DIFFERENTIAL INVARIANTS AND CURVED BERNSTEIN-GELFAND-GELFAND SEQUENCES

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Abstract. We give a simple construction of the Bernstein-Gelfand-Gelfand sequences of natural differential operators on a manifold equipped with a parabolic geometry. This method permits us to define the additional structure of a bilinear differential "cup product" on this sequence, satisfying a Leibniz rule up to curvature terms. It is not associative, but is part of an A_∞-algebra of multilinear differential operators, which we also obtain explicitly. We illustrate the construction in the case of conformal differential geometry, where the cup product provides a wide-reaching generalization of helicity raising and lowering for conformally invariant field equations.

Introduction

In a sequence of pioneering papers [1, 2, 3], Robert Baston introduced a number of general methods to study invariant differential operators on conformal manifolds, and a related class of parabolic geometries, which he called "almost hermitian symmetric (AHS) structures". In particular, he suggested that certain complexes of natural differential operators, dual to generalized Bernstein-Gelfand-Gelfand (BGG) resolutions of parabolic Verma modules, could be extended from the homogeneous context (generalized flag manifolds) to curved manifolds modelled on these spaces. He provided a construction of such a BGG sequence (no longer a complex in general) for AHS structures [2], and introduced (in [3]) a class of induced modules, now called semiholonomic Verma modules [23].

Baston's work fits into the programme of parabolic invariant theory initiated by Fefferman and Graham [24, 25]. Several authors have joined in an endeavour to complete these ideas and hence provide a theory of invariant operators in all parabolic geometries, which include conformal geometry, projective geometry, quaternionic geometry, projective contact geometry, CR geometry and quaternionic CR geometry. In [23], Eastwood and Slovák began the study of semiholonomic Verma modules and classified the Verma module homomorphisms lifting to the semiholonomic modules in the conformal case. The AHS structures have been extensively studied by Čap, Slovák and Souček in [17, 18, 19], and in the last paper of this series they construct a large class of invariant differential operators for these geometries. Then, in [20], Čap, Slovák and Souček clarified Baston's construction of the BGG sequences in the AHS case, and in the process, generalized it to all parabolic geometries. Hence we now know that all standard homomorphisms of parabolic Verma modules induce differential operators also in the curved setting, providing us with a huge supply of invariant linear differential operators.

This paper has two main objectives: to simplify the construction of the BGG sequences given in [20], and to equip these sequences of linear differential operators with bilinear differential pairings, inducing, in the flat case, a cup product on cohomology.

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The key tool for the construction of the BGG sequences is an invariant differential operator from relative Lie algebra homology bundles to twisted differential forms, denoted L in [2] and [20]. However, when one tries to produce a cup product, one needs a differential operator defined on the whole bundle of twisted differential forms which induces L on the homology bundles. The search for such an operator (within the dual homogeneous formalism of Verma modules) led the second author to a procedure which, in addition to providing a cup coproduct on the BGG resolutions, gives a simpler construction of the resolutions themselves. These developments are described in [21]. It is straightforward to dualize this procedure and one readily sees that it generalizes from the homogeneous context to arbitrary parabolic geometries, although the presence of curvature introduces phenomena that do not arise in the flat case, and also suggests further constructions of multilinear differential operators. We present, in geometric language, these constructions and phenomena here. That is, we give a simple self-contained approach to the curved BGG sequences and cup product, in their natural geometrical context, and introduce an A_∞-algebra of invariant multilinear differential operators.

We begin by recalling some basic facts from Cartan geometry, emphasizing the simple first order constructions a Cartan connection provides. Our approach is mainly influenced by [1, 20, 39, 40]. In section 2, we define parabolic geometries as Cartan geometries associated to a semisimple Lie algebra $\mathfrak{g}$ with a parabolic Lie subalgebra $\mathfrak{p}$. We summarize the basic representation-theoretic facts that we will need and give some examples.

The most substantial piece of representation theory we need is Lie algebra homology, and we discuss this in section 3. In order to keep the paper as self-contained as possible, we give proofs for all the basic properties of Lie algebra homology, although we only indicate briefly how Kostant's Hodge decomposition may be established. Additionally, there are some non-standard aspects to our treatment: firstly we concentrate on Lie algebra homology, rather than cohomology, since it is the Lie algebra homology that is $\mathfrak{p}$-equivariant, and secondly, we eschew the lamentably inverted notation ∂, ∂^* for the Lie algebra coboundary and boundary operators (for some reason, ∂, although a boundary operator in [32], is the coboundary operator in [1, 15, 18, 43]). Instead, following Kostant in part [32], and by analogy with the deRham complex, we use d and δ, with subscripts to indicate that it is the Lie algebraic rather than differential operators we are considering. This analogy with the deRham complex is central to our proof. After stating the main theorem to be proved at the end of section 3, we begin the study, in section 4, of the twisted deRham complex. As observed in [20], there are in fact two natural deRham complexes one might consider, which differ if the parabolic geometry has torsion.

We prove an explicit version of our main result in section 5. There we give a construction, using a Neumann series, of the BGG sequences of differential operators found in [20], and at the same time construct the bilinear differential pairings. Our method enables us to compute explicitly the curvature terms entering into the squares of the BGG differentials and into the Leibniz rule for the pairings. The BGG sequence of [20] is based on the the twisted deRham complex with torsion. We show that under a weak additional assumption, there is another BGG sequence based on the torsion-free twisted deRham sequence. The operators involved are in some ways more complicated because of the corrections needed to "remove" the torsion, but we believe they are more natural and illustrate this by interpreting curvature terms for normal regular parabolic geometries. For torsion-free parabolic geometries, of course, the two BGG sequences agree.

At the end of section 5 and in the following section, we introduce multilinear differential operators and establish that they form a (curved) A_∞-algebra. We study adjointness properties of the BGG operators and cup product in section 7, introducing a dual BGG sequence and a cap product. In section 8, potential applications, such as deformation and moduli space problems, are discussed, mostly in a rather open-ended fashion, since working out the details in many cases is a substantial research project. We attempt to be more concrete in the final section, where we give examples in conformal geometry, and show how the cup product generalizes helicity raising and lowering by Penrose twistors in four dimensional conformal geometry to arbitrary twistors in arbitrary dimensions.

Finally, one feature of our methods is that we work with spaces of smooth sections, and do not need the machinery of semiholonomic infinite jets and Verma modules. However, for the convenience of the reader, we sketch an approach to this machine in an appendix.

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1. Cartan geometries and invariant differentiation

Cartan geometries are geometries modelled on a homogeneous space G/P (for a modern introduction, see [39]). Such a homogeneous space has a natural principal P-bundle $G \to G/P$, equipped with $\mathfrak{g}$-valued 1-form $\omega\colon TG \to \mathfrak{g}$, namely the Maurer-Cartan form which trivializes TG by the right-invariant vector fields.

In order to avoid fixing G, we work with a pair $(\mathfrak{g}, P)$ consisting of a Lie algebra $\mathfrak{g}$ and a group P acting on $\mathfrak{g}$ by automorphisms such that the Lie algebra $\mathfrak{p}$ of P is a subalgebra of $\mathfrak{g}$ and the action of P on $\mathfrak{g}$ induces the adjoint action of P on $\mathfrak{p}$ and of $\mathfrak{p}$ on $\mathfrak{g}$.

1.1. Definition (*Cartan geometry*)**.** Let M be a manifold of the same dimension as $\mathfrak{g}/\mathfrak{p}$.

(i) A *Cartan geometry* of type $(\mathfrak{g}, P)$ on M is a principal P-bundle $\pi\colon \mathcal{G} \to M$, together with a P-equivariant $\mathfrak{g}$-valued 1-form $\eta\colon T\mathcal{G} \to \mathfrak{g}$ such that for each $u \in \mathcal{G}$, $\eta_u\colon T_u\mathcal{G} \to \mathfrak{g}$ is an isomorphism restricting to the canonical isomorphism between $T_u(\mathcal{G}_{\pi(u)})$ and $\mathfrak{p}$.

(ii) A *Kleinian* or *homogeneous model* of a Cartan geometry of type $(\mathfrak{g}, P)$ is a homogeneous space G/P, for a Lie group G with subgroup P and Lie algebra $\mathfrak{g}$.

(iii) The *curvature* $K\colon \Lambda^2 T\mathcal{G} \to \mathfrak{g}$ of a Cartan geometry is defined by

$$K(U,V) = d\eta(U,V) + [\eta(U), \eta(V)].$$

It induces a curvature function $\kappa\colon \mathcal{G} \to \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ via

$$\kappa(u)(\xi,\chi) = K_u\big(\eta^{-1}(\xi), \eta^{-1}(\chi)\big) = [\xi,\chi] - \eta_u[\eta^{-1}(\xi), \eta^{-1}(\chi)],$$

where $u \in \mathcal{G}$ and the latter bracket is the Lie bracket of vector fields on $\mathcal{G}$.

(iv) Associated to a P-module $\mathbb{E}$ is a vector bundle $E = \mathcal{G} \times_P \mathbb{E}$. In particular, the Cartan connection η identifies the tangent bundle of M with $\mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$. The *adjoint* *bundle* of a Cartan geometry is $\mathfrak{g}_M = \mathcal{G} \times_P \mathfrak{g}$. The quotient map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$ induces a surjective bundle map $\pi_{\mathfrak{g}} \colon \mathfrak{g}_M \to TM$.

Note that the associated bundle construction identifies sections of $E = \mathcal{G} \times_P \mathbb{E}$ over M with P-equivariant maps from $\mathcal{G}$ to $\mathbb{E}$:

$$\mathrm{C}^\infty(M, E) = \mathrm{C}^\infty(\mathcal{G}, \mathbb{E})^P.$$

We shall make frequent use of this identification, often without comment.

The curvature K of a Cartan geometry measures the failure of the Cartan 1-form η to induce a Lie algebra homomorphism. This is the obstruction to finding a local isomorphism between $\mathcal{G} \to M$ and a homogeneous model $G \to G/P$.

The following definition is essentially given in [17, 20], except that we do not restrict the derivative to horizontal tangent vectors, and hence we do not lose P-equivariance. The same idea appears in [14, 39], the latter reference attributing it to Cartan [13].

1.2. Definition (*Invariant derivative*)**.** Let $(\mathcal{G}, \eta)$ be a Cartan geometry of type $(\mathfrak{g}, P)$ on M, and let $\mathbb{E}$ be a P-module with associated vector bundle $E = \mathcal{G} \times_P \mathbb{E}$. Then the *invariant derivative* on E is defined by

$$\nabla^\eta \colon \mathrm{C}^\infty(\mathcal{G}, \mathbb{E}) \to \mathrm{C}^\infty(\mathcal{G}, \mathfrak{g}^* \otimes \mathbb{E})$$
$$\nabla^\eta_\xi f = df\big(\eta^{-1}(\xi)\big)$$

for all ξ in $\mathfrak{g}$. It is P-equivariant and so maps $\mathrm{C}^\infty(\mathcal{G}, \mathbb{E})^P$ into $\mathrm{C}^\infty(\mathcal{G}, \mathfrak{g}^* \otimes \mathbb{E})^P$. Identifying P-equivariant maps to a P-module with sections of the associated vector bundle therefore provides a linear map $\nabla^\eta \colon \mathrm{C}^\infty(M, E) \to \mathrm{C}^\infty(M, \mathfrak{g}_M^* \otimes E)$.

We now build up some simple properties of this operation.

1.3. Proposition (*1-jets*)**.** *Let $(\mathcal{G}, \eta)$ be a Cartan geometry of type $(\mathfrak{g}, P)$ on M.*

(i) *The curvature K is a horizontal 2-form and so induces $K_M \in \mathrm{C}^\infty(M, \Lambda^2 T^*M \otimes \mathfrak{g}_M)$. Equivalently $\kappa(\xi, .) = 0$ for $\xi \in \mathfrak{p}$, so $\kappa \in \mathrm{C}^\infty(\mathcal{G}, \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g})^P$.*

(ii) *The invariant derivative of a P-equivariant map $f \colon \mathcal{G} \to \mathbb{E}$ is vertically trivial in the sense that $(\nabla^\eta_\xi f)(u) + \xi \cdot \big(f(u)\big) = 0$ for all $\xi \in \mathfrak{p}$ and $u \in \mathcal{G}$. In particular if P acts trivially on $\mathbb{E}$ and $f \colon M \to E$, then $\nabla^\eta f = df \circ \pi_{\mathfrak{g}}$.*

(iii) *If $f_1 \colon \mathcal{G} \to \mathbb{E}_1$ and $f_2 \colon \mathcal{G} \to \mathbb{E}_2$ then $\nabla^\eta_\xi(f_1 \otimes f_2) = (\nabla^\eta_\xi f_1) \otimes f_2 + f_1 \otimes (\nabla^\eta_\xi f_2)$.*

(iv) *The invariant derivative satisfies the "Ricci identity":*

$$\nabla^\eta_\xi(\nabla^\eta_\chi f) - \nabla^\eta_\chi(\nabla^\eta_\xi f) = \nabla^\eta_{[\xi,\chi]} f - \nabla^\eta_{\kappa(\xi,\chi)} f.$$

(v) *The map $\mathrm{C}^\infty(M, E) \to \mathrm{C}^\infty\big(M, E \oplus (\mathfrak{g}_M^* \otimes E)\big)$ sending s to $(s, \nabla^\eta s)$ defines an injective bundle map from $J^1 E$ to $E \oplus \big(\mathfrak{g}_M^* \otimes E\big)$ whose image is $\mathcal{G} \times_P J^1_0 \mathbb{E}$ where $J^1_0 \mathbb{E} = \{(\phi_0, \phi_1) \in \mathbb{E} \oplus \big(\mathfrak{g}^* \otimes \mathbb{E}\big) : \phi_1(\xi) + \xi \cdot \phi_0 = 0 \text{ for all } \xi \in \mathfrak{p}\}$.*

Proof. These are straightforward calculations.

(i) For $\xi \in \mathfrak{p}$, we have by definition that $\eta^{-1}(\xi)$ is a vertical vector field generated by the right P-action. Now the map $\chi \mapsto \eta^{-1}(\chi)$ is P-equivariant for any $\chi \in \mathfrak{g}$, from which it follows by differentiating that $[\eta^{-1}(\xi), \eta^{-1}(\chi)] = \eta^{-1}[\xi, \chi]$.

(ii) Differentiate the P-equivariance condition $p \cdot (f(up)) = f(u)$.

(iii) This is just the product rule for $d(f_1 \otimes f_2)$.

(iv) The Ricci identity holds because both sides are equal to $df([\eta^{-1}(\xi), \eta^{-1}(\chi)])$.

(v) Certainly the map on smooth sections only depends on the 1-jet at each point, and it is injective since the symbol of ∇^η is the inclusion $T^*M \otimes E \to \mathfrak{g}^*_M \otimes E$, as one easily sees from the product rule (for $\mathbb{E}_1$ trivial and $\mathbb{E}_2 = \mathbb{E}$). It maps into $\mathcal{G} \times_P J^1_0\mathbb{E}$ by vertical triviality, so the result follows by comparing the ranks of the bundles. □

The final term in the Ricci identity is first order in general, due to the presence of torsion. The *torsion* is defined to be TM-valued 2-form $\pi_{\mathfrak{g}}(K_M)$ obtained by projecting the curvature K onto $\mathfrak{g}/\mathfrak{p}$ and a Cartan geometry is said to be *torsion-free* if K takes values in $\mathfrak{p}$ so that $\pi_{\mathfrak{g}}(K_M) = 0$ and $\kappa \in \mathrm{C}^\infty(\mathcal{G}, \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p})$. In this case, for any P-equivariant $f\colon \mathcal{G} \to \mathbb{E}$, we have $-\nabla^\eta_{\kappa(\xi,\chi)} f = \kappa(\xi,\chi)\cdot f$.

We end this section by considering the invariant derivative when the P-module is also a $\mathfrak{g}$-module.

1.4. Definition. A $(\mathfrak{g}, P)$-*module* is a vector space $\mathbb{W}$ carrying a representation of P and a P-equivariant representation of $\mathfrak{g}$, such that the induced actions of $\mathfrak{p}$ coincide.

For a Lie group G with Lie algebra $\mathfrak{g}$ and subgroup P, any G-module is a $(\mathfrak{g}, P)$-module.

1.5. Definition (*Twistor connection*)**.** Let $\mathbb{W}$ be a $(\mathfrak{g}, P)$-module and define

$$\nabla^{\mathfrak{g}}\colon \mathrm{C}^\infty(\mathcal{G}, \mathbb{W}) \to \mathrm{C}^\infty(\mathcal{G}, \mathfrak{g}^* \otimes \mathbb{W})$$
$$\nabla^{\mathfrak{g}}_\xi f = \nabla^\eta_\xi f + \xi \cdot f\,.$$

Then for P-equivariant f, $\nabla^{\mathfrak{g}}_\xi f$ vanishes for $\xi \in \mathfrak{p}$, so $\nabla^{\mathfrak{g}} f$ takes values in $(\mathfrak{g}/\mathfrak{p})^* \otimes \mathbb{W}$ and hence $\nabla^{\mathfrak{g}}$ induces a covariant derivative on $W = \mathcal{G} \times_P \mathbb{W}$ which will be called the *twistor connection* on W. Its curvature is easily computed to be the action of K_M on W.

If G is a Lie group with subgroup P and Lie algebra $\mathfrak{g}$, then the principal G-bundle $\widetilde{\mathcal{G}} = \mathcal{G} \times_P G$ has a principal bundle connection induced by η, and, on a G-module $\mathbb{W}$, $\nabla^{\mathfrak{g}}$ is simply the covariant derivative induced by this connection. Although these basic ideas from the theory of Cartan connections are well-established, the systematic use of a *linear* representation of the Cartan connection seems to first appear in twistor theory, where $\mathbb{W}$ is the *local twistor bundle* and $\nabla^{\mathfrak{g}}$ defines *local twistor transport*. Following Baston [1], we adapt this terminology to more general situations.

