Local Cohomology at Monomial Ideals

by

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Introduction

Let B be an ideal in a polynomial ring $R = k[X_1, \ldots, X_n]$ in n variables over a field k. The local cohomology of R at B is defined by

$$\mathrm{H}^i_B(R) = \lim \mathrm{Ext}^i_R(R/B^d, R).$$

In general, this limit is not well behaved: the natural maps

$$\mathrm{Ext}^i_R(R/B^d, R) \longrightarrow \mathrm{H}^i_B(R)$$

are not injective and it is difficult to understand how their images converge to $\mathrm{H}^i_B(R)$ (see Eisenbud, Mustaţă and Stillman [1998] for a discussion of related problems).

However, in the case when B is a monomial ideal we will see that the situation is especially nice if instead of the sequence $\{B^d\}_{d\geq 1}$ we consider the cofinal sequence of ideals $\{B_0^{[d]}\}_{d\geq 1}$, consisting of the "Frobenius powers" of the ideal $B_0 = \mathrm{radical}(B)$. They are defined as follows: if $m_1, \ldots, m_r$ are monomial generators of B_0, then

$$B_0^{[d]} = (m_1^d, \ldots, m_r^d).$$

Our first main result is that the natural map

$$\mathrm{Ext}^i_R(R/B_0^{[d]}, R) \longrightarrow \mathrm{H}^i_B(R)$$

is an isomorphism onto the submodule of $H^i_B(R)$ of elements of multidegree α, with $\alpha_j \geq -d$ for all j.

The second main result gives a filtration of $\mathrm{Ext}^i_R(R/B, R)$ for a squarefree monomial ideal B. For $\alpha \in \{0,1\}^n$, let $\mathrm{supp}(\alpha) = \{j \mid \alpha_j = 1\}$ and $P_\alpha = (X_j \mid j \in \mathrm{supp}(\alpha))$.

We describe a canonical filtration of

$$\mathrm{Ext}^i_R(R/B, R) : 0 = M_0 \subset \ldots \subset M_n = \mathrm{Ext}^i_R(R/B, R)$$

such that for every l,

$$M_l/M_{l-1} \cong \bigoplus_{|\alpha|=l} (R/P_\alpha(\alpha))^{\beta_{l-i,\alpha}(B^\vee)}.$$

The numbers $\beta_{l-i,\alpha}(B^\vee)$ are the Betti numbers of $B^\vee$, the Alexander dual ideal of B (see section 3 below for the related definitions). For an interpretation of this filtration in terms of Betti diagrams, see Remark 1 after Theorem 3.3, below.

In a slightly weaker form, this result has been conjectured by David Eisenbud.

Let's see this filtration for a simple example: $R = k[a,b,c,d]$, $B = (ab, cd)$ and $i = 2$. Since B is a complete intersection, we get $\mathrm{Ext}^2_R(R/B, R) \cong R/B(1,1,1,1)$. Our filtration is $M_0 = M_1 = 0$, $M_2 = R\overline{ac} + R\overline{ad} + R\overline{bc} + R\overline{bd}$, $M_3 = R\overline{a} + R\overline{b} + R\overline{c} + R\overline{d}$ and $M_4 = \mathrm{Ext}^2_R(R/B, R)$.

From the description of $\mathrm{Ext}^2_R(R/B, R)$ it follows that

$$M_2/M_1 = M_2 \cong R/(b,d)(0,1,0,1) \oplus R/(b,c)(0,1,1,0) \oplus$$

$$R/(a,d)(1,0,0,1) \oplus R/(a,c)(1,0,1,0),$$

$$M_3/M_2 \cong R/(b,c,d)(0,1,1,1) \oplus R/(a,c,d)(1,0,1,1) \oplus$$

$$R/(a,b,d)(1,1,0,1) \oplus R/(a,b,c)(1,1,1,0),$$

$$M_4/M_3 \cong R/(a,b,c,d)(1,1,1,1).$$

On the other hand, $B^\vee = (bd, bc, ad, ac)$. If $F_\bullet$ is the minimal multigraded resolution of $B^\vee$, then

$$F_0 = R(0,-1,0,-1) \oplus R(0,-1,-1,0) \oplus R(-1,0,0,-1) \oplus R(-1,0,-1,0),$$

$$F_1 = R(0,-1,-1,-1) \oplus R(-1,0,-1,-1) \oplus R(-1,-1,0,-1) \oplus R(-1,-1,-1,0),$$

$$F_2 = R(-1,-1,-1,-1).$$

We see that for each $\alpha \in \{0,1\}^4$ such that $R(-\alpha)$ appears in F_{l-2}, there is a corresponding summand $R/P_\alpha(\alpha)$ in M_l/M_{l-1}.

In order to prove this result about the filtration of $\mathrm{Ext}^i_R(R/B, R)$ we will study the multigraded components of this module and how an element of the form $X_j \in R$ acts on these components. As we have seen, it is enough to study the same problem for $H^i_B(R)$.

We give two descriptions for the degree α part of $H^i_B(R)$, as simplicial cohomology groups of certain simplicial complexes depending only on B and the signs of the components of α. The first complex is on the set of minimal generators of B and the second one is a full subcomplex of the simplicial complex associated to $B^\vee$ via the Stanley-Reisner correspondence. The module structure on $H^i_B(R)$ is described by the maps induced in cohomology by inclusion of simplicial complexes.

