New Solutions of the Einstein-Dirac Equation in Dimension $n = 3$. *

a short announcement by Thomas Friedrich (Berlin)

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Consider a Riemannian spin manifold of dimension $n \geq 3$ and denote by D the Dirac operator acting on spinor fields. A solution of the Einstein-Dirac equation is a spinor field ψ solving the equations

$$Ric - \frac{1}{2}\, S \cdot g = \pm \frac{1}{4}\, T_\psi \quad , \quad D(\psi) = \lambda\psi.$$

Here S denotes the scalar curvature of the space, λ is a real constant and T_ψ is the energy-momentum tensor of the spinor field ψ defined by the formula

$$T_\psi(X, Y) = (X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi, \psi).$$

Any weak Killing spinor ψ^* (WK-spinor)

$$\nabla_X \psi^* = \frac{n}{2(n-1)}\, dS(X)\psi^* + \frac{2\lambda}{(n-2)S}\, Ric(X) \cdot \psi^* - \frac{\lambda}{n-2}\, X \cdot \psi^* + \frac{1}{2(n-1)S}\, X \cdot dS \cdot \psi^*$$

yields a solution ψ of the Einstein-Dirac equation after normalization

$$\psi = \sqrt{\frac{(n-2)|S|}{|\lambda||\psi^*|^2}}\, \psi^*.$$

In fact, in dimension $n = 3$ the Einstein-Dirac equation is essentially equivalent to the weak Killing equation (see [KimF]). Up to now the following 3-dimensional Riemannian manifolds admitting WK-spinors are known:

1. the flat torus T^3 with a parallel spinor;
2. the sphere S^3 with a Killing spinor;
3. two non-Einstein Sasakian metrics on the sphere S^3 admitting WK-spinors. The scalar curvature of these two left-invariant metrics equals $S = 1 \pm \sqrt{5}$.

The aim of this short note is to announce the existence of a one-parameter family of left-invariant metrics on S^3 admitting WK-spinors. This family contains the two non-Einstein Sasakian metrics with WK-spinors on S^3, but does not contain the standard sphere S^3 with Killing spinors. Moreover, any simply-connected, complete Riemannian manifold $X^3 \neq S^3$ with WK-spinors such that the eigenvalues of the Ricci tensor are

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constant is isometric to a space of this one-parameter family.
In order to formulate the result precisely, we fix real parameters $K, L, M \in \mathbb{R}$ and denote by $X^3(K, L, M)$ the 3-dimensional, simply-connected and oriented Riemannian manifold defined by the following structure equations:

$$\omega_{12} = K\sigma^3 \quad , \quad \omega_{13} = L\sigma^2 \quad , \quad \omega_{23} = M\sigma^1,$$

or, equivalently:

$$d\sigma^1 = (L - K)\sigma^2 \wedge \sigma^3 \quad , \quad d\sigma^2 = (M + K)\sigma^1 \wedge \sigma^3 \quad , \quad d\sigma^3 = (L - M)\sigma^1 \wedge \sigma^2.$$

The 1-forms $\sigma^1, \sigma^2, \sigma^3$ are the dual forms of an orthonormal frame of vector fields. Using this frame the Ricci tensor of $X^3(K, L, M)$ is given by the matrix

$$Ric = \begin{pmatrix} -2KL & 0 & 0 \\ 0 & 2KM & 0 \\ 0 & 0 & -2LM \end{pmatrix}.$$

Theorem: *Let $X^3 \neq S^3$ be a complete, simply-connected Riemannian manifold such that:*

a) the eigenvalues of the Ricci tensor are constant;

b) the scalar curvature $S \neq 0$ does not vanish.

