$, depending on whether N_j is L_{i_j} or $L_{i_j}^c$. In any case, we have $N_{j-1} \supseteq N_j^-$, and congruence number i_j relates either the sets N_j^- and N_j or their complements. Therefore, $N_{j-1} \succeq N_j^- \succeq N_j$ for all j. Since $\succeq$ is transitive and $N_J = N_0$, we have $N_j^- \succeq N_{j-1}$ for $0 < j \le J$. Since the system of congruences is consistent, N_j^- cannot be a proper subset of N_{j-1}, so we must have $N_j^- = N_{j-1}$ for $0 < j \le J$; this means that all of the edges are good. $\square$

Now construct the undirected graph $\mathcal{G}_0$ by treating each pair of oppositely-directed good edges in $\mathcal{G}$ as a single undirected edge between L_i and R_i or between L_i^c and R_i^c.

Claim 2. The undirected graph $\mathcal{G}_0$ is acyclic (i.e., its connected components are trees).

Proof. Note that sets N, N' are in the same component of $\mathcal{G}_0$ if and only if the congruence $\bigcup\{A_k : k \in N\} \cong \bigcup\{A_k : k \in N'\}$ follows from the given system of congruences. In particular, N and N^c cannot be in the same component of $\mathcal{G}_0$, because the system is weak. Also, note that congruence number i gives rise to two edges of $\mathcal{G}_0$, one between L_i and R_i and one between L_i^c and R_i^c; these edges must be in different components of $\mathcal{G}_0$.

Now suppose we have a nontrivial cycle in $\mathcal{G}_0$; by taking a minimal such cycle, we may ensure that there are no repeated edges in the cycle. Let one of the edges in the cycle be an edge from L to R, coming from congruence number i. Then the rest of this cycle cannot use this edge and cannot use the other edge coming from congruence number i (since this is not even in the same component), so it consists entirely of edges coming from other congruences. But the rest of the cycle gives a path from L to R, so $\bigcup\{A_k : k \in L\} \cong \bigcup\{A_k : k \in R\}$ is deducible from the system without using congruence number i. So congruence number i is deducible from the others, contradicting the assumption that the system is nonredundant. $\square$

Using these two claims, we get:

Claim 3. Every path of length 2^r in the digraph $\mathcal{G}$ contains a pair of consecutive edges with labels f_i and f_i^{-1}, or vice versa, for some i.

Proof. Suppose we have a path of length 2^r in $\mathcal{G}$. Since there are fewer than 2^r vertices in $\mathcal{G}$, some vertex must be visited more than once, so we get a nontrivial subpath which starts and ends at the same vertex (i.e., a cycle). By Claim 1, this subpath consists entirely of good edges, so it induces a corresponding path in the graph $\mathcal{G}_0$ which also starts and ends at the same place. By Claim 2, this latter path cannot be a nontrivial cycle, so it must double back on itself (use the same edge twice in succession); hence, the original path uses both edges of a pair of oppositely-directed good edges successively, which gives the desired conclusion. $\square$

Now, for any $g \in G$, x_0 is connected to $g(x_0)$ by a chain of links, and this chain can be read off from the reduced form of g. In order to prove (4) for the sets $\hat{B}_k$, it will suffice to show that, if $M_g \neq \varnothing$, then either all of the links in this chain are active for the sets B_k^n, or the chain has fewer than 2^r links; once we know this, (4) for B_k^n implies that there are only finitely many points $g(x_0)$ such that $M_g \neq \varnothing$ (equivalently, since G acts freely on the orbit of x_0, the set of g such that $M_g \neq \varnothing$ is finite), so only finitely many new links are activated when B_k^n is enlarged to $\hat{B}_k$, so (4) for B_k^n implies (4) for $\hat{B}_k$.

So suppose $M_g \neq \varnothing$ and the above chain has at least 2^r links. Then $M_h \neq \varnothing$ for all of the intermediate points $h(x_0)$ on the chain. It must now be true that, given any 2^r consecutive links in the chain, at least one of the $2^r + 1$ endpoints of these links is in one of the sets B_k^n, because otherwise the sets M_h at these $2^r + 1$ endpoints would give a counterexample to Claim 3. (If none of these points $h(x_0)$ is in any of the sets B_k^n, then we always have $M_h^+ = M_h$. Now, if h and $h' = \rho \circ h$ are final subwords of the reduced word for g, where ρ is f_i or f_i^{-1}, then the way in which $M_{h'}$ is computed from M_h shows that there is an edge in $\mathcal{G}$ from M_h to $M_{h'}$ labeled ρ. The resulting path of length 2^r cannot include consecutive edges labeled f_i and f_i^{-1} or vice versa because we are working with the reduced form of g.) It follows that all 2^r of the links are active for B_k^n; since this was an arbitrary subchain of the chain, all of the links in the chain are active for B_k^n. This completes the proof of (4) for $\hat{B}_k$.

Now that we have (2)–(4) for $\hat{B}_k$, we can enlarge these sets to get open sets. Let S be the set of $g \in G$ such that x_0 is connected to $g(x_0)$ by a chain of links which are active for the sets $\hat{B}_k$, and let S' be the set of $g' \in G$ such that $g'(x_0)$ is connected to $g(x_0)$ for some $g \in S$ by a chain of at most $2^r + 1$ links. Then $T_k \subseteq S$ for all k, $S \subseteq S'$, and S and S' are finite by (4). Let U_0 be an open neighborhood of x_0 so small that the images $g(U_0)$ for $g \in S'$ are pairwise disjoint and each of them is either included in or disjoint from each of the sets B_k^n. (This is possible because, by the choice of x_0, no point in S' is on the boundary of any of the sets B_k^n.) Now let $B_k^{n+1} = B_k^n \cup \bigcup\{g(U_0)\colon g \in T_k\}$ for each k; we must see that these sets satisfy properties (2)–(4).

From the definition of B_k^{n+1} and the disjointness of the sets $g(U_0)$ for $g \in S'$, the following two statements follow easily: If $x \in g(U_0)$ for some $g \in S'$, then $x \in B_k^{n+1}$ if and only if $g(x_0) \in \hat{B}_k$. If $x \in \mathcal{X}$ is not in any of the sets $g(U_0)$ for $g \in S$, then $x \in B_k^{n+1}$ if and only if $x \in B_k^n$.

We can now prove (2)–(4) for B_k^{n+1}.

(2): If a point x is in one of the neighborhoods $g(U_0)$ where $g \in S$, then $g(x_0) \notin \bigcap_{k=1}^r \hat{B}_k$ by (2) for $\hat{B}_k$, so $x \notin \bigcap_{k=1}^r B_k^{n+1}$; if x is not in one of these neighborhoods, then $x \notin \bigcap_{k=1}^r B_k^n$ by (2) for B_k^n, so $x \notin \bigcap_{k=1}^r B_k^{n+1}$.

(3): We prove $f_i(\bigcap_{k\in L_i} B_k^{n+1}) \subseteq \bigcap_{k\in R_i} B_k^{n+1}$; the other parts are similar. Suppose $x \in \bigcap_{k\in L_i} B_k^{n+1}$. If $x \in g(U_0)$ for some $g \in S$, then $g(x_0) \in \bigcap_{k\in L_i} \hat{B}_k$, so $f_i(g(x_0)) \in \bigcap_{k\in R_i} \hat{B}_k$ by (3) for $\hat{B}_k$; but $f_i \circ g \in S'$ and $f_i(x) \in f_i(g(U_0))$, so $f_i(x) \in \bigcap_{k\in R_i} B_k^{n+1}$. If x is not in $g(U_0)$ for any $g \in S$, then $x \in \bigcap_{k\in L_i} B_k^n$, so $f_i(x) \in \bigcap_{k\in R_i} B_k^n$ by (3) for B_k^n.