As a first consequence of these results and using also a formula of Hochster [1977], we obtain an isomorphism

$$\mathrm{Ext}^i_R(R/B, R)_{-\alpha} \cong \mathrm{Tor}^R_{|\alpha|-i}(B^\vee, k)_\alpha,$$

for every $\alpha \in \{0,1\}^n$.

This result is equivalent to the fact that in our filtration the numbers are as stated above. This isomorphism has been obtained also by Yanagawa [1998]. It can be considerd as a strong form of the inequality of Bayer, Charalambous and Popescu [1998] between the Betti numbers of B and those of $B^\vee$. As shown in that paper, this implies that B and $B^\vee$ have the same extremal Betti numbers, extending results of Eagon and Reiner [1996] and Terai [1997].

As a final application of our analysis of the graded pieces of $\mathrm{Ext}^i_R(R/B, R)$, we give a topological description for the associated primes of $\mathrm{Ext}^i_R(R/B, R)$. In the terminology of Vasconcelos [1998], these are the homological associated primes of R/B. In particular, we characterize the minimal associated primes of $\mathrm{Ext}^i_R(R/B, R)$ using only the Betti numbers of $B^\vee$.

We mention here the recent work of Terai [1998] on the Hilbert function of the modules $H^i_B(R)$. It is easy to see that using the results in our paper one can deduce Terai's formula for this Hilbert function.

The problem of effectively computing the local cohomology modules with respect to an arbitrary ideal is quite difficult since these modules are not finitely generated. The general approach is to use the D-module structure for the local cohomology (see, for example, Walther [1999]). However, in the special case of monomial ideals our results show that it is possible to make this computation with elementary methods.

Our main motivation for studying local cohomology at monomial ideals comes from the applications in the context of toric varieties. Via the homogeneous coordinate ring, the cohomology of sheaves on such a variety can be expressed as local cohomology of modules at the "irrelevant ideal", which is a squarefree monomial ideal. For a method of computing the cohomology of sheaves on toric varieties in this way, see Eisenbud, Mustaţă and Stillman [1998]. For applications to vanishing theorems on toric varieties and related results, see Mustaţă [1999].

The main reference for the definitions and the results that we use is Eisenbud [1995]. For the basic facts about the cohomology of simplicial complexes, see Munkres [1984]. Cohomology of simplicial complexes is always taken to be reduced cohomology. Notice also that we make a distinction between the empty complex which contains just the empty set (which has nontrivial cohomology in degree -1) and the void complex which doesn't contain any set (whose cohomology is trivial in any degree).

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§1. Local cohomology as a union of Ext modules

Let $B \subset R = k[X_1, \ldots, X_n]$ be a squarefree monomial ideal. All the modules which appear are $\mathbf{Z}^n$-graded. We partially order the elements of $\mathbf{Z}^n$ by setting $\alpha \geq \beta$ iff $\alpha_j \geq \beta_j$, for all j.

Theorem 1.1. *For each i and d, the natural map*

$$\mathrm{Ext}^i_R(R/B^{[d]}, R) \longrightarrow \mathrm{H}^i_B(R)$$

is an isomorphism onto the submodule of $\mathrm{H}^i_B(R)$ of elements of degree $\geq (-d, \ldots, -d)$.

Proof. We will compute $\mathrm{Ext}^i_R(R/B^{[d]}, R)$ using the Taylor resolution $F^d_\bullet$ of $R/B^{[d]}$ (see Eisenbud [1995], exercise 17.11). The inclusion $B^{[d+1]} \longrightarrow B^{[d]}$, $d \geq 1$ induces a morphism of complexes $\phi^d : F^{d+1}_\bullet \longrightarrow F^d_\bullet$. The assertions in the theorem are consequences of the more precise lemma below.

Lemma 1.2. *If $(\phi^d)^* : (F^d_\bullet)^* \longrightarrow (F^{d+1}_\bullet)^*$ is the dual $\mathrm{Hom}_R(\phi^d, R)$ of the above map, then in a multidegree $\alpha \in \mathbf{Z}^n$ we have:*
(a) If $\alpha \geq (-d, \ldots, -d)$, then $(\phi^d)^*_\alpha$ is an isomorphism of complexes.
(b) If $\alpha_j < -d$ for some j, $1 \leq j \leq n$, then $(F^d_\bullet)^*_\alpha = 0$, so $(\phi^d)^*_\alpha$ is the zero map.

Proof of the lemma. Let $m_1, \ldots, m_r$ be monomial minimal generators of B. For any subset I of $\{1, \ldots, r\}$ we set

$$m_I = \mathrm{LCM}\,\{m_i \mid i \in I\}.$$

As each m_I is square-free, $\deg m_I \in \mathbf{Z}^n$ is a vector of ones and zeros.

Recall from Eisenbud [1995] that $F^d_\bullet$ is a free R-module with basis $\{f^d_I \mid I \subset \{1, \ldots, r\}\}$, where $\deg(f^d_I) = d \deg(m_I)$. Therefore, the degree α part of $(F^d_\bullet)^*$ has a vector space basis consisting of elements of the form ne^d_I where $n \in R$ is a monomial, $e^d_I = (f^d_I)^*$ has degree equal to $-d \deg(m_I)$, and $\deg(n) - d \deg(m_I) = \alpha$.