1.6. Proposition. *Let $\Psi_{\mathbb{W}}$ the P-equivariant automorphism of $(\mathbb{E} \otimes \mathbb{W}) \oplus (\mathfrak{g}^* \otimes \mathbb{E} \otimes \mathbb{W})$, for any P-module $\mathbb{E}$, sending $(\phi_0 = e \otimes w, \phi_1)$ to $(\phi_0, \widetilde{\phi}_1)$ where $\widetilde{\phi}_1(\chi) = \phi_1(\chi) + e \otimes \chi \cdot w$ for any $\chi \in \mathfrak{g}$. Then $\Psi_{\mathbb{W}}$ restricts to an isomorphism from $J^1_0(\mathbb{E} \otimes \mathbb{W})$ to $J^1_0(\mathbb{E}) \otimes \mathbb{W}$.*

Proof. For any $\chi \in \mathfrak{p}$ and $(\phi_0 = e \otimes w, \phi_1) \in J^1_0(\mathbb{E} \otimes \mathbb{W})$, we have

$$\widetilde{\phi}_1(\chi) + (\chi \cdot e) \otimes w = \phi_1(\chi) + (\chi \cdot e) \otimes w + e \otimes \chi \cdot w = \phi_1(\chi) + \chi \cdot (e \otimes w) = 0,$$

and so $\Psi_{\mathbb{W}}$ maps $J^1_0(\mathbb{E} \otimes \mathbb{W})$ into $J^1_0(\mathbb{E}) \otimes \mathbb{W}$. □

The operator $\Psi_{\mathbb{W}}$ formalises the process of twisting a first order operator (on a P-module $\mathbb{E}$) by the twistor connection on $\mathbb{W}$. We apply this to the exterior derivative in section 4.

2. Parabolic geometries

Parabolic geometries can be described as Cartan geometries of type $(\mathfrak{g}, P)$ where $\mathfrak{g}$ is semisimple and the Lie algebra $\mathfrak{p}$ of P is a parabolic subalgebra, i.e., a subalgebra containing a maximal solvable subalgebra of $\mathfrak{g}$. We need a few facts about parabolic subalgebras, all of which are straightforward: we refer to [4, 15, 40, 43] for proofs.

The parabolic subalgebra splits naturally as the semidirect sum of a reductive subalgebra $\mathfrak{g}_0$ and a nilpotent ideal $\mathfrak{m}^*$, where $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{p}$ in $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$—the nilpotent part of $\mathfrak{p}$ is identified with $\mathfrak{m}^*$ using this Killing form. Because of this duality, $\mathfrak{p}^* := \mathfrak{g}_0 \ltimes \mathfrak{m}$ is also a parabolic subalgebra of $\mathfrak{g}$. By choosing a Cartan subalgebra of the semisimple part of $\mathfrak{g}_0$ and extending it to a Cartan subalgebra of $\mathfrak{g}$ inside $\mathfrak{g}_0$, one can show (in the complexified setting) that parabolic subalgebras of semisimple Lie algebras correspond, up to isomorphism, to Dynkin diagrams with crossed nodes, where a node is crossed if the corresponding root lies in the centre of $\mathfrak{g}_0$. Real forms are classified in a similar way (using, for instance, Satake diagrams). The distinction between real and complex geometries does not cause any difficulties at this level: some of the statements in the following require minor modification in the real case, but we make little further comment on this.

We note that $[\mathfrak{g}_0, \mathfrak{m}] \subseteq \mathfrak{m}$, $[\mathfrak{g}_0, \mathfrak{m}^*] \subseteq \mathfrak{m}^*$ and hence $\mathfrak{m}$ and $\mathfrak{m}^*$ may be decomposed into graded nilpotent algebras by the action of the centre of $\mathfrak{g}_0$. This centre is nontrivial: in particular there exists an element E in the centre of $\mathfrak{g}_0$ such that $ad E$ has positive integer eigenvalues on $\mathfrak{m}$ and negative integer eigenvalues on $\mathfrak{m}^*$, which may be normalized by the requirement that 1 is an eigenvalue. If E acts by a scalar on a $\mathfrak{g}_0$-module (as it does on an irreducible $\mathfrak{g}_0$-module), then this scalar will be called the *geometric weight*. By decomposing into irreducibles we can talk about the geometric weights of any semisimple $\mathfrak{g}_0$-module, and hence of any element or function with values in that module. An important observation in parabolic geometry is that although the grading of a $\mathfrak{g}$ or $\mathfrak{p}$-module by geometric weight is not $\mathfrak{p}$-equivariant, it induces a $\mathfrak{p}$-equivariant filtration. The associated graded vector space is the corresponding $\mathfrak{g}_0$-module.

If P is a Lie group with Lie algebra $\mathfrak{p}$ then we define G_0 to be the subgroup $\{p \in P : Ad_p(\mathfrak{g}_0) \leqslant \mathfrak{g}_0\}$; this has Lie algebra $\mathfrak{g}_0$. We need to restrict the freedom in the choice of P by assuming throughout that $P = G_0 \exp \mathfrak{m}^*$. This holds automatically if G is a Lie group with Lie algebra $\mathfrak{g}$ and $P = \{g \in G : Ad_g(\mathfrak{p}) \leqslant \mathfrak{p}\}$. The reason for this assumption is that if we need to show a manifestly G_0-invariant construction is P-invariant, we only need to check $\mathfrak{m}^*$-invariance. We refer to such a $(\mathfrak{g}, P)$, satisfying in addition the assumptions of the first section, as a parabolic pair.

2.1. Definition. A *parabolic geometry* on M is a Cartan geometry whose type is a parabolic pair $(\mathfrak{g}, P)$. If $\mathfrak{m}$ is abelian, then this is called the *abelian* or *almost Hermitian symmetric* case. A parabolic geometry is said to be *semiregular* if the geometric weights of the curvature κ are all nonpositive, and *regular* if they are all negative.

In the abelian case, the centre is one dimensional, and the geometric weight determines the action of the centre on an irreducible $\mathfrak{g}_0$-module. In fact $\mathfrak{m}$ itself has geometric weight 1, and so an abelian parabolic geometric is regular. Note that $\Lambda^2\mathfrak{m}^* \otimes \mathfrak{p}$ has negative geometric weights (at most -2), so the (semi)regularity condition is a condition on the torsion alone. Regularity ensures that the Lie bracket of vector fields on M is compatible with the Lie bracket in $\mathfrak{m}$—see [15, 40, 43].

In practice, parabolic geometries are defined in terms of more primitive data, which has to be prolonged (i.e., differentiated) to obtain the Cartan geometry. It is natural to impose a further constraint on the curvature of Cartan connections arising in this way, see 5.9. Here we give some examples of geometric structures inducing such "normal" parabolic geometries.

Conformal geometry. It is well known that conformal geometry in $n \geqslant 3$ dimensions (or Möbius geometry in dimension two [10]) can be described by a Cartan geometry with $\mathfrak{g} \cong \mathfrak{so}(n+1,1)$. We fix a Lorentzian vector space V of signature $(n+1,1)$. Then the space of null lines in V is the n-sphere viewed as a conformal manifold, and the Lorentzian transformations act conformally. We choose a point in S^n and denote its tangent space, which is a conformal vector space, by $\mathfrak{m}$. The isotropy group fixing this null line is isomorphic to $\mathrm{CO}(\mathfrak{m}) \ltimes \mathfrak{m}^*$, with the conformal group $\mathrm{CO}(\mathfrak{m})$ acting on $\mathfrak{m}^*$ in the obvious way. The Lorentzian Lie algebra $\mathfrak{g}$, which is semisimple for any $n \geqslant 1$, is a vector space direct sum $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{co}(\mathfrak{m}) \oplus \mathfrak{m}^*$. The geometric weight is the conformal weight.

Possible choices for P are $\mathrm{CO}(\mathfrak{m}) \ltimes \mathfrak{m}^*$, where $\mathrm{CO}(\mathfrak{m})$ may or may not include the orientation reversing transformations, or $\mathrm{CSpin}(\mathfrak{m}) \ltimes \mathfrak{m}^*$. These parabolic geometries are called (oriented) conformal geometry and conformal spin geometry respectively.

A more primitive definition of a conformal manifold is a manifold equipped with an L^2-valued metric on the tangent bundle, where L^1 is the density line bundle. We shall briefly describe how that Cartan connection arises geometrically. A conformal manifold has a distinguished family of torsion-free connections called *Weyl connections*, which form an affine space on the space of 1-forms. We can define the bundle of Weyl geometries $\mathcal{W}$ as the bundle of splittings of $J^1TM \to TM$ determined by the Weyl connections. This is an affine bundle modelled on T^*M and the Weyl connections are its sections. The principal bundle $\mathcal{G}$ is the pullback of $\mathcal{W}$ to the bundle of conformal frames $\mathrm{CO}(M)$. The Cartan connection arises from the observation that a 0-jet of a section of $\mathcal{W}$ can be extended uniquely to a 1-jet of a section with vanishing Ricci tensor. Usually a more algebraic description is given: for a more detailed discussion, with proofs, see [1, 18, 37].

We describe the following examples even more briefly, our aim being only to indicate the scope of parabolic geometry.

Projective geometry. This is a parabolic geometry of type $\big(\mathfrak{sl}(n{+}1,\mathbb{R}), \mathrm{GL}(n,\mathbb{R}) \ltimes \mathbb{R}^n\big)$. The structure is purely second order, being given by a projective equivalence class of torsion-free connections on the tangent bundle.

Quaternionic geometry. This is a parabolic geometry in $n = 4m$ dimensions of type $\big(\mathfrak{sl}(m+1,\mathbb{H}), S(GL(1,\mathbb{H}) \times \mathrm{GL}(m,\mathbb{H})) \ltimes \mathbb{H}^m\big)$. A manifold is equipped with an (almost) quaternionic structure iff there is a chosen rank 3 Lie subalgebra bundle of $\mathrm{End}(TM)$ pointwise isomorphic to the imaginary quaternions. A quaternionic structure is an almost quaternionic structure with vanishing torsion.

Projective contact geometry. There is a a contact geometry associated with each semisimple Lie algebra. Projective contact geometry is a parabolic geometry of type $\big(\mathfrak{sp}(2(m+1),\mathbb{R}), \mathrm{Sp}(2m,\mathbb{R}) \ltimes \mathcal{H}^{2m+1}\big)$, where $\mathcal{H}^{2m+1}$ is the Heisenberg group with Lie algebra $\mathbb{R}^{2m} \oplus \mathbb{R}$, the Lie bracket being the standard symplectic form on $\mathbb{R}^{2m}$. A projective contact manifold turns out to be a contact manifold together with a chosen class of "projectively equivalent" partial connections.

CR geometry. The geometry induced on strictly pseudoconvex hypersurfaces in complex manifolds is a parabolic geometry of type $\big(\mathfrak{su}(m+1,1), \mathrm{CU}(m) \ltimes \mathcal{H}^{2m+1}\big)$, where the Heisenberg Lie algebra is now identified with $\mathbb{C}^m \oplus \mathbb{R}$ and the Lie bracket is the imaginary part of the standard Hermitian form on $\mathbb{C}^n$. In fact a partial integrability condition on an almost CR structure suffices to define the Cartan geometry [15].

Quaternionic CR geometry. We define quaternionic CR geometry to be a parabolic geometry of type $\big(\mathfrak{sp}(m+1,1), \mathbb{R}^+ \times \mathrm{Sp}(1)\,\mathrm{Sp}(m) \ltimes \tilde{\mathcal{H}}^{4m+3}\big)$, where $\tilde{\mathcal{H}}^{4m+3}$ is the Lie group of the nilpotent Lie algebra structure on $\mathbb{H}^m \oplus \mathbb{R}^3$ whose Lie bracket is the direct sum of the three symplectic forms on $\mathbb{H}^m$.

Pfaffian systems in five variables. One of the first nontrivial Cartan geometries discovered (by Cartan, of course [12]) turns out to be an exceptional parabolic geometry. One starts from a Lie algebra $\mathfrak{m}$ with basis $\{X_1, Y_1, Z_2, X_3, Y_3\}$ such that $[X_1, Y_1] = Z_2$, $[X_1, Z_2] = X_3$, $[Y_1, Z_2] = Y_3$ and all other brackets are trivial. Here the subscripts denote the geometric weight. A derivation of this algebra is determined by its action on X_1 and Y_1, so the derivations preserving geometric weight form a Lie algebra isomorphic to $\mathfrak{gl}(2,\mathbb{R})$ and E is the identity matrix. A more lengthy computation shows that there is only a two dimensional space of derivations from $\mathfrak{m}$ to $\mathfrak{m} \rtimes \mathfrak{gl}(2,\mathbb{R})$ lowering the geometric weight by one. Prolonging twice more gives a Lie algebra structure on $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{gl}(2,\mathbb{R}) \oplus \mathfrak{m}^*$, which turns out to be a real form of the exceptional Lie algebra $\mathfrak{g}_2$. Hence if one equips a 5-manifold M with a rank two subbundle H of the tangent bundle such that Lie brackets of vector fields in H generate a rank three subbundle, and that further Lie brackets with H generate TM, then one obtains, by [15, 43], a parabolic geometry of type $\big(\mathfrak{g}_2, GL(2,\mathbb{R}) \ltimes \hat{\mathcal{H}}^5\big)$ where $\hat{\mathcal{H}}^5$ is the Lie group of $\mathfrak{m}$.

This example is in some sense typical: most "normal" parabolic geometries are determined by a Pfaffian system on the tangent bundle [44]. The preceding examples (apart from quaternionic CR geometry) are unusual in this respect: the geometric structure involves additional data.

In the main example of conformal geometry, we noted that Weyl geometries are closely related to the Cartan principal bundle. Turning this around, we can define a *Weyl connection*, for an arbitrary parabolic geometry, to be a G_0-equivariant section of $\mathcal{G} \to \mathcal{G}_0$ where $\mathcal{G}_0$ is the principal G_0-bundle $\mathcal{G}/\exp\mathfrak{m}^*$. Since $\mathcal{G}$ is a principal $\exp\mathfrak{m}^*$-bundle over $\mathcal{G}_0$, such a section is equivalently given by a P-equivariant map $\mathcal{G} \to \exp\mathfrak{m}^*$, i.e., a section of the associated bundle $\mathcal{W} := \mathcal{G} \times_P \exp\mathfrak{m}^* \cong \mathcal{G}/G_0$ over M, where P acts on $\exp\mathfrak{m}^*$ by the regular representation, not the adjoint representation. Hence this is an affine bundle, the *bundle of Weyl geometries*, modelled on $T^*M = \mathcal{G} \times_P \mathfrak{m}^*$—note that the adjoint action of P on $\mathfrak{m}^*$ is equivalent to its adjoint action on $\exp\mathfrak{m}^*$. For any P-module $\mathbb{E}$, a Weyl connection (i.e., a section of $\mathcal{W}$ over M) identifies $\mathcal{G} \times_P \mathbb{E}$, filtered by geometric weight, with $\mathcal{G}_0 \times_{G_0} \mathbb{E}$, which is the associated graded bundle. We shall use this observation later: for further information, see [16].

2.2. Definition (*Parabolic twistors*)**.** Let $\mathbb{W}$ be a $(\mathfrak{g}, P)$-module. Then $\mathfrak{m}^*$ acts on $\mathbb{W}$ and the image $\mathfrak{m}^* \cdot \mathbb{W}$ of this action is a P-subrepresentation since $\mathfrak{m}^*$ is an ideal of $\mathfrak{p}$. Hence there is a natural P-equivariant map $\mathbb{W} \to \mathbb{W}_{\mathfrak{m}^*}$ where $\mathbb{W}_{\mathfrak{m}^*} := \mathbb{W}/(\mathfrak{m}^* \cdot \mathbb{W})$ is the space of *coinvariants* of $\mathfrak{m}^*$ acting on $\mathbb{W}$. Consequently, on a parabolic geometry, sections s of W induce sections of $W_{T^*M} := \mathcal{G} \times_P \mathbb{W}_{\mathfrak{m}^*}$. Parallel sections of W will be called *parabolic twistors* and the induced sections of W_{T^*M} will be called *parabolic twistor fields.*

Missing from this description is a *twistor operator*: we shall see later that there is a differential operator acting on sections of W_{T^*M} which characterizes the parabolic twistor fields. The twistor operator is the first operator in the curved BGG sequence which we shall construct. To do this we need some Lie algebra homology.

3. Lie algebra homology and cohomology

Any Lie algebra $\mathfrak{l}$ possesses naturally defined homology and cohomology theories with coefficients in an arbitrary $\mathfrak{l}$-module $\mathbb{W}$. These homology and cohomology groups can be constructed using a (projective or injective) resolution of $\mathbb{W}$ by a Koszul complex of $\mathbb{W}$-valued alternating forms. We shall apply this to parabolic geometries by taking $\mathbb{W}$ to be a $\mathfrak{g}$-module and letting $\mathfrak{l} = \mathfrak{m}$ or $\mathfrak{m}^*$, using the vector space direct sum $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_0 \oplus \mathfrak{m}^*$. We focus on $\Lambda\mathfrak{m}^* \otimes \mathbb{W}$, which leads to the homology of $\mathfrak{m}^*$ or the cohomology of $\mathfrak{m}$ with values in $\mathbb{W}$.

$\mathfrak{m}^*$ homology with values in $\mathbb{W}$. We first interpret the space $\Lambda^k\mathfrak{m}^* \otimes \mathbb{W}$ as the space $C_k(\mathfrak{m}^*, \mathbb{W})$ of *k-chains* on $\mathfrak{m}^*$ with values in $\mathbb{W}$. This space carries a natural representation of $\mathfrak{p}$: the action on $\mathbb{W}$ is the restriction of the $\mathfrak{g}$ action, while on $\Lambda^k\mathfrak{m}^*$ we have:

$$\xi \cdot \beta := \sum_i [\xi, \varepsilon^i] \wedge (e_i \lrcorner \beta) \tag{3.1}$$

for $\xi \in \mathfrak{p}$, where e_i denotes a basis of $\mathfrak{m}$ with dual basis ε^i. In the abelian case this action of $\mathfrak{m}^* \subseteq \mathfrak{p}$ on $\Lambda^k\mathfrak{m}^*$ is trivial. In general it is compatible with exterior multiplication by $\alpha \in \mathfrak{m}^*$ in the sense that:

$$\xi \cdot (\alpha \wedge \beta) = \alpha \wedge (\xi \cdot \beta) + [\xi, \alpha] \wedge \beta. \tag{3.2}$$

There is also a compatibility with interior multiplication by $v \in \mathfrak{m}$:

$$v \lrcorner (\xi \cdot \beta) = \xi \cdot (v \lrcorner \beta) + \langle [\xi, \varepsilon^i], v \rangle e_i \lrcorner \beta \tag{3.3}$$

and so
$$\xi \cdot (v \lrcorner \beta) = v \lrcorner (\xi \cdot \beta) + [\xi, v]_{\mathfrak{m}} \lrcorner \beta,$$

where $[\xi, v]_{\mathfrak{m}}$ denotes the $\mathfrak{m}$ component of the Lie bracket in $\mathfrak{g}$, which is the coadjoint action of ξ on $v \in \mathfrak{m} \leqslant \mathfrak{g}^*$, or equivalently the natural action on $v \in \mathfrak{m} \cong \mathfrak{g}/\mathfrak{p}$.