(4): Let w be any point of $\mathcal{X}$, and consider the set of all points connected to w by a path of links which are active for the sets B_k^{n+1}. If this set contains no point which is in $g(U_0)$ for any $g \in S$, then all of the links connecting the set were in fact active for B_k^n. (Note: If the link from x to x' is activated by x'', because there is a chain of at most 2^r links connecting x'' to x or to x', then all of the links in this chain are also activated by x''.) Hence, the set is finite by (4) for B_k^n. So suppose $y \in g(U_0)$ is connected by active links to w, and $g \in S$. A point is connected to w if and only if it is connected to y, so it will suffice to show that only finitely many points are connected to y.

Suppose y is actively linked to y', say $y' = f_i(y)$ (the case $y' = f_i^{-1}(y)$ is similar). Let y'' be a point in one of the sets B_k^{n+1} such that y'' is connected to either y or y' by a chain of at most 2^r links. Then there is an element h of G such that $h(y) = y''$, and the reduced form of h in terms of the generators f_I has length at most $2^r + 1$ (and, if it has length $2^r + 1$, then the rightmost component is f_i). Therefore, $h \circ g \in S'$. We now have $y'' \in h(g(U_0))$, so, since $y'' \in B_k^{n+1}$, we must have $h(g(x_0)) \in \hat{B}_k$. This means that the link from $g(x_0)$ to $f_i(g(x_0))$ is active for the sets $\hat{B}_k$, so $f_i(y) \in f_i(g(x_0))$ and $f_i \circ g \in S$.

Now this argument can be repeated starting at y', and so on; the result is that, for any chain of active (for the sets B_k^{n+1}) links starting at y, all of the links in the corresponding chain starting at $g(x_0)$ are also active (for the sets $\hat{B}_k$). Furthermore, if y is connected to two different points y' and y'' by such chains of links, this will give $y' = h'(y)$ and $y'' = h''(y)$ for some distinct elements h', h'' of G, and the corresponding points reached from $g(x_0)$ will be $h'(g(x_0))$ and $h''(g(x_0))$; since G acts freely on the orbit of x_0, these two points will also be different. Therefore, since $g(x_0)$ is connected to only finitely many points, y (and hence w) must be connected to only finitely many points. This completes the proof of (4) for the sets B_k^{n+1}.

This completes the induction. ■

One can use this result to give a new proof of Theorem 4.8 from Dougherty and Foreman [6]:

Corollary 2.2. *Suppose $\mathcal{X}$ is a Polish space and G is a countable group of homeomorphisms of $\mathcal{X}$ which acts freely on a comeager subset of $\mathcal{X}$. Then, for any $N \geq 3$, if elements f_i ($1 \leq i \leq N$) of G are free generators for a free subgroup of G, then there is an open subset A of $\mathcal{X}$ such that the sets $f_i(A)$ are disjoint and their union is dense in $\mathcal{X}$. In fact, if $f_{ij} \in G$ for $1 \leq i \leq j$ and $3 \leq j \leq N$ are free generators for a free subgroup of G, then there is an open set A such that, for each j, the sets $f_{ij}(A)$ for $i \leq j$ are disjoint and have dense union.*

Proof. We will prove the second part; the proof of the first part can be obtained from this by omitting most of the congruences (in fact, the first part is essentially a special case of the second).

Let R be the set of sequences $s = \langle s_j : 3 \le j \le N \rangle$ such that $1 \le s_j \le j$ for each j; we will construct a system of congruences between sets A_s for $s \in R$. (Of course, one can relabel the sets to make the index set $\{1, 2, \dots, r\}$, where $r = |R| = N!/2$.) The congruences are: $\bigcup\{A_s : s(N) = 1\} \cong \bigcup\{A_s : s(j) = i\}$ for each pair $(i, j) \ne (1, N)$ such that $3 \le j \le N$ and $1 \le i \le j$. The only proper congruences that can be deduced from this system are those of the form $\bigcup\{A_s : s(j) = i\} \cong \bigcup\{A_s : s(j') = i'\}$ and their complementary versions; it follows easily that the system is weak and consistent. It is also easy to check that the system is nonredundant. Therefore, by Theorem 2.1, one can find a quasi-solution to the system using open sets A_s for $s \in R$, where the congruence for (i, j) is witnessed by the element $f_{ij} \circ f_{1N}^{-1}$ of G. (Since the elements f_{ij} are free generators for their subgroup, the elements $f_{ij} \circ f_{1N}^{-1}$ for $(i, j) \ne (1, N)$ are free generators for *their* subgroup.)

Now let $A = f_{1N}^{-1}(\bigcup\{A_s : s(N) = 1\})$. Then, for each (i, j), $f_{ij}(A)$ differs from $\bigcup\{A_s : s(j) = i\}$ by a meager set; it follows that, for each j, the sets $f_{ij}(A)$ for $i \le j$ are quasi-disjoint and their union is a comeager (hence dense) subset of $\mathcal{X}$. Since quasi-disjoint open sets must actually be disjoint, we are done. ■

The trick used here to get congruent rather than quasi-congruent sets is quite specific; many weak consistent systems of congruences do not have quasi-solutions in open sets if one actually requires congruences instead of quasi-congruences. This will be explored further in a later paper [5].

In order to get results concerning sets with the property of Baire, we will combine the preceding results about open sets with the Robinson-Dekker results on arbitrary sets, using the following lemma, which is a variant of Lemma 2.4 from Dougherty and Foreman [6]:

Lemma 2.3. *Suppose $\mathcal{X}$ is a Polish space, and $f_1, \dots, f_m$ are homeomorphisms from $\mathcal{X}$ to $\mathcal{X}$. Also suppose that we have a system of m congruences such that: there is a solution to the system in $\mathcal{X}$ with f_i witnessing congruence number i for $i = 1, 2, \dots, m$; and there is a quasi-solution to the system in $\mathcal{X}$ using nonmeager sets with the property of Baire so that f_i witnesses congruence number i. Then there is a solution to the system in $\mathcal{X}$ using nonmeager sets with the property of Baire so that f_i witnesses congruence number i.*

Proof. Let G be the countable group of homeomorphisms generated by $f_1, \dots, f_m$. Suppose the quasi-solution consists of sets A'_k with the property of Baire for $1 \le k \le r$, while the solution is given by sets A''_k, $1 \le k \le r$. Let D be the union of $\mathcal{X} \setminus \bigcup_{k=1}^{r} A'_k$, the intersections $A'_k \cap A'_{k'}$ for $k \ne k'$, and the differences $f_i(\bigcup_{k \in L_i} A'_k) \triangle \bigcup_{k \in R_i} A'_k$ for $i \le m$. Then D is meager, and so is the union D' of all of the images of D under the elements of G. Now let $A_k = (A'_k \setminus D') \cup (A''_k \cap D')$. These sets have the property of Baire (since A'_k has the property of Baire and D' and $A''_k \cap D'$ are meager), and they are easily seen to be disjoint; using the G-invariance of D', it is easy to verify that $f_i(\bigcup_{k \in L_i} A_k) = \bigcup_{k \in R_i} A_k$ for each i. Also, since A'_k is nonmeager, A_k is nonmeager. Therefore, the sets A_k are as desired. ■