Part (b) of the Lemma follows at once. For part (a), note that $(\phi^k)^* : (F^d_\bullet)^* \longrightarrow (F^{d+1}_\bullet)^*$ takes e^d_I to $m_I e^{d+1}_I$. The vector $\deg(e^{d+1}_I) = -(d+1)\deg(m_I)$ has entry $-(d+1)$ wherever $\deg(m_I)$ has entry 1, so any element ne^{d+1}_I of degree $\alpha \geq (-d, -d, \ldots, -d)$ must have n divisible by m_I. It is thus of the form $(\phi^d)^*(x)$ for the unique element $x = (n/m_I)e^d_I$, as required. ∎

§2. Local cohomology as simplicial cohomology

To describe $\mathrm{H}^i_B(R)$ in a multidegree $\alpha \in \mathbf{Z}^n$, we will use two simplicial complexes associated with B and α. We will assume that $B \neq (0)$.

By computing local cohomology using the Taylor complex we will express $H^i_B(R)_\alpha$ as the simplicial cohomology of a complex on the set of minimal generators of B. We will interpret this later as the cohomology of an other complex, this time on the potentially smaller set $\{1,\ldots,n\}$. This one is a full subcomplex of the complex associated to the dual ideal $B^\vee$ via the Stanley-Reisner correspondence. In fact, this is the complex used in the computation of the Betti numbers of $B^\vee$ (see the next section for the definitions). We will use this result to derive the relation between $\mathrm{Ext}(R/B,R)$ and $\mathrm{Tor}^R(B^\vee,k)$ in Corolary 3.1 below.

Let $m_1,\ldots,m_r$ be the minimal monomial generators of B. As above, for $J \subset \{1,\ldots,r\}$, m_J will denote $\mathrm{LCM}\,(m_j; j \in J)$.

For $i \in \{1,\ldots,n\}$, we define

$$T_i := \{J \subset \{1,\ldots,r\} \,|\, X_i \not| m_J\}.$$

For every subset $I \subset \{1,\ldots,n\}$, we define $T_I = \bigcup_{i\in I} T_i$. When $I = \emptyset$, we take T_I to be the void complex. It is clear that each T_i is a simplicial complex on the set $\{1,\ldots,r\}$, and therefore so is T_I.

For $\alpha \in \mathbf{Z}^n$, we take $I_\alpha = \{i \,|\, \alpha_i \leq -1\} \subset \{1,\ldots,n\}$. Note that the complex T_{I_α} depends only on the signs of the components of α (and, of course, on B).

If $e_1,\ldots,e_n$ is the canonical basis of $\mathbf{Z}^n$ and $\alpha' = \alpha + e_l$, we have obviously $I_{\alpha'} \subset I_\alpha$, with equality iff $\alpha_l \neq -1$. Therefore, $T_{I_{\alpha'}}$ is a subcomplex of T_{I_α}.

Theorem 2.1.

(a) *With the above notation, we have*

$$\mathrm{H}^i_B(R)_\alpha \cong \mathrm{H}^{i-2}(T_{I_\alpha}; k).$$

(b) Via the isomorphisms given in (a), the multiplication by X_l:

$$\nu_{X_l} : \mathrm{H}^i_B(R)_\alpha \longrightarrow \mathrm{H}^i_B(R)_{\alpha'}$$

corresponds to the morphism:

$$\mathrm{H}^{i-2}(T_{I_\alpha}; k) \longrightarrow \mathrm{H}^{i-2}(T_{I_{\alpha'}}; k),$$

induced in cohomology by the inclusion $T_{I_{\alpha'}} \subset T_{I_\alpha}$. In particular, if $\alpha_l \neq -1$, then ν_{X_l} is an isomorphism.

Proof. We have seen in Lemma 1.2 that

$$\mathrm{Ext}^i_R(R/B^{[d]},R)_\alpha \cong \mathrm{H}^i_B(R)_\alpha$$

if $\alpha \geq (-d, \ldots, -d)$. We fix such a d. With the notations in Lemma 1.2 , we have seen that the degree α part of $(F^d_\bullet)^*$ has a vector space basis consisting of elements of the form ne^d_J, where $n \in R$ is a monomial and $\deg(n) - d\deg(m_J) = \alpha$. Therefore, the basis of $(F^d_p)^*_\alpha$ is indexed by those $J \subset \{1, \ldots, r\}$ with $|J| = p$ and $\alpha + d\deg(m_J) \geq (0, \ldots, 0)$. Because $\alpha_j \leq -1$ iff $j \in I_\alpha$ and $\alpha \geq (-d, \ldots, -d)$, the above inequality is equivalent to $X_j | m_J$ for every $j \in I_\alpha$ i.e. to $J \notin T_{I_\alpha}$.

Let $G^\bullet$ be the cochain complex computing the relative cohomology of the pair (D, T_{I_α}) with coefficients in k, where D is the full simplicial complex on the set $\{1, \ldots, r\}$.

If $I_\alpha \neq \emptyset$, then the degree α part of $(F^d_p)^*$ is equal to G^{p-1} for every p. Moreover, the maps are the same and therefore we get $H^i_B(R)_\alpha \cong \mathrm{H}^{i-1}(D, T_{I_\alpha}; k)$. Since D is contractible, the long exact sequence in cohomology of the pair (D, T_{I_α}) yields $\mathrm{H}^i_B(R)_\alpha \cong \mathrm{H}^{i-2}(T_{I_\alpha}; k)$.

If $I_\alpha = \emptyset$, then $(F^d_\bullet)^*$ in degree α is up to a shift the complex computing the reduced cohomology of D with coefficients in k. Since D is contractible, we get $\mathrm{H}^i_B(R)_\alpha = 0 = \mathrm{H}^{i-2}(T_{I_\alpha}; k)$, which completes the proof of part (a).

