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Level-one Highest Weight Representation of $U_q[sl(\widehat{N}|1)]$ and Bosonization of the Multi-component Super $t-J$ Model

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Abstract

We study the level-one irreducible highest weight representations of the quantum affine superalgebra $U_q[sl(\widehat{N}|1)]$, and calculate their characters and supercharacters. We obtain bosonized q-vertex operators acting on the irreducible $U_q[sl(\widehat{N}|1)]$-modules and derive the exchange relations satisfied by the vertex operators. We give the bosonization of the multi-component super $t-J$ model by using the bosonized vertex operators.

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I Introduction

The purpose of this paper is two-fold. One is to study irreducible highest weight representations and q-vertex operators [1] of the quantum affine superalgebra $U_q[sl(\widehat{N}|1)]$, $N > 2$. Another one is to apply these results to bosonize the multi-component super $t-J$ model on an infinite lattice.

We shall adapt the bosonization technique initiated in [2, 3], which turns out to be very powerful in constructing highest weight representations and q-vertex operators. Recently, free bosonic realizations of the level-one representations and "elementary" q-vertex operators have been obtained for $U_q[sl(\widehat{M}|N)]$, $M \neq N$ [4] and $U_q[gl(\widehat{N}|N)]$ [5]. However, these free boson representations are not irreducible in general. Moreover, the elementary q-vertex operators obtained in [4, 5] were determined solely from their commutation relations with the bosonized Drinfeld generators [6] of the relevant algebras, and thus one can ask on which representations these bosonized q-vertex operators act. To construct irreducible highest weight representations and q-vertex operators acting on them, we need to study in details the structure of the bosonic Fock space generated by the free boson fields. This has been done for $U_q[sl(\widehat{2}|1)]$ [4, 7] and $U_q[gl(\widehat{N}|N)]$, $N \leq 2$ [8]. In this paper we treat the $U_q[sl(\widehat{N}|1)]$ ($N > 2$) case.

Irreducible highest weight representations and bosonized q-vertex operators acting on them play an essential role in the algebraic analysis method of lattice integrable models, which was invented by the Kyoto group and collaborators [9, 10]. In this approach, the following assumption is the vital key :

$$\text{"}\textbf{the physical space of states of the model}\text{"} = \bigoplus_{\alpha,\alpha'} V(\lambda_\alpha) \otimes V(\lambda_{\alpha'})^{*S} \qquad \text{(I.1)}$$

where $V(\lambda_\alpha)$ is the level-one irreducible highest weight module of the underlying quantum affine algebras and $V(\lambda_\alpha)^{*S}$ is the dual module of $V(\lambda_\alpha)$. By this method, various integrable models have been analysed such as the higher spin XXZ chains [11, 12, 13], the higher rank cases [14, 15], the twisted $A_2^{(2)}$ case [16], and the face type statistical models [17, 18].

Spin chain models with quantum superalgebra symmetries have been the focus of recent studies in the context of strongly correlated fermion systems [19, 20, 21, 22, 23]. It is natural to generalize the algebraic analysis method to treat super spin chains on an infinite lattice. In [7], the q-deformed supersymmetric $t-J$ model which has $U_q'[sl(\widehat{2}|1)]$ as its non-abelian symmetry has been analysed. However, the super case is fundamentally different from the non-super case. Unlike the latter, $U_q[sl(\widehat{2}|1)]$ has infinite number of level-one irreducible highest weight representations and the bosonized q-vertex operators act in all of them. This leads to [7] the assumption that for the q-deformed supersymmetric t-J model α, α' in (I.1) take infinite number of integer values.

In this paper we extend the work [7] to treat the multi-component $t-J$ model with $U_q'[sl(\widehat{N}|1)]$ ($N > 2$) symmetry. As we shall see, the level-one irreducible highest weight representations of $U_q[sl(\widehat{N}|1)]$ ($N > 2$) have similar structures as the $N = 2$ case. So we shall make the assumption that the physical space of states of the multi-component $t-J$ model on an infinite lattice is of the form (I.1) with $\alpha,\ \alpha'$ being any integers.

This paper is organized as follows. After presenting some neccessary preliminaries, we in section 3 construct the level-one irreducible highest weight representations of $U_q[sl(\widehat{N}|1)]$ and calculate their (super)characters by means of the BRST resolution. In section 4, we compute the exchange relations of the q-vertex operators and show that they form the graded Faddeev-Zamolodchikov algebra. In section 5, we consider the application of these results to the multi-component super $t-J$ model on an infinite lattice. Generalizing the Kyoto group's work [9], we give the bosonization of this model using the bosonized vertex operators of $U_q[sl(\widehat{N}|1)]$. Finally, we compute the one-point correlation functions of the local operators and give an integral expression of the correlation functions.

II Preliminaries

II.1 Quantum affine superalgebra $U_q[sl(\widehat{N}|1)]$

Let us introduce orthonormal basis $\{\epsilon'_i|i=1,2,\cdots,N+1\}$ with the bilinear form $(\epsilon'_i,\epsilon'_j)=\nu_i\delta_{ij}$, where $\nu_i=1$ for $i\neq N+1$ and $\nu_{N+1}=-1$. The classical fundamental weights are defined by $\bar{\Lambda}_i=\sum_{j=1}^{i}\epsilon_j$ $(i=1,2,\cdots,N)$, with $\epsilon_i=\epsilon'_i-\frac{\nu_i}{N-1}\sum_{j=1}^{N+1}\epsilon'_j$. Introduce the affine weight Λ_0 and the null root δ having $(\Lambda_0,\epsilon'_i)=(\delta,\epsilon'_i)=0$ for $i=1,2,\cdots,N+1$ and $(\Lambda_0,\Lambda_0)=(\delta,\delta)=0$, $(\Lambda_0,\delta)=1$. The affine simple roots and fundamental weights are given by

$$\alpha_i=\nu_i\epsilon'_i-\nu_{i+1}\epsilon'_{i+1},\quad i=1,2,\cdots,N,\qquad \alpha_0=\delta-\sum_{i=1}^{N}\alpha_i\ ,$$
$$\Lambda_0=\Lambda_0,\qquad \Lambda_i=\Lambda_0+\bar{\Lambda}_i,\quad i=1,2,\cdots,N. \tag{II.1}$$

The Cartan matrix of the affine superalgebra $sl(\widehat{N}|1)$ reads as

$$(a_{ij})=\begin{pmatrix} 0 & -1 & & & & 1\\ -1 & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1\\ 1 & & & & -1 & 0 \end{pmatrix}\qquad (i,j=0,1,2,\cdots,N). \tag{II.2}$$

The Quantum affine superalgebra $U_q[sl(\widehat{N}|1)]$ is a q-analogue of the universal enveloping algebra of $sl(\widehat{N}|1)$ generated by the Chevalley generators $\{e_i,f_i,q^{h_i},d|i=0,1,2,\cdots,N\}$, where d is the usual derivation operator. The Z_2-grading of the generators are $[e_0]=[f_0]=[e_N]=[f_N]=1$ and zero otherwise. The defining relations are

$$[h_i,h_j]=0,\quad h_id=dh_i,\quad [d,e_i]=\delta_{i,0}e_i,\quad [d,f_i]=-\delta_{i,0}f_i,$$
$$q^{h_i}e_jq^{-h_i}=q^{a_{ij}}e_j,\quad q^{h_i}f_jq^{-h_i}=q^{-a_{ij}}f_j,\quad [e_i,f_j]=\delta_{ij}\frac{q^{h_i}-q^{-h_i}}{q-q^{-1}},$$
$$[e_i,e_j]=[f_i,f_j]=0,\quad \text{for}\quad a_{ij}=0,$$
$$[e_j,[e_j,e_i]_{q^{-1}}]_q=0,\quad [f_j,[f_j,f_i]_{q^{-1}}]_q=0,\ \text{for}\ \ |a_{ij}|=1,\quad j\neq 0,N.$$

Here and throughout, $[a,b]_x\equiv ab-(-1)^{[a][b]}xba$ and $[a,b]\equiv[a,b]_1$. We do not write down the extra q-Serre relations which can be obtained by using Yamane's Dynkin diagram procedure [24].

$U_q[sl(\widehat{N}|1)]$ is a $\mathbf{Z}_2$-graded quasi-triangular Hopf algebra endowed with the following coproduct Δ, counit ϵ and antipode S:

$$\Delta(h_i)\ =\ h_i\otimes 1+1\otimes h_i,\qquad \Delta(d)=d\otimes 1+1\otimes d,$$

$$\begin{aligned} \Delta(e_i) &= e_i\otimes 1+q^{h_i}\otimes e_i, \qquad \Delta(f_i)=f_i\otimes q^{-h_i}+1\otimes f_i,\\ \epsilon(e_i) &= \epsilon(f_i)=\epsilon(h)=0,\\ S(e_i) &= -q^{-h_i}e_i, \qquad S(f_i)=-f_iq^{h_i}, \qquad S(h)=-h, \end{aligned} \tag{II.3}$$

where $i=0,1,\cdots,N$. Notice that the antipode S is a $\mathbf{Z}_2$-graded algebra anti-homomorphism. Namely, for any homogeneous elements $a,b\in U_q[sl(\widehat{N}|1)]$ $S(ab)=(-1)^{[a][b]}S(b)S(a)$, which extends to inhomogeneous elements through linearity. Moreover,

$$S^2(a)=q^{-2\rho}\,a\,q^{2\rho}, \qquad \forall a\in U_q[sl(\widehat{N}|1)], \tag{II.4}$$

where ρ is an element in the Cartan subalgebra such that $(\rho,\alpha_i)=(\alpha_i,\alpha_i)/2$ for any simple root $\alpha_i,\ i=0,1,2,\cdots,N$. Explicitly,

$$\rho=(N-1)d+\bar\rho=(N-1)d+\frac{1}{2}\sum_{k=1}^{N}(N-2k)\epsilon'_k-\frac{1}{2}N\epsilon'_{N+1}, \tag{II.5}$$

which $\bar\rho$ is the half-sum of positive roots of $sl(N|1)$. The multiplication rule on the tensor products is $\mathbf{Z}_2$-graded: $(a\otimes b)(a'\otimes b')=(-1)^{[b][a']}(aa'\otimes bb')$ for any homogeneous elements $a,b,a',b'\in U_q[sl(\widehat{N}|1)]$.