Next we define the boundary operator or codifferential $\delta_{\mathfrak{m}^*}$, where the subscript denotes the Lie algebra which effectively acts in the following definition:

$$\begin{aligned} &\delta_{\mathfrak{m}^*} : C_k(\mathfrak{m}^*, \mathbb{W}) \to C_{k-1}(\mathfrak{m}^*, \mathbb{W}) \\ &\delta_{\mathfrak{m}^*}(\beta \otimes w) = \sum_i \big(\tfrac{1}{2}\varepsilon^i \cdot (e_i \lrcorner \beta) \otimes w + e_i \lrcorner \beta \otimes \varepsilon^i \cdot w \big). \end{aligned} \tag{3.4}$$

For $k = 0, 1$ this definition means $\delta_{\mathfrak{m}^*} w = 0$ and $\delta_{\mathfrak{m}^*}(\alpha \otimes w) = \alpha \cdot w$.

3.1. Lemma (*Cartan's identity*)**.** *For $\alpha \in \mathfrak{m}^*$, $\beta \in \Lambda^k\mathfrak{m}^*$, $w \in \mathbb{W}$ and $c \in C_k(\mathfrak{m}^*, \mathbb{W})$ we have*

$$\sum_i e_i \lrcorner (\alpha \wedge \beta) \otimes \varepsilon^i \cdot w + \sum_i \alpha \wedge (e_i \lrcorner \beta) \otimes \varepsilon^i \cdot w = \beta \otimes \alpha \cdot w$$

$$\sum_i \varepsilon^i \cdot \big(e_i \lrcorner (\alpha \wedge \beta) \big) \otimes w + \sum_i \alpha \wedge \varepsilon^i \cdot (e_i \lrcorner \beta) \otimes w = 2\alpha \cdot \beta \otimes w$$

and consequently
$$\delta_{\mathfrak{m}^*}(\alpha \wedge c) + \alpha \wedge (\delta_{\mathfrak{m}^*} c) = \alpha \cdot c\,.$$

Proof. The first part is immediate from the fact that $e_i \lrcorner (\alpha \wedge \beta) = \alpha(e_i)\beta - \alpha \wedge (e_i \lrcorner \beta)$. For the second part, we compute, using (3.1) and (3.2),

$$\begin{aligned} \sum_i \varepsilon^i \cdot (e_i \lrcorner \alpha \wedge \beta) \otimes w &= \alpha \cdot \beta \otimes w - \sum_i \varepsilon^i \cdot (\alpha \wedge e_i \lrcorner \beta) \otimes w \\ &= \alpha \cdot \beta \otimes w - \sum_i \big([\varepsilon^i, \alpha] \wedge (e_i \lrcorner \beta) + \alpha \wedge \varepsilon^i \cdot (e_i \lrcorner \beta) \big) \otimes w \end{aligned}$$

$$= 2\alpha\cdot\beta\otimes w - \sum_i \alpha\wedge\varepsilon^i\cdot(e_i \lrcorner \beta)\otimes w.$$

The final formula (Cartan's identity) follows from the first two by taking $c = \beta\otimes w$. $\square$

Cartan's identity perhaps best explains the curious factor of $1/2$ in the definition of the codifferential. This factor is also crucial in the following.

3.2. Lemma (*Boundary property*)**.** *The Lie algebra codifferential defines a complex, i.e.,* $\delta_{\mathfrak{m}^*}\circ\delta_{\mathfrak{m}^*} = 0$.

Proof. If $\mathbb{W}$ is a trivial representation, then the definition of the codifferential reduces to the term $\sum_i \frac{1}{2}\varepsilon^i\cdot(e_i\lrcorner\beta)\otimes w$. We first show that the square of this term vanishes separately by virtue of (3.3) and the Jacobi identity for $\mathfrak{m}^*$:

$$\begin{aligned}\sum_{i,j}\varepsilon^i\cdot(e_i\lrcorner\varepsilon^j\cdot(e_j\lrcorner\beta)) &= \sum_{j,k}[\varepsilon^j,\varepsilon^k]\cdot(e_k\lrcorner e_j\lrcorner\beta) + \sum_{i,j}\varepsilon^i\cdot\big(\varepsilon^j\cdot(e_i\lrcorner e_j\lrcorner\beta)\big)\\ &= \frac{1}{2}\sum_{i,j}[\varepsilon^i,\varepsilon^j]\cdot(e_j\lrcorner e_i\lrcorner\beta) = \frac{1}{2}\sum_{i,j,k}[[\varepsilon^i,\varepsilon^j],\varepsilon^k]\wedge(e_k\lrcorner e_j\lrcorner e_i\lrcorner\beta)\\ &= 0.\end{aligned}$$

We can now compute $\delta_{\mathfrak{m}^*}\circ\delta_{\mathfrak{m}^*}$ in general:

$$\begin{aligned}\delta_{\mathfrak{m}^*}\big(\delta_{\mathfrak{m}^*}(\beta\otimes w)\big) &= \frac{1}{4}\sum_{i,j}\varepsilon^i\cdot(e_i\lrcorner\varepsilon^j\cdot(e_j\lrcorner\beta))\otimes w + \sum_{i,j} e_i\lrcorner e_j\lrcorner\beta\otimes\varepsilon^i\cdot\varepsilon^j\cdot w\\ &\quad+\frac{1}{2}\sum_{i,j} e_i\lrcorner\varepsilon^j\cdot(e_j\lrcorner\beta)\otimes\varepsilon^i\cdot w + \frac{1}{2}\sum_{i,j}\varepsilon^i\cdot(e_i\lrcorner e_j\lrcorner\beta)\otimes\varepsilon^j\cdot w\\ &= 0 + \frac{1}{2}\sum_{i,j} e_i\lrcorner e_j\lrcorner\beta\otimes[\varepsilon^i,\varepsilon^j]\cdot w\\ &\quad+\frac{1}{2}\sum_{i,j}\big(e_i\lrcorner\varepsilon^j\cdot(e_j\lrcorner\beta) - \varepsilon^j\cdot(e_i\lrcorner e_j\lrcorner\beta)\big)\otimes\varepsilon^i\cdot w,\end{aligned}$$

which vanishes by (3.3). $\square$

3.3. Lemma ($\mathfrak{p}$*-equivariance*)**.** *For* $\alpha\in\mathfrak{m}^*$ *and* $c\in C_k(\mathfrak{m}^*,\mathbb{W})$, $\delta_{\mathfrak{m}^*}(\alpha\cdot c) = \alpha\cdot(\delta_{\mathfrak{m}^*}c)$.

Proof. Both sides equal $\delta_{\mathfrak{m}^*}\big(\alpha\wedge(\delta_{\mathfrak{m}^*}c)\big)$ by the previous two lemmas. $\square$

It follows that $\delta_{\mathfrak{m}^*}$ is $\mathfrak{p}$-equivariant (since it is clearly $\mathfrak{g}_0$-equivariant).

3.4. Definition (*Homology*)**.** The cycles, boundaries and homology of $\delta_{\mathfrak{m}^*}$ are given by:

$$\begin{aligned}Z_k(\mathfrak{m}^*,\mathbb{W}) &:= \mathit{ker}\,\delta_{\mathfrak{m}^*}: C_k(\mathfrak{m}^*,\mathbb{W})\to C_{k-1}(\mathfrak{m}^*,\mathbb{W}),\\ B_k(\mathfrak{m}^*,\mathbb{W}) &:= \mathit{im}\,\delta_{\mathfrak{m}^*}: C_{k+1}(\mathfrak{m}^*,\mathbb{W})\to C_k(\mathfrak{m}^*,\mathbb{W}),\\ H_k(\mathfrak{m}^*,\mathbb{W}) &:= Z_k(\mathfrak{m}^*,\mathbb{W})/B_k(\mathfrak{m}^*,\mathbb{W}).\end{aligned}$$

These are all $\mathfrak{p}$-modules, and by Cartan's identity, $\mathfrak{m}^*$ maps $Z_k(\mathfrak{m}^*,\mathbb{W})$ into $B_k(\mathfrak{m}^*,\mathbb{W})$ and hence acts trivially on the homology $H_k(\mathfrak{m}^*,\mathbb{W})$.

Note that the zero homology $H_0(\mathfrak{m}^*,\mathbb{W})$ is the space of coinvariants of $\mathbb{W}$ with respect to $\mathfrak{m}^*$, since in that case $Z_0(\mathfrak{m}^*,\mathbb{W}) = \mathbb{W}$ and $B_0(\mathfrak{m}^*,\mathbb{W}) = \mathit{im}(\cdot:\mathfrak{m}^*\otimes\mathbb{W}\to\mathbb{W}) = \mathfrak{m}^*\cdot\mathbb{W}$, i.e., $H_0(\mathfrak{m}^*,\mathbb{W}) = \mathbb{W}/(\mathfrak{m}^*\cdot\mathbb{W}) = \mathbb{W}_{\mathfrak{m}^*}$.

3.5. *Examples.* The homology for the trivial representation in the abelian case gives back the usual multilinear forms:

$$\begin{aligned} \mathbb{W} &= \mathbb{R}, \\ H_k(\mathfrak{m}^*, \mathbb{W}) &= \Lambda^k \mathfrak{m}^*. \end{aligned}$$

In the case of conformal geometry, the Lorentzian vector space $V = L^1 \oplus \mathfrak{m}^0 \oplus L^{-1}$, where $\mathfrak{m}^0 = L^1 \otimes \mathfrak{m}^*$, is itself a $\mathfrak{g}$-module:

$$\begin{aligned} \mathbb{W} &= L^1 \oplus \mathfrak{m}^0 \oplus L^{-1}, \\ H_0(\mathfrak{m}^*, \mathbb{W}) &= L^1, \\ H_k(\mathfrak{m}^*, \mathbb{W}) &= \Lambda^k \mathfrak{m}^* \odot \mathfrak{m}^0, \\ H_n(\mathfrak{m}^*, \mathbb{W}) &= \Lambda^n \mathfrak{m}^* \otimes L^{-1}. \end{aligned}$$

Here elements in the *Cartan product* $\Lambda^k \mathfrak{m}^* \odot \mathfrak{m}^*$ can be thought of as tensors in $\Lambda^k \mathfrak{m}^* \otimes \mathfrak{m}^*$ which are alternating-free and trace-free (i.e., in the kernel of the natural maps to $\Lambda^{k+1}\mathfrak{m}^*$ and $\Lambda^{k-1}\mathfrak{m}^*$)—this is the component generated by the tensor product of highest weight vectors (in the complexified representations if necessary).

Similarly for the adjoint representation $\mathbb{W} = \mathfrak{g}$ we find:

$$\begin{aligned} \mathbb{W} &= \mathfrak{m} \oplus \mathfrak{co}(\mathfrak{m}) \oplus \mathfrak{m}^*, \\ H_0(\mathfrak{m}^*, \mathbb{W}) &= \mathfrak{m}, \\ H_1(\mathfrak{m}^*, \mathbb{W}) &= \mathfrak{m}^* \odot \mathfrak{m}, \\ H_k(\mathfrak{m}^*, \mathbb{W}) &= \Lambda^k \mathfrak{m}^* \odot \mathfrak{so}(\mathfrak{m}), \\ H_{n-1}(\mathfrak{m}^*, \mathbb{W}) &= \Lambda^{n-1} \mathfrak{m}^* \odot \mathfrak{m}^*, \\ H_n(\mathfrak{m}^*, \mathbb{W}) &= \Lambda^n \mathfrak{m}^* \otimes \mathfrak{m}^*. \end{aligned}$$

Again elements in the Cartan product $\Lambda^k \mathfrak{m}^* \odot \mathfrak{so}(\mathfrak{m})$ can be thought of as elements in the usual tensor product which are in the kernel of natural $\mathfrak{co}(\mathfrak{m})$-equivariant contractions and alternations. Elements in the homology for $k = 0, 1, 2$ have immediate geometric interpretations as vectors, linearized conformal metrics and Weyl curvature tensors.

$\mathfrak{m}$ cohomology with values in $\mathbb{W}$. Secondly we view $\Lambda^k \mathfrak{m}^* \otimes \mathbb{W}$ as the space $C^k(\mathfrak{m}, \mathbb{W})$ of *k-cochains* on $\mathfrak{m}$ with values in $\mathbb{W}$. From this point of view it carries a natural $\mathfrak{p}^*$-action, where the action of $\chi \in \mathfrak{p}^*$ on $\beta \in \Lambda^k \mathfrak{m}^*$ is

$$\chi \cdot \beta = \sum_i \varepsilon^i \wedge \big([e^i, \chi] \lrcorner \beta\big) = \sum_i [\chi, \varepsilon^j]_{\mathfrak{m}^*} \wedge (e_j \lrcorner \beta),$$

with $[\chi, \varepsilon^j]_{\mathfrak{m}^*}$ denoting the $\mathfrak{m}^*$ component of the Lie bracket.

The coboundary operator or differential $d_{\mathfrak{m}} \colon C^k(\mathfrak{m}, \mathbb{W}) \to C^{k+1}(\mathfrak{m}, \mathbb{W})$ is defined by

$$d_{\mathfrak{m}}(\beta \otimes w) = \sum_i \big(\tfrac{1}{2}\varepsilon^i \wedge (e_i \cdot \beta) \otimes w + \varepsilon^i \wedge \beta \otimes e_i \cdot w\big).$$

This formula is minus the transpose of the formula for a boundary operator. To be precise, it means that $d_{\mathfrak{m}} = -(\delta_{\mathfrak{m}})^*$, where $\delta_{\mathfrak{m}}$ is the codifferential for $\mathfrak{m}$ homology with values in $\mathbb{W}^*$, whose k-chains are $C_k(\mathfrak{m}, \mathbb{W}^*) = C^k(\mathfrak{m}, \mathbb{W})^*$. It immediately follows that $d_{\mathfrak{m}}$ defines a complex and is $\mathfrak{p}^*$-equivariant (since $\delta_{\mathfrak{m}}$ is equivariant with respect to $\mathfrak{p}^* = \mathfrak{g}_0 \oplus \mathfrak{m}$ not $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{m}^*$). Cartan's identity becomes $d_{\mathfrak{m}}(v \lrcorner c) + v \lrcorner d_{\mathfrak{m}} c = v \cdot c$ for $v \in \mathfrak{m}$ and the cohomology $H^k(\mathfrak{m}, \mathbb{W})$ of $d_{\mathfrak{m}}$ is naturally a $\mathfrak{p}^*$-module with $\mathfrak{m}$ acting trivially.

Duality. In the above treatment of cohomology we made use of the fact that it is dual to homology. More precisely, the $\mathfrak{m}$ cohomology with values in $\mathbb{W}$ is dual to the $\mathfrak{m}$ homology with values in $\mathbb{W}^*$:

$$C^k(\mathfrak{m},\mathbb{W})^* = C_k(\mathfrak{m},\mathbb{W}^*), \qquad (d_{\mathfrak{m}})^* = -\delta_{\mathfrak{m}}, \qquad H^k(\mathfrak{m},\mathbb{W})^* = H_k(\mathfrak{m},\mathbb{W}^*).$$

Similarly, $\mathfrak{m}^*$ homology with values in $\mathbb{W}$ (the first homology theory above) is dual to $\mathfrak{m}^*$ cohomology with values in $\mathbb{W}^*$:

$$C_k(\mathfrak{m}^*,\mathbb{W})^* = C^k(\mathfrak{m}^*,\mathbb{W}^*), \qquad (\delta_{\mathfrak{m}^*})^* = -d_{\mathfrak{m}^*}, \qquad H_k(\mathfrak{m}^*,\mathbb{W})^* = H^k(\mathfrak{m}^*,\mathbb{W}^*).$$

There is also a kind of Poincaré duality between $\mathfrak{m}^*$ homology and cohomology (and similarly for $\mathfrak{m}$): if $\mathfrak{m}^*$ is n-dimensional then $C^k(\mathfrak{m}^*,\mathbb{W}^*) \cong \Lambda^n\mathfrak{m}\otimes C_{n-k}(\mathfrak{m}^*,\mathbb{W}^*)$ and one easily checks that this intertwines boundary and coboundary so that $H^k(\mathfrak{m}^*,\mathbb{W}^*) \cong \Lambda^n\mathfrak{m}\otimes H_{n-k}(\mathfrak{m}^*,\mathbb{W}^*)$.

We are interested primarily in $\delta_{\mathfrak{m}^*}$ and $d_{\mathfrak{m}}$, both of which are defined on $\Lambda\mathfrak{m}^*\otimes\mathbb{W}$. It is natural to ask how they are related. Since $\mathfrak{g}$ is semisimple, one can use a Cartan involution to find positive definite inner products on $\mathfrak{g}$ and $\mathbb{W}$ with respect to which $\delta_{\mathfrak{m}^*}$ is minus the adjoint of $d_{\mathfrak{m}}$. From this, one obtains Kostant's Hodge decomposition [32]:

$$\Lambda\mathfrak{m}^*\otimes\mathbb{W} = \mathit{im}\, d_{\mathfrak{m}} \oplus \big(\mathit{ker}\, d_{\mathfrak{m}} \cap \mathit{ker}\, \delta_{\mathfrak{m}^*}\big) \oplus \mathit{im}\, \delta_{\mathfrak{m}^*}.$$

Furthermore, $\mathit{ker}\, d_{\mathfrak{m}} \cap \mathit{ker}\, \delta_{\mathfrak{m}^*}$ may be identified with the kernel of Kostant's quabla operator $\Box = \delta_{\mathfrak{m}^*}d_{\mathfrak{m}} + d_{\mathfrak{m}}\delta_{\mathfrak{m}^*}$. The first two terms in the Hodge decomposition give $\mathit{ker}\, d_{\mathfrak{m}}$ and the last two terms give $\mathit{ker}\, \delta_{\mathfrak{m}^*}$ and hence

$$H^k(\mathfrak{m},\mathbb{W}) \cong \mathit{ker}\,\Box \cong H_k(\mathfrak{m}^*,\mathbb{W}).$$

This isomorphism is an isomorphism of $\mathfrak{g}_0$-modules, although the cohomology is viewed as a $\mathfrak{p}^*$-module with $\mathfrak{m}$ acting trivially, whereas the homology is viewed as a $\mathfrak{p}$-module with $\mathfrak{m}^*$ acting trivially. Similarly, the Hodge decomposition, as a direct sum, is only $\mathfrak{g}_0$-invariant, although the filtration $0 \leqslant \mathit{im}\, d_{\mathfrak{m}} \leqslant \mathit{ker}\, d_{\mathfrak{m}} \leqslant \Lambda\mathfrak{m}^*\otimes\mathbb{W}$ is $\mathfrak{p}^*$-invariant and the filtration $0 \leqslant \mathit{im}\, \delta_{\mathfrak{m}^*} \leqslant \mathit{ker}\, \delta_{\mathfrak{m}^*} \leqslant \Lambda\mathfrak{m}^*\otimes\mathbb{W}$ is $\mathfrak{p}$-invariant.