For part (b), we may suppose that $I_{\alpha'} \neq (0)$. With the above notations, ν_{X_l} is induced by the map $\phi_l : (F^d_p)^*_\alpha \longrightarrow (F^d_p)^*_{\alpha'}$, given by $\phi_l(ne^d_J) = X_l ne^d_J$.

If $G'^\bullet$ is constructed as above, but for α' instead of α, then via the isomorphisms:

$$(F^d_p)^*_\alpha \cong G^{p-1},$$

$$(F^d_p)^*_{\alpha'} \cong G'^{p-1},$$

the map ϕ_l corresponds to the canonical projection $G^{p-1} \longrightarrow G'^{p-1}$, which concludes the proof of part (b). ∎

Remark. *The last assertion in Theorem 2.1(b), that ν_{X_l} is an isomorphism if $\alpha_l \neq -1$ has been obtained also in Yanagawa [1998].*

The next corollary describes $\mathrm{H}^i_B(R)_\alpha$ as the cohomology of a simplicial complex with vertex set $\{1, \ldots, n\}$.

We first introduce the complex Δ defined by:

$$\Delta := \{F \subset \{1, \ldots, n\} \mid \prod_{j \notin F} X_j \in B\}.$$

In fact , by the Stanley-Reisner correspondence between square-free monomial ideals and simplicial complexes (see Bruns and Herzog [1993]), Δ corresponds to $B^\vee$.

For any subset $I \subset \{1, \ldots, n\}$, we define Δ_I to be the full simplicial subcomplex of Δ supported on I:

$$\Delta_I := \{F \subset \{1, \ldots, n\} \mid F \in \Delta, F \subset I\}.$$

When $I = \emptyset$, we take Δ_I to be the void complex. It is clear that if $I \subset I'$, then $\Delta_{I'}$ is a subcomplex of Δ_I. This is the case if $\alpha' = \alpha + e_l$, $I = I_\alpha$ and $I' = I_{\alpha'}$.

Corollary 2.2.

(a) *With the above notation, for any* $\alpha \in \mathbf{Z}^n$

$$\mathrm{H}^i_B(R)_\alpha \cong \mathrm{H}^{i-2}(\Delta_{I_\alpha}; k).$$

(b) Via the isomorphisms given by (a), the multiplication map ν_{X_l} corresponds to the morphism:

$$\mathrm{H}^{i-2}(\Delta_{I_\alpha}; k) \longrightarrow \mathrm{H}^{i-2}(\Delta_{I_{\alpha'}}; k),$$

induced in cohomology by the inclusion $\Delta_{I_{\alpha'}} \subset \Delta_{I_\alpha}$.

Proof. Using the notation in Theorem 2.1, if $I_\alpha \neq \emptyset$, then $T_{I_\alpha} = \bigcup_{i \in I_\alpha} T_i$.

If $i_1, \ldots, i_k \in I_\alpha$ and $\bigcap_{1 \le p \le k} T_{i_p} \neq \emptyset$, then

$$\bigcap_{1 \le p \le k} T_{i_p} = \{J \subset \{1, \ldots, r\} \mid X_{i_p} \not| m_J, 1 \le p \le k\}$$

is the full simplicial complex on those j with $X_{i_p} \not| m_j$, for every p, $1 \le p \le k$. Therefore it is contractible.

This shows that we can compute the cohomology of T_I as the cohomology of the nerve $\mathcal{N}$ of the cover $T_I = \bigcup_{i \in I} T_i$ (see Godement [1958]). But by definition, $\{i_1, \ldots, i_k\} \subset I$ is a simplex in $\mathcal{N}$ iff $\bigcap_{1 \le p \le k} T_{i_p} \neq \emptyset$ iff there is j such that $X_{i_p} \not| m_j$ for every p, $1 \le p \le k$. This shows that $\mathcal{N} = \Delta_I$ and we get that $\mathrm{H}^i_B(R)_\alpha \cong \mathrm{H}^{i-2}(\Delta_I; k)$ when $I \neq \emptyset$.

When $I = \emptyset$, $\mathrm{H}^i_B(R)_\alpha = 0$ by theorem 2.1 and also $\mathrm{H}^{i-2}(\Delta_I; k) = 0$ (the reduced cohomology of the void simplicial complex is zero).

Part (b) follows immediately from part (b) in Theorem 2.1 and the fact that the isomorphism between the cohomology of a space and that of the nerve of a cover as above is functorial. ∎

Remark. *The same type of arguments as in the proofs of Theorem 2.1 and of Corollary 2.2 can be used to give a topological description for* $\mathrm{Ext}^i_B(R/B, R)_\alpha$*, for a possibly non-reduced nonzero monomial ideal* B*. Namely, for* $\alpha \in \mathbf{Z}^n$*, we define the simplicial complex* Δ_α *on* $\{1, \ldots, n\}$ *by* $J \in \Delta_\alpha$ *iff there is a monomial* m *in* B *such that* $\deg(X^\alpha m)_j < 0$ *for* $j \in J$*. We make the convention that* Δ *is the void complex iff* $\alpha \ge 0$*. Then*

$$\mathrm{Ext}^i_R(R/B, R)_\alpha \cong \mathrm{H}^{i-2}(\Delta_\alpha; k).$$

Moreover, we can describe these k-vector spaces using a more geometric object. If we view $B \subset \mathbf{Z}^n \subset \mathbf{R}^n$, let P_α be the subspace of R^n supported on B, translated by α, minus the first quadrant. More precisely,

$$P_\alpha = \{x \in \mathbf{R}^n | x - \alpha \ge m, \text{for some}\, m \in B\} \setminus \mathbf{R}^n_+.$$

Then, using a similar argument to the one in the proof of corollary 1.4, one can show that

$$\mathrm{Ext}^i_R(R/B, R)_\alpha \cong \mathrm{H}^{i-2}(P_\alpha; k),$$

where the right-hand side is the reduced singular cohomology group. Here we have to make the convention that for $\alpha \ge 0$, P_α is the "void topological space", with trivial reduced cohomology (as oposed to the empty topological space which has nonzero reduced cohomology in degree -1).