If X^3 admits a WK-spinor, then X^3 is isometric to $X^3(K, L, M)$ and the parameters are a solution of the equation

$$-K^2L(L - M)^2M + L^3M^3 + KL^2M^2(M - L) + K^3(L - M)(L + M)^2 = 0 \qquad (*)$$

Conversely, any space $X^3(K, L, M)$ such that $(K, L, M) \neq (0, 0, 0)$ is a solution of $(*)$ admits two WK-spinors for one and only one WK-number λ. With respect to the fixed orientation of $X^3(K, L, M)$ we have the two cases:

$$\lambda = +\frac{S}{2\sqrt{2}} \sqrt{\frac{S}{S^2 - |Ric|^2}} \qquad \text{if } -K < M$$

$$\lambda = -\frac{S}{2\sqrt{2}} \sqrt{\frac{S}{S^2 - |Ric|^2}} \qquad \text{if } M < -K.$$

The spaces $X^3(K, L, M)$ are isometric to S^3 equipped with a left-invariant metric.

Remark: If the parameters $K = M$ coincide, the solution of the equation $(*)$ is given by

$$L = \frac{1}{4}K(1 - \sqrt{5}) \quad , \quad L = \frac{1}{4}K(1 + \sqrt{5})$$

and we obtain the Ricci tensors

$$Ric = \begin{pmatrix} \frac{1}{2}K^2(\sqrt{5}-1) & 0 & 0 \\ 0 & 2K^2 & 0 \\ 0 & 0 & \frac{1}{2}K^2(\sqrt{5}-1) \end{pmatrix}$$

or

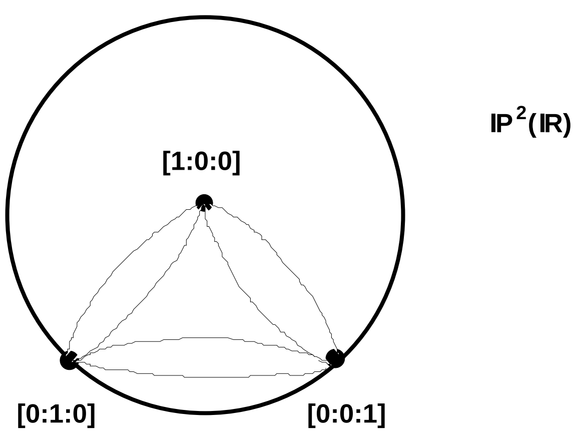
$$Ric = \begin{pmatrix} -\frac{1}{2}K^2(1+\sqrt{5}) & 0 & 0 \\ 0 & 2K^2 & 0 \\ 0 & 0 & -\frac{1}{2}K^2(1+\sqrt{5}) \end{pmatrix}.$$

The non-Einstein-Sasakian metrics on S^3 occur for the parameter $K = 1$ (see [KimF]).

Remark: Using the standard basis of the Lie algebra $\mathfrak{so}(3)$ we can write the left-invariant metric of the space $X^3(K, L, M)$ in the following way:

$$\begin{pmatrix} \frac{1}{|M-L||K+M|} & 0 & 0 \\ 0 & \frac{1}{|K-L||M-L|} & 0 \\ 0 & 0 & \frac{1}{|K-L||K+M|} \end{pmatrix}.$$

The equation $(*)$ is a homogeneous equation of order six. The transformation $(K, L, M) \to (\mu K, \mu L, \mu M)$ corresponds to a homothety of the metric. Therefore - up to a homothety - the moduli space of solutions is a subset of the real projective space $\mathbb{P}^2(\mathbb{R})$ given by the equation $(*)$. This subset is a configuration of six curves in $\mathbb{P}^2(\mathbb{R})$ connecting the three points $[K : L : M] = [1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$, corresponding to flat metrics.



In particular, we have constructed two paths of solutions of the Einstein-Dirac equation deforming the non-Einstein Sasakian metrics on S^3.

The proof of the Theorem as well as the complete computations will be published in a furthercoming paper of the author (see [F]).

References

[KimF] E.C. Kim and Th. Friedrich, The Einstein-Dirac equation on Riemannian spin manifolds, Journ. Geom. Phys. 33 (2000), 128-172.

[F] Th. Friedrich, New solutions of the Einstein-Dirac equation in dimension $n = 3$, to appear.