The Main Theorem. If M is a parabolic geometry of type $(\mathfrak{g},P)$ and $\mathbb{W}$ is a $(\mathfrak{g},P)$-module, then the Lie algebra homology groups are all P-modules and hence induce vector bundles on M. We shall write $H_k(W)$ for $\mathcal{G}\times_P H_k(\mathfrak{m}^*,\mathbb{W})$, where $W = \mathcal{G}\times_P\mathbb{W}$. We also write $\mathrm{C}^\infty(H_k(W))$ as a shorthand for $\mathrm{C}^\infty(M, H_k(W))$; more formally, we can interpret it as the sheaf $U \mapsto \mathrm{C}^\infty(U, H_k(W))$ for open subsets U of M. Since U is a parabolic geometry in its own right, this slight of hand is more apparent than real.

Our goal in the next two sections is to prove an explicit version of the following result, the first part of which is due to Čap, Slovák and Souček [20], although our construction will be less complicated.

3.6. Theorem. *Let (M,η) be a parabolic geometry of type $(\mathfrak{g},P)$ and let $\mathbb{W}$ be a finite dimensional $(\mathfrak{g},P)$-module. Then there is a naturally defined sequence*

$$\mathrm{C}^\infty(H_0(W)) \overset{\mathcal{D}_0^\eta}{\to} \mathrm{C}^\infty(H_1(W)) \overset{\mathcal{D}_1^\eta}{\to} \mathrm{C}^\infty(H_2(W)) \overset{\mathcal{D}_2^\eta}{\to} \cdots$$

of linear differential operators such that the kernel of the first operator is isomorphic to the parabolic twistors associated to W and the symbols of the differential operators depend only on $(\mathfrak{g},P,\mathbb{W})$ not (M,η). If M is flat then this is sequence is locally exact and hence *computes the cohomology of M with coefficients in the locally constant sheaf of parallel sections of W.*

Suppose further that $\mathbb{W}_1$, $\mathbb{W}_2$ and $\mathbb{W}_3$ are three finite dimensional $(\mathfrak{g}, P)$-modules with a nontrivial $(\mathfrak{g}, P)$-equivariant linear map $\mathbb{W}_1 \otimes \mathbb{W}_2 \to \mathbb{W}_3$ (for instance $\mathbb{W}_3 = \mathbb{W}_1 \otimes \mathbb{W}_2$). Then there are nontrivial bilinear differential pairings

$$\begin{array}{rcl} \mathrm{C}^\infty(H_k(W_1)) \times \mathrm{C}^\infty(H_\ell(W_2)) & \to & \mathrm{C}^\infty(H_{k+\ell}(W_3)) \\ (\alpha, \beta) & \mapsto & \alpha \sqcup_\eta \beta \end{array}$$

whose symbols depend only on $(\mathfrak{g}, P, \mathbb{W}_1, \mathbb{W}_2, \mathbb{W}_3)$ and which have the following properties if M is flat: for $k = \ell = 0$ the pairing extends the given pairing of twistors $\mathbb{W}_1 \otimes \mathbb{W}_2 \to \mathbb{W}_3$, while more generally the following Leibniz rule holds

$$\mathcal{D}^\eta_{k+\ell}(\alpha \sqcup_\eta \beta) = (\mathcal{D}^\eta_k \alpha) \sqcup_\eta \beta + (-1)^k \alpha \sqcup_\eta (\mathcal{D}^\eta_\ell \beta),$$

and hence the pairing descends to a cup product in cohomology.

4. The twisted deRham sequence

On a parabolic geometry M of type $(\mathfrak{g}, P)$ the chain complex of the previous section induces, provided $\mathbb{W}$ is a $(\mathfrak{g}, P)$-module, a complex of vector bundles on M. If W is the bundle induced by $\mathbb{W}$, then the bundle induced by the k-chains is $\Lambda^k T^*M \otimes W$. The codifferential $\delta_{\mathfrak{m}^*}$ induces a codifferential δ_{T^*M} on the k-chain bundles, since it is P-equivariant. On the other hand, $d_{\mathfrak{m}}$ is not P-equivariant and so does not define an operator on the k-chain bundles on M. There is, however, a first order differential operator, namely the exterior covariant derivative (twisted deRham differential)

$$d^{\mathfrak{g}} \colon J^1(\Lambda^k T^*M \otimes W) \to \Lambda^{k+1} T^*M \otimes W$$

which behaves in many ways like $d_{\mathfrak{m}}$. To construct $d^{\mathfrak{g}}$ formally, recall that the invariant derivative defines an isomorphism from $J^1(\Lambda^k T^*M \otimes W)$ to $\mathcal{G} \times_P J^1_0(\Lambda^k \mathfrak{m}^* \otimes \mathbb{W})$, which in turn is isomorphic to $\mathcal{G} \times_P J^1_0(\Lambda^k \mathfrak{m}^*) \otimes \mathbb{W}$. Hence we need to find the P-equivariant map $J^1_0(\Lambda^k \mathfrak{m}^*) \to \Lambda^{k+1} \mathfrak{m}^*$ induced by the exterior derivative. So let $\alpha \colon \mathcal{G} \to \Lambda^k \mathfrak{m}^*$ be P-equivariant. Then η identifies α with a horizontal P-equivariant k-form on $\mathcal{G}$. Since exterior differentiation commutes with pullback, we can compute the exterior derivative on $\mathcal{G}$. This gives the following formula for $d\alpha \colon \mathcal{G} \to \Lambda^{k+1} \mathfrak{m}^*$.

$$\begin{aligned} d\alpha(\xi_0, \dots \xi_k) &= \sum_i (-1)^i (\partial_{\eta^{-1}(\xi_i)} \alpha)(\xi_0, \dots \hat{\xi}_i, \dots \xi_k) \\ &\quad + \sum_{i<j} (-1)^{i+j} \alpha(\eta[\eta^{-1}(\xi_i), \eta^{-1}(\xi_j)], \xi_0, \dots \hat{\xi}_i, \dots \hat{\xi}_j, \dots \xi_k) \end{aligned}$$

and so $$d\alpha = \sum_i \varepsilon^i \wedge \nabla^\eta_{e_i} \alpha - \sum_{i<j} \varepsilon^i \wedge \varepsilon^j \wedge \big(\kappa(e_i, e_j) \lrcorner \alpha\big) + \tfrac{1}{2} \sum_i \varepsilon^i \wedge e_i \cdot \alpha\,.$$

Combining this with the formula for $\Psi_{\mathbb{W}}$ in Proposition 1.5, gives the following result.

4.1. Proposition (*Formal exterior derivatives*)**.** *Let $\mathbb{W}$ a $(\mathfrak{g}, P)$-module. Then the exterior covariant derivative induces the P-equivariant map*

$$\begin{aligned} &\sigma(d^{\mathfrak{g}}) \colon J^1_0(\Lambda^k \mathfrak{m}^* \otimes \mathbb{W}) \to \Lambda^{k+1} \mathfrak{m}^* \otimes \mathbb{W} \\ &(\phi_0, \phi_1) \mapsto \sum_i \varepsilon^i \wedge \phi_1(e_i) + d_{\mathfrak{m}} \phi_0 - \sum_{i<j} \varepsilon^i \wedge \varepsilon^j \wedge \big(\kappa(e_i, e_j) \lrcorner \phi_0\big). \end{aligned} \tag{4.1}$$

The last term vanishes if the parabolic geometry is torsion-free. In general it is P-equivariant, and so the P-equivariant map

$$\begin{aligned} \sigma(d^\eta)\colon J^1_0(\Lambda^k\mathfrak{m}^*\otimes\mathbb{W}) &\to \Lambda^{k+1}\mathfrak{m}^*\otimes\mathbb{W}\\ (\phi_0,\phi_1) &\mapsto \sum_i \varepsilon^i\wedge\phi_1(e_i)+d_{\mathfrak{m}}\phi_0 \end{aligned} \tag{4.2}$$

induces a exterior covariant derivative d^η with torsion (unless η is torsion-free).

Thus we see that although the zero order operator $d_{\mathfrak{m}}$ is not P-equivariant, we can correct it by a first order term: the symbol of the exterior derivative (wedge product). There are two ways to do this. The simplest formula (4.2) (with no torsion correction) gives an exterior covariant derivative with torsion, but an extra term can be added to remove the torsion (4.1). On the bundle level, these exterior derivatives are related by

$$d^{\mathfrak{g}}s = d^\eta s - \sum_{i<j}\varepsilon^i\wedge\varepsilon^j\wedge K_M(e_i,e_j)\lrcorner\, s$$

for any section s of $\Lambda^kT^*M\otimes W$: note that only the torsion part of K_M contributes to the contraction. This difference is also illustrated by the following result.

4.2. Proposition (*Curvature*)**.** *The composites $R^{\mathfrak{g}}=(d^{\mathfrak{g}})^2$ and $R^\eta := (d^\eta)^2$ acting on a section s of $\Lambda^kT^*M\otimes W$ are given by*

$$\begin{aligned} R^{\mathfrak{g}}s &= tr\big(X\mapsto K_M\wedge X\cdot s\big)\\ R^\eta s &= tr\big(X\mapsto -K_M\wedge\nabla^\eta_X s\big). \end{aligned}$$

Here $X\in\mathfrak{g}_M$: in the first formula, the $\mathfrak{g}_M$-values of K_M act on the W-values of s, while in the second formula, the $\mathfrak{g}_M$-values are contracted with the invariant derivative.

Proof. The first formula is clear: the square of the $d^{\mathfrak{g}}$ is the wedge product action of the curvature of $\nabla^{\mathfrak{g}}$. For the second formula, let $f\colon\mathcal{G}\to\Lambda^k\mathfrak{m}^*\otimes\mathbb{W}$ be P-equivariant. Then

$$d^\eta f = \sigma(d^\eta)(f,\nabla^\eta f) = d_{\mathfrak{m}}f+\sum_i\varepsilon^i\wedge\nabla^\eta_{e_i}f.$$

We must apply d^η to this expression. To do this, note that $d_{\mathfrak{m}}$ is constant on $\mathcal{G}$, so that

$$\sum_j\varepsilon^j\wedge\nabla^\eta_{e_j}d^\eta f = \sum_j\varepsilon^j\wedge d_{\mathfrak{m}}\nabla^\eta_{e_j}f+\sum_{i,j}\varepsilon^j\wedge\varepsilon^i\wedge\nabla^\eta_{e_j}(\nabla^\eta_{e_i}f)$$

and therefore

$$\begin{aligned} (d^\eta)^2 f &= (d_{\mathfrak{m}})^2 f+\sum_i d_{\mathfrak{m}}(\varepsilon^i\wedge\nabla^\eta_{e_i}f)+\sum_j\varepsilon^j\wedge d_{\mathfrak{m}}\nabla^\eta_{e_j}f+\sum_{i,j}\varepsilon^j\wedge\varepsilon^i\wedge\nabla^\eta_{e_j}(\nabla^\eta_{e_i}f)\\ &= 0+\frac12\sum_{i,j}\big(\varepsilon^j\wedge[e_j,\varepsilon^i]_{\mathfrak{m}^*}\wedge\nabla^\eta_{e_i}f+\varepsilon^j\wedge\varepsilon^i\wedge(\nabla^\eta_{e_j}(\nabla^\eta_{e_i}f)-\nabla^\eta_{e_i}(\nabla^\eta_{e_j}f))\big)\\ &= \frac12\sum_{i,j}\varepsilon^j\wedge\varepsilon^i\wedge\big(\nabla^\eta_{e_j}(\nabla^\eta_{e_i}f)-\nabla^\eta_{e_i}(\nabla^\eta_{e_j}f)-\nabla^\eta_{[e_j,e_i]}f\big)\\ &= \frac12\sum_{i,j}\varepsilon^j\wedge\varepsilon^i\wedge\nabla^\eta_{\kappa(e_i,e_j)}f = -\sum_i\langle\zeta^i,\kappa\rangle\wedge\nabla^\eta_{\chi_i}f, \end{aligned}$$

where χ_i is a basis of $\mathfrak{g}$ with dual basis ζ^i. □

These vanish if K is zero, or if K takes values in a subspace of $\mathfrak{p}$ acting trivially on $\mathbb{W}$.

5. The BGG sequence and cup product

The key tool for proving the main theorem is a family of differential operators

$$\Pi^\eta_k\colon \mathrm{C}^\infty(\Lambda^k T^*M\otimes W)\to \mathrm{C}^\infty(\Lambda^k T^*M\otimes W)$$

which vanish on $im\,\delta_{T^*M}$, map into $ker\,\delta_{T^*M}$, and induce the identity on homology. As motivation for the construction of such an operator, recall Kostant's quabla operator $\square=\delta_{\mathfrak{m}^*}d_{\mathfrak{m}}+d_{\mathfrak{m}}\delta_{\mathfrak{m}^*}$ (with $ker\,\square\cong H_k(\mathfrak{m}^*,\mathbb{W})$) and the Hodge decomposition:

$$\Lambda^k\mathfrak{m}^*\otimes\mathbb{W}=im\,d_{\mathfrak{m}}\oplus ker\,\square\oplus im\,\delta_{\mathfrak{m}^*}\,.$$

The projection onto $ker\,\square$ in this direct sum has image contained in $ker\,\delta_{\mathfrak{m}^*}$ and induces the identity on homology. Unfortunately, it is not $\mathfrak{p}$-equivariant. Ignoring this for the moment, note that $\square$ is invertible on its own image and so the projection onto $ker\,\square$ may be written $id-\square^{-1}\square$. A more refined formula may be obtained by observing that $\square$ commutes with $d_{\mathfrak{m}}$, and hence so does $\square^{-1}$ on the image of $\square$. Therefore:

$$id-\square^{-1}(\delta_{\mathfrak{m}^*}d_{\mathfrak{m}}+d_{\mathfrak{m}}\delta_{\mathfrak{m}^*})=id-\square^{-1}\delta_{\mathfrak{m}^*}d_{\mathfrak{m}}-d_{\mathfrak{m}}\square^{-1}\delta_{\mathfrak{m}^*}\,.$$

The advantage of this formula is that we only need the inverse of $\square$ on $im\,\delta_{\mathfrak{m}^*}$ (which is a $\mathfrak{p}$-module). Indeed, we only need the operator $\square^{-1}\delta_{\mathfrak{m}^*}$.

We now address the problem of $\mathfrak{p}$-equivariance. Of course $\square$ fails to be $\mathfrak{p}$-equivariant simply because $d_{\mathfrak{m}}$ is not $\mathfrak{p}$-equivariant. However, in the previous section we noted that one resolution is to replace $d_{\mathfrak{m}}$ with a first order differential operator: either d^η or $d^{\mathfrak{g}}$. We shall concentrate first on the former, but return to the latter at the end of the section.

5.1. Definition (*First order quabla operator*)**.** Let M be a parabolic geometry of type $(\mathfrak{g},P)$ and let $\mathbb{W}$ be a $(\mathfrak{g},P)$-module. Then the *quabla operator* on $\Lambda T^*M\otimes W$ is the first order differential operator $\square_\eta=\delta_{T^*M}d^\eta+d^\eta\delta_{T^*M}$.

Note that $\square_\eta$ commutes with δ_{T^*M} and also maps k-forms to k-forms, so it preserves $B_k(W)=im\,\delta_{T^*M}\colon\Lambda^{k+1}T^*M\otimes W\to\Lambda^kT^*M\otimes W$. In the flat case it also commutes with d^η, but in general $\square_\eta\circ d^\eta-d^\eta\circ\square_\eta=\delta_{T^*M}\circ R^\eta-R^\eta\circ\delta_{T^*M}$.

5.2. Theorem. *Suppose M is a parabolic geometry of type $(\mathfrak{g},P)$ and $\mathbb{W}$ is a finite dimensional $(\mathfrak{g},P)$-module. Then $\square_\eta\colon\mathrm{C}^\infty(B_k(W))\to\mathrm{C}^\infty(B_k(W))$ is invertible. Furthermore the inverse is a differential operator of finite order.*

Proof. To prove that $\square_\eta$ has a two-sided differential inverse, we choose a Weyl connection, i.e., a section of the bundle of Weyl geometries $\mathcal{W}$. Such a section always exists locally on M—in the smooth, rather than analytic, category, they exist globally, since $\mathcal{W}$ is an affine bundle, but we shall only need local sections, since we are constructing a local operator and, by the uniqueness of two-sided inverses, the local inverses patch together. Hence we assume we have a section over all of M, which identifies the k-chain bundles, filtered by geometric weight, with the associated graded bundles. The operators $\square$ and $d_{\mathfrak{m}}$, which are G_0-invariant, but not P-invariant, define operators on associated graded bundles, and hence, using the Weyl connection, on the k-chain bundles themselves.

5.3. Lemma. $\square^{-1}(\square_\eta-\square)\colon\mathrm{C}^\infty(B_k(W))\to\mathrm{C}^\infty(B_k(W))$ *is nilpotent.*

Proof of Lemma. Since $\mathbb{W}$ is finite dimensional, the $\mathfrak{p}$-module $B_k(\mathfrak{m}^*,\mathbb{W})$ decomposes into finitely many irreducible $\mathfrak{g}_0$-submodules and clearly the action of $\mathfrak{m}^*$ lowers the geometric weight. Suppose that $s\colon M \to B_k(W)$ takes values in an subbundle associated to an irreducible $\mathfrak{g}_0$-submodule. Now

$$\begin{aligned}(\Box_\eta - \Box)s &= (\delta_{T^*M}(d^\eta - d_{\mathfrak{m}}) + (d^\eta - d_{\mathfrak{m}})\delta_{T^*M})s\\ &= \sum_i \big(\delta_{T^*M}(\varepsilon^i \wedge \nabla^\eta_{e_i} s) + \varepsilon^i \wedge \delta_{T^*M}\nabla^\eta_{e_i} s\big)\\ &= \sum_i \varepsilon^i \cdot \nabla^\eta_{e_i} s\end{aligned}$$

which has lower geometric weight, since the covariant derivative preserves the filtration, while the action of T^*M lowers the weight. Finally, $\Box^{-1}$ is $\mathfrak{g}_0$-equivariant and so it preserves the geometric weight. Hence $\Box^{-1}(\Box_\eta - \Box)$ lowers the geometric weight, so it is nilpotent. $\Box$

Writing $\mathcal{N} = -\Box^{-1}(\Box_\eta - \Box)$, we have $\Box_\eta = \Box(id - \mathcal{N})$. Therefore the two-sided inverse $\Box_\eta^{-1}$ is given by a Neumann series:

$$\Box_\eta^{-1} = (id - \mathcal{N})^{-1}\Box^{-1} = \Big(\sum_{k\geqslant 0} \mathcal{N}^k\Big)\Box^{-1}.$$

This inverse is a differential operator whose order is the degree of nilpotency of the first order differential operator $\mathcal{N}$. Hence it is an inverse on any open subset of M. Our construction involved the choice of Weyl connection, but the inverse constructed is of course independent of this choice. $\Box$

We can now define differential operators Q_η and Π^η from $\mathrm{C}^\infty(\Lambda T^*M \otimes W)$ to itself:

$$Q_\eta = \Box_\eta^{-1}\delta_{T^*M}\,, \qquad \Pi^\eta = id - d^\eta \circ Q_\eta - Q_\eta \circ d^\eta.$$

Clearly Π^η preserves the degree of the form. This gives our operators Π^η_k.