Theorem 2.4. *Suppose $\mathcal{X}$ is a Polish space and G is a countable group of homeomorphisms of $\mathcal{X}$ which acts freely on a comeager subset of $\mathcal{X}$ and locally commutatively on all of $\mathcal{X}$, and which has a subgroup which is free on m generators ($m \ge 1$). Suppose that a system of m congruences is specified by pairs (L_i, R_i) ($1 \le i \le m$) of subsets of $\{1, 2, \dots, r\}$; also suppose that this system is weak and consistent. Then there is a sequence of nonmeager sets $A_k \subseteq \mathcal{X}$ ($k \le r$) with the property of Baire which is a solution to the system. Furthermore, if the system is nonredundant, and elements f_i ($1 \le i \le m$) of G are free generators for a free subgroup of G, then there is a* *sequence of sets A_k as above such that, for each $i \leq m$, f_i witnesses that $\bigcup_{k\in L_i} A_k$ is congruent to $\bigcup_{k\in R_i} A_k$.*

Proof. The second part follows immediately from Lemma 2.3, Theorem 2.1, and the results of Robinson and Dekker, while the first follows from the second as in the proof of Theorem 2.1. ∎

We now recall that a free group on two generators has subgroups which are free on any finite number of generators [9, Problem 1.4.12]. This allows us to simplify the statement of the following corollary, which follows from Theorem 2.4 just as Corollary 2.2 follows from Theorem 2.1:

Corollary 2.5 [6, Theorem 5.4]. *Suppose $\mathcal{X}$ is a Polish space and G is a countable group of homeomorphisms of $\mathcal{X}$ which acts freely on a comeager subset of $\mathcal{X}$ and locally commutatively on all of $\mathcal{X}$, and which has a subgroup which is free on more than one generator. Then, for any $N \geq 3$, $\mathcal{X}$ can be partitioned into N G-congruent pieces with the property of Baire; in fact, there is a set $A \subseteq \mathcal{X}$ with the property of Baire such that, for $3 \leq j \leq N$, $\mathcal{X}$ can be partitioned into j pieces congruent to A.* ∎

3. Negative results

We will now see why the systems of congruences used in Theorems 2.1 and 2.4 must be weak and consistent in order to get the desired conclusions for all suitable $\mathcal{X}$ and G. In fact, it suffices to look at 2.1 only; if one has a solution (or even a quasi-solution) to a system of congruences using sets A_k with the property of Baire, and if A'_k is an open set differing from A_k by a meager set, then the sets A'_k are a quasi-solution to the same system of congruences. We will look at the case of the sphere S^2 acted on by a free group of rotations. (For other spaces or groups, more systems of congruences might be solvable. For example, if G is a free group on countably infinitely many generators and we put the discrete metric on G, then we get a Polish space acted on by G in which every proper system of congruences has a solution [12, Cor. 4.12], and this solution will automatically consist of open sets because the space is discrete.)

We first see why consistency is necessary, at least if we want quasi-solutions involving nonempty open sets. It is easy to verify that, if the sets A_k are a quasi-solution to the system, and the subcongruence $\bigcup_{k\in R} A_k \preceq \bigcup_{k\in R'} A_k$ is deducible from the system, then $\bigcup_{k\in R} A_k$ actually is quasi-congruent to a subset of $\bigcup_{k\in R'} A_k$. We now use the following fact, a slight variant of Proposition 5.5 from Dougherty and Foreman [6]:

Proposition 3.1. *If B and C are quasi-disjoint subsets of S^2 with the property of Baire such that $B \cup C$ is quasi-congruent to a subset of B, then C is meager.*

Proof. Let σ be an isometry witnessing the quasi-congruence, and let B' and C' be the unique regular-open sets such that $B \triangle B'$ and $C \triangle C'$ are meager. (For the definition and properties of regular-open sets, see Oxtoby [10, Ch. 4]. The relevant fact is that B' is the largest open set such that $B \triangle B'$ is meager, and similarly for C'.) Since $\sigma(B\cup C) \setminus B$ is meager, $\sigma(B' \cup C') \setminus B'$ must be meager; but B' is regular-open, so we have $\sigma(B' \cup C') \subseteq B'$. Also, since B and C are quasi-disjoint, $B' \cap C'$ is a meager open set and hence empty. If λ is the standard rotation-invariant probability measure on S^2, then $\lambda(B') \geq \lambda(\sigma(B' \cup C')) = \lambda(B' \cup C') = \lambda(B') + \lambda(C')$, so $\lambda(C') \leq 0$, so C' must be empty, so C is meager. ∎

Therefore, if the open sets $A_k \subseteq S^2$ are a quasi-solution for a system of congruences, and $\bigcup_{k\in R} A_k \preceq \bigcup_{k\in R'} A_k$ is deducible from the system where R' is a proper subset of R, then A_k must be meager and hence empty for each $k \in R \setminus R'$. Hence, if one insists on a solution using nonempty open sets, then the system of congruences must be consistent. If it does not matter that some of the sets are empty, then, given a system of congruences, one should proceed as follows: Find all inconsistencies $\bigcup_{k\in R} A_k \preceq \bigcup_{k\in R'} A_k$, $R' \subset R$ deducible from the system, list all indices k occurring in $R \setminus R'$, and delete the corresponding sets A_k from the congruences (i.e., delete these indices from $\{1, 2, \ldots, r\}$ and from all sets L_i, R_i to get new sets L'_i, R'_i defining a new system of congruences). This may produce new inconsistencies; if so, repeat the process, and continue until no inconsistencies remain. If nothing is left (all sets A_k have been declared empty), then the original system had no quasi-solutions using open subsets of S^2. Otherwise, the final system is consistent. If it is also weak, then the final system, and hence the original system, has solutions for any $\mathcal{X}$ and G as in Theorem 2.4; if the final system is not weak, then we will see below that the final system has no quasi-solutions using open subsets of S^2 with a free group G of rotations, and it follows that the original system also has no quasi-solutions in this case.

Suppose G is a free group of rotations of S^2, and we restrict ourselves to congruences which are witnessed in G; we will now see that only weak systems can have quasi-solutions using open sets in this case. (The corresponding statement about solutions using sets with the property of Baire, or even using arbitrary sets, is trivial because any rotation has fixed points and hence cannot map a set to its complement.) To see this, we use a lemma about open subsets of S^2 which are quasi-invariant under a rotation of infinite order. (A set A is said to be *quasi-invariant* under a homeomorphism f if the symmetric difference $f(A)\triangle A$ is meager.)

Lemma 3.2. *If an open subset A of S^2 is invariant under a rotation of infinite order around an axis ℓ, then A is invariant under all rotations around ℓ. The same applies to quasi-invariance.*

Proof. Let σ be a rotation of infinite order around ℓ under which A is invariant. Fix $x \in A$, and let C be the circle generated by rotating x *continuously* around ℓ; we must show that $C \subseteq A$.