We leave the details of the proof to the interested reader.

§3. The filtration on the Ext modules

The Alexander dual of a reduced monomial ideal B is defined by

$$B^{\vee} = (X^F | F \subset \{1, \dots n\}, X^{F^c} \notin B),$$

where $F^c := \{1, \dots, n\} \setminus F$ (see Bayer, Charalambous and Popescu [1998] for interpretation in terms of Alexander duality). Note that $(B^{\vee})^{\vee} = B$.

We will derive first a relation between $\mathrm{Ext}_R(R/B, R)$ and $\mathrm{Tor}^R(B^{\vee}, k)$. This can be seen as a stronger form of the inequality in Bayer, Charalambous and Popescu [1998] between the Betti numbers of B and $B^{\vee}$.

For $\alpha \in \mathbf{Z}^n$, we will denote $|\alpha| = \sum_i \alpha_i$.

Corollary 3.1. *Let $B \subset R = k[X_1, \dots, X_n]$ be a reduced monomial ideal and $\alpha \in \mathbf{Z}^n$ a multidegree. If $\alpha \notin \{0,1\}^n$, then $\mathrm{Tor}_i^R(B^{\vee}, k)_\alpha = 0$, and if $\alpha \in \{0,1\}^n$, then*

$$\mathrm{Tor}_i^R(B^{\vee}, k)_\alpha \cong \mathrm{Ext}_R^{|\alpha|-i}(R/B, R)_{-\alpha}.$$

Proof. We will use Hochster's formula for the Betti numbers of reduced monomial ideals (see, for example, Hochster [1977] or Bayer, Charalambous and Popescu [1998]). It says that if $\alpha \notin \{0,1\}^n$, then $\mathrm{Tor}_i^R(B^{\vee}, k)_\alpha = 0$ and if $\alpha \in \{0,1\}^n$, then

$$\mathrm{Tor}_i^R(B^{\vee}, k)_\alpha \cong \mathrm{H}^{|\alpha|-i-2}(\Delta_I; k),$$

where I is the support of α.

Obviously, we may suppose that $B \neq (0)$. If $\alpha \in \{0,1\}^n$, then corollary 2.2 gives

$$\mathrm{H}^{|\alpha|-i-2}(\Delta_I; k) \cong \mathrm{H}_B^{|\alpha|-i}(R)_{-\alpha}$$

and theorem 1.1 gives

$$\mathrm{H}_B^{|\alpha|-i}(R)_{-\alpha} \cong \mathrm{Ext}_R^{|\alpha|-i}(R/B, R)_{-\alpha}.$$

Putting together these isomorphisms, we get the assertion of the corollary. ■

We recall that the multigraded Betti numbers of B are defined by

$$\beta_{i,\alpha}(B) := \mathrm{dim_k Tor_i^R(B, k)}_\alpha.$$

Equivalently, if $F_\bullet$ is a multigraded minimal resolution of B, then

$$F_i \cong \sum_{\alpha \in \mathbf{Z}^n} R(-\alpha)^{\beta_{i,\alpha}(B)}.$$

One says that (i, α) is extremal (or that $\beta_{i,\alpha}$ is extremal) if $\beta_{j,\alpha'}(B) = 0$ for all $j \geq i$ and $\alpha' > \alpha$ such that $|\alpha'| - |\alpha| \geq j - i$.

Remark. *Using Theorems 1.1, 2.1(b) and Corollary 3.1 one can give a formula for the Hilbert function of $\mathrm{H}_B^i(R)$ using the Betti numbers of $B^{\vee}$. This formula is equivalent to the one which appears in Terai [1998].*

As a consequence of the above corollary, we obtain the inequality between the Betti numbers of B and $B^{\vee}$ from Bayer, Charalambous and Popescu [1998]. It implies the equality of extremal Betti numbers from that paper, in particular the equality $\mathrm{reg}\, B = \mathrm{pd}(R/B^{\vee})$ from Terai [1997].

Corollary 3.2. *If $B \subset R$ is a reduced monomial ideal, then*

$$\beta_{i,\alpha}(B) \leq \sum_{\alpha \leq \alpha'} \beta_{|\alpha|-i-1,\alpha'}(B^\vee),$$

for every $i \geq 0$ and every $\alpha \in \{0,1\}^n$. If $\beta_{|\alpha|-i-1,\alpha}(B^\vee)$ is extremal, then so is $\beta_{i,\alpha}(B)$ and

$$\beta_{i,\alpha}(B) = \beta_{|\alpha|-i-1,\alpha}(B^\vee).$$

Proof. Since $\beta_{i,\alpha}(B) = \mathrm{dim_k Tor_i^R(B,k)_\alpha}$, by the previous corollary we get

$$\beta_{i,\alpha}(B) = \mathrm{dim_k Ext_R^{|\alpha|-i}(R/B^\vee, R)_{-\alpha}} = \mathrm{dim_k\, H^{|\alpha|-i}(Hom(F_\bullet, R))_{-\alpha}},$$

where $F_\bullet$ is the minimal free resolution of $R/B^\vee$.