$U_q[sl(\widehat{N}|1)]$ can also be realized in terms of the Drinfeld generators [6] $\{X^{\pm,i}_m,\ H^i_n,\ q^{\pm H^i_0},\ c,\ d|m\in\mathbf{Z},\ n\in\mathbf{Z}-\{0\},\ i=1,2,\cdots,N\}$. The $\mathbf{Z}_2$-grading of the Drinfeld generators is given by $[X^{\pm,N}_m]=1$ for $m\in\mathbf{Z}$ and zero otherwise. The relations satisfied by the Drinfeld generators read [24, 25]

$$\begin{aligned} &[c,a]=[d,H^i_0]=[H^i_0,H^j_n]=0, \qquad [d,H^i_n]=nH^i_n, \qquad \forall a\in U_q[sl(\widehat{N}|1)]\\ &[d,X^{\pm,i}_n]=nX^{\pm,i}_n, \qquad q^{H^j_0}X^{\pm,i}_nq^{-H^j_0}=q^{\pm a_{ij}}X^{\pm,i}_n,\\ &[H^i_n,H^j_m]=\delta_{n+m,0}\frac{[a_{ij}n]_q[nc]_q}{n}, \qquad [H^i_n,X^{\pm,j}_m]=\pm\frac{[a_{ij}n]_q}{n}X^{\pm,j}_{n+m}q^{\pm|n|c/2},\\ &[X^{+,i}_n,X^{-,j}_m]=\frac{\delta_{ij}}{q-q^{-1}}\left(q^{\frac{c}{2}(n-m)}\psi^{+,i}_{n+m}-q^{-\frac{c}{2}(n-m)}\psi^{-,i}_{n+m}\right),\\ &[X^{\pm,i}_n,X^{\pm,j}_m]=0, \qquad \text{for } a_{ij}=0,\\ &[X^{\pm,i}_{n+1},X^{\pm,j}_m]_{q^{\pm a_{ij}}}-[X^{\pm,j}_{m+1},X^{\pm,i}_n]_{q^{\pm a_{ij}}}=0, \quad \text{for } \ a_{ij}\neq 0,\\ &Sym_{l,m}[X^{\pm,i}_l,[X^{\pm,i}_m,X^{\pm,j}_n]_{q^{-1}}]_q=0, \quad \text{for } \ a_{ij}=0, i\neq N, \end{aligned} \tag{II.6}$$

where $\sum_{n\in\mathbf{Z}}\psi^{\pm,j}_nz^{-n}=q^{\pm H^j_0}\exp\left(\pm(q-q^{-1})\sum_{n>0}H^j_{\pm n}z^{\mp n}\right)$, and the symbol $Sym_{k,l}$ means symmetrization with respect to k and l. We used the standard notation $[x]_q=(q^x-q^{-x})/(q-q^{-1})$. The Chevalley generators are related to the Drinfeld generators by the formulas:

$$\begin{aligned} &h_i=H^i_0, \quad e_i=X^{+,i}_0, \quad f_i=X^{-,i}_0, \quad i=1,2,\cdots,N, \quad h_0=c-\sum_{k=1}^{N}H^k_0,\\ &e_0=-[X^{-,N}_0,[X^{-,N-1}_0,\cdots,[X^{-,2}_0,X^{-,1}_1]_{q^{-1}}]_{q^{-1}}\cdots]_{q^{-1}}\,q^{-\sum_{k=1}^{N}H^k_0},\\ &f_0=q^{\sum_{k=1}^{N}H^k_0}[[\cdots[[X^{+,1}_{-1},X^{+,2}_0]_q,\cdots,X^{+,N-1}_0]_q,X^{+,N}_0]_q. \end{aligned} \tag{II.7}$$

II.2 Free Bosonic realization of the quantum affine superalgebra $U_q[sl(\widehat{N}|1)]$ at level one

Introduce bosonic oscillators $\{a^i_n,\ b_n,\ c_n,\ Q_{a^i}, Q_b, Q_c\ |n \in \mathbf{Z}, i = 1,2,\cdots,N,\}$ which satisfy the commutation relations

$$[a^i_n, a^j_m] = \delta_{n+m,0}\delta_{ij}\frac{[n]_q[n]_q}{n}, \qquad [a^i_0, Q_{a^j}] = \delta_{ij},$$
$$[b_n, b_m] = -\delta_{n+m,0}\frac{[n]_q^2}{n}, \qquad [b_0, Q_b] = -1,$$
$$[c_n, c_m] = \delta_{n+m,0}\frac{[n]_q^2}{n}, \qquad [c_0, Q_c] = 1. \tag{II.8}$$

The remaining commutation relations are zero. Define $\{h^i_m | i = 1,2,\cdots,N,\ m \in \mathbf{Z}\}$:

$$h^i_m = a^i_m q^{-|m|/2} - a^{i+1} q^{|m|/2}, \quad Q_{h_i} = Q_{a^i} - Q_{a^{i+1}}, \quad i = 1,2,\cdots,N-1,$$
$$h^N_m = a^N_m q^{-|m|/2} + b_m q^{-|m|/2}, \qquad Q_{h_N} = Q_{a^N} + Q_b. \tag{II.9}$$

Let us introduce the notation $h^j(z;\kappa) = Q_{h_j} + h^j_0 \ln z - \sum_{n\neq 0}\frac{h^j_n}{[n]_q} q^{\kappa|n|} z^{-n}$. The bosonic fields $c(z;\beta)$, $b(z;\beta)$ and $h^*_j(z;\beta)$ are defined in the same way. Define the Drinfeld currents, $X^{\pm,i}(z) = \sum_{n\in\mathbf{Z}} X^{\pm,i}_n z^{-n-1}$, $i = 1,2,\cdots,N$, and the q-differential operator $\partial_z f(z) = \frac{f(qz)-f(q^{-1}z)}{(q-q^{-1})z}$. Then, the Drinfeld generators of $U_q[sl(\widehat{N}|1)]$ at level one can be realized by the free boson fields as [4]

$$c = 1, \quad H^i_m = h^i_m, \quad X^{+,N}(z) =: e^{h^N(z;-\frac{1}{2})} e^{c(z;0)} : e^{-\sqrt{-1}\pi\sum_{i=1}^{N-1} a^i_0},$$
$$X^{-,N}(z) =: e^{-h^N(z;\frac{1}{2})} \partial_z\{e^{-c(z;0)}\} : e^{\sqrt{-1}\pi\sum_{i=1}^{N-1} a^i_0},$$
$$X^{\pm,i}(z) = \pm : e^{\pm h^i(z;\mp\frac{1}{2})} : e^{\pm\sqrt{-1}\pi a^i_0}, \qquad i = 1,2,\cdots,N-1. \tag{II.10}$$

II.3 Bosonization of level-one vertex operators

In order to construct the vertex operators of $U_q[sl(\widehat{N}|1)]$, we firstly consider the level-zero representations (i.e. the evaluation representations) of $U_q[sl(\widehat{N}|1)]$.

Let $E_{i,j}$ be the $(N+1)\times(N+1)$ matrix whose (i,j)-element is unity and zero elsewhere. Let $\{v_1, v_2, \cdots, v_{N+1}\}$ be the basis vectors of the (N+1)-dimensional graded vector space V. The $\mathbf{Z}_2$-grading of these basis vectors is chosen to be $[v_i] = (\nu_i + 1)/2$. The $(N+1)$-dimensional level-zero representation V_z of $U_q[sl(\widehat{N}|1)]$ is given by

$$e_i = E_{i,i+1}, \qquad f_i = \nu_i E_{i+1,i}, \qquad t_i = q^{\nu_i E_{i,i} - \nu_{i+1} E_{i+1,i+1}},$$
$$e_0 = -zE_{N+1,1}, \qquad f_0 = z^{-1}E_{1,N+1}, \qquad t_0 = q^{-E_{1,1} - E_{N+1,N+1}}, \tag{II.11}$$

where $i = 1,\cdots,N$. Let V^{*S}_z be the left dual module of V_z, defined by

$$\pi_{V^{*S}_z}(a) = \pi_{V_z}(S(a))^{st}, \quad \forall a \in U_q[sl(\widehat{N}|1)], \tag{II.12}$$

where st denotes the supertansposition.

Now, we study the level-one vertex operators [1] of $U_q[sl(\widehat{N}|1)]$. Let $V(\lambda)$ be the highest weight $U_q[sl(\widehat{N}|1)]$-module with the highest weight λ and the highest weight vector $|\lambda>$. Consider the following intertwiners of $U_q[sl(\widehat{N}|1)]$-modules [10]:

$$\Phi^{\mu V}_\lambda(z): \ V(\lambda) \longrightarrow V(\mu)\otimes V_z, \qquad \Phi^{\mu V^*}_\lambda(z): \ V(\lambda) \longrightarrow V(\mu)\otimes V^{*S}_z,$$
$$\Psi^{V\mu}_\lambda(z): \ V(\lambda) \longrightarrow V_z\otimes V(\mu), \qquad \Psi^{V^*\mu}_\lambda(z): \ V(\lambda) \longrightarrow V^{*S}_z\otimes V(\mu). \tag{II.13}$$

They are intertwiners in the sense that for any $x \in U_q[sl(\widehat{N}|1)]$

$$\Xi(z) \cdot x = \Delta(x) \cdot \Xi(z), \quad \Xi(z) = \Phi_\lambda^{\mu V}(z),\ \Phi_\lambda^{\mu V^*}(z),\ \Psi_\lambda^{V\mu}(z),\ \Psi_\lambda^{V^*\mu}(z). \tag{II.14}$$

We expand the vertex operators as [10]

$$\Phi_\lambda^{\mu V}(z) = \sum_{j=1}^N \Phi_{\lambda,j}^{\mu V}(z) \otimes v_j, \quad \Phi_\lambda^{\mu V^*}(z) = \sum_{j=1}^N \Phi_{\lambda,j}^{\mu V^*}(z) \otimes v_j^*,$$
$$\Psi_\lambda^{V\mu}(z) = \sum_{j=1}^N v_j \otimes \Psi_{\lambda,j}^{V\mu}(z), \quad \Psi_\lambda^{V^*\mu}(z) = \sum_{j=1}^N v_j^* \otimes \Psi_{\lambda,j}^{V^*\mu}(z). \tag{II.15}$$

The intertwiners are even, which implies $[\Phi_{\lambda,j}^{\mu V}(z)] = [\Phi_{\lambda,j}^{\mu V^*}(z)] = [\Psi_{\lambda,j}^{V\mu}(z)] = [\Psi_{\lambda,j}^{V^*\mu}(z)] = [v_j] = \frac{\nu_j+1}{2}$. According to [10], $\Phi_\lambda^{\mu V}(z)$ $\left(\Phi_\lambda^{\mu V^*}(z)\right)$ is called type I (dual) vertex operator and $\Psi_\lambda^{V\mu}(z)$ $\left(\Psi_\lambda^{V^*\mu}(z)\right)$ type II (dual) vertex operator.

Introduce the bosonic operators $\phi_j(z)$, $\phi_j^*(z)$, $\psi_j(z)$ and $\psi_j^*(z)$ [4]:

$$\begin{aligned}
&\phi_{N+1}(z) =: e^{-h_N^*(q^N z;\frac{1}{2})} e^{c(q^N z;0)} (q^N z)^{\frac{N-2}{2(N-1)}} : e^{\sqrt{-1}\pi \sum_{i=1}^N \frac{1-i}{N-1} a_0^i},\\
&\nu_l \phi_l(z)(-1)^{[f_l]([v_l]+[v_{l+1}])} = [\phi_{l+1}(z)\ ,\ f_l]_{q^{\nu_{l+1}}},\\
&\phi_1^*(z) =: e^{h_1^*(qz;\frac{1}{2})} (q^N z)^{\frac{N-2}{2(N-1)}} : e^{-\sqrt{-1}\pi \sum_{i=1}^N \frac{1-i}{N-1} a_0^i},\\
&-\nu_l q^{\nu_l} \phi_{l+1}^*(z)(-1)^{[f_l]([v_l]+[v_{l+1}])} = [\phi_l^*(z)\ , f_l]_{q^{\nu_l}},\\
&\psi_1(z) =: e^{-h_1^*(qz;-\frac{1}{2})} (q^N z)^{\frac{N-2}{2(N-1)}} : e^{\sqrt{-1}\pi \sum_{i=1}^N \frac{1-i}{N-1} a_0^i},\\
&\psi_{l+1}(z) = [\psi_l(z)\ , e_l]_{q^{\nu_l}},\\
&\psi_{N+1}^*(z) =: e^{h_N^*(q^{2-N} z;-\frac{1}{2})} \partial_z \{e^{-c(q^{2-N} z;0)}\} (q^N z)^{\frac{N-2}{2(N-1)}} : e^{-\sqrt{-1}\pi \sum_{i=1}^N \frac{1-i}{N-1} a_0^i},\\
&-\nu_l \nu_{l+1} q^{-\nu_l} \psi_l^*(z) = [\psi_{l+1}^*(z)\ ,\ e_l]_{q^{\nu_{l+1}}},
\end{aligned} \tag{II.16}$$