Let $2\pi\theta$ be the angle through which σ rotates S^2; since σ does not have finite order, θ is irrational. It follows that the fractional parts of $j \cdot \theta$ for positive integers j are dense in the interval $(0, 1)$; equivalently, the rotations through positive integer multiples of $2\pi\theta$ arbitrarily well approximate any rotation around ℓ. In particular, for any $y \in C$, the rotation around ℓ which takes y to x can be approximated by a rotation σ^j so well that $\sigma^j(y)$ is in the open set A; this means that $y \in \sigma^{-j}(A) = A$. Since y was arbitrary, we have $C \subseteq A$, as desired.

Now suppose the set A is just quasi-invariant under σ. Let A' be the regular-open set that differs from A by a meager set; then $\sigma(A')$ is a regular-open set that differs from $\sigma(A)$ by a meager set. But $\sigma(A)$ differs from A by a meager set, so $\sigma(A')$ is a regular-open set differing from A by a meager set, so it must be equal to A'. Therefore, A' is invariant under σ, so it is invariant under any rotation τ around ℓ; since $A\triangle A'$ and $\tau(A)\triangle\tau(A')$ are meager, A must be quasi-invariant under τ. ■

Now, suppose a given system has a quasi-solution in S^2 using a free group G of rotations, but is not weak; fix a set $L \subset \{1, \ldots, r\}$ such that the congruence $\bigcup_{k\in L} A_k \cong \bigcup_{k\in L^c} A_k$ can be deduced from the system. Then this quasi-congruence is witnessed by some $\sigma \in G$, which clearly is not the identity and therefore must be a rotation of infinite order. But then σ^2 is also a rotation of infinite order, and $\bigcup_{k\in L} A_k$ is quasi-invariant under σ^2, so it is quasi-invariant under all rotations around the axis of σ^2. In particular, $\bigcup_{k\in L} A_k$ is quasi-invariant under σ, so $\bigcup_{k\in L} A_k$ differs from $\bigcup_{k\in L^c} A_k$ by a meager set, which is impossible because $\bigcup_{k\in L} A_k$ differs from the complement of $\bigcup_{k\in L^c} A_k$ by a meager set, and S^2 is not meager. This contradiction shows that the non-weak system had no quasi-solution after all.

This shows why weakness and consistency are required in Theorem 2.1. Next, we consider the requirement of nonredundancy for a system of congruences to be satisfied with specified witnesses to the congruences. Even for a simple redundant system such as $A_1 \cong A_1$, $A_1 \cong A_1$, $A_1 \cong A_2$, $A_1 \cong$

A_3, one cannot arbitrarily specify the witnesses for the congruences: if the first two congruences are witnessed by rotations of infinite order around different axes, then Lemma 3.2 implies that A_1 must be quasi-invariant under any rotation around either of these axes, so A_1 must be either empty or comeager. (By considering the regular-open set which differs from A_1 by a meager set, we can reduce this claim to the corresponding claim about invariant sets: if A is a nonempty open proper subset of S^2, then A cannot be invariant under all rotations around either of two different axes. To see this, note that a connected component of A must have nonempty boundary. If A is invariant around an axis, then this boundary consists of one or two parts, each of which is a point on the axis or a circle obtained by revolving a point around the axis, so the axis can be reconstructed given the boundary.) Either of these makes the rest of the system impossible to satisfy. With a little more work, one can show that requiring even a single redundant congruence to be witnessed by a rotation of infinite order, such as in the system $A_1 \cong A_1$, $A_1 \cong A_2$, $A_1 \cong A_3$, can make a system unsatisfiable. (It would require A_1 to be a union of spherical disks and annuli with a common axis, and A_2 and A_3 would also have to be such unions but around different axes; such sets cannot fit together closely enough to cover a dense subset of S^2.)

Note that, if G is a countable free group of rotations of S^2, then, since each element of G other than the identity has only two fixed points, G acts freely on $S^2 \setminus D$ for some countable set D. But $S^2 \setminus D$ is a G_δ set in S^2 and is therefore a Polish space itself [8, §33 VI], and the preceding paragraphs hold for this new space as well. Hence, even in a Polish space on which a free group of homeomorphisms acts freely, one cannot guarantee that a system of congruences has a quasi-solution using open sets unless the reduced form of that system (after deleting inconsistencies as above) is weak and consistent.

Another way to modify the space S^2 is as follows: Let G be a free group of rotations of S^2 on $\aleph_0$ generators, and let z be a point of S^2 such that G acts freely on the G-orbit of z; fix another point z' of S^2 which is neither z nor the point opposite z. For each $g \in G$, consider the space $\mathcal{X}_g$ which is the union of S^2 and its tangent ray at $g(z)$ in the direction of the (shortest) great-circle arc from $g(z)$ to $g(z')$, with the standard Euclidean metric from $\mathbf{R}^3$. Note that $h(\mathcal{X}_g) = \mathcal{X}_{h \circ g}$ if we view the rotations as acting on all of $\mathbf{R}^3$. Now take a copy of $\mathcal{X}_g$ for each g, and identify corresponding points of S^2 to get a space $\mathcal{X}$. (Tangent rays that happen to intersect will *not* have their common points identified. If $x, y \in \mathcal{X}$ are in different tangent rays, then the distance from x to y in $\mathcal{X}$ is the length of the shortest path in $\mathbf{R}^3$ from x to y *via a point of* S^2.) Then $\mathcal{X}$ is a Polish space, and its group of isometries is precisely G; G acts locally commutatively on $\mathcal{X}$ and freely on $\mathcal{X} \setminus D$ for a countable meager set D, and the negative results given above for S^2 also apply to $\mathcal{X}$, so we can get such results even when using the entire isometry group of a suitable Polish space.

4. Congruences on the sphere using all isometries

We now know what congruences have solutions using subsets of S^2 with the property of Baire and using free rotations; it is natural to ask what can be done if arbitrary isometries of S^2 are allowed. The results in the preceding section concerning consistency apply just as well for arbitrary isometries, so even here a system must be consistent (or at least reducible to a consistent system by deletion of some sets) in order to have such a solution. However, it turns out that the restriction of weakness can be removed if we allow arbitrary isometries to witness the congruences.

As usual, one of the ingredients needed for the proof is a corresponding result for arbitrary subsets of S^2; this result is due to Adams [1] (see also Wagon [12, Theorem 4.16]). Unfortunately, Adams' proof cannot be used here; the particular isometries he uses to witness the congruences cannot be used to get a corresponding result concerning open sets. (Adams' construction causes a complementary congruence $A \cong A^c$ to be witnessed by an isometry τ such that τ^2 is a rotation of infinite order; then A is invariant under τ^2, and we saw in the preceding section why this cannot work for open sets.) We therefore give a revised proof of this result.

Theorem 4.1 (Adams). *Any system of proper congruences has a solution in S^2, if arbitrary isometries can be used as witnesses for the congruences.*

Proof. We first transform the system into an equivalent system having a useful form. Call two systems of congruences (on the same index set $\{1, 2, \ldots, r\}$) *equivalent* if any congruence in one can be deduced from the other, and vice versa; clearly equivalent systems have the same solutions. By moving to an equivalent system if necessary, we may assume that the system is presented with as few congruences as possible (i.e., there is no equivalent system with fewer congruences).