Since $F_{|\alpha|-i} = \oplus_{\alpha' \in \mathbf{Z}^n} R(-\alpha')^{\beta_{|\alpha|-i-1,\alpha'}(B^\vee)}$, we get

$$\beta_{i,\alpha}(B) \leq \sum_{\alpha' \in \mathbf{Z}^n} \beta_{|\alpha|-i-1,\alpha'}(B^\vee)\mathrm{dim_k(R(\alpha')_{-\alpha})} = \sum_{\alpha \leq \alpha'} \mathrm{\beta_{|\alpha|-i-1,\alpha'}(B^\vee)}.$$

If $\beta_{|\alpha|-i-1,\alpha}(B^\vee)$ is extremal, the above inequality becomes $\beta_{i,\alpha}(B) \leq \beta_{|\alpha|-i-1,\alpha}(B^\vee)$. Applying the same inequality for $j \geq i$ and $\alpha' > \alpha$ such that $|\alpha'| - |\alpha| \geq j - i$ and the fact that $\beta_{|\alpha|-i-1,\alpha}(B^\vee)$ is extremal, we get that $\beta_{i,\alpha}(B)$ is extremal.

Applying the previous inequality with B replaced by $B^\vee$, we obtain $\beta_{|\alpha|-i-1,\alpha}(B^\vee) \leq \beta_{i,\alpha}(B)$, which concludes the proof. ∎

We fix some notations for the remaining of this section. Let $[n] = \{0,1\}^n$ and $[n]_l = \{\alpha \in [n] \,|\, |\alpha| = l\}$, for every l, $0 \leq l \leq n$. For $\alpha \in [n]$, let $\mathrm{supp}(\alpha) = \{j \,|\, \alpha_j = 1\}$ and $P_\alpha = (X_j \,|\, j \in \mathrm{supp}(\alpha))$. The ideals P_α, $\alpha \in [n]$ are exactly the monomial prime ideals of R.

The following theorem gives the canonical filtration of $\mathrm{Ext}_R^i(R/B, R)$ announced in the Introduction.

Theorem 3.3. *Let $B \subset R$ be a squarefree monomial ideal. For each l, $0 \leq l \leq n$, let M_l be the submodule of $\mathrm{Ext}_R^i(R/B, R)$ generated by all $\mathrm{Ext}_R^i(R/B, R)_{-\alpha}$, for $\alpha \in [n]$, $|\alpha| \leq l$. Then $M_0 = 0$, $M_n = \mathrm{Ext}_R^i(R/B, R)$ and for every l, $0 \leq l \leq n$,*

$$M_l/M_{l-1} \cong \bigoplus_{\alpha \in [n]_l} (R/P_\alpha(\alpha))^{\beta_{l-i,\alpha}(B^\vee)}.$$

Proof. Clearly we may suppose $B \neq 0$. The fact that $M_0 = 0$ follows from Corollary 2.2(a).

Let's see first that $M_n = \mathrm{Ext}_R^i(R/B, R)$. For this it is enough to prove that all the minimal monomial generators of $\mathrm{Ext}_R^i(R/B, R)$ are in degrees $-\alpha$, $\alpha \in [n]$.

Indeed, if $\alpha_j \leq -1$ for some j, then the multiplication by X_j defines an isomorphism

$$\mathrm{Ext}_R^i(R/B, R)_{-\alpha-e_j} \longrightarrow \mathrm{Ext}_R^i(R/B, R)_{-\alpha}$$

by Corollary 2.2(b) and Theorem 1.1. In particular, there are no minimal generators in degree $-\alpha$.

On the other hand, by Theorem 1.1, $\mathrm{Ext}^i_R(R/B,R)_{-\alpha}=0$ if $\alpha_j\geq 2$, for some j. Therefore we have $M_n=\mathrm{Ext}^i_R(R/B,R)$.

Suppose now that we have homogeneous elements $f_1,\dots,f_r$ with $\deg(f_q)\in[n]_{l'}$, $l'\leq l$, for every q, $1\leq q\leq r$. We suppose that they are linearly independent over k and that their linear span contains $\mathrm{Ext}^i_R(R/B,R)_{-\alpha}$, for every $\alpha\in[n]_{l'}$, $l'\leq l-1$. We will suppose also that $\deg(f_r)=-\alpha$, $|\alpha|=l$. If $T:=\sum_{1\leq q\leq r-1}R\,f_q$, let $\overline{f}_r$ be the image of f_r in M_l/T.

Claim. With the above notations, $\mathrm{Ann}_R(\overline{f}_r)=P_\alpha$.

Let $F=\mathrm{supp}(\alpha)$. If $j\in F$, then $\deg(X_jf_r)=-(\alpha-e_j)$, $\alpha-e_j\in[n]$. By our assumption, it follows that $X_jf_r\in T$, so that $P_\alpha\subset\mathrm{Ann}_R(\overline{f}_r)$.