where

$$h_n^{*i} = \sum_{j=1}^N \frac{[\alpha_{ij} m]_q [\beta_{ij} m]_q}{[(N-1)m]_q [m]_q} h_n^j\ ,\quad Q_{h^i}^* = \sum_{j=1}^N \frac{\alpha_{ij}\beta_{ij}}{N-1} Q_{h^j}\ ,\quad h_0^{*i} = \sum_{j=1}^N \frac{\alpha_{ij}\beta_{ij}}{N-1} h^j,$$

with $\alpha_{ij} = min(i,j)$, and $\beta_{ij} = N-1-max(i,j)$. Define the even operators $\phi(z),\ \phi^*(z),\ \psi(z)$ and $\psi^*(z)$ by $\phi(z) = \sum_{j=1}^{N+1} \phi_j(z) \otimes v_j$, $\phi^*(z) = \sum_{j=1}^{N+1} \phi_j^*(z) \otimes v_j^*$, $\psi(z) = \sum_{j=1}^{N+1} v_j \otimes \psi_j(z)$ and $\psi^*(z) = \sum_{j=1}^{N+1} v_j^* \otimes \psi_j^*(z)$. Then the vertex operators $\Phi_\lambda^{\mu V}(z),\ \Phi_\lambda^{\mu V^*}(z),\ \Psi_\lambda^{V\mu}(z)$ and $\Psi_\lambda^{V^*\mu}(z)$, if they exist, are bosonized by $\phi(z),\ \phi^*(z),\ \psi(z)$ and $\psi^*(z)$, respectively [4]. We remark that our vertex operators differ from those of Kimura et al [4] by a scalar factor $(q^N z)^{\frac{N-2}{2(N-1)}}$, which is needed in order for the vertex operators also satisfy (II.14) for the element $x = d$. $\phi(z),\ \phi^*(z),\ \psi(z)$ and $\psi^*(z)$ are referred to as the "elementary q-vertex operators" of $U_q[sl(\widehat{N}|1)]$.

III Highest weight $U_q[sl(\widehat{N}|1)]$-modules

We begin by defining the Fock module. Denote by $F_{\lambda_1,\lambda_2,\cdots,\lambda_{N+1};\lambda_{N+2}}$ the bosonic Fock space generated by $a_{-m}^i, b_{-m}, c_{-m} (m > 0)$ over the vector $|\lambda_1, \lambda_2, \cdots, \lambda_{N+1}; \lambda_{N+2}>$:

$$F_{\lambda_1,\lambda_2,\cdots,\lambda_{N+1};\lambda_{N+2}} = \mathbf{C}[a_{-1}^i, a_{-2}^i, \cdots; b_{-1}, b_{-2}, \cdots; c_{-1}, c_{-2}, \cdots] |\lambda_1, \lambda_2, \cdots, \lambda_{N+1}; \lambda_{N+2}>,$$

where

$$|\lambda_1, \lambda_2, \cdots, \lambda_{N+1}; \lambda_{N+2} >= e^{\sum_{i=1}^N \lambda_i Q_{a^i} + \lambda_{N+1} Q_b + \lambda_{N+2} Q_c} |0> .$$

The vacuum vector $|0>$ is defined by $a^i_m|0>= b_m|0>= c_m|0>= 0$ for $i = 1, 2, \cdots, N$, and $m \geq 0$. Obviously,

$$a^i_m|\lambda_1, \lambda_2, \cdots, \lambda_{N+1}; \lambda_{N+2} >= 0, \quad \text{for } i = 1, 2, \cdots, N \text{ and } \ m > 0,$$
$$b_m|\lambda_1, \lambda_2, \cdots, \lambda_{N+1}; \lambda_{N+2} >= c_m|\lambda_1, \lambda_2, \cdots, \lambda_{N+1}; \lambda_{N+2} >= 0, \quad \text{for } m > 0.$$

To obtain the highest weight vectors of $U_q[sl(\widehat{N}|1)]$, we impose the conditions:

$$e_i|\lambda_1, \cdots, \lambda_{N+1}; \lambda_{N+2} >= 0, \ i = 0, 1, 2, \cdots, N,$$
$$h_i|\lambda_1, \cdots, \lambda_{N+1}; \lambda_{N+2} >= \lambda^i|\lambda_1, \cdots, \lambda_{N+1}; \lambda_{N+2} >, \quad i = 0, 1, 2, \cdots, N. \quad \text{(III.1)}$$

Solving these equations, we obtain two classess of solutions:

1. $(\lambda_1, \cdots, \lambda_i, \lambda_{i+1}, \cdots, \lambda_{N+1}; \lambda_{N+2}) = (\beta{+}1, \cdots, \underbrace{\beta+1,\ \beta}_{i,\, i+1}, \cdots, \beta; 0)$, where $i = 1, \cdots, N$, and β is arbitrary. It follows that $(\lambda^0, \lambda^1, \cdots, \lambda^i, \lambda^{i+1} \cdots, \lambda^N)$=$(0, 0, \cdots, \underbrace{0,\ 1,\ 0}_{i-1,\, i,\, i+1}, \cdots, 0)$ and we have the identification $|\Lambda_i >=|\beta+1, \cdots, \underbrace{\beta+1,\ \beta}_{i,\, i+1}, \cdots, \beta; 0>$.

2. $(\lambda_1, \cdots, \lambda_N, \lambda_{N+1}; \lambda_{N+2}) = (\beta, \cdots, \beta, \beta-\alpha; -\alpha)$, where α, β are arbitrary. We have $(\lambda^0, \lambda^1, \cdots, \lambda^{N-1}, \lambda^N)$=$(1-\alpha, 0, \cdots, 0, \alpha)$ and $|(1-\alpha)\Lambda_0 + \alpha\Lambda_N >=|\beta, \cdots, \beta, \beta - \alpha; -\alpha >$.

Associated to the above two classes of solutions are the following Fock spaces:

$$\mathcal{F}^m_\beta = \bigoplus_{\{i_1,\cdots,i_N\}\in\mathbf{Z}} F_{\beta+1+i_1,\beta+1-i_1+i_2,\cdots,\beta+1-i_{m-1}+i_m,\beta-i_m+i_{m+1},\cdots,\beta-i_{N-1}+i_N,\beta+i_N;i_N},$$
$$\mathcal{F}_{(\alpha;\beta)} = \bigoplus_{\{i_1,\cdots,i_N\}\in\mathbf{Z}} F_{\beta+i_1,\beta-i_1+i_2,\cdots,\beta-i_{N-1}+i_N,\beta-\alpha+i_N;-\alpha+i_N},$$

where $m = 1, 2, \cdots, N$, and it should be understood that $i_0 \equiv 0$. However, it is easily seen that $\mathcal{F}^m_\beta = F_{(m;\beta)}$, $m = 1, \cdots, N$. Thus, it is sufficient to study the Fock space $\mathcal{F}_{(\alpha;\beta)}$. In the following we shall also restrict ourselves to the $\alpha \in \mathbf{Z}$ case.

It can be shown that the bosonized action of $U_q[sl(\widehat{N}|1)]$ (II.10) on $\mathcal{F}_{(\alpha;\beta)}$ is closed :

$$U_q[sl(\widehat{N}|1)]\mathcal{F}_{(\alpha;\beta)} = \mathcal{F}_{(\alpha;\beta)}.$$

Hence each Fack space $\mathcal{F}_{(\alpha;\beta)}$ constitutes a $U_q[sl(\widehat{N}|1)]$-module. However, these modules are not irreducible in general. To obtain irreducible subspaces, we introduce a pair of ghost fields [4]

$$\eta(z) = \sum_{n\in\mathbf{Z}} \eta_n z^{-n-1} =: e^{c(z)} :, \qquad \xi(z) = \sum_{n\in\mathbf{Z}} \xi_n z^{-n} =: e^{-c(z)} : .$$

The mode expansion of $\eta(z)$ and $\xi(z)$ is well defined on $\mathcal{F}_{(\alpha;\beta)}$ for $\alpha \in \mathbf{Z}$, and the modes satisfy the relations

$$\xi_m\xi_n + \xi_n\xi_m = \eta_m\eta_n + \eta_n\eta_m = 0, \qquad \xi_m\eta_n + \eta_n\xi_m = \delta_{m+n,0}. \qquad \text{(III.2)}$$

Since $\eta_0\xi_0$ and $\xi_0\eta_0$ qualify as projectors, we use them to decompose $\mathcal{F}_{(\alpha;\beta)}$ into a direct sum $\mathcal{F}_{(\alpha;\beta)} = \eta_0\xi_0\mathcal{F}_{(\alpha;\beta)} \oplus \xi_0\eta_0\mathcal{F}_{(\alpha;\beta)}$ for $\alpha \in \mathbf{Z}$. $\eta_0\xi_0\mathcal{F}_{(\alpha;\beta)}$ is referred to as Ker_{η_0} and $\xi_0\eta_0\mathcal{F}_{(\alpha;\beta)} = \mathcal{F}_{(\alpha;\beta)}/\eta_0\xi_0\mathcal{F}_{(\alpha;\beta)}$ as $Coker_{\eta_0}$. Since η_0 commutes (or anticommutes) with the bosonized action of $U_q[sl(\widehat{N|1})]$, Ker_{η_0} and $Coker_{\eta_0}$ are both $U_q[sl(\widehat{N|1})]$-modules for $\alpha \in \mathbf{Z}$.

III.1 Character and supercharacter

We want to determine the character and supercharacter formulae of the $U_q[sl(\widehat{N|1})]$-modules constructed in the bosonic Fock space. We first of all bosonize the derivation operator d as

$$d = -\sum_{m\geq 1}\frac{m^2}{[m]_q^2}\{\sum_{i=1}^{N} h^i_{-m}h^{*i}_m + c_{-m}c_m\} - \frac{1}{2}\{\sum_{i=1}^{N} h^i_0 h^{*i}_0 + c_0(c_0+1)\}. \tag{III.3}$$

It obeys the commutation relations

$$[d,h_i]=0, \quad [d,h^i_m]=mh^i_m, \quad [d,X^{\pm,i}_m]=mX^{\pm,i}_m, \quad i=1,2,\cdots,N,$$

as required. Moreover, $[d,\xi_0]=[d,\eta_0]=0$.

The character and supercharacter of a $U_q[sl(\widehat{N|1})]$-module M are defined by

$$\begin{aligned} Ch_M(q;x_1,x_2,\cdots,x_N) &= tr_M(q^{-d}x_1^{h_1}x_2^{h_2}\cdots x_N^{h_N}),\\ Sch_M(q;x_1,x_2,\cdots,x_N) &= Str_M(q^{-d}x_1^{h_1}x_2^{h_2}\cdots x_N^{h_N})\\ &= tr_M((-1)^{N_f}q^{-d}x_1^{h_1}x_2^{h_2}\cdots x_N^{h_N}), \end{aligned} \tag{III.4}$$

respectively. The Fermi-number operaor N_f can be bosonized as

$$N_f = \begin{cases} (N-1)b_0 & if \;\; N \; even, \; i.e. \;\; N=2L \\ L(\sum_{k=1}^{N} a^i_0 - b_0) + c_0 & if \;\; N \; odd, \; i.e. \;\; N=2L+1 \end{cases}. \tag{III.5}$$

Indeed, N_f satisfies

$$(-1)^{N_f}\Theta(z) = (-1)^{[\Theta(z)]}\Theta(z)(-1)^{N_f},$$

where $\Theta(z) = X^{\pm,i}(z)$, $\phi_i(z)$, $\phi^*_i(z)$, $\psi_i(z)$ and $\psi^*_i(z)$.