Now, suppose the system (call it S_0) is not weak. Let M_0 be a subset of $\{1, 2, \ldots, r\}$ such that the congruence $\bigcup_{k\in M_0} A_k \cong \bigcup_{k\in M_0^c} A_k$ is deducible from S_0; choose M_0 so that this deduction requires as few steps as possible. Such a deduction gives a sequence $M_0, M_1, \ldots, M_n$ of subsets of $\{1, 2, \ldots, r\}$ such that $M_n = M_0^c$ and each pair (M_i, M_{i+1}) appears as one of the congruences in S_0, perhaps in the complemented form (M_i^c, M_{i+1}^c). Because the deduction is minimal, no set appears more than once in the list $M_0, M_1, \ldots, M_n$, and the only case where both a set and its complement appear in the list is $M_n = M_0^c$. But this easily implies that no congruence in S_0 is used more than once during the deduction: if it were used twice in the same form, this would require a duplication in the list, while if it were used once in the given form and once in the complemented form (assuming these are different), there would be more than one instance of a set and its complement appearing in the list. Let S_1 be S_0 with the last congruence used in the above deduction (the one between M_{n-1} and M_n, or maybe their complements) deleted, and let S_0' be S_1 together with the congruence $\bigcup_{k\in M_0} A_k \cong \bigcup_{k\in M_0^c} A_k$. Since the congruence between M_0 and M_{n-1} is deducible from S_1, the congruence between M_n and M_{n-1} is deducible from S_0', so S_0' is equivalent to S_0.

Now look at S_1, ignoring the new self-complement congruence. If S_1 is not weak, one can repeat the above process to change S_1 into an equivalent system S_1' with the same number of congruences, where S_1' is S_2 together with another self-complement congruence. Repeat this process as many times as possible, until we reach a system S_j which is weak. Let the congruences in S_j be given by pairs (L_i, R_i) for $1 \le r \le \bar{m}$, and let the self-complement congruences be given as (L_i, R_i) (with $R_i = L_i^c$) for $\bar{m} + 1 \le i \le m$. (If the original system was weak, then $\bar{m} = m$.) We have now found a system equivalent to the original system, with a minimal number m of congruences (the same number as in S_0), such that the first $\bar{m}$ congruences form a weak system and the remaining $m - \bar{m}$ congruences are between a set and its complement. Since m is minimal, this system is nonredundant.

The main result of Dekker [4] states that any reasonable-sized (continuum or smaller) free product of cyclic groups can be embedded in the rotation group of S^2. Therefore, we can choose rotations σ_i $(1 \le i \le \bar{m})$ and τ_i' $(\bar{m} < i \le m)$ such that each σ_i has infinite order, each τ_i' has order 4, and the group G' generated by all of these rotations is the free product of the cyclic groups generated by the rotations individually. Let ζ be the antipodal isometry which maps each point of S^2 to the point opposite it. Let $\tau_i = \zeta \circ \tau_i'$ for each i, and let G be the group generated by the isometries σ_i and τ_i. We will construct a solution to the revised system of congruences so that σ_i $(i \le \bar{m})$ or τ_i $(i > \bar{m})$ witnesses congruence number i.

Clearly ζ^2 is the identity on S^2. Since isometries of S^2 preserve oppositeness of points, ζ commutes with all isometries of S^2. Using this, we see that τ_i has order 4 in G, and the homomorphism from G' to G which sends σ_i to σ_i and τ_i' to τ_i (which exists and is unique by the definition of free products) is in fact an isomorphism. Therefore, G is also a free product of $\bar{m}$ copies of $\mathbf{Z}$ and $m - \bar{m}$ copies of $\mathbf{Z}_4$. Also, $G \cap G'$ is a group of index 2 in G and in G', consisting of those words in G' such that the total number of occurrences of the generators τ_i' is even.

We will now show that much of Robinson's work on free groups, as presented in Chapter 4 of Wagon [12], can be carried out as well for free products of $\mathbf{Z}$'s and $\mathbf{Z}_4$'s. The rest of this proof will follow the relevant parts of that chapter rather closely.

First, we look at the structure of the group G (of course, the same results will apply to G', which is isomorphic to G). Any element of G has a unique expression as a reduced word. Here a 'word' is a product (possibly of length 0) of elements $\sigma_i^{\pm 1}$ and $\tau_i^{\pm 1}$; a word is reduced if there is no occurrence of $\sigma_i \sigma_i^{-1}$, $\sigma_i^{-1}\sigma_i$, τ_i^4, or τ_i^{-1} (which is equal to τ_i^3).

Given such a reduced word g, we can express it in the form $h_1 h_2 h_3$ with $h_3 = h_1^{-1}$ where h_1 and h_3 include as much of the word g as possible. (For a reduced word h_1, the inverse reduced word h_1^{-1} is obtained by reversing the word and then replacing σ_i with σ_i^{-1}, σ_i^{-1} with σ_i, and maximal consecutive blocks τ_i^j with τ_i^{4-j}.) To do this, start by setting h_1 and h_3 to be the identity and h_2 to be g. If the current h_2 starts with σ_i and ends with σ_i^{-1}, transfer the σ_i to h_1 and the σ_i^{-1} to h_3; similarly if h_2 starts with σ_i^{-1} and ends with σ_i. If the current h_2 both starts and ends with one or more copies of τ_i, with a total of at least 4 such copies (but h_2 is not just a power of τ_i), then transfer j copies from the start of h_2 to h_1 and $4 - j$ copies from the end of h_2 to h_3, where j is 1, 2, or 3, as appropriate. (If there are more than 4 such copies at the ends of h, so that more than one choice of j is possible, then use $j = 2$, for a reason to be seen below.) Repeat until h_2 cannot be reduced further.

When g is expressed as above, it is easy to see that the reduced form of g^n is $h_1 h_2^n h_3$ for any positive n, unless h_2 is of the form τ_i^k, in which case the reduced form of g^n is the null word (the identity) if kn is divisible by 4, $h_1 \tau_i^{kn \bmod 4} h_3$ otherwise. (If h_2 is not of the form τ_i^k, then the fact that h_2 cannot be reduced further as above shows that h_2^n is a reduced word.) In fact, the same statement also holds for negative n, because we used $j = 2$ whenever possible in the preceding paragraph. Furthermore, if g^n is broken into three pieces as above, then the three pieces are precisely h_1, h_2^n, and h_3.

From these facts, it follows immediately that the only elements of G of finite order are conjugates of powers τ_i^k. In particular, the only elements of G of order 2 have the form $g\tau_i^2 g^{-1}$ for some g and i. (A similar statement holds for G', of course.)

One more fact we will need is that the only abelian subgroups of G are the cyclic subgroups (so, if two elements of G commute, then they are powers of a single element of G). This follows from the Kurosh Subgroup Theorem [9, Cor. 4.9.1].

We now start to work out the analogues for G of Robinson's results for free groups.

Lemma 4.2. *The group G above can be partitioned into subsets $A_1, A_2, \ldots, A_r$ satisfying the given system of congruences, with σ_i ($i \le \bar{m}$) or τ_i ($i > \bar{m}$) witnessing congruence number i for each $i \le m$. In fact, for any word w from G in which the total number of occurrences of the generators τ_i is even, there is such a partition of G which puts w in the same set A_k as the identity element 1 of G.*

Proof. First, we show that the subsets of $\{1, 2, \ldots, r\}$ can be colored with two colors so that: for any set L, L and L^c have opposite colors; for any $i \le \bar{m}$, L_i and R_i have the same color. To do this, define an equivalence relation on subsets of $\{1, 2, \ldots, r\}$ as follows: L is equivalent to L' iff the congruence $L \cong L'$ can be deduced from the first $\bar{m}$ congruences of the given system. Clearly, if L is equivalent to L', then L^c is equivalent to L'^c. Also, L is never equivalent to L^c, since the first $\bar{m}$ of the given congruences form a weak system. Therefore, the equivalence classes under this relation come in complementary pairs; if we assign colors to the equivalence classes so that complementary classes get opposite colors, then the induced coloring of the subsets of $\{1, 2, \dots, r\}$ will be as desired.