5.4. *Remark.* The first order quabla operator $\Box_\eta$ maps sections of $Z_k(W)$ into $B_k(W)$. This means in particular that it preserves $B_k(W)$, but also that it descends to an operator on $C_k(W)/Z_k(W)$. In the construction of Q_η and Π^η, we only used the invertibility of $\Box_\eta$ on $B_k(W)$, but in fact it is also invertible on $C_k(W)/Z_k(W)$: in the algebraic setting this holds for Kostant's $\Box$ and exactly the same Neumann series argument goes through. Hence one might prefer to define $\tilde{Q}_\eta = \delta_{T^*M}\Box_\eta^{-1}$, where $\Box_\eta^{-1}$ is now the inverse on $C_k(W)/Z_k(W)$ and $\tilde{Q}_\eta$ acts on $C_k(W)$ by first passing to the quotient—clearly this is well defined since $Z_k(W) = C_k(W) \cap \ker \delta_{T^*M}$ by definition.

Now observe that Q_η and $\tilde{Q}_\eta$ are both given by isomorphisms from $C_k(W)/Z_k(W)$ to $B_{k-1}(W)$. Composing on each side by $\Box_\eta$ gives $\Box_\eta Q_\eta \Box_\eta = \delta_{T^*M}\Box_\eta$ and $\Box_\eta \tilde{Q}_\eta \Box_\eta = \Box_\eta \delta_{T^*M}$. Since $\Box_\eta$ is an isomorphism on $C_k(W)/Z_k(W)$ and on $B_{k-1}(W)$, and it commutes with δ_{T^*M}, we deduce that $\tilde{Q}_\eta = Q_\eta$.

We now establish the fundamental properties of Π^η.

5.5. Proposition (*Calculus of* Π*-operators*)**.** *The operator* $\Pi^\eta_k\colon \mathrm{C}^\infty(\Lambda^k T^*M \otimes W) \to \mathrm{C}^\infty(\Lambda^k T^*M \otimes W)$ *has the following properties.*

(i) Π^η_k *vanishes on* $\operatorname{im}\delta_{T^*M}$: $\Pi^\eta_k \circ \delta_{T^*M} = 0$.
(ii) Π^η_k *maps into* $\ker \delta_{T^*M}$: $\delta_{T^*M} \circ \Pi^\eta_k = 0$.
(iii) *On* $\ker \delta_{T^*M}$, $\Pi^\eta_k \cong id \mod \operatorname{im}\delta_{T^*M}$, *i.e.,* Π^η_k *induces the identity on homology.*
(iv) $d^\eta \circ \Pi^\eta_k - \Pi^\eta_{k+1} \circ d^\eta = Q_\eta \circ R^\eta - R^\eta \circ Q_\eta$.

(v) $(\Pi^\eta_k)^2 = \Pi^\eta_k + Q_\eta \circ R^\eta \circ Q_\eta$ *and so* Π^η_k *is a projection in the flat case, and for* $k = 0$.

(vi) $\Pi^\eta \circ \Box_\eta = Q_\eta \circ R^\eta \circ \delta_{T^*M}$ *and* $\Box_\eta \circ \Pi^\eta = \delta_{T^*M} \circ R^\eta \circ Q_\eta$.

Thus in the flat case Π^η_k *is a differential projection onto a subspace of* $\ker \delta_{T^*M}$ *complementary to* $\operatorname{im} \delta_{T^*M}$ *and is a chain map from the deRham complex to itself;* Q_η *is a chain homotopy between* Π^η_k *and* id.

Proof. The first three results follow from $\mathit{ker}\, \delta_{T^*M} = \mathit{ker}\, Q_\eta$ and $\mathit{im}\, Q_\eta = \mathit{im}\, \delta_{T^*M}$ (since $\Box_\eta^{-1}$ is the inverse on $\mathit{im}\, \delta_{T^*M}$). The fourth fact follows easily from the definition of Π^η and using this, the fifth fact is an immediate calculation:

$$\begin{aligned}(\Pi^\eta_k)^2 &= \Pi^\eta_k(id - d^\eta \circ Q_\eta) = \Pi^\eta_k - (d^\eta \circ \Pi^\eta_{k-1} - Q_\eta \circ R^\eta + R^\eta \circ Q_\eta) \circ Q_\eta \\ &= \Pi^\eta_k + Q_\eta \circ R^\eta \circ Q_\eta.\end{aligned}$$

The last part also follows easily from the definition of Π^η. ∎

The first two properties allow us to define two further operators:

$$\Pi^\eta_k \circ \mathit{repr} \colon \mathrm{C}^\infty(H_k(W)) \to \mathrm{C}^\infty(\Lambda^k T^*M \otimes W) \tag{5.1}$$

$$\mathit{proj} \circ \Pi^\eta_k \colon \mathrm{C}^\infty(\Lambda^k T^*M \otimes W) \to \mathrm{C}^\infty(H_k(W)) \tag{5.2}$$

where proj denotes the projection from the kernel of δ_{T^*M} to homology and repr means the choice of a representative of the homology class. Thus $\Pi^\eta_k \circ \mathit{repr}$ gives a canonical differential representative for homology classes and $\mathit{proj} \circ \Pi^\eta_k$ is a canonical differential projection onto homology. By Proposition 5.5 (vi), the canonical representative is the unique representative in $\mathit{ker}\, \Box_\eta$, whereas the canonical projection vanishes on $\mathit{im}\, \Box_\eta$.

The first operator (5.1) was originally constructed by Baston in the abelian case [2], and Čap, Slovák and Souček in general [20]. Note that on $\mathit{ker}\, \delta_{T^*M}$, $\Pi^\eta = id - \Box_\eta^{-1}\delta_{T^*M} d^\eta$. It is interesting to note that in the flat abelian case, Baston obtained the Neumann series formula for this in [2], equation (8).

We define $\mathcal{D}^\eta_k = \mathit{proj} \circ \Pi^\eta_{k+1} \circ d^\eta \circ \Pi^\eta_k \circ \mathit{repr}$. Since $d^\eta \circ \Pi^\eta_k$ maps $\mathit{ker}\, \delta_{T^*M}$ into $\mathit{ker}\, \delta_{T^*M}$ already, this equals $\mathit{proj} \circ d^\eta \circ \Pi^\eta_k \circ \mathit{repr}$, so we only actually need (5.1). For the pairings we really need (5.1) and (5.2) to define

$$\sqcup_\eta = \mathit{proj} \circ \Pi^\eta_{k+\ell} \circ \wedge \circ (\Pi^\eta_k \circ \mathit{repr}, \Pi^\eta_\ell \circ \mathit{repr})$$

where $\wedge$ denotes wedge product of forms contracted by the pairing $W_1 \otimes W_2 \to W_3$.

The main theorem is now straightforward (apart from the independence result for the symbols—see the appendix): in the flat case we have a locally exact resolution because Π^η, as a chain map on the deRham resolution by sheaves of smooth sections, is homotopic to the identity, and the Leibniz rule follows from the corresponding Leibniz rule for the wedge product. In the curved case we have the following results.

5.6. Proposition (*Composition*)**.** $\mathcal{D}^\eta_{k+1} \circ \mathcal{D}^\eta_k = \mathit{proj} \circ \Pi^\eta_{k+2} \circ R^\eta \circ \Pi^\eta_k \circ \mathit{repr}$.

Proof. By definition $\mathcal{D}^\eta_{k+1} \circ \mathcal{D}^\eta_k = \mathit{proj} \circ d^\eta \circ \Pi^\eta_{k+1} \circ d^\eta \circ \Pi^\eta_k \circ \mathit{repr}$. Now commute d^η_{k+1} past Π^η_{k+1} using the Π-operator calculus of the previous proposition. ∎

5.7. Proposition (*Leibniz rule*)**.** *For* $\alpha \in \mathrm{C}^\infty(H_k(W_1))$ *and* $\beta \in \mathrm{C}^\infty(H_\ell(W_2))$,

$$\begin{aligned}&\mathcal{D}^\eta_{k+\ell}(\alpha \sqcup_\eta \beta) = \mathcal{D}^\eta_k\alpha \sqcup_\eta \beta + (-1)^k \alpha \sqcup_\eta \mathcal{D}^\eta_\ell \beta \\ &\quad + \Big[\Pi^\eta_{k+\ell+1}\Big(\big(Q_\eta R^\eta \Pi^\eta_k \alpha\big) \wedge \Pi^\eta_\ell \beta + (-1)^k \Pi^\eta_k \alpha \wedge \big(Q_\eta R^\eta \Pi^\eta_\ell \beta\big) - R^\eta Q_\eta\big(\Pi^\eta_k \alpha \wedge \Pi^\eta_\ell \beta\big)\Big)\Big].\end{aligned}$$

Here, and henceforth, we write $[\dots]$ *for the projection to homology, and* Π_k^η *for* $\Pi_k^\eta \circ \mathrm{repr}$.

Proof. This again follows easily from Proposition 5.5:

$$\begin{aligned}\mathcal{D}^\eta_{k+\ell}(\alpha \sqcup_\eta \beta) &= [\Pi^\eta_{k+\ell+1} d^\eta \Pi^\eta_{k+\ell}(\Pi^\eta_k \alpha \wedge \Pi^\eta_\ell \beta)] \\ &= [\Pi^\eta_{k+\ell+1} d^\eta (\Pi^\eta_k \alpha \wedge \Pi^\eta_\ell \beta)] - [\Pi^\eta_{k+\ell+1} R^\eta Q_\eta (\Pi^\eta_k \alpha \wedge \Pi^\eta_\ell \beta)].\end{aligned}$$

The first term can be expanded using the Leibniz rule for the exterior derivative:

$$d^\eta(\Pi^\eta_k \alpha \wedge \Pi^\eta_\ell \beta) = d^\eta \Pi^\eta_k \alpha \wedge \Pi^\eta_\ell \beta + (-1)^k \Pi^\eta_k \alpha \wedge d^\eta \Pi^\eta_\ell \beta.$$

We insert the projections Π^η_{k+1}, $\Pi^\eta_{\ell+1}$ using the definition $id = \Pi^\eta + d^\eta \circ Q_\eta + Q_\eta \circ d^\eta$. The first correction term does not contribute, since $d^\eta \Pi^\eta_k \alpha$ and $d^\eta \Pi^\eta_\ell \beta$ are in $\mathit{ker}\, \delta_{T^*M}$, while the second correction gives two further curvature terms as stated. □

We now consider the other choice of exterior covariant derivative: the torsion-free operator $d^{\mathfrak{g}}$. This change makes no difference if the parabolic geometry is torsion-free. In the presence of torsion, we can construct what we believe is a more natural curved analogue of the BGG complex, although to do this, we need to assume that the parabolic geometry is regular, i.e., the geometric weights of the curvature κ are negative (which is a condition on the torsion). Under this assumption the extra torsion correction in the formula for $d^{\mathfrak{g}}$ does not cause any problems in the proof of nilpotency in Theorem 5.2, since its action still lowers the geometric weight, and all other details of the proofs are unchanged. We thus obtain operators $\Box_{\mathfrak{g}}$, Q, Π_k, $\mathcal{D}_k$, $\sqcup$ satisfying the same formulae with d^η replaced by $d^{\mathfrak{g}}$ and R^η by $R^{\mathfrak{g}}$.

The "torsion-free" BGG sequences seem to us to be more natural, because $R^{\mathfrak{g}}$ is always zero order, given by wedge product with the curvature K_M. Another reason for preferring $d^{\mathfrak{g}}$ is the differential Bianchi identity.

5.8. Proposition. *Let* M *be a Cartan geometry of type* $(\mathfrak{g}, P)$ *with curvature* $K_M \in C^\infty(\Lambda^2 T^*M \otimes \mathfrak{g}_M)$. *Then* $d^{\mathfrak{g}} K_M = 0$.

We combine $d^{\mathfrak{g}} K_M = 0$ with a well-known definition.

5.9. Definition. A parabolic geometry is said to be *normal* if $\delta_{T^*M} K_M = 0$.

5.10. Theorem. *Let* $(\mathcal{G}, \eta)$ *be a normal regular parabolic geometry of type* $(\mathfrak{g}, P)$ *on* M. *Then the curvature* K_M *is uniquely determined by its homology class* $[K_M]$, *via the formula*

$$K_M = \Pi_2[K_M]$$

and $[K_M]$ *therefore satisfies* $\mathcal{D}_2[K_M] = 0$, *where* $\mathcal{D}_2$ *is the operator in the torsion-free BGG sequence associated to* $\mathfrak{g}$.

Furthermore the composite of two operators in the torsion-free BGG sequence associated to $\mathbb{W}$ *is given by*

$$\mathcal{D}_{k+1} \mathcal{D}_k \alpha = [K_M] \sqcup \alpha,$$

where the cup product is contracted by the pairing $\mathfrak{g} \otimes \mathbb{W} \to \mathbb{W}$ *given by the* $\mathfrak{g}$*-action.*

Proof. $\Pi_2[K_M]$ is the unique element of $\mathit{ker}\, \delta_{T^*M} \cap \mathit{ker}\, \Box_{\mathfrak{g}}$ whose homology class is $[K_M]$. But K_M itself satisfies $d^{\mathfrak{g}} K_M = 0$ and $\delta_{T^*M} K_M = 0$ and hence $\Box_{\mathfrak{g}} K_M = 0$.

For the second part, observe that

$$[K_M] \sqcup \alpha = \big[\Pi_{k+2}\big(\Pi_2[K_M] \wedge \Pi_k\alpha\big)\big] = \big[\Pi_{k+2}(K_M \wedge \Pi_k\alpha)\big] = \big[\Pi_{k+2}R^{\mathfrak{g}}\Pi_k\alpha\big]$$

which is the composite $\mathit{proj} \circ \Pi_{k+2} \circ R^{\mathfrak{g}} \circ \Pi_k \circ \mathit{repr}$ acting on α. □

For conformal geometry in four dimensions or more, the curvature of the Cartan connection is obtained by applying a first order operator to the Weyl curvature, as is well known. Even in the general context, the observation that the curvature is uniquely determined by its (co)homology class is an old one: see [15, 43]. Our approach reveals that the proofs in these references appear technical because they amount to the construction of $\Pi_2 \circ \mathit{repr}$ in this special case. Also, by working with Lie algebra homology, rather than cohomology, the explicit differential operator reconstructing the full curvature is realized as an operator on M, rather than $\mathcal{G}$, as in the conformal case.

In the second part of this theorem, it is slightly awkward that the action of $\mathfrak{g}$ needs to be specified. There is a convenient device to make this happen automatically. Suppose that all $(\mathfrak{g}, P)$-modules of interest belong to the symmetric algebra or the tensor algebra of $\mathbb{W} = \mathbb{W}_1 \oplus \mathbb{W}_2 \oplus \cdots$; for instance, $\mathbb{W}$ could be the standard representation or the direct sum of the fundamental representations of $\mathfrak{g}$. Now work either in the universal enveloping algebra of $\mathbb{W} \rtimes \mathfrak{g}$ (with trivial bracket on $\mathbb{W}$), or in the tensor algebra of $\mathbb{W} \oplus \mathfrak{g}$ modulo the ideal generated by $X \otimes \phi - \phi \otimes X - X \cdot \phi$ for $X \in \mathfrak{g}$ and $\phi \in \mathbb{W} \oplus \mathfrak{g}$ (a semiholonomic enveloping algebra—see the appendix). This algebra is filtered by finite dimensional $\mathfrak{g}$-modules, where the action is induced by the action on $\mathfrak{g} \oplus \mathbb{W}$, *and definitely* not *by left multiplication with elements of* $\mathfrak{g}$. It follows that for any differential form α with values in the associated algebra bundle, $R^{\mathfrak{g}}\alpha = K_M \wedge \alpha - \alpha \wedge K_M$, where the curvature K_M is viewed as a 2-form with values in the copy of $\mathfrak{g}_M$ in this algebra bundle. The properties of $\mathcal{K} = [K_M]$ and $\mathcal{D}$ established in the above theorem may now be rewritten:

$$\mathcal{D}_2\mathcal{K} = 0, \qquad \mathcal{D}_{k+1}\mathcal{D}_k\alpha = \mathcal{K} \sqcup \alpha - \alpha \sqcup \mathcal{K}. \tag{5.3}$$

The curvature terms in the Leibniz rule may be rewritten in a similar way:

$$\mathcal{D}_{k+\ell}(\alpha \sqcup \beta) = \mathcal{D}_k\alpha \sqcup \beta + (-1)^k\alpha \sqcup \mathcal{D}_\ell\beta - \langle\mathcal{K}, \alpha, \beta\rangle + \langle\alpha, \mathcal{K}, \beta\rangle - \langle\alpha, \beta, \mathcal{K}\rangle, \tag{5.4}$$

where the triple products are defined by

$$\langle\mathcal{K}, \alpha, \beta\rangle = \Big[\Pi_{k+\ell+1}\Big(\Pi_2\mathcal{K} \wedge Q\big(\Pi_k\alpha \wedge \Pi_\ell\beta\big) - Q\big(\Pi_2\mathcal{K} \wedge \Pi_k\alpha\big) \wedge \Pi_\ell\beta\Big)\Big]$$

and similarly for the other two products, although the first term acquires a sign $(-1)^k$. The contractions with $\mathcal{K}$ happen automatically in this combination of triple products. If α and β belong to BGG subsequences associated to $\mathbb{W}_1, \mathbb{W}_2$ then the formula can be contracted further using any $(\mathfrak{g}, P)$-equivariant linear map $\mathbb{W}_1 \otimes \mathbb{W}_2 \to \mathbb{W}_3$.