We calculate the characters and supercharacters by using the BRST resolution [7]. Let us define the Fock spaces, for $l \in \mathbf{Z}$

$$\mathcal{F}^{(l)}_{(\alpha;\beta)} = \bigoplus_{\{i_1,\cdots,i_N\}\in\mathbf{Z}} F_{\beta+i_1,\beta-i_1+i_2,\cdots,,\beta-i_{N-1}+i_N,\beta-\alpha+i_N;-\alpha+i_N+l}.$$

We have $\mathcal{F}^{(0)}_{(\alpha;\beta)} = \mathcal{F}_{(\alpha;\beta)}$. It can be shown that η_0 and ξ_0 intertwine these Fock spaces as follows:

$$\eta_0: \mathcal{F}^{(l)}_{(\alpha;\beta)} \longrightarrow \mathcal{F}^{(l+1)}_{(\alpha;\beta)}, \quad \xi_0: \mathcal{F}^{(l)}_{(\alpha;\beta)} \longrightarrow \mathcal{F}^{(l-1)}_{(\alpha;\beta)}.$$

We have the following BRST complexes:

$$\begin{array}{ccccccc} \cdots & \stackrel{Q_{l-1}=\eta_0}{\longrightarrow} & \mathcal{F}^{(l)}_{(\alpha;\beta)} & \stackrel{Q_l=\eta_0}{\longrightarrow} & \mathcal{F}^{(l+1)}_{(\alpha;\beta)} & \stackrel{Q_{l+1}=\eta_0}{\longrightarrow} & \cdots \\ & & |\mathbf{O} & & |\mathbf{O} & & \\ \cdots & \stackrel{Q_{l-1}=\eta_0}{\longrightarrow} & \mathcal{F}^{(l)}_{(\alpha;\beta)} & \stackrel{Q_l=\eta_0}{\longrightarrow} & \mathcal{F}^{(l+1)}_{(\alpha;\beta)} & \stackrel{Q_{l+1}=\eta_0}{\longrightarrow} & \cdots \end{array} \tag{III.6}$$

where $\mathbf{O}$ is an operator such that $\mathcal{F}^{(l)}_{(\alpha;\beta)} \longrightarrow \mathcal{F}^{(l)}_{(\alpha;\beta)}$. Noting the fact that $\eta_0\xi_0+\xi_0\eta_0=1$, and $\eta_0\xi_0$ ($\xi_0\eta_0$) is the projection operator from $\mathcal{F}^{(l)}_{(\alpha;\beta)}$ to Ker_{Q_l} ($Coker_{Q_l}$), we get

$$\begin{aligned} &Ker_{Q_l} = Im_{Q_{l-1}}, \quad \text{for any } l\in \mathbf{Z},\\ &tr(\mathbf{O})|_{Ker_{Q_l}} = tr(\mathbf{O})|_{Im_{Q_{l-1}}} = tr(\mathbf{O})|_{Coker_{Q_{l-1}}}. \end{aligned} \tag{III.7}$$

By the above results, we can write the trace over Ker or $Coker$ as the sum of trace over $\mathcal{F}^{(l)}_{(\alpha;\beta)}$, and compute the latter by using the technique introduced in [26]. The results are

$$\begin{aligned} Ch_{Ker_{\mathcal{F}_{(\alpha;\beta)}}}(q;x_1,\cdots,x_N) &= \frac{q^{\frac{1}{2}\alpha(\alpha-1)}}{\prod_{n=1}^{\infty}(1-q^n)^{N+1}}\sum_{l=1}^{\infty}(-1)^{l+1}q^{\frac{1}{2}\{l^2+l(2\alpha-1)\}}\\ &\times\sum_{\{i_1,\cdots,i_N\}\in\mathbf{Z}} q^{\frac{1}{2}\{i_N^2+i_N(1-2\alpha-2l)\}}q^{\frac{1}{2}\Delta(i_1,\cdots i_N)}\\ &\times x_1^{2i_1-i_2}x_2^{2i_2-i_1-i_3}\cdots x_{N-1}^{2i_{N-1}-i_N-i_{N-2}}x_N^{\alpha-i_N},\\ Ch_{Coker_{\mathcal{F}_{(\alpha;\beta)}}}(q;x_1,\cdots,x_N) &= \frac{q^{\frac{1}{2}\alpha(\alpha-1)}}{\prod_{n=1}^{\infty}(1-q^n)^{N+1}}\sum_{l=1}^{\infty}(-1)^{l+1}q^{\frac{1}{2}\{l^2+l(1-2\alpha)\}}\\ &\times\sum_{\{i_1,\cdots,i_N\}\in\mathbf{Z}} q^{\frac{1}{2}\{i_N^2+i_N(1-2\alpha+2l)\}}q^{\frac{1}{2}\Delta(i_1,\cdots i_N)}\\ &\times x_1^{2i_1-i_2}x_2^{2i_2-i_1-i_3}\cdots x_{N-1}^{2i_{N-1}-i_N-i_{N-2}}x_N^{\alpha-i_N}, \end{aligned}$$

where $\Delta(i_1,\cdots,i_N)=\sum_{l,l'=1}^{N}\frac{\alpha_{ll'}\beta_{ll'}}{N-1}\lambda^l_{i_1,\cdots,i_N}\lambda^{l'}_{i_1,\cdots,i_N}$ and

$$\left\{\begin{array}{ll} \lambda^l_{i_1,\cdots,i_N} = 2i_l-i_{l-1}-i_{l+1}, & 2\le l\le N-1\\ \lambda^1_{i_1,\cdots,i_N} = 2i_1-i_2, & \lambda^N_{i_1,\cdots,i_N} = \alpha-i_N \end{array}\right. . \tag{III.8}$$

Similary, the supercharacters of $Ker_{\mathcal{F}_{(\alpha;\beta)}}$ and $Coker_{\mathcal{F}_{(\alpha;\beta)}}$ are given by

1. For $N=2L$:

$$\begin{aligned} Sch_{Ker_{\mathcal{F}_{(\alpha;\beta)}}}(q;x_1,\cdots,x_N) &= \frac{(-1)^{\alpha}q^{\frac{1}{2}\alpha(\alpha-1)}}{\prod_{n=1}^{\infty}(1-q^n)^{N+1}}\sum_{l=1}^{\infty}(-1)^{l+1}q^{\frac{1}{2}\{l^2+l(2\alpha-1)\}}\\ &\times\sum_{\{i_1,\cdots,i_N\}\in\mathbf{Z}} (-1)^{i_N}q^{\frac{1}{2}\{i_N^2+i_N(1-2\alpha-2l)\}}q^{\frac{1}{2}\Delta(i_1,\cdots i_N)}\\ &\times x_1^{2i_1-i_2}x_2^{2i_2-i_1-i_3}\cdots x_{N-1}^{2i_{N-1}-i_N-i_{N-2}}x_N^{\alpha-i_N},\\ Sch_{Coker_{\mathcal{F}_{(\alpha;\beta)}}}(q;x_1,\cdots,x_N) &= \frac{(-1)^{\alpha}q^{\frac{1}{2}\alpha(\alpha-1)}}{\prod_{n=1}^{\infty}(1-q^n)^{N+1}}\sum_{l=1}^{\infty}(-1)^{l+1}q^{\frac{1}{2}\{l^2+l(1-2\alpha)\}}\\ &\times\sum_{\{i_1,\cdots,i_N\}\in\mathbf{Z}} (-1)^{i_N}q^{\frac{1}{2}\{i_N^2+i_N(1-2\alpha+2l)\}}q^{\frac{1}{2}\Delta(i_1,\cdots i_N)}\\ &\times x_1^{2i_1-i_2}x_2^{2i_2-i_1-i_3}\cdots x_{N-1}^{2i_{N-1}-i_N-i_{N-2}}x_N^{\alpha-i_N}, \end{aligned}$$

2. For $N=2L+1$:

$$\begin{aligned} Sch_{Ker_{\mathcal{F}_{(\alpha;\beta)}}}(q;x_1,\cdots,x_N) &= -\frac{(-1)^{(L+1)\alpha}q^{\frac{1}{2}\alpha(\alpha-1)}}{\prod_{n=1}^{\infty}(1-q^n)^{N+1}}\sum_{l=1}^{\infty}q^{\frac{1}{2}\{l^2+l(2\alpha-1)\}}\\ &\times\sum_{\{i_1,\cdots,i_N\}\in\mathbf{Z}} (-1)^{i_N}q^{\frac{1}{2}\{i_N^2+i_N(1-2\alpha-2l)\}}q^{\frac{1}{2}\Delta(i_1,\cdots i_N)} \end{aligned}$$

$$\times x_1^{2i_1-i_2}x_2^{2i_2-i_1-i_3}\cdots x_{N-1}^{2i_{N-1}-i_N-i_{N-2}}x_N^{\alpha-i_N},$$

$$\begin{aligned} Sch_{Coker_{\mathcal{F}_{(\alpha;\beta)}}}(q;x_1,\cdots,x_N) &= -\frac{(-1)^{(L+1)\alpha}q^{\frac{1}{2}\alpha(\alpha-1)}}{\prod_{n=1}^{\infty}(1-q^n)^{N+1}}\sum_{l=1}^{\infty}q^{\frac{1}{2}\{l^2+l(1-2\alpha)\}} \\ &\times\sum_{\{i_1,\cdots,i_N\}\in\mathbf{Z}}(-1)^{i_N}q^{\frac{1}{2}\{i_N^2+i_N(1-2\alpha+2l)\}}q^{\frac{1}{2}\Delta(i_1,\cdots.i_N)} \\ &\times x_1^{2i_1-i_2}x_2^{2i_2-i_1-i_3}\cdots x_{N-1}^{2i_{N-1}-i_N-i_{N-2}}x_N^{\alpha-i_N}. \end{aligned}$$

Since $\mathcal{F}^{(1)}_{(\alpha-(N-1);\beta+1)}=\mathcal{F}_{(\alpha;\beta)}$ and by (III.7), we have

$$Ch_{Coker_{\mathcal{F}_{(\alpha-(N-1);\beta+1)}}}=Ch_{Ker_{\mathcal{F}_{(\alpha;\beta)}}},\quad Sch_{Coker_{\mathcal{F}_{(\alpha-(N-1);\beta+1)}}}=Sch_{Ker_{\mathcal{F}_{(\alpha;\beta)}}}. \tag{III.9}$$

Relations (III.9) can also be checked by using the above explicit formulae of the (super)characters.