We can view the given m congruences as $2m$ formal inclusions: the equation $\sigma_i(\bigcup_{k\in L_i} A_k) = \bigcup_{k\in R_i} A_k$ can be expressed as the two inclusions $\sigma_i(\bigcup_{k\in L_i} A_k) \subseteq \bigcup_{k\in R_i} A_k$ and $\sigma_i^{-1}(\bigcup_{k\in R_i} A_k) \subseteq \bigcup_{k\in L_i} A_k$, and similarly for τ_i. We will therefore use the terms 'domain of σ_i' and 'range of σ_i' for the sets $\bigcup_{k\in L_i} A_k$ and $\bigcup_{k\in R_i} A_k$, respectively, and define 'domain of τ_i,' 'range of τ_i^{-1},' and so on similarly.

Suppose $w = \rho_n\rho_{n-1}\dots\rho_1$, where each ρ_k is $\sigma_i^{\pm 1}$ or τ_i for some i (and this is the reduced form of w). We will first assign the end segments $1, \rho_1, \rho_2\rho_1, \dots, w$ to suitable sets A_k, and then handle the remaining elements of G.

First, suppose that, for some $j \le n$, the range of ρ_j is neither the domain of ρ_{j+1} nor the complement of the domain of ρ_{j+1} (here we let $\rho_{n+1} = \rho_1$). Then either the range of ρ_j meets both the domain of ρ_{j+1} and its complement, or the complement of the range of ρ_j meets both the domain of ρ_{j+1} and its complement; let S be the domain of ρ_j in the former case, the complement of this domain in the latter case. Assign $\rho_{j-1}\dots\rho_1$ to one of the sets in S. Next, assign $\rho_{j-2}\dots\rho_1$ to an appropriate set A_k; this will be a set in the domain of ρ_{j-1} if $\rho_{j-1}\dots\rho_1$ is in the range of ρ_{j-1}, and a set not in this domain if $\rho_{j-1}\dots\rho_1$ is not in this range. Repeat this process to assign all of the shorter end segments of w, down to 1, to sets A_k. Put w in the same set as 1, and then assign $\rho_{n-1}\dots\rho_1$ and so on; continue until only $\rho_j\dots\rho_1$ remains unassigned. This word must be assigned to the range of ρ_j if S is the domain of ρ_j, the complement of this range otherwise; it also must be placed in the domain of ρ_{j+1} if $\rho_{j+1}\dots\rho_1$ is in the range of ρ_{j+1}, the complement of this domain otherwise. By the definition of S, these two requirements can both be met. (This must be reworded slightly in the case $j = n$, but the basic argument remains the same.)

Now, suppose that the preceding case does not hold; for every j, the range of ρ_j is either the domain of ρ_{j+1} or its complement. Let S_0 be the domain of ρ_1. Given a set S_{j-1} which is either the domain of ρ_j or the complement of the domain of ρ_j, let S_j be the range of ρ_j in the former case, the complement of the range of ρ_j in the latter. Then S_n must be either S_0 or the complement of S_0. Note that each S_j is a union of sets A_k, say $\bigcup_{k\in N_j} A_k$, and therefore S_j (actually, N_j) has had a color assigned earlier in the proof of the Lemma. Furthermore, if ρ_j is $\sigma_i^{\pm 1}$, then S_{j-1} and S_j have the same color; if ρ_j is τ_i, then S_{j-1} and S_j have opposite colors. Since the number of j's for which ρ_j is a generator τ_i is even (by hypothesis on w), S_n must have the same color as S_0, so S_n must be S_0 rather than the complement of S_0. We now easily assign each end segment $\rho_j\dots\rho_1$ to one of the sets A_k included in S_j, making sure to put 1 and w in the same set included in S_0; these assignments are compatible with the required inclusions.

Now that we have assigned the end segments of w to sets A_k, the remaining elements g of G can be assigned by an easy recursion on the reduced form of g. Suppose this reduced form starts with ρ, where ρ is $\sigma_i^{\pm 1}$ or τ_i, and $g = \rho g'$ where g' has already been assigned to a set A_k. Then, if g' is in the domain of ρ, assign g to the range of ρ; if g' is in the complement of the domain of ρ, assign g to the complement of the range of ρ.

We must verify that, if $g = \rho g'$ where ρ is σ_i or τ_i, then g is in the range of ρ if and only if g' is in the domain of ρ. Let v and v' be the reduced words for g and g'. If $v = \rho v'$ and v is an end segment of w, then the way in which the end segments of w were assigned to sets A_k gives the desired result here; the same applies if $v' = \rho^{-1}v$ and v' is an end segment of w. If $v = \rho v'$ and v is not an end segment of w, then we get the desired result from the recursive definition of the preceding paragraph; this also holds if $v' = \rho^{-1}v$ and v' is not an end segment of w. The only remaining case is when neither $\rho v'$ nor $\rho^{-1}v$ is reduced. This can happen only when ρ is τ_i and

$v' = \tau_i^3 v$. But then, by the preceding cases, we have $v \in L_i$ iff $\tau_i v \in L_i^c$ iff $\tau_i^2 v \in L_i$ iff $\tau_i^3 v \in L_i^c$; hence, we have the desired result in this case as well. Therefore, the sets A_k satisfy the system of congruences. ■

Another useful fact is that, if w is a word in the generators of G which has an odd number of occurrences of the generators τ_i (including as inverses), then the corresponding isometry of S^2 has no fixed points. Let w' be the element of G' corresponding to w (i.e., replace all generators τ_i with τ_i'). Since $\tau_i = \zeta\tau_i'$, ζ commutes with all elements of G', ζ^2 is the identity, and the number of occurrences of the generators τ_i in w is odd, we can compute that $w = \zeta w'$. Now w' is a rotation which cannot be of order 2, since the only elements of G' of order 2 are the conjugates of $\tau_I'^2$, which all have even numbers of generators τ_i'. But it is easy to see that the only rotations of S^2 which map some point to its antipodal point are rotations of order 2. Therefore, w' does not map any point to its antipodal point, so $w = \zeta w'$ has no fixed points.

We now resume the proof of Theorem 4.1. In order to get the desired partition of S^2, it will suffice to get such a partition for each G-orbit in S^2 and put them together (using the axiom of choice to choose one such partition for each orbit). So consider one such orbit $\mathcal{O}$. If G acts freely on $\mathcal{O}$, then fixing any element x of $\mathcal{O}$ determines a bijection $g \mapsto g(x)$ from G to $\mathcal{O}$ which preserves the action of G, so any partition of G as in Lemma 4.2 can be transferred to $\mathcal{O}$, giving a partition of $\mathcal{O}$ with the desired properties.