These triple products may seem ad hoc, but in fact this is the first appearance of natural trilinear differential pairings closely related to Massey products. For any $(\mathfrak{g}, P)$-equivariant linear map $\mathbb{W}_1 \otimes \mathbb{W}_2 \otimes \mathbb{W}_3 \to \mathbb{W}_4$, one can define a trilinear differential pairing from $\mathrm{C}^\infty(H_k(W_1)) \times \mathrm{C}^\infty(H_\ell(W_2)) \times \mathrm{C}^\infty(H_m(W_3))$ to $\mathrm{C}^\infty(H_{k+\ell+m-1}(W_4))$ by

$$\langle\alpha, \beta, \gamma\rangle = \Big[\Pi_{k+\ell+m-1}\Big((-1)^k\Pi_k\alpha \wedge Q\big(\Pi_\ell\beta \wedge \Pi_m\gamma\big) - Q\big(\Pi_k\alpha \wedge \Pi_\ell\beta\big) \wedge \Pi_m\gamma\Big)\Big].$$

This measures the failure of the cup product to be associative: one may compute that

$$\begin{aligned}\mathcal{D}_{k+\ell+m-1}\big\langle\alpha,\beta,\gamma\big\rangle &= (\alpha\sqcup\beta)\sqcup\gamma-\alpha\sqcup(\beta\sqcup\gamma)\\ &\quad-\big\langle\mathcal{D}_k\alpha,\beta,\gamma\big\rangle-(-1)^k\big\langle\alpha,\mathcal{D}_\ell\beta,\gamma\big\rangle-(-1)^{k+\ell}\big\langle\alpha,\beta,\mathcal{D}_m\gamma\big\rangle\\ &\quad+\big\langle\mathcal{K},\alpha,\beta,\gamma\big\rangle-\big\langle\alpha,\mathcal{K},\beta,\gamma\big\rangle+\big\langle\alpha,\beta,\mathcal{K},\gamma\big\rangle-\big\langle\alpha,\beta,\gamma,\mathcal{K}\big\rangle\end{aligned}\tag{5.5}$$

where the quadruple products each have five terms. In the flat case, this formula verifies that the cup product is associative in BGG cohomology. Note, though, that in practice, one often destroys this associativity by using incompatible (nonassociative) pairings to define the cup products: it is crucial above that the same map $\mathbb{W}_1\otimes\mathbb{W}_2\otimes\mathbb{W}_3\to\mathbb{W}_4$ is used for $(\alpha\sqcup\beta)\sqcup\gamma$ and $\alpha\sqcup(\beta\sqcup\gamma)$.

The relation, in the flat case, with a Massey product is as follows: if $\mathcal{D}_k\alpha=\mathcal{D}_\ell\beta=\mathcal{D}_m\gamma=0$ and if also $\alpha\sqcup\beta=\mathcal{D}_{k+\ell-1}A$ and $\beta\sqcup\gamma=\mathcal{D}_{\ell+m-1}C$, then

$$\mathcal{D}_{k+\ell+m-1}\big(A\sqcup\gamma-(-1)^k\alpha\sqcup C-\big\langle\alpha,\beta,\gamma\big\rangle\big)=0$$

and hence we obtain a partially defined triple product of BGG cohomology classes, with an ambiguity coming from the choice of A and C. Again the role of $\big\langle\alpha,\beta,\gamma\big\rangle$ is to correct the failure of $\sqcup$ to be associative on the BGG cochain complex. Note that the two terms in $\big\langle\alpha,\beta,\gamma\big\rangle$ modify the lifts ΠA and ΠC of A and C from Lie algebra homology.

6. Curved A_∞-algebras

The formulae (5.3), (5.4) and (5.5) give the first four defining relations of a curved A_∞-algebra. In the case of vanishing curvature, such algebras were introduced by Stasheff [42] nearly forty years ago. An A_∞-algebra is a graded vector space A equipped with a sequence of multilinear maps $\mu_k\colon\otimes^k A\to A$ of degree $2-k$ satisfying some identities. (In fact only parity really matters, and μ_k has parity k mod 2.) In the original formulation, $\mu_0=0$, μ_1 is a differential, and μ_2 is "strongly homotopy associative", i.e., it is associative up to a homotopy given by μ_3, which in turn satisfies higher order associativity conditions. In the presence of curvature, we require that for each $m\geqslant 0$,

$$\sum_{\substack{j+k=m+1\\ j\geqslant1,k\geqslant0}}\sum_{\ell=0}^{j-1}(-1)^{k+\ell+k\ell+k|\alpha_1\ldots\alpha_\ell|}\mu_j\big(\alpha_1,\ldots\alpha_\ell,\mu_k(\alpha_{\ell+1},\ldots\alpha_{k+\ell}),\alpha_{k+\ell+1},\ldots\alpha_m\big)=0,$$

for all $\alpha_1,\ldots\alpha_m\in A$ of homogeneous degree, where $|\alpha_1\ldots\alpha_\ell|$ denotes the sum of the degrees. The usual definition, with the sign conventions of [35], is recovered by putting $\mu_0=0$. For the Lie analogue of L_∞-algebras, the general curved case has been introduced by Zwiebach [45] within the context of String Field Theory, where the presence of μ_0 is interpreted as a non-conformal background, related to (genus 0) vacuum vertices. In our setting, μ_0 is the (background) curvature, and we now indicate briefly how such a curved A_∞-algebra structure arises. Following [29, 30, 36], we first work on the level of the chain bundles and define λ_m inductively, for $m\geqslant2$, by

$$\lambda_m(a_1,\ldots a_m)=\sum_{\substack{j+k=m\\ j,k\geqslant1}}(-1)^{(k-1)(j+|a_1\ldots a_j|)}Q\lambda_j(a_1,\ldots a_j)\wedge Q\lambda_k(a_{j+1},\ldots a_m)$$

where we formally set $Q\lambda_1=-id$. Note that the number of terms in λ_m is given by the Catalan number $\frac{1}{m+1}\binom{2m}{m}$. On the homology bundles, we then define: $\mu_0=\mathcal{K}$, $\mu_1(\alpha_1)=\mathcal{D}\alpha_1$ and $\mu_m(\alpha_1,\ldots\alpha_m)=[\Pi\lambda_m(\Pi\alpha_1,\ldots\Pi\alpha_m)]$ for $m\geqslant2$.

In order to prove that this is a curved A_∞-algebra, it is convenient to make use of the observation that an A_∞-algebra structure on a vector space A is equivalently an odd coderivation of square zero on the tensor coalgebra of A (with the grading of A shifted to get the signs right)—see for instance [35], Example 1.9. Although the tensor coalgebra of the sheaf of sections of the homology bundles of an enveloping algebra makes us a bit dizzy, we are only using this formalism as a way to compute identities for multilinear differential operators which avoids dealing with huge expressions and complicated signs.

To obtain the coderivation, put $\mu = \sum_{m\geqslant 0} \mu_m \colon \bigotimes A \to A$, let $p_i \colon \bigotimes A \to \otimes^i A$ be the projection, and define μ^c by $p_0\mu^c = 0$, $p_1\mu^c = \mu$ and $\Delta \circ \mu^c = (id \otimes \mu^c + \mu^c \otimes id) \circ \Delta$, where $\Delta(a_1 \otimes \ldots \otimes a_k) = \sum_j (a_1 \otimes \ldots \otimes a_j) \otimes (a_{j+1} \otimes \ldots \otimes a_k)$ is the coproduct. The defining relations of an A_∞-algebra are now equivalent to $(\mu^c)^2 = 0$, although it suffices to check that $\mu\mu^c = 0$, since $(\mu^c)^2$ is the coderivation $(\mu\mu^c)^c$.

In our case, we have $\mu = \mathit{proj}\,\Pi(K_M + d^{\mathfrak{g}} + \lambda)\Pi\,\mathit{repr}$, where $\lambda = \sum_{m\geqslant 2} \lambda_m$ and Π is extended to the tensor coalgebra as $\sum \Pi \otimes \cdots \otimes \Pi$. The proof that the induced coderivation has square zero follows [36], except that we must deal with curvature terms. Such terms appear in five ways: the curvature explicitly in the definition of μ; the term $(d^{\mathfrak{g}})^2 = R^{\mathfrak{g}}$ in $\mu\mu^c$; from $\Pi^2 = \Pi + QR^{\mathfrak{g}}Q$; from $\mathit{proj}\,\Pi d^{\mathfrak{g}}\Pi = \mathit{proj}\,\Pi(d^{\mathfrak{g}} - R^{\mathfrak{g}}Q)$; and from $\Pi d^{\mathfrak{g}}\Pi\,\mathit{repr} = (d^{\mathfrak{g}} - QR^{\mathfrak{g}})\Pi\,\mathit{repr}$. Note that the shift in the grading changes some signs and that the recursive definition of λ_m is equivalent to $\lambda = \lambda_2\big((Q\lambda - p_1) \otimes (Q\lambda - p_1)\big)\Delta$. Omitting the lift and projection to homology, we have

$$
\begin{aligned}
\mu\mu^c &= \Pi(K_M + d^{\mathfrak{g}} + \lambda)\Pi\big(\Pi(K_M + d^{\mathfrak{g}} + \lambda)\Pi\big)^c \\
&= \Pi d^{\mathfrak{g}}\Pi^2(K_M + d^{\mathfrak{g}} + \lambda)\Pi + \Pi\lambda\big(\Pi^2(K_M + d^{\mathfrak{g}} + \lambda)\big)^c\Pi \\
&= \Pi R^{\mathfrak{g}}p_1\Pi + \Pi(d^{\mathfrak{g}} - R^{\mathfrak{g}}Q)\lambda\Pi + \Pi\lambda\big(K_M + d^{\mathfrak{g}} - QR^{\mathfrak{g}}p_1 + \Pi\lambda + QR^{\mathfrak{g}}Q\lambda\big)^c\Pi \\
&= \Pi\big(d^{\mathfrak{g}}\lambda + \lambda(d^{\mathfrak{g}})^c + \lambda(\Pi\lambda)^c\big)\Pi + \Pi\lambda_2\big(K_M \otimes (Q\lambda - p_1) + (Q\lambda - p_1) \otimes K_M\big)\Pi \\
&\qquad\qquad + \Pi\lambda_2\big((Q\lambda - p_1) \otimes (Q\lambda - p_1)\big)(K_M^c \otimes id + id \otimes K_M^c)\Delta\Pi \\
&\qquad\qquad + \Pi\lambda\big(Q\lambda_2\big(K_M \otimes (p_1 - Q\lambda) + (p_1 - Q\lambda) \otimes K_M\big)\big)^c\Pi \\
&= \Pi\big(\lambda\lambda^c + d^{\mathfrak{g}}\lambda + \lambda(d^{\mathfrak{g}})^c - \lambda([d^{\mathfrak{g}}, Q]\lambda)^c\big)\Pi \\
&\quad + \Pi\lambda_2\big(Q\lambda K_M^c \otimes (Q\lambda - p_1) + (Q\lambda - p_1) \otimes Q\lambda K_M^c\big)\Delta\Pi \\
&\quad - \Pi\lambda\big(Q\lambda_2\big(K_M \otimes (Q\lambda - p_1) + (Q\lambda - p_1) \otimes K_M\big)\big)^c\Pi.
\end{aligned}
$$

Next we compute that $\lambda\lambda^c = \lambda_2\big(Q\lambda\lambda^c \otimes (Q\lambda - p_1) + (Q\lambda - p_1) \otimes Q\lambda\lambda^c\big)\Delta$—the term $\lambda_2\big(\lambda \otimes (Q\lambda - p_1) + (Q\lambda - p_1) \otimes \lambda\big)\Delta$ vanishes by expanding λ and using the associativity of λ_2. It follows by induction that $\lambda\lambda^c = 0$. Similarly, $d^{\mathfrak{g}}\lambda + \lambda(d^{\mathfrak{g}})^c - \lambda([d^{\mathfrak{g}}, Q]\lambda)^c = \lambda_2\Big(Q\big(d^{\mathfrak{g}}\lambda + \lambda(d^{\mathfrak{g}})^c - \lambda([d^{\mathfrak{g}}, Q]\lambda)^c\big) \otimes (Q\lambda - p_1) + (Q\lambda - p_1) \otimes Q\big(d^{\mathfrak{g}}\lambda + \lambda(d^{\mathfrak{g}})^c - \lambda([d^{\mathfrak{g}}, Q]\lambda)^c\big)\Big)\Delta$, and so it also follows that $d^{\mathfrak{g}}\lambda + \lambda(d^{\mathfrak{g}})^c - \lambda([d^{\mathfrak{g}}, Q]\lambda)^c = 0$. One more recursive argument shows that the curvature terms cancel as well.

6.1. *Remark.* J. Stasheff has pointed out to us that this sort of result can also be obtained using the techniques of Homological Perturbation Theory, at least in the flat case. The crucial idea is that Q defines strong deformation retraction data for the coderivation determined by $d^{\mathfrak{g}}$. The methods of [28] may then be used to transfer the perturbation of this coderivation induced by the wedge product to the Lie algebra homology bundles.

Finally, we remark that restricting the above to the (super)symmetric coalgebra gives an L_∞-algebra, in which one can work with $\mathbb{W} \rtimes \mathfrak{g}$ instead of its enveloping algebra.

7. The dual BGG sequences

The BGG cochain sequence of Lie algebra homology bundles $H_k(W)$ is dual to a chain sequence of Lie algebra cohomology bundles, generalizing the deRham chain complex of exterior divergences. To fix notations, recall that the latter is a complex

$$\mathrm{C}^\infty(L^{-n}) \xleftarrow{\delta} \mathrm{C}^\infty(L^{-n}\otimes TM) \xleftarrow{\delta} \mathrm{C}^\infty(L^{-n}\otimes TM) \xleftarrow{\delta} \cdots$$

where L^{-n} is the oriented line bundle of densities and δ is the exterior divergence, i.e., on vector field densities $\delta = div$, the natural divergence, and in general it is adjoint to d in the sense that for $\alpha \in \mathrm{C}^\infty(\Lambda^k T^*M)$ and $a \in \mathrm{C}^\infty(L^{-n}\otimes \Lambda^{k+1}TM)$, we have

$$div(\alpha \lrcorner a) = \langle d\alpha, a\rangle + \langle \alpha, \delta a\rangle,$$

where $\langle \theta, \alpha \lrcorner a\rangle = \langle \theta\wedge\alpha, a\rangle$ for any 1-form θ. For compactly supported sections, the complex can be augmented by $\int : \mathrm{C}^\infty_0(L^{-n}) \to \mathbb{R}$, giving a homology theory.

A simple way to obtain a dual BGG sequence is to twist the BGG sequence of the dual $(\mathfrak{g}, P)$-module $\mathbb{W}^*$ by the flat line bundle of pseudoscalars $L^{-n}\otimes\Lambda^n TM$, where $n = \dim M$: this is the orientation line bundle of M and is associated to a one dimensional $(\mathfrak{g}, P)$-module, on which only $G_0 \leqslant P$ might act nontrivially. By Poincaré duality for Lie algebra (co)homology, such a twist amounts to replacing $H_k(W^*)$ with $L^{-n}\otimes H^{n-k}(W^*)$, where $H^{n-k}(W^*) = H_{n-k}(W)^*$. Writing $\mathcal{D}^{n-1-k}_\eta$ for this twist of $\mathcal{D}^\eta_k$, we obtain a sequence

$$\mathrm{C}^\infty(L^{-n}\otimes H^0(W^*)) \xleftarrow{\mathcal{D}^0_\eta} \mathrm{C}^\infty(L^{-n}\otimes H^1(W^*)) \xleftarrow{\mathcal{D}^1_\eta} \mathrm{C}^\infty(L^{-n}\otimes H^2(W^*)) \xleftarrow{\mathcal{D}^2_\eta} \cdots$$

of linear differential operators. In the flat case, this is, by construction, an injective resolution of the sheaf of parallel sections of $L^{-n}\otimes\Lambda^n TM\otimes W^*$, beginning with $\mathcal{D}^{n-1}_\eta$, but it is natural to view it instead as a projective resolution of the dual of the sheaf of parallel sections of W by working with compactly supported sections and defining, for $a \in \mathrm{C}^\infty_0(L^{-n}\otimes H^0(W^*))$, $\langle \int a, w\rangle = \int\langle a, [w]\rangle$ for any parallel section w of W.

This point of view is further amplified by constructing the dual BGG sequences directly from the sequence of exterior divergences twisted by the twistor connection on W^*. In the presence of torsion, there are two possibilities: $\delta^\eta = -(d^\eta)^*$ or $\delta^{\mathfrak{g}} = -(d^{\mathfrak{g}})^*$. As in the previous section, we define $\hat{\Box}_\eta = d_{\mathfrak{m}^*}\delta^\eta + \delta^\eta d_{\mathfrak{m}^*}$ and find that it is invertible on $\mathrm{C}^\infty(L^{-n}\otimes B^k(W^*))$ and $\mathrm{C}^\infty(L^{-n}\otimes C^k(W^*)/Z^k(W^*))$, where $C^k(W^*) = \mathcal{G}\times_P C^k(\mathfrak{m}^*, \mathbb{W}^*) = \Lambda^k TM\otimes W^*$. Hence we obtain operators $\hat{Q}_\eta$ and $\hat{\Pi}^\eta$. Furthermore, this construction is adjoint to the construction of the previous section: since $\delta^\eta = -(d^\eta)^*$ and $d_{\mathfrak{m}^*} = -(\delta_{\mathfrak{m}^*})^*$, we have $\hat{\Box}_\eta = (\Box_\eta)^*$, $B^k(W^*) = (C_k(W)/Z_k(W))^*$, $C^k(W^*)/Z^k(W^*) = B_k(W)^*$, and hence, by Remark 5.4, $\hat{Q}_\eta = -(Q_\eta)^*$, so that $\hat{\Pi}^\eta = (\Pi^\eta)^*$.

The dual BGG operators obtained above by Poincaré duality are therefore equivalently defined by $\mathcal{D}^k_\eta = proj\circ\delta^\eta\circ\hat{\Pi}^\eta_k\circ repr$. Associated to a pairing $\mathbb{W}_1\otimes\mathbb{W}_2 \to \mathbb{W}_3$, the analogue of the cup product is a "cap product" between cochains and chains:

$$\begin{array}{ccc} \mathrm{C}^\infty(H_k(W_1))\times\mathrm{C}^\infty(L^{-n}\otimes H^{k+\ell}(W_3^*)) & \to & \mathrm{C}^\infty(L^{-n}H^\ell(W_2^*)) \\ (\alpha,b) & \mapsto & \alpha\sqcap_\eta b \end{array}$$

satisfying a Leibniz rule up to curvature terms. This can be defined by twisting the cup product by $L^{-n}\otimes\Lambda^n TM$ and using Poincaré duality, or by the formula

$$\sqcap_\eta = proj\circ\hat{\Pi}^\eta_{k+\ell}\circ\lrcorner\circ(\Pi^\eta_k\circ repr, \hat{\Pi}^\eta_\ell\circ repr)$$

where $\neg$ denotes the contraction of forms with multivectors together with the pairing $W_1 \otimes W_3^* \to W_2^*$. Here $\alpha \neg a$ for $\alpha \in \mathrm{C}^\infty(\Lambda^k T^*M)$ and $a \in \mathrm{C}^\infty(L^{-n} \otimes \Lambda^{k+\ell} TM)$ is defined by $\langle \theta, \alpha \neg a \rangle = \langle \theta \wedge \alpha, a \rangle$ for any ℓ-form θ.