III.2 $U_q[sl(\widehat{N}|1)]$-module structure of $\mathcal{F}_{(\alpha;\,\beta-\frac{1}{N-1}\alpha)}$

Set $\lambda_\alpha=(1-\alpha)\Lambda_0+\alpha\Lambda_N$ and

$$\begin{aligned} &|\lambda_\alpha>=|\beta,\cdots,\beta,\beta-\alpha;-\alpha>\in\mathcal{F}_{(\alpha;\beta)},\quad \alpha\in\mathbf{Z}, \\ &|\Lambda_m>=|\beta+1,\cdots,\beta+1,\beta,\cdots,\beta;0>\in\mathcal{F}_{(m;\beta)},\quad m=1,\cdots,N, \end{aligned}$$

The above vectors play the role of the highest weight vectors of $U_q[sl(\widehat{N}|1)]$-modules. one can check that

$$\left\{\begin{array}{ll} \eta_0|\lambda_\alpha>=0, & \text{for } \alpha=0,-1,\cdots \\ \eta_o|\Lambda_m>=0, & \text{for } m=1,\cdots,N \\ \eta_0|\lambda_\alpha>\neq 0, & \text{for } \alpha=1,2,\cdots \end{array}\right. . \tag{III.10}$$

It follows that the modules

$$\begin{aligned} &Coker_{\mathcal{F}_{(\alpha;\beta)}}\ (\alpha=1,2,\cdots),\quad Ker_{\mathcal{F}_{(\alpha;\beta)}}\ (\alpha=0,-1,-2,\cdots), \\ &Ker_{\mathcal{F}_{(m;\beta)}}\ (m=1,2,\cdots,N), \end{aligned}$$

are highest weight $U_q[sl(\widehat{N}|1)]$-modules. Denote them by $\overline{V}(\lambda_\alpha)$ and $\overline{V}(\Lambda_m)$, respectively. From (III.10) and (III.9), we have the following identifications of the highest weight $U_q[sl(\widehat{N}|1)]$-modules:

$$\begin{aligned} \overline{V}(\lambda_\alpha) &\cong Ker_{\mathcal{F}_{(\alpha;\beta-\frac{1}{N-1}\alpha)}}\equiv Coker_{\mathcal{F}_{(\alpha-(N-1);\beta-\frac{1}{N-1}\alpha+1)}},\quad \text{for } \alpha=0,-1,-2,\cdots, \\ &\cong Coker_{\mathcal{F}_{(\alpha;\beta-\frac{1}{N-1}\alpha)}}\equiv Ker_{\mathcal{F}_{(\alpha+(N-1);\beta-\frac{1}{N-1}\alpha-1)}},\quad \text{for } \alpha=1,2,\cdots, \qquad \text{(III.11)} \\ \overline{V}(\Lambda_m) &\cong Ker_{\mathcal{F}_{(m;\beta-\frac{1}{N-1}m)}}\equiv Coker_{\mathcal{F}_{(m-(N-1);\beta-\frac{1}{N-1}m+1)}},\quad \text{for } m=1,\cdots,N. \qquad \text{(III.12)} \end{aligned}$$

It is easy to see that the vertex operators (II.16) also commute (or anti-commute) with η_0. It follows from (III.11)-(III.12) that each Fock space $\mathcal{F}_{(\alpha;\beta-\frac{1}{N-1}\alpha)}$ is decomposed into a direct sum of the highest weight $U_q[sl(\widehat{N}|1)]$-modules:

$$
\begin{array}{rcccc}
 & Ker & & Coker & \\
\cdot & \cdot & & \cdot & \\
\cdot & \cdot & & \cdot & \\
F_{(-N;\,\beta+1+\frac{1}{N-1})} = & \overline{V}(\lambda_{-N}) & \oplus & \overline{V}(\lambda_{-1}) & \\
 & \phi(z)\uparrow\downarrow\phi^*(z) & & \phi(z)\uparrow\downarrow\phi^*(z) & \\
F_{(-N+1;\,\beta+1)} = & \overline{V}(\lambda_{-N+1}) & \oplus & \overline{V}(\Lambda_0) & \\
 & \phi(z)\uparrow\downarrow\phi^*(z) & & \phi(z)\uparrow\downarrow\phi^*(z) & \\
F_{(-N+2;\,\beta+1-\frac{1}{N-1})} = & \overline{V}(\lambda_{-N+2}) & \oplus & \overline{V}(\Lambda_1) & \\
 & \phi(z)\uparrow\downarrow\phi^*(z) & & \phi(z)\uparrow\downarrow\phi^*(z) & \\
\cdot & \cdot & & \cdot & \\
\cdot & \cdot & & \cdot & \\
F_{(-2;\,\beta+1-\frac{N-3}{N-1})} = & \overline{V}(\lambda_{-2}) & \oplus & \overline{V}(\Lambda_{N-3}) & \\
 & \phi(z)\uparrow\downarrow\phi^*(z) & & \phi(z)\uparrow\downarrow\phi^*(z) & \\
F_{(-1;\,\beta+1-\frac{N-2}{N-1})} = & \overline{V}(\lambda_{-1}) & \oplus & \overline{V}(\Lambda_{N-2}) & \\
 & \phi(z)\uparrow\downarrow\phi^*(z) & & \phi(z)\uparrow\downarrow\phi^*(z) & \\
F_{(0;\,\beta)} = & \overline{V}(\Lambda_0) & \oplus & \overline{V}(\Lambda_{N-1}) & \\
 & \phi(z)\uparrow\downarrow\phi^*(z) & & \phi(z)\uparrow\downarrow\phi^*(z) & . \\
F_{(1;\,\beta-\frac{1}{N-1})} = & \overline{V}(\Lambda_1) & \oplus & \overline{V}(\Lambda_N) & \\
 & \phi(z)\uparrow\downarrow\phi^*(z) & & \phi(z)\uparrow\downarrow\phi^*(z) & \\
F_{(2;\,\beta-\frac{2}{N-1})} = & \overline{V}(\Lambda_2) & \oplus & \overline{V}(\lambda_2) & \\
 & \phi(z)\uparrow\downarrow\phi^*(z) & & \phi(z)\uparrow\downarrow\phi^*(z) & \\
\cdot & \cdot & & \cdot & \\
\cdot & \cdot & & \cdot & \\
F_{(N-2;\,\beta-\frac{N-2}{N-1})} = & \overline{V}(\Lambda_{N-2}) & \oplus & \overline{V}(\lambda_{N-2}) & \\
 & \phi(z)\uparrow\downarrow\phi^*(z) & & \phi(z)\uparrow\downarrow\phi^*(z) & \\
F_{(N-1;\,\beta-1)} = & \overline{V}(\Lambda_{N-1}) & \oplus & \overline{V}(\Lambda_{N-1}) & \\
 & \phi(z)\uparrow\downarrow\phi^*(z) & & \phi(z)\uparrow\downarrow\phi^*(z) & \\
F_{(N;\,\beta-1-\frac{1}{N-1})} = & \overline{V}(\Lambda_N) & \oplus & \overline{V}(\lambda_N) & \\
 & \phi(z)\uparrow\downarrow\phi^*(z) & & \phi(z)\uparrow\downarrow\phi^*(z) & \\
F_{(N+1;\,\beta-1-\frac{2}{N-1})} = & \overline{V}(\lambda_2) & \oplus & \overline{V}(\lambda_{N+1}) & \\
\cdot & \cdot & & \cdot & \\
\cdot & \cdot & & \cdot &
\end{array}
\qquad \text{(III.13)}
$$

It is expected that $\overline{V}(\lambda_\alpha)$ ($\alpha \in \mathbf{Z}$) and $\overline{V}(\Lambda_m)$ ($m = 1, 2, \cdots, N-1$) are irreducible highest weight $U_q[sl(\widehat{N}|1)]$-modules with the highest weights λ_α and Λ_m, respectively. Thus we conjecture that

$$\overline{V}(\lambda_\alpha) = V(\lambda_\alpha), \quad \overline{V}(\Lambda_m) = V(\Lambda_m). \qquad \text{(III.14)}$$

IV Exchange Relations of Vertex Operators

In this section, we derive the exchange relations of the type I and type II bosonized vertex operators of $U_q[sl(\widehat{N}|1)]$. As expected, these vertex operators satisfy the graded Faddeev-Zamolodchikov algebra.

IV.1 The R-matrix

Throughout, we use the abbreviation

$$(z;x_1,\cdots,x_m)_\infty = \prod_{\{n_1,\cdots,n_m\}=0}^{\infty}(1-zx_1^{n_1}\cdots x_m^{n_m}),$$
$$\{z\}_\infty \stackrel{def}{=} (z;q^{2(N-1)},q^{2(N-1)})_\infty. \tag{IV.1}$$

Let $\bar{R}(z)\in End(V\otimes V)$ be the R-matrix of $U_q[sl(\widehat{N}|1)]$,

$$\bar{R}(z)(v_i\otimes v_j)=\sum_{k,l=1}^{2N}\bar{R}_{kl}^{ij}(z)v_k\otimes v_l \quad ,\forall v_i,v_j,v_k,v_l\in V, \tag{IV.2}$$

where the matrix elements of $\bar{R}(z)$ are given by

$$\begin{aligned}
&\bar{R}_{i,i}^{i,i}(z)=-1,\quad \bar{R}_{N+1,N+1}^{N+1,N+1}(z)=-\frac{zq^{-1}-q}{zq-q^{-1}},\quad i=1,2,\cdots,N,\\
&\bar{R}_{ij}^{ij}(z)=\frac{z-1}{zq-q^{-1}},\quad i\neq j,\\
&\bar{R}_{ij}^{ji}(z)=\frac{q-q^{-1}}{zq-q^{-1}}(-1)^{[i][j]},\quad i<j,\\
&\bar{R}_{ij}^{ji}(z)=\frac{(q-q^{-1})z}{zq-q^{-1}}(-1)^{[i][j]},\quad i>j,\\
&\bar{R}_{kl}^{ij}(z)=0,\quad \text{otherwise.}
\end{aligned}$$

Define the R-matrices $R^{(I)}(z)$ and $R^{(II)}(z)$ by

$$R^{(I)}(z)=r(z)\bar{R}(z),\qquad R^{(II)}(z)=\bar{r}(z)\bar{R}(z), \tag{IV.3}$$

where

$$\begin{aligned}
r(z)&=z^{\frac{2-N}{N-1}}\frac{(zq^2;q^{2(N-1)})_\infty(z^{-1}q^{2N-2};q^{2(N-1)})_\infty}{(z^{-1}q^2;q^{2(N-1)})_\infty(zq^{2N-2};q^{2(N-1)})_\infty},\\
\bar{r}(z)&=-z^{-\frac{1}{N-1}}\frac{(zq^{2N-4};q^{2(N-1)})_\infty(z^{-1}q^{2N-2};q^{2(N-1)})_\infty}{(z^{-1}q^{2N-4};q^{2(N-1)})_\infty(zq^{2N-2};q^{2(N-1)})_\infty}.
\end{aligned}$$

These R-matrices satisfy the graded Yang-Baxter equation on $V\otimes V\otimes V$:

$$R_{12}^{(i)}(z)R_{13}^{(i)}(zw)R_{23}^{(i)}(w)=R_{23}^{(i)}(w)R_{13}^{(i)}(zw)R_{12}^{(i)}(z),\quad i=I,II.$$

Moreover, they enjoy **(i)** the initial condition $R^{(i)}(1)=P,\ i=I,II$, where P is the graded permutation operator; **(ii)** the unitarity condition $R_{12}^{(i)}(\frac{z}{w})R_{21}^{(i)}(\frac{w}{z})=1,\quad i=I,II$, where $R_{21}^{(i)}(z)=PR_{12}^{(i)}(z)P$; **(iii)** the crossing-unitarity

$$(R^{(i)})^{-1,st_1}(z)\left((q^{-2\bar{\rho}}\otimes 1)R^{(i)}(zq^{2(1-N)})(q^{2\bar{\rho}}\otimes 1)\right)^{st_1}=1,\quad i=I,II,$$

where

$$\begin{aligned}
q^{2\bar{\rho}} &\equiv diag(q^{2\rho_1},q^{2\rho_2},\cdots,q^{2\rho_N},q^{2\rho_{N+1}})\\
&= diag(q^{N-2},q^{N-4},\cdots,q^{-N},q^{-N}).
\end{aligned}$$