So suppose G does not act freely on $\mathcal{O}$. Let w be a non-identity reduced word of G, as short as possible, such that w has a fixed point in $\mathcal{O}$; let x be such a fixed point. Then w cannot start with σ_i^{-1} and end with σ_i, because, if it did, then the reduced form of $\sigma_i \circ w \circ \sigma_i^{-1}$ would be shorter than w and would have a fixed point $\sigma_i(x) \in \mathcal{O}$. Similarly, w cannot start with σ_i and end with σ_i^{-1}; and w cannot start with τ_i^j and end with $\tau_i^{j'}$ where $j + j' \geq 4$, except in the case that w is just a power of τ_i. In fact, if w is not a power of τ_i, then we may assume that w does not both start and end with τ_i; if it ends with τ_i^j, then we can replace w with the reduced form of $\tau_i^j \circ w \circ \tau_i^{-j}$, which still starts with τ_i but ends with something else, and has the fixed point $\tau_i^j(x)$. We also know that w has an even number of occurrences of generators τ_i (and hence represents a rotation of S^2); in particular, if w is a power of some τ_i, then w must be τ_i^2.

Let ρ be the leftmost term in the reduced word w (either σ_i, σ_i^{-1}, or τ_i for some i). Define ρ' to be σ_i^{-1} if $\rho = \sigma_i$, σ_i if $\rho = \sigma_i^{-1}$, and τ_i if $\rho = \tau_i$; we have ensured that w does not end with ρ', unless $w = \tau_i^2$ for some i. Therefore, the word w^n is already in reduced form for positive n, and the reduced form of w^n for negative n does not begin with ρ, unless $w = \tau_i^2$.

The next thing to show is that the only elements of G which fix x are the powers of w. Suppose v is a nonidentity member of G such that $v(x) = x$. Then v must also have an even number of occurrences of generators τ_i, and is therefore a rotation. Since the rotation group acts locally commutatively on S^2, v and w must commute, so together they generate an abelian subgroup of G. We noted earlier that any abelian subgroup of G is cyclic, so there must be an element u of G such that v and w are both powers of u. We may assume that w is a *positive* power of u (replace u with u^{-1} if necessary). If u has finite order, then $u = g^{-1}\tau_i^j g$ for some g, i, and j; since v and w are nonidentity rotations and are powers of u, we must have $v = w = g^{-1}\tau_i^2 g$, so v is a power of w. Now suppose u is of infinite order. From the general arguments about the structure of G given earlier (specifically, the expression of u in the form $h_1h_2h_3$ so that the reduced form of u^n is $h_1h_2^nh_3$ for any $n > 0$), we see that, if $n > k > 0$, then the reduced form of u^n is longer than the reduced form of u^k. Suppose $w = u^n$ and $v = u^j$, and let k be the greatest common divisor of n and j. Then u^k can be expressed as a power of v times a power of w (by applying the extended Euclidean algorithm to n and j), so $u^k(x) = x$. We clearly have $k \leq n$, but we cannot have $k < n$, since otherwise u^k would be shorter than w, contradicting the choice of w as the shortest possible word with a fixed point in $\mathcal{O}$. Therefore, $k = n$, so j is divisible by n, so v is a power of w, as desired.

Using the above, we now show that every element of the orbit $\mathcal{O}$ has a unique expression of the form $g(x)$, where g is an element of G whose reduced form does not end in w and does not end in ρ'. (Exception: if $w = \tau_i^2$, then the reduced form of g is allowed to end in $\rho' = \tau_i$, but not in τ_i^2.) Let y be any point in this orbit, and let v be a shortest reduced word in G such that $v(x) = y$. Clearly v cannot end in w (otherwise the reduced form of vw^{-1} is shorter). If w is not of the form τ_i^2, and v ends in ρ', then vw does not end in w, and it does not end in ρ' either, since w does not end in ρ'. (If the entire w cancels out when vw is transformed to reduced form, then vw has a shorter reduced form than v, contradicting the choice of v.) Therefore, we can take g to be either v or vw. To see that g is unique, suppose u and v are distinct and $u(x) = v(x) = y$. Then $(u^{-1}v)(x) = x$, so $u^{-1}v$ is a power of w; by interchanging u and v if necessary, we may ensure that $u^{-1}v$ is a *positive* power of w, say w^j. Then either $v = uw^j$ ends in w, or there is some cancellation when u is multiplied by w^j; in the latter case, u must end in ρ' (in τ_i^2 if $w = \tau_i^2$, since in this case w^j must be w). This completes the proof that g is unique.

Now, apply Lemma 4.2 to partition G into sets $A_1, A_2, \ldots, A_r$ satisfying the congruences, so that w is in the same set A_k as 1. This lets us partition $\mathcal{O}$ into sets $B_1, B_2, \ldots, B_r$ as follows: for any point $y \in \mathcal{O}$, find the unique expression $g(x)$ for y as above, and put $y \in B_k$ iff $g \in A_k$. We must see that the sets B_k satisfy the system of congruences.

First, suppose $y \in \mathcal{O}$, $i \le \bar{m}$, and $z = \sigma_i(y)$; we must see that $y \in \bigcup_{k\in L_i} B_k$ if and only if $z \in \bigcup_{k\in R_i} B_k$. Express y and z as $g(x)$ and $h(x)$, where g and h do not end in w and (if w is not of the form τ_I^2) do not end in ρ'. Then $y \in \bigcup_{k\in L_i} B_k$ iff $g \in \bigcup_{k\in L_i} A_k$, and $z \in \bigcup_{k\in R_i} B_k$ iff $h \in \bigcup_{k\in R_i} A_k$. Also, since the sets A_k satisfy the congruences, we have $g \in \bigcup_{k\in L_i} A_k$ iff $\sigma_i g \in \bigcup_{k\in R_i} A_k$, and $\sigma_i^{-1}h \in \bigcup_{k\in L_i} A_k$ iff $h \in \bigcup_{k\in R_i} A_k$. Therefore, we are done if $h = \sigma_i g$. So suppose $h \ne \sigma_i g$. Then the reduced form of $\sigma_i g$ must end in w or in ρ', while the reduced form of g does not. There are only two cases in which this can happen: either $\rho = \sigma_i$ and $\sigma_i g = w$, or $\rho' = \sigma_i$ and $g = 1$. In the first of these cases we have $h = 1$, and since 1 and w lie in the same set A_k, we have $h \in \bigcup_{k\in R_i} A_k$ iff $\sigma_i g \in \bigcup_{k\in R_i} A_k$, and this gives the desired result. In the second case, we have $\sigma_i^{-1}h = w$, so $g \in \bigcup_{k\in L_i} A_k$ iff $\sigma_i^{-1}h \in \bigcup_{k\in L_i} A_k$, and again we are done.

Now suppose $y \in \mathcal{O}$, $i > \bar{m}$, and $z = \tau_i(y)$. Define g and h as above. Repeating this argument, we again see that we are done unless $h \ne \tau_i g$. Again this happens in only two cases: either $\rho = \tau_i$ and $\tau_i g = w$, or $\rho' = \tau_i$ and $g = 1$ (and $w \ne \tau_i^2$). These two cases are handled just as before.

This completes the construction of the desired partition for an arbitrary orbit of S^2 under G, so we are done. ■

The corresponding result for open sets is:

Theorem 4.3. *Any consistent system of proper congruences has a quasi-solution in S^2 using nonempty open sets (and arbitrary isometries).*

Proof. Revise the system as in the first three paragraphs of the proof of Theorem 4.1, and define isometries σ_i, τ_i', ζ, and τ_i and groups G and G' as in the fourth paragraph of that proof; we will use the same isometries as witnesses here.

The proof will follow that of (the second part of) Theorem 2.1 quite closely, so we will just give the differences here. Let f_i be σ_i if $i \le \bar{m}$, τ_i if $i > \bar{m}$.