The Leibniz rule, for $\alpha \in \mathrm{C}^\infty(H_k(W_1))$ and $b \in \mathrm{C}^\infty(L^{-n} \otimes H^{k+\ell}(W_3^*))$, is:

$$\begin{aligned}\mathcal{D}^\ell_\eta(\alpha \sqcap_\eta b) &= \alpha \sqcap_\eta (\mathcal{D}^{k+\ell}_\eta b) - (-1)^\ell (\mathcal{D}^\eta_k \alpha) \sqcap_\eta b \\ &+ \Big[\Pi^\eta_{\ell-1}\Big(\Pi^\eta_k \alpha \neg \big(\hat{Q}_\eta \hat{R}^\eta \hat{\Pi}^\eta_{k+\ell} b\big) - (-1)^\ell \big(Q_\eta R^\eta \Pi^\eta_k \alpha\big) \neg \hat{\Pi}^\eta_{k+\ell} b - \hat{R}^\eta \hat{Q}_\eta \big(\Pi^\eta_k \alpha \neg \hat{\Pi}^\eta_\ell b\big)\Big)\Big].\end{aligned}$$

Similar results hold for the torsion-free sequence $\mathcal{D}^k = \mathit{proj} \circ \delta^{\mathfrak{g}} \circ \hat{\Pi}_k \circ \mathit{repr}$ (in the regular case) and if the parabolic geometry is also normal, the composite of dual BGG operators is given by cap product with $[K_M]$ and the correction terms to the Leibniz rule are given by triple products of $[K_M]$, α and b.

The cap product gives a neat way to see the duality between $\mathcal{D}_k$ and $\mathcal{D}^k$. Consider the pairing $\mathbb{W} \otimes \mathbb{R} \to \mathbb{W}$, with cap product

$$\mathrm{C}^\infty(M, H_k(W)) \times \mathrm{C}^\infty(M, L^{-n} \otimes H^{k+\ell}(W^*)) \to \mathrm{C}^\infty(M, L^{-n} \otimes H^\ell(\mathbb{R})).$$

For $\ell = 0$, $H^0(\mathbb{R}) = \mathbb{R}$ and this pairing is the duality pairing of $H_k(W)$ and $H^k(W^*)$, tensored with the density bundle L^{-n}. When $\ell = 1$, $H^1(\mathbb{R})$ is the subbundle of TM associated to $\mathfrak{g}_1$, the geometric weight 1 subspace of $\mathfrak{g}$. Hence we obtain a pairing with values in vector densities. The claim is that the Leibniz rule for $\ell = 1$ becomes

$$\mathit{div}\, \alpha \sqcap b = \langle \mathcal{D}_k \alpha, b \rangle + \langle \alpha, \mathcal{D}^k b \rangle$$

with no curvature corrections. The inner curvature corrections vanish because $Z_k(W) = B^k(W^*)^0$ and $Z^k(W^*) = B_k(W)^0$, so that the contractions of Π with $\hat{Q}$ and $\hat{\Pi}$ with Q are zero. The outer correction vanishes because the curvature is acting on the trivial representation. A similar result holds for the analogue of the triple product rule (5.5), showing that for adjoint pairings of $(\mathfrak{g}, P)$-modules, $c \mapsto \beta \sqcap c$ is adjoint to $\alpha \mapsto \alpha \sqcup \beta$.

8. General applications

We turn now to potential applications of the BGG sequence and cup product. We restrict ourselves first to general discussions: more explicit examples are given in the next section.

Twistor operators. The exterior covariant derivatives $d^{\mathfrak{g}}$ and d^η on W-valued 0-forms are both simply the covariant derivative $\nabla^{\mathfrak{g}}$ on sections of W. Also $\mathit{ker}\, \delta_{T^*M} = W$ and so a parabolic twistor f is the natural representative $\Pi_0[f]$ of its homology class, which is a parabolic twistor field. Hence $\Pi_0 \circ \mathit{repr}$ is a "jet operator" which assigns a parabolic twistor to the corresponding parabolic twistor field. In the flat case, the *twistor operator* $\mathcal{D}_0 = \mathit{proj} \circ \nabla^{\mathfrak{g}} \circ \Pi_0 \circ \mathit{repr}$ characterizes parabolic twistor fields as solutions of a differential equation. In the curved case it is natural to define parabolic twistor fields by the kernel of this operator, but $\mathcal{D}_0 \phi = 0$ only implies that $\nabla^{\mathfrak{g}} \Pi_0 \phi = Q R^{\mathfrak{g}} \Pi_0 \phi$, and so $\Pi_0 \phi$ might not be parallel in general.

Twistor algebra. $\mathfrak{g}$-modules form an algebra under direct sum and tensor product. The cup product $\mathrm{C}^\infty(M, H_0(W_1)) \times \mathrm{C}^\infty(M, H_0(W_2)) \to \mathrm{C}^\infty(M, H_0(W_1 \otimes W_2))$ defines an algebra structure on sections of the corresponding homology bundles. The Leibniz rule shows that in the flat case the cup product algebra extends the algebra of twistors. A similar observation can be made for $\mathfrak{g}$-modules under Cartan product (provided one is careful with identifications between isomorphic representations)—in this case the cup product is zero order, given by the Cartan product of zeroth homologies.

Deformation theory and moduli spaces. Suppose that M is a compact manifold admitting a flat parabolic geometry of type $(\mathfrak{g}, P)$. What is the moduli space of flat parabolic geometries of type $(\mathfrak{g}, P)$ on M? A first approximation to this question is to study deformations of the given flat structure $(\mathcal{G}, \eta)$. We discuss briefly deformations of regular normal parabolic geometries, with emphasis on deformations of flat structures.

Fixing the principal P-bundle $\mathcal{G} \to M$, Cartan connections of type $(\mathfrak{g}, P)$ form an open subset of an affine space modelled on the P-equivariant horizontal 1-forms $T\mathcal{G} \to \mathfrak{g}$ (it is an open subset because of the condition that η is an isomorphism on each tangent space). Therefore a small deformation of η may be written $\eta_\varepsilon = \eta + \tilde{\alpha}_\varepsilon$ where $\tilde{\alpha}_\varepsilon$ is a curve of such P-equivariant horizontal 1-forms with $\tilde{\alpha}_0 = 0$. The curvature of η_ε is $K^\varepsilon(U, V) = d\eta_\varepsilon(U, V) + [\eta_\varepsilon(U), \eta_\varepsilon(V)]$ and passing to associated bundles gives

$$K^\varepsilon_M = K_M + d^{\mathfrak{g}}\alpha_\varepsilon + \alpha_\varepsilon \wedge \alpha_\varepsilon, \tag{8.1}$$

where $\alpha_\varepsilon\colon TM \to \mathfrak{g}_M$ (with $\alpha_0 = 0$) and we think of $\mathfrak{g}_M$ as a subbundle of its universal enveloping algebra bundle, so that $(\alpha_\varepsilon \wedge \alpha_\varepsilon)(X, Y) = \alpha_\varepsilon(X) \cdot \alpha_\varepsilon(Y) - \alpha_\varepsilon(Y) \cdot \alpha_\varepsilon(X) = [\alpha_\varepsilon(X), \alpha_\varepsilon(Y)] \in \mathfrak{g}_M$ for $X, Y \in TM$ (equivalently, we can work directly with the Lie bracket in $\mathfrak{g}_M$). Differentiating with respect to ε at $\varepsilon = 0$, gives, up to third order,

$$\dot{K}_M = d^{\mathfrak{g}}\dot{\alpha}, \qquad \ddot{K}_M = d^{\mathfrak{g}}\ddot{\alpha} + 2\dot{\alpha} \wedge \dot{\alpha}, \qquad \dddot{K}_M = d^{\mathfrak{g}}\dddot{\alpha} + 3(\ddot{\alpha} \wedge \dot{\alpha} + \dot{\alpha} \wedge \ddot{\alpha}).$$

Suppose now that $K_M = 0$. Then, in order for η_ε to be normal, we need $\delta_{T^*M}\dot{K}_M = 0$. Also $d^{\mathfrak{g}}\dot{K}_M = (d^{\mathfrak{g}})^2\dot{\alpha} = 0$, and so $\dot{K}_M = \Pi_2[\dot{K}_M]$ and $\mathcal{D}_2[\dot{K}_M] = 0$. Adding $d^{\mathfrak{g}}s$ to $\dot{\alpha}$ does not alter $\dot{K}_M$, and so one can assume $\delta_{T^*M}\dot{\alpha} = 0$. Hence $\Box_{\mathfrak{g}}\dot{\alpha} = 0$ and $\dot{\alpha}$ represents a homology class $A = [\dot{\alpha}]$. We then have $\dot{\alpha} = \Pi_1 A$, $[\dot{K}_M] = [d^{\mathfrak{g}}\dot{\alpha}] = \mathcal{D}_1 A$. This does not completely fix the freedom to add $d^{\mathfrak{g}}s$ to $\dot{\alpha}$: we can still add $\mathcal{D}_0 f$ to A.

To summarize, we see that the linearized theory is controlled by the BGG complex with $\mathbb{W} = \mathfrak{g}$: an infinitesimal deformation of η (as a regular normal parabolic geometry) is given by a section A of $H_1(\mathfrak{g}_M)$ and $\mathcal{D}_1 A$ is the linearized curvature. Since $\mathcal{D}_1\mathcal{D}_0 f = 0$, $A = \mathcal{D}_0 f$ as just an infinitesimal gauge transformation. Hence the formal tangent space to the moduli space is the first cohomology of the complex

$$\mathrm{C}^\infty(M, H_0(\mathfrak{g}_M)) \xrightarrow{\mathcal{D}_0} \mathrm{C}^\infty(M, H_1(\mathfrak{g}_M)) \xrightarrow{\mathcal{D}_1} \mathrm{C}^\infty(M, H_2(\mathfrak{g}_M)) \xrightarrow{\mathcal{D}_2} \mathrm{C}^\infty(M, H_3(\mathfrak{g}_M)).$$

This is only the actual tangent space if all infinitesimal deformations can be integrated. We first consider second order deformations. If $\dot{K}_M = 0$ (i.e., $\mathcal{D}_1 A = 0$), then normality implies that $\delta_{T^*M}\ddot{K}_M = 0$, and since also $d^{\mathfrak{g}}\ddot{K}_M = 0$, $\ddot{K}_M = \Pi_2[\ddot{K}_M]$. Hence it suffices to consider $[\Pi_2 d^{\mathfrak{g}}\ddot{\alpha}] + [\Pi_2(\dot{\alpha} \wedge \dot{\alpha})]$ and the second term is $A \sqcup A$. As before, we can assume $\delta_{T^*M}\ddot{\alpha} = 0$, so that the first term is $\mathcal{D}_1[\ddot{\alpha}]$. Hence we have a quadratic obstruction to solving $\ddot{K}_M = 0$: we need $A \sqcup A$ to be in the image of $\mathcal{D}_1$. The Leibniz rule gives $\mathcal{D}_2(A \sqcup A) = 2(\mathcal{D}_1 A) \sqcup A = 0$ and so the obstruction is the class of $A \sqcup A$ in the second cohomology of above complex.

The obstructions to building a formal power series all lie in this second cohomology space, but the construction involves the A_∞-algebra of multilinear operators, not just the cup product (alternatively we can work in the L_∞-algebra associated to $\mathfrak{g}$). The reason for this can be seen at third order: $\Box_{\mathfrak{g}}\ddot{\alpha} = \delta_{T^*M} d^{\mathfrak{g}}\ddot{\alpha}$ is not zero in general, and

$\ddot\alpha = \Pi_1\ddot\alpha + Qd^{\mathfrak g}\ddot\alpha = \Pi_1\ddot\alpha - 2Q(\dot\alpha\wedge\dot\alpha)$. Hence if $A_1 = [\dot\alpha]$ and $A_2 = \frac{1}{2}[\Pi_1\ddot\alpha]$, we have

$$\dot\alpha = \Pi_1 A_1 \qquad \text{and} \qquad \ddot\alpha = 2\big(\Pi_1 A_2 - Q(\Pi_1 A_1 \wedge \Pi_1 A_1)\big),$$

and therefore $$[\Pi_2(\ddot\alpha\wedge\dot\alpha + \dot\alpha\wedge\ddot\alpha)] = 2\big(A_2 \sqcup A_1 + A_1 \sqcup A_2 + \langle A_1, A_1, A_1\rangle\big).$$

The Leibniz and triple product rules (5.4), (5.5) imply that this is in the kernel of $\mathcal{D}_2$, and its cohomology class is the second obstruction.

In general if $A = \sum_{j=1}^k A_j\varepsilon^j$ satisfies the equation $\mathcal{D}_1 A + A \sqcup A + \langle A, A, A\rangle + \cdots = 0$ to order k in ε, then $\mathcal{D}_2(A \sqcup A + \langle A, A, A\rangle + \cdots)$ vanishes to order $k+1$ in ε and the cohomology class of the degree $k+1$ term of $A \sqcup A + +\langle A, A, A\rangle + \cdots$ is the obstruction to finding A_{k+1} such that $\tilde A = \sum_{j=1}^{k+1} A_j\varepsilon^j$ satisfies the equation to order $k+1$ in ε.

This deformation theory parallels numerous examples in algebra and geometry which have been studied since [27, 31]. It would be interesting to extend it to half-flat geometries such as selfdual conformal 4-manifolds [22].

Linear field theories. In the flat case (when the BGG sequence is a complex) it is natural to view it as a linear gauge theory. The beginning of the sequence gives the kinematics, while the end gives the dynamics; equivalently, the dynamics are given by the beginning of the dual BGG sequence.

$$\begin{array}{ccccccc}
\text{charges} & \text{gauges} & & \text{potentials} & & \text{kinematic fields} \\
0 \to \mathbb{W} \to & C^\infty(H_0(W)) & \xrightarrow[\mathit{Twistor}]{\mathcal{D}_0} & C^\infty(H_1(W)) & \xrightarrow[\mathit{Potential}]{\mathcal{D}_1} & C^\infty(H_2(W)) \\
0 \leftarrow \mathbb{W}^* \leftarrow & C^\infty(L^{-n}H^0(W^*)) & \xleftarrow[\mathit{ConsLaw}]{\mathcal{D}^0} & C^\infty(L^{-n}H^1(W^*)) & \xleftarrow[\mathit{FieldEqn}]{\mathcal{D}^1} & C^\infty(L^{-n}H^2(W^*)) \\
\text{dual charges} & \text{fluxes} & & \text{sources} & & \text{dynamic fields}
\end{array}$$

The prototype for such a sequence is the deRham complex describing electromagnetism. We shall present some further justification for this point of view on linear field theory in the following, but we refer to [21] for a more thorough discussion. The assumption that the BGG sequence is a complex means that potentials coming from a gauge via $\mathcal{D}_0$ have no kinematic field, and that sources coming from a dynamic field automatically satisfy the conservation law $\mathcal{D}^0$. Not shown is the kinematic field equation $\mathcal{D}_2$ which is automatically satisfied by kinematic fields coming from a potential.

Given a $(\mathfrak{g}, P)$-equivariant pairing $\mathbb{W}_1\otimes\mathbb{W}_2 \to \mathbb{W}_3$, we have in particular a cup product $C^\infty(M, H_0(W_1)) \times C^\infty(M, H_k(W_2)) \to C^\infty(M, H_k(W_3))$. This means that twistors in W_1 can be used to transform objects in the field theory associated to $\mathbb{W}_2$ to the field theory associated to $\mathbb{W}_3$. The Leibniz rule implies that this transformation will be compatible with the operators in the sequence. This is often called the translation principle.

Relations to deRham complexes. We restrict attention here to the abelian case, so the BGG sequence of the trivial representation $\mathbb{R}$ is the deRham complex of exterior derivatives, and the dual BGG sequence is the dual deRham complex of exterior divergences. For any $\mathfrak{g}$-module $\mathbb{W}$, we always have a pairing $\mathbb{W}\otimes\mathbb{R} \to \mathbb{W}$, and so, given a twistor in W, we can construct potentials and kinematic fields in the $\mathbb{W}$-theory from differential forms using the cup product ("kinematic helicity raising"), while the cap product gives multivector densities from dynamic fields and sources ("dynamic helicity lowering"). Similarly, the pairing $\mathbb{W}^*\otimes\mathbb{R} \to \mathbb{W}^*$ shows that a twistor in W^* can be used to construct dynamic fields and sources from multivector densities via cap product ("dynamic helicity raising"), and differential forms from potentials and kinematic fields via cup product ("kinematic helicity lowering"). Let us focus on the cap product

$$\mathrm{C}^\infty(H_k(W)) \times \mathrm{C}^\infty(L^{-n} \otimes H^{k+\ell}(W^*)) \to \mathrm{C}^\infty(L^{-n} \otimes \Lambda^\ell TM)$$

associated to the pairing $\mathbb{W} \otimes \mathbb{R} \to \mathbb{W}$; this has a rich physical and geometric interpretation. We have already seen in the previous section that for $\ell = 0$, this is the natural duality between H_k and H^k, while the Leibniz rule for $\ell = 1$ shows that $\mathcal{D}_k$ and $\mathcal{D}^k$ are adjoint. For $\ell = 2$, we have a bivector density, integrable over (cooriented) codimension two submanifolds and the Leibniz rule therefore shows that in the flat case, $\mathcal{D}_k$ and $\mathcal{D}^{k+1}$ are "adjoint in codimension one". In particular, if $k = 0$, $\ell = 2$, this is an example of dynamic helicity lowering, using a twistor to construct a bivector density from a dynamic field. In the flat case, the Leibniz rule shows that this will be divergence-free wherever the dynamic field is source-free. Integrating over a cooriented compact codimension two submanifold then gives a conserved quantity. Hence on a simply connected region, the dynamic field and codimension two submanifold define a real valued linear map on $\mathbb{W}$, i.e., an element of $\mathbb{W}^*$. This is one motivation for viewing twistors as "charges".

Curvature corrections. If the geometry is not flat, then some of the above statements only hold up to curvature terms. However, the curvature corrections appearing in 5.6, 5.7 are sometimes trivial even for non-flat geometries. For instance, when the bilinear pairings and operators are zero or first order, then there may not be enough derivatives for curvature to contribute. Also, some parts of the curvature might not act on certain modules, so that partial flatness assumptions suffice to kill the curvature terms. Finally, there is the simple observation that if $\mathcal{D}_k\alpha = 0$ then $[K_M] \sqcup \alpha = 0$, which can be quite a strong condition if the cup product has low order.