The various supertranspositions of the R-matrix are given by

$$\begin{aligned}
&(R^{st_1}(z))_{ij}^{kl}=R_{kj}^{il}(z)(-1)^{[i]([i]+[k])},\qquad (R^{st_2}(z))_{ij}^{kl}=R_{il}^{kj}(z)(-1)^{[j]([l]+[j])}\ ,\\
&(R^{st_{12}}(z))_{ij}^{kl}=R_{kl}^{ij}(z)(-1)^{([i]+[j])([i]+[j]+[k]+[l])}=R_{kl}^{ij}(z)\ .
\end{aligned}$$

IV.2 The graded Faddeev-Zamolodchikov algebra

We now calculate the exchange relations of the type I and type II bosonic vertex operators of $U_q[sl(\widehat{N}|1)]$. Define

$$\oint dz f(z) = Res(f) = f_{-1}, \quad \text{for a formal series function } f(z) = \sum_{n\in\mathbf{Z}} f_n z^n.$$

Then, the Chevalley generators of $U_q[sl(\widehat{N}|1)]$ can be expressed by the integrals

$$e_i = \oint dz X^{+,i}(z), \qquad f_i = \oint dz X^{-,i}(z), \; i = 1, 2, \cdots, N.$$

One can also get the integral expressions of the bosonic vertex operators $\phi(z)$, $\phi^*(z)$, $\psi(z)$ and $\psi^*(z)$. Using these integral expressions and the relations given in appendices A and B, we find that the bosonic vertex operators defined in (II.16) satisfy the graded Faddeev-Zamolodchikov algebra

$$\begin{aligned}
\phi_j(z_2)\phi_i(z_1) &= \sum_{k,l=1}^{N+1} R^{(I)}(\frac{z_1}{z_2})_{ij}^{kl}\phi_k(z_1)\phi_l(z_2)(-1)^{[i][j]}, \\
\psi_i^*(z_1)\psi_j^*(z_2) &= \sum_{k,l=1}^{N+1} R^{(II)}(\frac{z_1}{z_2})_{kl}^{ij}\psi_l^*(z_2)\psi_k^*(z_1)(-1)^{[i][j]}, \\
\psi_i^*(z_1)\phi_j(z_2) &= \tau(\frac{z_1}{z_2})\phi_j(z_2)\psi_i^*(z_1)(-1)^{[i][j]},
\end{aligned} \qquad \text{(IV.4)}$$

where

$$\tau(z) = -z^{\frac{2-N}{N-1}}\frac{(zq;q^{2(N-1)})_\infty(z^{-1}q^{2N-3};q^{2(N-1)})_\infty}{(z^{-1}q;q^{2(N-1)})_\infty(zq^{2N-3};q^{2(N-1)})_\infty}.$$

By

$$: e^{-h_N^*(zq^N;\frac{1}{2})+h_1^*(zq;\frac{1}{2})-h^1(zq^2;\frac{1}{2})-h^2(zq^3;\frac{1}{2})\cdots-h^N(zq^{N+1};\frac{1}{2})} := 1,$$

we obtain the first invertibility relations

$$\phi_i(z)\phi_j^*(z) = g^{-1}(-1)^{[i]}\delta_{ij}, \qquad \sum_{k=1}^{N+1}(-1)^{[k]}\phi_k^*(z)\phi_k(z) = g^{-1}, \qquad \text{(IV.5)}$$

and the second invertibility relations

$$\phi_i^*(zq^{2(N-1)})\phi_j(z) = -g^{-1}q^{2\rho_i}\delta_{ij}, \qquad \sum_{k=1}^{N+1} q^{-2\rho_k}\phi_k(z)\phi_k^*(zq^{2(N-1)}) = -g^{-1}, \qquad \text{(IV.6)}$$

where $g = e^{\frac{\sqrt{-1}\pi N}{2(N-1)}}\frac{(q^2;q^{2(N-1)})_\infty}{(q^{2(N-1)};q^{2(N-1)})_\infty}$. Using the fact that $\eta_0\xi_0$ is a projection operator, we can make the following identifications:

$$\begin{aligned}
\Phi_i(z) &= \eta_0\xi_0\phi_i(z)\eta_0\xi_0, \qquad \Phi_i^*(z) = \eta_0\xi_0\phi_i^*(z)\eta_0\xi_0, \\
\Psi_i(z) &= \eta_0\xi_0\psi_i(z)\eta_0\xi_0, \qquad \Psi_i^*(z) = \eta_0\xi_0\psi_i^*(z)\eta_0\xi_0.
\end{aligned} \qquad \text{(IV.7)}$$

Set

$$\mu_\alpha = \begin{cases} \Lambda_\alpha, & \alpha = 0, 1, \cdots, N \\ \lambda_{\alpha-(N-1)}, & \text{for } \alpha > N \\ \lambda_\alpha, & \text{for } \alpha < 0 \end{cases} . \qquad \text{(IV.8)}$$

It is easy to see that the vertex operators $\phi(z)$, $\phi^*(z)$, $\psi(z)$ and $\psi^*(z)$ commute (or anti-commute) with the BRST charge η_0. It follows from (III.13) and (III.14) that the vertex operators (IV.7) intertwine all the level-one irreducible highest weight $U_q[sl(\widehat{N}|1)]$-modules $V(\mu_\alpha)$ ($\alpha \in \mathbf{Z}$) as follows

$$\begin{aligned}
&\Phi(z) : V(\mu_\alpha) \longrightarrow V(\mu_{\alpha-1}) \otimes V_z, \quad \Phi^*(z) : V(\mu_\alpha) \longrightarrow V(\mu_{\alpha+1}) \otimes V_z^{*S}, \\
&\Psi(z) : V(\mu_\alpha) \longrightarrow V_z \otimes V(\mu_{\alpha-1}), \quad \Psi^*(z) : V(\mu_\alpha) \longrightarrow V_z^{*S} \otimes V(\mu_{\alpha+1}).
\end{aligned} \tag{IV.9}$$

¿From (IV.4), we have

$$\begin{aligned}
\Phi_j(z_2)\Phi_i(z_1) &= \sum_{k,l=1}^{N+1} R^{(I)}(\frac{z_1}{z_2})_{ij}^{kl} \Phi_k(z_1)\Phi_l(z_2)(-1)^{[i][j]}, \\
\Psi_i^*(z_1)\Psi_j^*(z_2) &= \sum_{k,l=1}^{N+1} R^{(II)}(\frac{z_1}{z_2})_{kl}^{ij} \Psi_l^*(z_2)\Psi_k^*(z_1)(-1)^{[i][j]}, \\
\Psi_i^*(z_1)\Phi_j(z_2) &= \tau(\frac{z_1}{z_2})\Phi_j(z_2)\Psi_i^*(z_1)(-1)^{[i][j]}.
\end{aligned} \tag{IV.10}$$

Moreover, we have the following invertibility relations:

$$\begin{aligned}
&\Phi_i(z)\Phi_j^*(z) = g^{-1}(-1)^{[i]}\delta_{ij} id_{V(\mu_\alpha)}, \\
&\sum_{k=1}^{N+1} (-1)^{[k]}\Phi_k^*(z)\Phi_k(z) = g^{-1} id_{V(\mu_\alpha)}, \\
&\Phi_i^*(zq^{2(N-1)})\Phi_j(z) = -g^{-1}q^{2\rho_i}\delta_{ij} id_{V(\mu_\alpha)}, \\
&\sum_{k=1}^{N+1} q^{-2\rho_k}\Phi_k(z)\Phi_k^*(zq^{2(N-1)}) = -g^{-1} id_{V(\mu_\alpha)}.
\end{aligned} \tag{IV.11}$$

V Multi-component super t-J model

In this section, we give a mathematical definition of the multi-component super t-J model on an infinite lattice.

V.1 Space of states

By means of the R-matrix (IV.2) of $U_q[sl(\widehat{N}|1)]$, one defines a spin chain model, referred to as the multi-component super $t - J$ model, on the infinte lattice $\cdots \otimes V \otimes V \otimes V \cdots$. Let h be the operator on $V \otimes V$ such that

$$P\bar{R}(\frac{z_1}{z_2}) = 1 + uh + \cdots, \qquad u \longrightarrow 0,$$

$$P : \text{the graded permutation operator}, \quad e^u \equiv \frac{z_1}{z_2}.$$

The Hamiltonian H of this model is given by

$$H = \sum_{l \in Z} h_{l+1,l}. \tag{V.1}$$

H acts formally on the infinite tensor product,

$$\cdots V \otimes V \otimes V \cdots. \tag{V.2}$$

It can be easily checked that

$$[U_q'(\widehat{sl}(N|1)), H] = 0,$$

where $U_q'[\widehat{sl}(N|1)]$ is the subalgebra of $U_q[sl(\widehat{N}|1)]$ with the derivation operator d being dropped. So $U_q'[\widehat{sl}(N|1)]$ plays the role of infinite dimensional *non-abelian symmetry* of the multi-component super t-J model on the infinite lattice.

¿From the intertwining relation (IV.9), one have the following composition of the type I vertex operators:

$$V(\mu_\alpha) \stackrel{\Phi(1)}{\longrightarrow} V(\mu_{\alpha-1}) \otimes V \stackrel{\Phi(1)\otimes id}{\longrightarrow} V(\mu_{\alpha-1}) \otimes V \otimes V \stackrel{\Phi(1)\otimes id \otimes id}{\longrightarrow} \cdots \longrightarrow W_l, \qquad \text{(V.3)}$$

where $W_l \stackrel{def}{=} \cdots \otimes V \otimes V$, i.e the left half-infinite tensor product. We conjecture that such a composition converges to a map :

$$i : V(\mu_\alpha) \longrightarrow W_l.$$

Such a map i satisfies $i(xv) = \Delta^{(\infty)}(x)i(v)$, $x \in U_q[sl(\widehat{N}|1)]$ and $v \in V(\mu_\alpha)$. Following [9], we could replace the infinite tensor product (V.2) by the level-zero $U_q[sl(\widehat{N}|1)]$-module,

$$F_{\alpha\alpha'} = \mathrm{Hom}(V(\mu_\alpha), V(\mu_{\alpha'})) \cong V(\mu_\alpha) \otimes V(\mu_{\alpha'})^*,$$

where $V(\mu_\alpha)$ is level-one irreducible highest weight $U_q[sl(\widehat{N}|1)]$-module and $V(\mu_{\alpha'})^*$ is the dual module of $V(\mu_{\alpha'})$. By (III.13), this homomorphism can be realized by applying the type I vertex operators repeatedly. So we shall make the (hypothetical) identification:

$$\text{“}\textbf{the space of physical states}\text{”} = \bigoplus_{\alpha,\alpha' \in \mathbf{Z}} V(\mu_\alpha) \otimes V(\mu_{\alpha'})^*.$$

Namely, we take

$$F \equiv End(\bigoplus_{\alpha \in \mathbf{Z}} V(\mu_\alpha)) \cong \bigoplus_{\alpha,\alpha' \in \mathbf{Z}} F_{\alpha\alpha'}$$

as the space of states of the multi-component super t-J model on the infinite lattice. The left action of $U_q[sl(\widehat{N}|1)]$ on F is defined by

$$x.f = \sum x_{(1)} \circ f \circ S(x_{(2)})(-1)^{[f][x_{(2)}]}, \quad \forall x \in U_q[sl(\widehat{N}|1)],\ f \in F,$$

where we have used notation $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$. Note that $F_{\alpha\alpha}$ has the unique canonical element $id_{V(\mu_\alpha)}$. We call it the vacuum [10] and denote it by $|vac>_\alpha$.

V.2 Local structure and local operators

Following Jimbo et al [10], we use the type I vertex operators and their variants to incorporate the local structure into the space of physical states F, that is to formulate the action of local operators of the multi-component super t-J model on the infinite tensor product (V.2) in terms of their actions on $F_{\alpha\alpha'}$.