Conversely, consider now $m=\prod X_j^{m_j}\in\mathrm{Ann}\,\overline{f}_r$ and suppose that $m\notin(X_j\,|\,j\in F)$. We can suppose that m has minimal degree. Let j be such that $m_j\geq 1$. Then $j\notin F$ and therefore $m_j-\alpha_j=m_j\geq 1$. Since $m\,f_r\in T$, we can write

$$m\,f_r=\sum_{q<r}c_qn_qf_q,$$

where n_q are monomials and $c_q\in k$. Since $\deg(f_q)\leq 0$ for every q, in the above equality we may assume that $X_j|n_q$ for every q such that $c_q\neq 0$. But by Corollary 2.2(b) and Theorem 1.1, the multiplication by X_j is an isomorphism:

$$\mathrm{Ext}^i_R(R/B,R)_{-\alpha+\deg\,m-e_j}\longrightarrow\mathrm{Ext}^i_R(R/B,R)_{-\alpha+\deg\,m}.$$

Therefore $m/X_j\in\mathrm{Ann}\overline{f}_r$, in contradiction with the minimality of m. We get $\mathrm{Ann}\overline{f}_r=(X_j\,|\,j\in F)$, which completes the proof of the claim.

The first consequence is that for every nonzero $f\in M_l$, $\deg(f)=-\alpha$, $\alpha\in[n]_l$, if $\overline{f}$ is the image of f in M_l/M_{l-1}, then $\mathrm{Ann}_R(\overline{f})=P_\alpha$, so that $R\,\overline{f}\cong R/P_\alpha(\alpha)$.

Let's consider now a homogeneous basis $f_1,\dots,f_N$ of $\oplus_{\alpha\in[n]_l}\mathrm{Ext}^i_R(R/B,R)_{-\alpha}$. By Corollary 3.1,

$$\mathrm{dim_k Ext^i_R(R/B,R)}_{-\alpha}=\beta_{\mathrm{l-i},\alpha}(\mathrm{B}^\vee).$$

Therefore, to complete the proof of the theorem, it is enough to show that

$$M_l/M_{l-1}\cong\bigoplus_{1\leq j\leq N}R\overline{f}_j.$$

Here $\overline{f}_j$ denotes the image of f_j in M_l/M_{l-1}.

Since $M_l=M_{l-1}+\sum_{1\leq j\leq N}Rf_j$, we have only to show that if $\sum_{1\leq j\leq N}n_jf_j\in M_{l-1}$, then $n_jf_j\in M_{l-1}$ for every j, $1\leq j\leq N$.

Let $\{g_1,\dots,g_{N'}\}$ be the union of homogeneous bases for $\mathrm{Ext}^i_R(R/B,R)_{-\alpha}$, for $\alpha\in[n]_{l'}$, $l'\leq l-1$.

Let's fix some j, with $1\leq j\leq N$. If $\deg(f_j)=-\alpha$, by applying the above claim to f_j, as part of $\{f_p\,|\,1\leq p\leq N\}\cup\{g_{p'}\,|\,1\leq p'\leq N'\}$, we get that $n_j\in P_\alpha$. But we have already seen that $P_\alpha f_j\subset M_{l-1}$ and therefore the proof is complete. ■

Remark 1. *We can interpret the statement of Theorem 3.3 using the multigraded Betti diagram of* $B^\vee$*. This is the diagram having at the intersection of the* i^{th} *row with the* j^{th} *column the Betti numbers* $\beta_{j,\alpha}(B^\vee)$*, for* $\alpha \in \mathbf{Z}^n$*,* $|\alpha| = i + j$*.*

For each i and j we form a module corresponding to (i, j):

$$E_{i,j} = \bigoplus_{\alpha \in [n]_{i+j}} (R/P_\alpha(\alpha))^{\beta_{j,\alpha}(B^\vee)}.$$

Theorem 3.3 gives a filtration of $\mathrm{Ext}^i_R(R/B, R)$ having as quotients the modules constructed above corresponding to the i^{th} row: $E_{i,j}$, $j \in \mathbf{Z}$.

Notice that by definition, $\mathrm{Tor}^R_i(B^\vee, k)$ is obtained by a "dual" procedure applied to the i^{th} column (in this case the extensions being trivial). Indeed, if for (j, i) we put

$$E'_{j,i} = \bigoplus_{\alpha \in [n]_{i+j}} k(-\alpha)^{\beta_{i,\alpha}(B^\vee)},$$

then $\mathrm{Tor}^R_i(B^\vee, k) \cong \oplus_{j \in \mathbf{Z}} E'_{j,i}$.

Remark 2. *Using Theorem 3.3 one can compute the Hilbert series of* $\mathrm{Ext}^i_R(R/B, R)$ *in terms of the Betti numbers of* $B^\vee$*. Using local duality, one can derive the fomula, due to Hochster [1997], for the Hilbert series of the local cohomology modules* $H^{n-i}_{\underline{m}}(R/B)$*, where* $\underline{m} = (X_1, \ldots, X_n)$ *(see also Bruns and Herzog [1993], Theorem 5.3.8).*

We describe now the set of homological associated primes of R/B i.e. the set

$$\cup_{i \geq 0} \mathrm{Ass}(\mathrm{Ext}^i_R(R/B, R))$$

(see Vasconcelos [1998]). Since the module $\mathrm{Ext}^i_R(R/B, R)$ is $\mathbf{Z}^n$-graded, its associated primes are of the form P_α, for some $\alpha \in [n]$. In fact, Theorem 3.3 shows that

$$\mathrm{Ass}(\mathrm{Ext}^i_R(R/B, R)) \subset \{P_\alpha \,|\, \beta_{|\alpha|-i,\alpha}(B^\vee) \neq 0\}.$$

The next result gives the necessary and sufficient condition for a prime ideal P_α to be in $\mathrm{Ass}(\mathrm{Ext}^i_R(R/B, R))$. In particular, we get the characterization of the minimal associated primes of this module using only the Betti numbers of $B^\vee$.