9. Explicit examples in conformal geometry

The BGG calculus permits one to carry out many calculations without worrying too much what the linear and bilinear differential operators are. Nevertheless, in applications, it is sometimes desirable to determine the operators explicitly, in terms, for instance, of a chosen Weyl connection, although this will only be feasible if the operators and pairings have low order. The Neumann series definition for Q, together with the results of Kostant [32], provides one method to carry out these calculations. However, for many examples, particularly in conformal geometry, one can proceed more directly, by guessing what the operators and pairings are, using the considerable collective experience in the literature [2, 3, 9, 17, 19, 21, 25, 26, 40]. The work of [20] and this paper contributes to this process by asserting the existence of the BGG operators, pairings, and Leibniz rules, so that one knows what to look for.

Twistor invariants. Let ϕ be a twistor spinor, i.e., a solution of the twistor equation for the spin representation of $\mathfrak{g} = \mathfrak{so}(V)$. Then ϕ is a spinor field with conformal weight $1/2$ satisfying the equation $D_X\phi = \frac{1}{n}c(X)\not{D}\phi$ for vector fields X, where D is an arbitrary Weyl connection (the equation is independent of this choice), n is the dimension of the conformal manifold, $\not{D} = c \circ D$ is the Dirac operator and $c(X)$ is Clifford multiplication (with $c(X)^2 = \langle X, X\rangle id$). The twistor operator $\mathcal{D}_0$ in this case is given by the difference $D\phi - \frac{1}{n}c(.)\not{D}\phi$; the Lie algebra homology bundle here is the Cartan product of the cotangent bundle and the spinor bundle, which is the kernel of Clifford multiplication by 1-forms.

In four or more dimensions $[K_M]$ is given by the Weyl curvature $W^{\mathsf{c}} \in \mathrm{C}^\infty(\Lambda^2 T^*M \odot \mathfrak{so}(TM))$. If ϕ is a twistor spinor then $0 = \mathcal{D}_1\mathcal{D}_0\phi = W^{\mathsf{c}} \sqcup \phi$. This pairing is zero order: one can check directly that twistor spinors satisfy $W^{\mathsf{c}}_{X,Y} \cdot \phi = 0$ for any vector fields X, Y.

The zero order cup products of two solutions ϕ, ψ take values in bundles of conformal weight one. These include $\omega(\phi,\psi) = \phi \odot \psi$, which is a (weight 1) middle-dimensional form, $X(\phi,\psi) = \langle c(.)\phi, \psi\rangle$, which is a vector field, and $\mu(\phi,\psi) = \langle \phi,\psi\rangle$, which is a weight 1 scalar. With our convention for Clifford multiplication these are all symmetric in ϕ and ψ. One readily verifies that ω satisfies a first order twistor equation, that X is a conformal vector field, and that μ has vanishing conformal trace-free Hessian (i.e., $sym_0(D^2\mu + r^D\mu) = 0$, where r^D is the normalized Ricci tensor of D). Only the last of these has enough derivatives for Weyl curvature to enter, but it does not act on twistor spinors. One can check explicitly:

$$\begin{aligned}
D^2_{X,Y}\langle\phi,\psi\rangle &= \tfrac{1}{n}\big(\langle c(Y)D_X \not{D}\phi,\psi\rangle + \langle \phi, c(Y)D_X\not{D}\psi\rangle\big)\\
&\quad + \tfrac{1}{n^2}\big(\langle c(Y)\not{D}\phi, c(X)\not{D}\psi\rangle + \langle c(X)\not{D}\phi, c(Y)\not{D}\psi\rangle\big)\\
&= -\tfrac{1}{2}\big(\langle c(Y)c(r^D(X))\phi,\psi\rangle + \langle\phi, c(Y)c(r^D(X))\psi\rangle\big) + \tfrac{2}{n^2}\langle X,Y\rangle\langle \not{D}\phi,\not{D}\psi\rangle\\
&= -r^D(X,Y)\langle\phi,\psi\rangle + \tfrac{2}{n^2}\langle X,Y\rangle\langle\not{D}\phi,\not{D}\psi\rangle.
\end{aligned}$$

As an application, wherever $\langle\phi,\psi\rangle$ is nonzero, it defines a compatible Einstein metric [5].

There is also a first order cup product taking values in the trivial representation, given by $C(\phi,\psi) = \langle\not{D}\phi,\psi\rangle - \langle\phi,\not{D}\psi\rangle$. One can again check directly that $dC = 0$, as a consequence of the twistor equation. Note that, with our convention for Clifford multiplication, $div^D X(\phi,\psi) = \langle\not{D}\phi,\psi\rangle + \langle\phi,\not{D}\psi\rangle$, so the Dirac operator is skew adjoint. This convention also makes $C(\phi,\psi)$ skew in ϕ and ψ. However, complexifying if necessary, we may assume the spinor bundle has an orthogonal complex structure J, and define $C(\phi) = C(\phi, J\phi)$. This is the quadratic invariant of Friedrich and Lichnerowicz [5, 34].

Friedrich also found a quartic invariant (see [5]). Many similar invariants can be obtained by iterating the cup product, i.e., for suitable pairings, $\phi \sqcup (\phi \sqcup (\phi \sqcup \phi))$ will be a nontrivial scalar. The details are quite complicated, but one obtains a hierarchy of quartic invariants by considering the cup products factoring through k-forms, for $k = 0, 1, 2, \ldots$. The first of these is Friedrich's quartic invariant.

Helicity. Twistor spinors have been systematically exploited to study massless field equations (see [38]). The focus has been mainly on first order field equations, where zero and first order pairings with twistor spinors are used to raise or lower the helicity of massless fields. Most of these pairings are cup products: apart from the helicity 0 and helicity $\pm 1/2$ equations (given by the conformal Laplacian and Dirac operator respectively), the massless field equation is the dynamic equation in a BGG sequence, and the helicity is $\pm(w+1)$ where w is the conformal weight of the twistor bundle $H_0(W)$. Helicity ± 1 corresponds to the deRham complex and electromagnetic fields.

Geometrical field theories. We focus on helicity ± 2: this is the expected helicity of linear theories of gravity, and because of the close links between gravity and geometry, these differential equations are of particular interest in conformal differential geometry. There are (at least) three such theories in four (or more) dimensions: the massless field equations studied by Penrose, the linearization of Bach's theory of gravity, and a conformal version of a field theory due to Fierz. The corresponding representations of $\mathfrak{g} = \mathfrak{so}(V)$ are $\Lambda^3 V$, $\mathfrak{g} = \mathfrak{so}(V) = \Lambda^2 V$ and V. We have given the Lie algebra homologies in 3.5 and the BGG sequences begin:

$$\begin{aligned}
&(9.1) \quad C^\infty(L^1\,\mathfrak{so}(TM)) &&\to C^\infty(L^1 T^*M \odot \mathfrak{so}(TM)) &&\to C^\infty(L^1\Lambda^2 T^*M \odot \mathfrak{so}(TM))\\
&(9.2) \quad C^\infty(TM) &&\to C^\infty(\mathrm{Sym}_0\, TM) &&\to C^\infty(\Lambda^2 T^*M \odot \mathfrak{so}(TM))\\
&(9.3) \quad C^\infty(L^1) &&\to C^\infty(L^{-1}\,\mathrm{Sym}_0\, TM) &&\to C^\infty(L^{-1} T^*M \odot \mathfrak{so}(TM))
\end{aligned}$$

Here $\mathrm{Sym}_0\, TM = T^*M \odot TM$ denotes the symmetric traceless endomorphisms—note also that $\mathfrak{so}(TM) \cong L^2\Lambda^2 T^*M \cong L^{-2}\Lambda^2 TM$.

In Penrose gravity (9.1), the operators are both first order, and the cup product of twistors with Weyl curvature is zero order, giving an integrability condition for solving the twistor equation. In four dimensions, the sequence decomposes into selfdual and antiselfdual parts, each part being a complex iff the Weyl curvature is antiselfdual or selfdual respectively. This sequence has been used to give a simple proof of the classification of compact selfdual Einstein metrics of positive scalar curvature [8]. It also yields a selfduality result for Einstein-Weyl structures on selfdual 4-manifolds [11]. In Minkowski space, the Weyl tensor of the Schwarzschild metric defines a natural dynamic field, which may be viewed as a linearization of the Schwarzschild solution [21].

Since Bach gravity (9.2) is associated to the adjoint representation, this sequence is the one arising in the study of deformations and moduli spaces. The twistor operator here is the first order conformal Killing operator, whose kernel consists of conformal vector fields. It takes values in $\mathrm{Sym}_0\, TM \cong L^2 S^2_0 T^*M$, the bundle of linearized conformal metrics. The second operator is the linearized Weyl curvature, taking values in the bundle of Weyl tensors. In four dimensions, the adjoint of the linearized Weyl operator is sometimes called the Bach operator: it can be applied to the Weyl curvature itself to give the Bach tensor of the conformal structure.

The composite of the conformal Killing operator and the linear Weyl operator is a first order cup product with the Weyl curvature, which one readily computes to be a multiple of $\mathcal{L}_X W^{\mathsf{c}}$: obviously a conformal vector field has to preserve the Weyl curvature. This cup product is associated to the Lie bracket pairing $\mathfrak{so}(V) \otimes \mathfrak{so}(V) \to \mathfrak{so}(V)$. There is also an inner product pairing, which gives, for example, a 2-form-valued first order cup product between vector fields K and Weyl tensors W:

$$(K \sqcup W)(X, Y) = (n-2)\langle W_{X,Y}, DK\rangle - \langle (\delta^D W)_{X,Y}, K\rangle.$$

The twistor equation in Fierz gravity (9.3) is second order, and is the conformal tracefree Hessian mentioned above. Its kernel consists of Einstein (pseudo)gauges, i.e., a nonvanishing solution gives a length scale for a compatible Einstein metric. The cup product with curvature is a first order pairing, sometimes called the Cotton-York operator, since it assigns a Cotton-York tensor to a (pseudo)gauge. This corresponds to the fact that Einstein metrics have vanishing Cotton-York tensor.

Helicity lowering. A typical example of helicity lowering occurs in Penrose gravity. Here the natural zero order contraction of a dynamic Weyl field W and a Penrose twistor 2-form ω gives a bivector density and there is a simple Leibniz rule:

$$\delta\langle W, \omega\rangle = \langle \delta W, \omega\rangle + \langle W, \mathit{Twist}\,\omega\rangle.$$

Even within the framework of conformal geometry, the cup and cap products considerably generalize these ideas. For example, the analogous process in Fierz gravity requires first order pairings:

$$\delta\big(F(D\mu,.,.) - \tfrac{1}{2}\mu(\delta^D F)(.,.)\big) = \mu\,\mathrm{div}^D(Sdiv\, F) - (Sdiv\, F)(D\mu,.) \\ + \langle sym_0(D^2 + r^D)\mu, F\rangle + \tfrac{1}{2}\langle W^{\mathsf{c}}, F\rangle\mu$$

where $F \in \mathrm{C}^\infty((L^{1-n}TM \odot \mathfrak{so}(TM))$, $\mu \in \mathrm{C}^\infty(L^1)$, $Sdiv\, F$ is the symmetric divergence in $L^{1-n}\,\mathrm{Sym}_0\, TM$, $\delta^D F$ is the skew divergence in $L^{-1-n}\Lambda^2 TM$, and W^{c} is the Weyl curvature. Higher order pairings rapidly become very complicated, although a few examples involving second order pairings can be computed explicitly.

Appendix: semiholonomic jets and Verma modules

In this appendix we recall the link with semiholonomic Verma modules. On any Cartan geometry of type $(\mathfrak{g}, P)$, iterating the invariant derivative defines the following.

9.1. Proposition. *The map sending a section s to $\big(s, \nabla^\eta s, (\nabla^\eta)^2 s, \ldots (\nabla^\eta)^k s\big)$ takes values in the subbundle $\mathcal{G} \times_P \hat{J}^k_0\mathbb{E}$ of $\bigoplus_{j=0}^k\big((\otimes^j \mathfrak{g}^*_M) \otimes E\big)$, where $\hat{J}^k_0\mathbb{E}$ is the set of all $(\phi_0, \phi_1, \ldots \phi_k)$ in $\bigoplus_{j=0}^k\big((\otimes^j \mathfrak{g}^*) \otimes \mathbb{E}\big)$ satisfying (for $1 \leqslant i < j \leqslant k$) the equations*

$$\phi_j(\xi_1, \ldots \xi_i, \xi_{i+1}, \ldots \xi_j) - \phi_j(\xi_1, \ldots \xi_{i+1}, \xi_i, \ldots \xi_j) = \phi_{j-1}(\xi_1, \ldots [\xi_i, \xi_{i+1}], \ldots \xi_j)$$
$$\phi_i(\xi_1, \ldots \xi_i) + \xi_i{\cdot}\big(\phi_{i-1}(\xi_1, \ldots \xi_{i-1})\big) = 0$$

for all $\xi_1, \ldots \xi_j \in \mathfrak{g}$ with $\xi_i \in \mathfrak{p}$.

Proof. The equations are those given by the vertical triviality and the Ricci identity, bearing in mind the horizontality of the curvature κ (see Proposition 1.3). □

This map is sometimes called a "semiholonomic jet operator", since it identifies the semiholonomic jet bundle $\hat{J}^k E$ with the associated bundle $\mathcal{G} \times_P \hat{J}^k_0\mathbb{E}$. In particular, $\hat{J}^k_0\mathbb{E}$ is itself the fibre of the semiholonomic jet bundle at $0 = [P] \in G/P$. This is a minor variation of the construction of a semiholonomic jet operator given in [17, 20, 23], except that we have presented $\hat{J}^k_0\mathbb{E}$ as a complicated subspace of an easily defined $\mathfrak{p}$-module, whereas in these references, $\hat{J}^k_0\mathbb{E}$ is given as a complicated $\mathfrak{p}$-module structure on an easy vector space, namely $\bigoplus_{j=0}^k\big(\otimes^j(\mathfrak{g}/\mathfrak{p})^*\big) \otimes \mathbb{E}$. The equations defining $\hat{J}^k_0\mathbb{E}$ have a natural algebraic interpretation in the dual language of semiholonomic Verma modules.

9.2. Definition. [3, 23] Let $\mathfrak{g}$ be a Lie algebra with subalgebra $\mathfrak{p}$.

(i) The *semiholonomic universal enveloping algebra* $U(\mathfrak{g}, \mathfrak{p})$ of $\mathfrak{g}$ with respect to $\mathfrak{p}$ is defined to be the quotient of the tensor algebra $\bigotimes(\mathfrak{g})$ by the ideal generated by

$$\{\xi \otimes \chi - \chi \otimes \xi - [\xi, \chi] : \xi \in \mathfrak{p}, \chi \in \mathfrak{g}\}.$$

We denote by $U^k(\mathfrak{g}, \mathfrak{p})$ the filtration given by the image of $\bigoplus_{j=0}^k(\otimes^j \mathfrak{g})$, which is compatible with the algebra structure.

Note that $U(\mathfrak{g}, \mathfrak{g})$ is the usual universal enveloping algebra $U(\mathfrak{g})$, that $U(\mathfrak{p})$ is a subalgebra of $U(\mathfrak{g}, \mathfrak{p})$, and that $U(\mathfrak{g}, \mathfrak{p})$ is a $U(\mathfrak{g})$-bimodule.

(ii) Let $\mathbb{E}^*$ be a $\mathfrak{p}$-module. Then the *semiholonomic Verma module* associated to $\mathbb{E}^*$ is the $U(\mathfrak{g})$-module given by $\hat{V}(\mathbb{E}^*) = U(\mathfrak{g}, \mathfrak{p}) \otimes_{U(\mathfrak{p})} \mathbb{E}^*$. We denote by $\hat{V}^k(\mathbb{E}^*)$ the filtration given by the image of $U^k(\mathfrak{p}) \otimes \mathbb{E}^*$.

As a filtered vector space, $\hat{V}(\mathbb{E}^*)$ is naturally isomorphic to $\big(\bigotimes(\mathfrak{g}/\mathfrak{p})\big)\otimes\mathbb{E}^*$. The induced action of $\mathfrak{g}$ on $\big(\bigotimes(\mathfrak{g}/\mathfrak{p})\big)\otimes\mathbb{E}^*$ is most easily described by choosing a complement $\mathfrak{m}$ to $\mathfrak{p}$ in $\mathfrak{g}$ so that $\mathfrak{g}/\mathfrak{p}$ is isomorphic to $\mathfrak{m}$. Then the action of $\xi\in\mathfrak{g}$ on $v_1\otimes\cdots\otimes v_k\otimes z\in\otimes^k\mathfrak{m}\otimes\mathbb{E}^*$ is obtained by tensoring on the left with ξ, then commuting the $\mathfrak{p}$ component $\xi_{\mathfrak{p}}$ past all the v_j's so that it can act on z. This introduces Lie bracket terms $[\xi_{\mathfrak{p}}, v_j]$, whose $\mathfrak{p}$-components must in turn be commuted to the right. This process is then repeated until no elements of $\mathfrak{p}$ remain in the tensor product.

If we define $\hat{J}^\infty_0\mathbb{E}$ be the inverse limit with respect to the natural maps $\hat{J}^k_0\mathbb{E}\to\hat{J}^{k-1}_0\mathbb{E}$ then the equations defining $\hat{J}^k_0\mathbb{E}$ imply that $\hat{J}^\infty_0\mathbb{E}$ is the subspace of $\big(\bigotimes\mathfrak{g}^*\big)\otimes\mathbb{E}$ such that the pairing with $\big(\bigotimes\mathfrak{g}\big)\otimes\mathbb{E}^*$ descends to $\hat{V}(\mathbb{E}^*)$. Comparing dimensions, we see that $\hat{V}^k(\mathbb{E}^*)\cong(\hat{J}^k_0\mathbb{E})^*$. This is why the dual of $\mathbb{E}$ is used in the definition of the Verma module.

The advantage of semiholonomic jets is that they are defined purely in terms of the 1-jet functor J^1, the natural transformation $J^1E\to E$, and some abstract nonsense. The construction of the Π-operators was entirely first order, and so can be carried out formally on the infinite semiholonomic jet bundle, rather than on smooth sections. This is perhaps the easiest way to see that the symbols of the operators are the same for all geometries of a given type, since the semiholonomic Verma module homomorphisms are the same. One also sees that the operators of the BGG sequences are all strongly invariant in the sense that they are defined by semiholonomic Verma module homomorphisms, and so can be twisted with an arbitrary $\mathfrak{g}$-module [20].

In the flat case, the first equation in Proposition 9.1 holds for $\xi_i\in\mathfrak{g}$ (not just $\mathfrak{p}$) and so one can work with the usual holonomic jets and Verma modules. Working dually with $V(\mathbb{E}^*)$, instead of $J^\infty E$ or $\mathrm{C}^\infty(E)$ as we have done here, leads to a cup coproduct on BGG resolutions of parabolic Verma modules, as described in [21]. The constructions of section 6 now equip the family of resolutions with an A_∞-coalgebra structure.

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