The definition of 'active link' is changed slightly: a link from x to x' will be considered active for the sets B_k^n if there is a point in one or more of these sets which is connected to x or to x' by a chain of at most 2^{r+1} (rather than 2^r) links.

The next change is at the proof that, if $g = f_i \circ g'$, then $L_i \subseteq M^+_{g'}$ iff $R_i \subseteq M^+_g$, and $L^c_i \subseteq M^+_{g'}$ iff $R^c_i \subseteq M^+_g$. If f_i is σ_i, then the argument is unchanged, but if f_i is τ_i, then the cases are slightly different. If the reduced form of g' does not start with τ_i^3, then M_g is defined from $M_{g'}$, and we get the desired result as before. If the reduced form of g' does start with τ_i^3, so that g' is $\tau_i^3 g$, then $M_{g'}$ is defined from $M_{\tau_i^2 g}$, which is defined from $M_{\tau_i g}$, which is defined from M_g. Now, using the preceding case and the fact that $R_i = L^c_i$, we get $L_i \subseteq M^+_{g'}$ iff $L^c_i \subseteq M^+_{\tau_i^2 g}$ iff $L_i \subseteq M^+_{\tau_i g}$ iff $L^c_i \subseteq M^+_g$; similarly, $L^c_i \subseteq M^+_{g'}$ iff $L_i \subseteq M^+_g$, as desired.

Next, we must give revised forms of the Claims. Define a labeled directed graph $\mathcal{G}$ from the system of congruences as before, except that the edges labeled f_i^{-1} for $i > \bar{m}$ are omitted. Again the digraph $\mathcal{G}$ has cycles of length 2 connecting pairs (L_i, R_i) or (L^c_i, R^c_i), for $i \leq \bar{m}$; each such cycle consists of an σ_i-edge and an σ_i^{-1}-edge. For $i > \bar{m}$ we also get 2-cycles between L_i and $R_i = L^c_i$; in this case both edges in the cycle will be labeled τ_i. Call the edges in all of these 2-cycles *good* edges, and call all other edges *bad* edges.

Since the system is still consistent, the same argument as before gives:

Claim 1. No cycle in $\mathcal{G}$ contains a bad edge. □

Again construct the undirected graph $\mathcal{G}_0$ by treating each pair of oppositely-directed good edges in $\mathcal{G}$ as a single undirected edge.

Claim 2. The undirected graph $\mathcal{G}_0$ is acyclic; furthermore, each component of $\mathcal{G}_0$ contains at most one edge coming from the self-complement congruences.

Proof. Suppose we have a nontrivial cycle in $\mathcal{G}_0$; as before, we may assume that this cycle does not use an edge more than once. If this cycle includes an edge coming from congruence number i for $i > \bar{m}$, then, since congruence number i produces only one edge of $\mathcal{G}_0$, the rest of the cycle must come from the other congruences; as before, this implies that congruence number i is deducible from the remaining congruences, contradicting nonredundancy. So all of the edges in the cycle come from the first $\bar{m}$ congruences; since these congruences form a weak system, we get a contradiction as in the old Claim 2.

Now, suppose there are two self-complement edges in the same component. Find a shortest possible path connecting endpoints of two such edges; this path (possibly of length 0) consists of distinct edges from the first $\bar{m}$ congruences. Say this path connects L to R, where $L \in \{L_i, L^c_i\}$, $R \in \{L_{i'}, L^c_{i'}\}$, and i and i' are distinct numbers greater than $\bar{m}$. Then there is a nontrivial cycle in $\mathcal{G}_0$ from L to R (the given path) to R^c (the $\tau_{i'}$-edge) to L^c (the complemented and reversed form of the given path) to L (the τ_i-edge), contradicting the preceding paragraph. □

Using these two claims, we can now get:

Claim 3. Every path of length 2^{r+1} in the digraph $\mathcal{G}$ contains either four consecutive edges with the same label τ_i for some $i > \bar{m}$ or a pair of consecutive edges with labels σ_i and σ_i^{-1}, or vice versa, for some $i \leq \bar{m}$.

Proof. Suppose we have a path of length 2^{r+1} in $\mathcal{G}$. Since there are fewer than 2^r vertices in $\mathcal{G}$, some vertex, say L, must be visited at least three times. Let p be the subpath from L to L to L. By Claim 1, this subpath consists entirely of good edges, so it induces a corresponding path p_0 in the graph $\mathcal{G}_0$ which also goes from L to L to L. By Claim 2, p_0 cannot include a nontrivial cycle, so each of its two L-to-L parts must double back on itself. If either doubles back on itself at a σ_i-edge, then p has a pair of consecutive edges with labels σ_i and σ_i^{-1}, or vice versa, so we are done. If neither part of p_0 doubles back on itself at a σ_i-edge, then they both must double back at a τ_i-edge. By Claim 2, there is only one such edge e in the component of $\mathcal{G}_0$ containing p_0, and there is a unique path q in $\mathcal{G}_0$ from L to the nearest endpoint of e which does not double back. Hence, p_0 must consist of q, an even number of traversals of e, q' (the reversal of q), q again, another even number of traversals of e, and q' again. If q is non-null, then p_0 doubles back on itself at a σ_j-edge (the last edge of q' followed by the first edge of q), so we are done as before; if q is null, then p_0 consists of at least four consecutive occurrences of e, so p contains four consecutive τ_i-edges, as desired. □

The next step is to show that, for any $g \in G$, if $M_g \neq \varnothing$, then either all of the links in the canonical chain from x_0 to $g(x_0)$ (i.e., the chain read off from the reduced form of g; note that there might be other chains from x_0 to $g(x_0)$, since G is no longer free) are active for the sets B_k^n, or this chain has fewer than 2^{r+1} links. The proof of this works as before (with 2^r replaced by 2^{r+1}), so properties (2)–(4) hold for the sets $\hat{B}_k$.

The construction of the sets B_k^{n+1} goes through as before, with two minor changes: one must replace 2^r with 2^{r+1} throughout, and one must not assume that there is a unique chain of links connecting two points in the G-orbit of x_0. The wording of the definition of the set S does not need to be changed, but one must note that it refers to arbitrary chains from x_0 to $g(x_0)$, rather than just the canonical chain. Also, in the second paragraph of the proof of (4) for the sets B_k^{n+1}, one does not necessarily use the *reduced* form of the element of h; instead, one just uses the fact that there is *some* expression of h as a product of elements f_I and their inverses (one is allowed to use f_I^{-1} even if $I > \bar{m}$) such that the product has length at most $2^{r+1}+1$, and if its length is equal to $2^{r+1}+1$, then the rightmost component is f_i. Everything else goes through as before.

This completes the induction. ■

Since the same isometries were used in the preceding two proofs to witness the congruences, Lemma 2.3 now gives:

Theorem 4.4. *Any consistent system of proper congruences has a solution in S^2 using nonmeager sets with the property of Baire (and arbitrary isometries).* ■

Actually, the proof of Theorem 4.3 goes through without change if the involution ζ is deleted, so that the group G' is used instead of G. This gives:

Proposition 4.5. *Any consistent system of proper congruences has a quasi-solution in S^2 using nonempty open sets, with rotations witnessing the congruences.* ■

However, this result does not lead to a result about sets with the property of Baire, because there is no corresponding result giving solutions using arbitrary sets (unless the system is weak).

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