Using the isomorphisms

$$\begin{aligned} \Phi(1) &: \quad V(\mu_\alpha) \longrightarrow V(\mu_{\alpha-1}) \otimes V, \\ \Phi^{*,st}(q^{2(N-1)}) &: \quad V \otimes V(\mu_\alpha)^* \longrightarrow V(\mu_{\alpha-1})^*, \end{aligned} \qquad \text{(V.4)}$$

were st is the supertransposition on the quantum space, we have the following identification:

$$V(\mu_\alpha)\otimes V(\mu_{\alpha'})^* \stackrel{\sim}{\to} V(\mu_{\alpha-1})\otimes V\otimes V(\mu_{\alpha'})^* \stackrel{\sim}{\to} V(\mu_{\alpha-1})\otimes V(\mu_{\alpha'-1})^*.$$

The resulting isomorphism can be identified with the super translation (or shift) operator defined by

$$T=-g\sum_i \Phi_i(1)\otimes \Phi_i^{*,st}(q^{2(N-1)})(-1)^{[i]}q^{-2\rho_i}.$$

Its inverse is given by

$$T^{-1}=g\sum_i \Phi_i^*(1)\otimes\Phi_i^{st}(1).$$

Thus we can define the local operators on V as operators on $F_{\alpha\alpha'}$ [10]. Let us label the tensor components from the middle as $1,2,\cdots$ for the left half and as $0,-1,-2,\cdots$ for the right half. The operators acting on the site 1 are defined by

$$E_{ij}\stackrel{def}{=}E_{ij}^{(1)}=g\Phi_i^*(1)\Phi_j(1)(-1)^{[j]}\otimes id. \qquad (V.5)$$

More generally we set

$$E_{ij}^{(n)}=T^{-(n-1)}E_{ij}T^{n-1} \qquad (n\in Z). \qquad (V.6)$$

Then, from the invertibility relations of the type I vertex operators of $U_q[sl(\widehat{N}|1)]$, we can show that the local operators $E_{ij}^{(n)}$ acting on $F_{\alpha\alpha'}$ satisfy the following relations:

$$E_{ij}^{(m)}E_{kl}^{(n)}=\left\{\begin{array}{ll}\delta_{jk}E_{il}^{(n)} & \text{if}\quad m=n\\ (-1)^{([i]+[j])([k]+[l])}E_{kl}^{(n)}E_{ij}^{(m)} & \text{if}\quad m\neq n\end{array}\right. .$$

This result implies that the local operators $E_{ij}^{(n)}$ are nothing but the $U_q[sl(N|1)]$ generators acting on the n-th component of $\cdots\otimes V\otimes V\otimes\cdots$. They include all the local operators in the multi-component super $t-J$ model [10].

As is expected from the physical point of view, the vacuum vectors $|vac>_\alpha$ are supertranslationally invariant and singlets (i.e. they belong to the trivial representation of $U_q[sl(\widehat{N}|1)]$):

$$T|vac>_\alpha=|vac>_{\alpha-1}, \qquad x.|vac>_\alpha=\epsilon(x)|vac>_\alpha,\quad \forall x\in U_q[sl(\widehat{N}|1)].$$

This is proved as follow. Let $u_l^{(\alpha)}$ $(u_l^{*(\alpha)})$ be a basis vectors of $V(\mu_\alpha)$ $(V(\mu_\alpha)^*)$ and

$$|vac>_\alpha\stackrel{def}{=}id_{V(\mu_\alpha)}=\sum_l u_l^{(\alpha)}\otimes u_l^{*(\alpha)}.$$

Then

$$T|vac>_\alpha=-g\sum_{m,l}q^{-2\rho_m}\Phi_m(1)u_l^{(\alpha)}\otimes\Phi_m^{*,st}(q^{2(N-1)})u_l^{*(\alpha)}(-1)^{[m]+[l][m]}.$$

We want to show $T|vac>_\alpha=|vac>_{\alpha-1}$. This is equivalent to proving

$$-g\sum_{m,l}q^{-2\rho_m}\Phi_m(1)u_l^{(\alpha)}\,\Phi_m^{*,st}(q^{2(N-1)})\cdot u_l^{*(\alpha)}(v)(-1)^{[m]+[l][m]}=v,\quad \forall v\in V(\mu_{\alpha-1}).$$

Now

$$\begin{aligned} l.h.s &= -g\sum_{m,l} q^{-2\rho_m}\Phi_m(1)u_l^{(\alpha)}u_l^{*(\alpha)}\left(\Phi^*_m(q^{2(N-1)})^{st})^{st}v\right)(-1)^{[m]} \\ &= -g\sum_{m,l} q^{-2\rho_m}\Phi_m(1)u_l^{(\alpha)}u_l^{*(\alpha)}(\Phi^*_m(q^{2(N-1)})v) \\ &= -g\sum_{m} q^{-2\rho_m}\Phi_m(1)\Phi^*_m(q^{2(N-1)})v = v, \end{aligned}$$

where we have used $(\Phi^*_m(z)^{st})^{st} = \Phi^*_m(z)(-1)^{[m]}$ and (IV.11). As to the second equation, we have

$$\begin{aligned} x\cdot|vac>_\alpha &= \sum x_{(1)}u_l^{(\alpha)}\otimes x_{(2)}u_l^{*(\alpha)}(-1)^{[l][x_{(2)}]} \\ &= \sum x_{(1)}u_l^{(\alpha)}\otimes \pi_{V(\mu_\alpha)^*}(x_{(2)})_{ml}u_m^{*(\alpha)}(-1)^{[l][x_{(2)}]} \\ &= \sum x_{(1)}u_l^{(\alpha)}\otimes \pi_{V(\mu_\alpha)}(S(x_{(2)}))_{lm}u_m^{*(\alpha)} \\ &= \sum x_{(1)}\pi_{V(\mu_\alpha)}(S(x_{(2)}))_{lm}u_l^{(\alpha)}\otimes u_m^{*(\alpha)} \\ &= \sum x_{(1)}S(x_{(2)})u_m^{(\alpha)}\otimes u_m^{*(\alpha)} = \epsilon(x)|vac>_\alpha . \end{aligned}$$

This completes the proof.

For any local operator $O\in F$, its vacuum expectation value is defined by

$$_\alpha<vac|O|vac>_\alpha \stackrel{def}{=} \frac{tr_{V(\mu_\alpha)}(q^{-2\rho}O)}{tr_{V(\mu_\alpha)}(q^{-2\rho})} = \frac{tr_{V(\mu_\alpha)}(q^{-2(N-1)d-2h_{\bar\rho}}O)}{tr_{V(\mu_\alpha)}(q^{-2(N-1)d-2h_{\bar\rho}})}, \quad \text{(V.7)}$$

where

$$2h_{\bar\rho} = \sum_{l=1}^{N} l(N-1-l)h_l.$$

We shall denote the correlator $_\alpha<vac|O|vac>_\alpha$ by $<O>_\alpha$.

VI Correlation functions

The aim of this section is to calculate $<E_{mn}>_\alpha$. The generalization to the calculation of the multi-point functions is straightforward.

Set

$$P^m_n(z_1,z_2|q|\alpha) = \frac{tr_{V(\mu_\alpha)}(q^{-2(N-1)d-2h_{\bar\rho}}\Phi^*_m(z_1)\Phi_n(z_2))}{tr_{V(\mu_\alpha)}(q^{-2(N-1)d-2h_{\bar\rho}})},$$

then $<E_{mn}>_\alpha = P^m_n(z,z|q|\alpha)$. By (IV.8), it is sufficient to calculate

$$F^{(\alpha)}_{mn}(z_1,z_2) = \frac{tr_{F_{(\alpha;\beta-\alpha)}}(q^{-2(N-1)d-2h_{\bar\rho}}\phi^*_m(z_1)\phi_n(z_2)\eta_0\xi_0)}{tr_{F_{(\alpha;\beta-\alpha)}}(q^{-2(N-1)d-2h_{\bar\rho}}\eta_0\xi_0)}. \quad \text{(VI.1)}$$

Using the Clavelli-Shapiro technique [26], we get

$$F^{(\alpha)}_{mn}(z_1,z_2) = \frac{\delta_{mn}}{\chi_\alpha}F^{(\alpha)}_m(z_1,z_2) \equiv \frac{\delta_{mn}}{\chi_\alpha}\sum_{l=1}^{\infty}(-1)^{l+1}F^{(\alpha)}_{m,-l}(z_1,z_2),$$

where

$$
\begin{aligned}
\chi_\alpha &= Ch_{Ker_{\mathcal{F}_{(\alpha;\beta)}}}(q^{2(N-1)}; q^{-(N-2)}, \cdots, q^{-l(N-1-l)}, \cdots, q^N),\\
F^{(\alpha)}_{m,l}(z_1,z_2) &= -e^{\frac{\sqrt{-1}\pi N}{2(N-1)}} C^*_1 C^*_N (C_1)^{N-1}(C_{N+1})^2\\
&\quad (z_1 q)^{\frac{1}{N-1}} \frac{\{\frac{z_1}{z_2}q^{2(N-1)}\}_\infty \{\frac{z_2}{z_1}q^{2(N-1)}\}_\infty}{\{\frac{z_1}{z_2}q^{2N}\}_\infty \{\frac{z_2}{z_1}q^{2N}\}_\infty} \oint dw_1 \cdots \oint dw_N\\
&\quad \times \{\prod_{k=1}^{m-1} \frac{(1-q^2)}{q w_{k-1}(\frac{w_k}{w_{k-1}}q; q^{2(N-1)})_\infty (\frac{w_{k-1}}{w_k}q; q^{2(N-1)})_\infty}\}\\
&\quad \times \frac{1}{w_{m-1}(\frac{w_m}{w_{m-1}}q; q^{2(N-1)})_\infty (\frac{w_{m-1}}{w_m}q^{2N-1}; q^{2(N-1)})_\infty}\\
&\quad \times \{\prod_{k=m+1}^{N} \frac{(1-q^2)}{w_k(\frac{w_k}{w_{k-1}}q; q^{2(N-1)})_\infty (\frac{w_{k-1}}{w_k}q; q^{2(N-1)})_\infty}\}\\
&\quad \times \sum_{\{i_1,\cdots,i_N\}\in \mathbf{Z}} I^{(a,\,l)}_{i_1,\cdots,i_N}(z_1,z_2|w_1,\cdots,w_N)\\
&\quad \times \left\{ \frac{(\frac{z_2}{w_N}q^{N-1})^{l-\alpha+i_N}}{w_N q(\frac{z_2}{w_N}q^{N-1}; q^{2(N-1)})_\infty (\frac{w_N}{z_2}q^{N-1}; q^{2(N-1)})_\infty} \right.\\
&\quad \left. + \frac{(\frac{z_2}{w_N}q^{N+1})^{l-\alpha+i_N}}{z_2 q^N(\frac{z_2}{w_N}q^{3N-1}; q^{2(N-1)})_\infty (\frac{w_N}{z_2}q^{-N-1}; q^{2(N-1)})_\infty} \right\},\\
&\quad \text{for} \quad m = 1, \cdots, N,
\end{aligned}
$$

$$
\begin{aligned}
F^{(\alpha)}_{N+1,l}(z_1,z_2) &= e^{\frac{\sqrt{-1}\pi N}{2(N-1)}} C^*_1 C^*_N (C_1)^{N}(C_{N+1})^2 (z_1 q)^{\frac{1}{N-1}} \frac{\{\frac{z_1}{z_2}q^{2(N-1)}\}_\infty \{\frac{z_2}{z_1}q^{2(N-1)}\}_\infty}{\{\frac{z_1}{z_2}q^{2N}\}_\infty \{\frac{z_2}{z_1}q^{2N}\}_\infty}\\
&\quad \times \oint dw_1 \cdots \oint dw_N \{\prod_{k=1}^{N} \frac{(1-q^2)}{q w_{k-1}(\frac{w_k}{w_{k-1}}q; q^{2(N-1)})_\infty (\frac{w_{k-1}}{w_k}q; q^{2(N-1)})_\infty}\}\\
&\quad \times \frac{1}{w_N(\frac{z_2}{w_N}q^{N+1}; q^{2(N-1)})_\infty (\frac{w_N}{z_2}q^{N-1}; q^{2(N-1)})_\infty}\\
&\quad \times \sum_{\{i_1,\cdots,i_N\}\in \mathbf{Z}} I^{(a,\,l)}_{i_1,\cdots,i_N}(z_1,z_2|w_1,\cdots,w_N)\\
&\quad \times \partial_{w_N}\{\frac{(\frac{z_2}{w_N}q^{N})^{l-\alpha+i_N}}{w_N(\frac{z_2}{w_N}q^{N}; q^{2(N-1)})_\infty (\frac{w_N}{z_2}q^{N-2}; q^{2(N-1)})_\infty}\}.
\end{aligned}
$$