Theorem 3.4. *Let* $B \subset R$ *be a nonzero square-free monomial ideal and* $\alpha \in [n]$*. Let* $F = \mathrm{supp}(\alpha)$*.*

(a) The ideal P_α belongs to $\mathrm{Ass}(\mathrm{Ext}^i_R(R/B, R))$ iff

$$\bigcap_{j \in F} \mathrm{Ker}(H^{i-2}(\Delta_F; k) \longrightarrow \mathrm{H}^{i-2}(\Delta_{F \setminus j}; k)) \neq 0.$$

(b) The ideal P_α is a minimal prime in $\mathrm{Ass}(\mathrm{Ext}^i_R(R/B, R))$ iff

$$\beta_{|\alpha|-i,\alpha}(B^\vee) \neq 0$$

and

$$\beta_{|\alpha'|-i,\alpha'}(B^\vee) = 0,$$

for every $\alpha' \in [n]$, $\alpha' \leq \alpha$, $\alpha' \neq \alpha$.

Proof. By Corollary 2.2, the condition in (a) is equivalent to the existence of $u \in \mathrm{Ext}^i_R(R/B,R)_{-\alpha}$, $u \neq 0$ such that $X_j u = 0$ for every $j \in F$. Since $\alpha_j = 0$ for $j \notin F$, Corollary 2.2(b) and Theorem 1.1 imply that for every monomial m, $m \notin P_\alpha$, the multiplication by m is injective on $\mathrm{Ext}^i_R(R/B,R)_{-\alpha}$.

Therefore, in the above situation we have $\mathrm{Ann}_R(u) = P_\alpha$, so that P_α is an element of $\mathrm{Ass}(\mathrm{Ext}^i_R(R/B,R))$.

Conversely, suppose that $P_\alpha \in \mathrm{Ass}(Ext^i_R(R/B,R))$. Since P_α and $\mathrm{Ext}^i_R(R/B,R)$ are $\mathbf{Z}^n$-graded, this is equivalent to the existence of $u \in \mathrm{Ext}^i_R(R/B,R)_{\alpha'}$, for some $\alpha' \in \mathbf{Z}^n$, such that $P_\alpha = \mathrm{Ann}_R(u)$. To complete the proof of part (a), it is enough to show that we can take $\alpha' = -\alpha$.

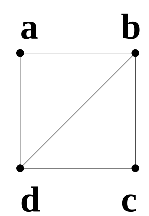
By Theorem 1.1, $\alpha' \geq (-1,\ldots,-1)$. Since $X_j u = 0$ for $j \in F$, multiplication by X_j on $\mathrm{Ext}^i_R(R/B,R)_\alpha$ is not injective so that by Corollary 2.2(b), we must have $\alpha'_j = -1$ for $j \in F$.

Let's consider some $j \notin F$. If $\alpha'_j \geq 1$, by Corollary 2.2(b) there is $u' \in \mathrm{Ext}^i_R(R/B,R)_{\alpha''}$, $\alpha'' = \alpha' - \alpha'_j e_j$ such that $X_j^{\alpha'_j} u' = u$ and $\mathrm{Ann}_R(u') = \mathrm{Ann}_R(u) = P_\alpha$. Therefore, we may suppose that $\alpha'_j \leq 0$.

If $\alpha'_j = -1$, since $X_j \notin \mathrm{Ann}_R(u)$, which is prime, we have $\mathrm{Ann}_R(X_j u) = \mathrm{Ann}_R(u) = P_\alpha$. This shows that we may suppose $\alpha'_j = 0$ for every $j \notin F$, so that $\alpha' = -\alpha$.

The sufficiency of the condition in part (b) follows directly from part (a) and Corollary 3.1. For the converse, it is enough to notice that if for some $G \subset \{1,\ldots,n\}$, there is $0 \neq u \in \mathrm{H}^{i-2}(\Delta_G;k)$, then there is $H \subset G$ such that $X^H u$ corresponds to a nonzero element in $\bigcap_{j\in G\setminus H} \mathrm{Ker}(\mathrm{H}^{i-2}(\Delta_{G\setminus H};k) \longrightarrow \mathrm{H}^{i-2}(\Delta_{G\setminus(H\cup j)};k))$. ■

Example 1. *Let $R = k[a,b,c,d]$ and $B = (ab,bc,cd,ad,ac)$. Then Δ is the simplicial complex:*

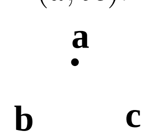


Theorem 3.4(a) gives easily that

$$\mathrm{Ass}(\mathrm{Ext}^3_R(R/B,R)) = \{(a,b,d),(b,c,d)\}.$$

Example 2. *In general, it is not sufficient for $\beta_{|\alpha|-i,\alpha}(B^\vee)$ to be nonzero in order to have $P_\alpha \in \mathrm{Ass}(\mathrm{Ext}^i_R(R/B,R))$.*

Let's consider $R = k[a,b,c]$ and $B = (a,bc)$. Then Δ is the simplicial complex:



Using Theorem 3.4(a), we get:

$$\mathrm{Ass}(\mathrm{Ext}^2_R(R/B, R)) = \{(a,b), (a,c)\},$$

while

$$\{F \mid \beta_{|\alpha_F|-2,\alpha_F}(B^\vee) \neq 0\} = \{\{a,b,c\}, \{a,b\}, \{a,c\}\}.$$

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