In the above equations, $w_0 \equiv z_1 q$, and

$$
\begin{aligned}
I^{(a,\,l)}_{i_1,\cdots,i_N}(z_1,z_2|w_1,\cdots,w_N) &= q^{(N-1)\alpha(\alpha-1)}(z_1 q)^{i_1-\frac{\alpha}{N-1}}(z_2 q^N)^{\frac{N}{N-1}\alpha - i_N}\\
&\quad \times q^{(N-1)\{l^2+l(1-2\alpha)+i_N^2+i_N(1-2\alpha+2l)+\Delta(i_1,\cdots i_N)\}}\\
&\quad \times \prod_{k=1}^{N}(w_k q^{k(N-1-k)})^{-\lambda^k_{i_1,\cdots,i_N}},
\end{aligned}
$$

$$
\begin{aligned}
&C^*_1 = \frac{\{q^{2N}\}_\infty}{\{q^{4N-4}\}_\infty}, \qquad C^*_N = \frac{\{q^{4N-2}\}_\infty}{\{q^{2(N-1)}\}_\infty},\\
&C_1 = (q^{2(N-1)}; q^{2(N-1)})_\infty (q^{2N}; q^{2(N-1)})_\infty, \quad C_{N+1} = (q^{2(N-1)}; q^{2(N-1)})_\infty.
\end{aligned}
$$

We now derive the difference equations satisfied by these one-point functions. Noticing that

$$x^d\phi_i(z)x^{-d}=\phi_i(zx^{-1}),\quad x^d\phi^*_i(z)x^{-d}=\phi^*_i(zx^{-1}),$$
$$x^d\psi_i(z)x^{-d}=\psi_i(zx^{-1}),\quad x^d\psi^*_i(z)x^{-d}=\psi^*_i(zx^{-1}),$$
$$x^d\eta_0x^{-d}=\eta_0,\quad x^d\xi_0x^{-d}=\xi_0,$$

we get the difference equations

$$F^{(\alpha)}_m(z_1,z_2q^{2(N-1)})=q^{-2\rho_m}\sum_k R(z_2,z_1)^{km}_{mk}F^{(\alpha-1)}_k(z_1,z_2)\,(-1)^{[m]+[k]+[m][k]}.$$

Since $\alpha\in\mathbf{Z}$, it is easily seen that this is a set of infinite number of difference equations.

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Appendix A

In this appendix, we give the normal ordered relations of the fundamental bosonic fields:

$$:e^{h^i(z;\beta_1)}::e^{h^j(w;\beta_2)}:=z^{a_{ij}}(1-\frac{w}{z}q^{\beta_1+\beta_2})^{a_{ij}}:e^{h^i(z;\beta_1)+h^j(w;\beta_2)}:,\quad i\neq j,$$
$$:e^{h^i(z;\beta_1)}::e^{h^i(w;\beta_2)}:=z^2(1-\frac{w}{z}q^{\beta_1+\beta_2-1})(1-\frac{w}{z}q^{\beta_1+\beta_2+1}):e^{h^i(z;\beta_1)+h^i(w;\beta_2)}:,\quad i\neq N,$$
$$:e^{h^N(z;\beta_1)}::e^{h^N(w;\beta_2)}:=:e^{h^N(z;\beta_1)+h^N(w;\beta_2)}:,$$
$$:e^{h^i(z;\beta_1)}::e^{h^*_j(w;\beta_2)}:=z^{\delta_{ij}}(1-\frac{w}{z}q^{\beta_1+\beta_2})^{\delta_{ij}}:e^{h^i(z;\beta_1)+h^*_j(w;\beta_2)}:,$$
$$:e^{h^*_i(z;\beta_1)}::e^{h^j(w;\beta_2)}:=z^{\delta_{ij}}(1-\frac{w}{z}q^{\beta_1+\beta_2})^{\delta_{ij}}:e^{h^*_i(z;\beta_1)+h^j(w;\beta_2)}:,$$
$$:e^{h^*_N(z;\beta_1)}::e^{h^*_N(w;\beta_2)}:=z^{-\frac{N}{N-1}}\frac{(\frac{w}{z}q^{\beta_1+\beta_2+2N-1};q^{2(N-1)})}{(\frac{w}{z}q^{\beta_1+\beta_2-1};q^{2(N-1)})}:e^{h^*_N(z;\beta_1)+h^*_N(w;\beta_2)}:,$$
$$:e^{h^*_1(z;\beta_1)}::e^{h^*_1(w;\beta_2)}:=z^{\frac{N-2}{N-1}}\frac{(\frac{w}{z}q^{\beta_1+\beta_2+1};q^{2(N-1)})}{(\frac{w}{z}q^{\beta_1+\beta_2+2N-3};q^{2(N-1)})}:e^{h^*_1(z;\beta_1)+h^*_1(w;\beta_2)}:,$$
$$:e^{h^*_1(z;\beta_1)}::e^{h^*_N(w;\beta_2)}:=z^{-\frac{1}{N-1}}\frac{(\frac{w}{z}q^{\beta_1+\beta_2+N};q^{2(N-1)})}{(\frac{w}{z}q^{\beta_1+\beta_2+N-2};q^{2(N-1)})}:e^{h^*_1(z;\beta_1)+h^*_N(w;\beta_2)}:,$$
$$:e^{h^*_N(z;\beta_1)}::e^{h^*_1(w;\beta_2)}:=z^{-\frac{1}{N-1}}\frac{(\frac{w}{z}q^{\beta_1+\beta_2+N};q^{2(N-1)})}{(\frac{w}{z}q^{\beta_1+\beta_2+N-2};q^{2(N-1)})}:e^{h^*_N(z;\beta_1)+h^*_1(w;\beta_2)}:,$$
$$:e^{c(z;\beta_1)}::e^{c(w;\beta_2)}:=z(1-\frac{w}{z}q^{\beta_1+\beta_2}):e^{c(z;\beta_1)+c(w;\beta_2)}:,$$

where a_{ij} is the Cartan-matrix of $sl(\widehat{N}|1)$ and $i,j=1,2,\cdots,N$.

Appendix B

By means of the bosonic realization (II.10) of $U_q[sl(\widehat{N}|1)]$, the integral expressions of the bosonized vertex operators (II.16) and the technique given in [18], one can check the following relations

- For the type I vertex operators:

$$\begin{aligned}
&[\phi_k(z), f_l] = 0 \text{ if } k \neq l, l+1, \quad [\phi_{l+1}(z), f_l]_{q^{\nu_{l+1}}} = \nu_l \phi_l(z)(-1)^{[f_l]([v_l]+[v_{l+1}])},\\
&[\phi_l(z), f_l]_{q^{-\nu_l}} = 0, \quad [\phi_l(z), e_l] = q^{h_l}\phi_{l+1}(z)(-1)^{[e_l]([v_l]+[v_{l+1}])},\\
&[\phi_k(z), e_l] = 0 \text{ if } k \neq l, \quad q^{h_l}\phi_l(z)q^{-h_l} = q^{-\nu_l}\phi_l(z),\\
&q^{h_l}\phi_k(z)q^{-h_l} = \phi_k(z) \text{ if } k \neq l, l+1, \quad q^{h_l}\phi_{l+1}(z)q^{-h_l} = q^{\nu_{l+1}}\phi_{l+1}(z),
\end{aligned}$$

$$\begin{aligned}
&[\phi^*_k(z), f_l] = 0 \text{ if } k \neq l, l+1, \quad [\phi^*_{l+1}(z), f_l]_{q^{-\nu_{l+1}}} = 0,\\
&[\phi^*_k(z), e_l] = 0 \text{ if } k \neq l+1, \quad [\phi^*_{l+1}(z), e_l] = -\nu_l\nu_{l+1}q^{h_l-\nu_l}\phi^*_l(z)(-1)^{[e_l]([v_l]+[v_{l+1}])},\\
&[\phi^*_l(z), f_l]_{q^{\nu_l}} = -\nu_l q^{\nu_l}\phi^*_{l+1}(z)(-1)^{[f_l]([v_l]+[v_{l+1}])}, \quad q^{h_l}\phi^*_l(z)q^{-h_l} = q^{\nu_l}\phi^*_l(z),\\
&q^{h_l}\phi^*_k(z)q^{-h_l} = \phi^*_k(z) \text{ if } k \neq l, l+1, \quad q^{h_l}\phi^*_{l+1}(z)q^{-h_l} = q^{-\nu_{l+1}}\phi^*_{l+1}(z).
\end{aligned}$$

- For the type II vertex operators:

$$\begin{aligned}
&[\psi_k(z), e_l] = 0 \text{ if } k \neq l, l+1, \quad [\psi_{l+1}(z), e_l]_{q^{-\nu_{l+1}}} = 0, \quad [\psi_l(z), e_l]_{q^{\nu_l}} = \psi_{l+1}(z),\\
&[\psi_k(z), f_l] = 0 \text{ if } k \neq l+1, \quad [\psi_{l+1}(z), f_l] = \nu_l q^{-h_l}\psi_l(z),\\
&q^{h_l}\psi_l(z)q^{-h_l} = q^{-\nu_l}\psi_l(z), \quad q^{h_l}\psi_{l+1}(z)q^{-h_l} = q^{\nu_{l+1}}\psi_{l+1}(z),\\
&q^{h_l}\psi_k(z)q^{-h_l} = \psi_k(z) \text{ if } k \neq l, l+1,
\end{aligned}$$

$$\begin{aligned}
&[\psi^*_k(z), e_l] = 0 \text{ if } k \neq l, l+1, \quad [\psi^*_l(z), e_l]_{q^{-\nu_l}} = 0,\\
&[\psi^*_k(z), f_l] = 0 \text{ if } k \neq l, \quad [\psi^*_l(z), f_l] = -\nu_l q^{-h_l+\nu_l}\psi^*_{l+1}(z),\\
&[\psi^*_{l+1}(z), e_l]_{q^{\nu_{l+1}}} = -\nu_l\nu_{l+1}q^{-\nu_l}\psi^*_l(z), \quad q^{h_l}\psi^*_l(z)q^{-h_l} = q^{\nu_l}\psi^*_l(z),\\
&q^{h_l}\psi^*_k(z)q^{-h_l} = \psi^*_k(z) \text{ if } k \neq l, l+1, \quad q^{h_l}\psi^*_{l+1}(z)q^{-h_l} = q^{-\nu_{l+1}}\psi^*_{l+1}(z).
\end{aligned}$$

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