Application of Tree-like Structure of Graph to Matrix Analysis.

Buslov V.A.

Abstract

Formulas for matrix determinants, algebraic adjunctions, characteristic polynomial coefficients, components of eigenvectors are obtained in the form of signless sums of matrix elements products taking by special graphs. Signless formulas are very important for singular and stochastic problems. They are also useful for spectral analysis of large very sparse matrices.

1   Introduction

Graph theory is a natural instrument for matrix determinants and combined features calculating. It is clear that determinants can be expressed in form of alternating sum of matrix elements products taking by some graphs corresponding to index permutation, each term sign is determined by permutation even. However, the alternation itself is an essential obstruction both in numerical calculating and in analytic research of spectrum properties. It concerns for example cases when matrix elements depend on small (large) parameter and possess on it different orders. Thus under singular perturbed equations research [4] (in particular such as Fokker-Plank equations) necessity appears for spectral analysis of big order matrix having exponentially small elements [7, 8]. The same situation appear under consideration of Markov's chains connected with diffusion process [5] where the questions of stochastic continuity, possible subprocess kinds and their structure properties run into the necessity of spectral analysis just such kind of matrixes.

Our aim is to obtain the formulas for matrix determinants, algebraic adjunctions, characteristic polynomial coefficients, components of eigen-vectors in the form of signless sums of matrix elements products taking by special graphs. It turns out that one can get such formulas in terms of so called "tree" structure of some graph corresponding to matrix.

The first step in this direction was made by Kirchhoff [1] who computed the number of connected subgraphs containing all vertices and containing no circuits (spanning trees). This number turned out to be equal to cofactor of any element of so called conductivity or Kirchhoff matrix of non-directed graph. Later this theorem was generalized for all coefficients of characteristic polynomial of this matrix, and also for cases of directed graphs where every arc (ordered pairs of vertices) possesses some quantity called weight [3]. Here we get the same type formulas not for special matrices (as Kirchhoff type) but for arbitrary ones.

2   Main definitions and designations

Unification of designations and even terminology proper is not complete yet in graph theory. So firstly we adduce the necessary definitions and notations.

Let G be digraph (directed graph). We use $\mathcal{V}G$ and $\mathcal{A}G$ to denote the set of vertices and arcs of G. The subgraph H of G is called factor if $\mathcal{V}H = \mathcal{V}G$. The outdegree (indegree) of the vertex i (the number of arcs going out of (into) i) we denote $d^+(i)$ $(d^-(i))$.

If G is digraph in which every arc has its own weight g_{ij} (weighted adjacencies digraph), corresponding matrix $\mathbf{G} = \{g_{ij}\}_{i,j=1}^N$ is called generalized adjacency matrix (the element $g_{ij} = 0$ if there is no arc (i,j) in G).

The sequence of following each other arcs along their orientations is called a way if all vertices besides possibly the uttermost ones are different. The way connecting the vertices m and n we denote $m \cdot n$. The cyclic way is called dicircuit. Linear digraph is digraph every vertex of which has unit in- and out-degree. So it consists of dicircuits.

We associate with any weighed adjacencies digraph G the quantity π_G by the rule

$$\pi_G = \prod_{(i,j)\in\mathcal{A}G} g_{ij}$$

which is naturally to call by productivity.

Later on forests are the main graph theory object we use. As known there are two forest kinds in digraph situation. Here we call by forest digraph without dicircuits in which every vertex outdegree is equal to 0 or 1 ($d^+(i) = 0, 1$). The only vertex of tree (component of forest) having zero outdegree ($d^+(i) = 0$) we call root.

Let $\mathcal{N}$ be finite set and $\mathcal{W}$ is some subset of $\mathcal{N}$. By $\mathcal{F}_{\mathcal{W}}^k$, $k \le |\mathcal{N}| - |\mathcal{W}|$ we denote the set of forests F obeying the following conditions

1) $\mathcal{V}F = \mathcal{N}$;

2) F consists of exactly $k + |\mathcal{W}|$ trees;

3) The set of roots of F contain $\mathcal{W}$ as a subset.

Suppose also $\mathcal{F}^k = \mathcal{F}_\emptyset^k$, $\mathcal{F}_\mathcal{W} = \mathcal{F}_\mathcal{W}^0$. Note that the set $\mathcal{F}^0$ is empty set and $\mathcal{F}^{|\mathcal{N}|}$ consists of the only empty forest having only roots and no arcs.

If some additional condition z is put on forests from $\mathcal{F}_\mathcal{U}^k$ we denote such set of forests as $\mathcal{F}_\mathcal{U}^k(z)$. Other necessary utilized notations we sign as necessary and also we use sometimes the term "graph" in wide sense designating by it digraphs with weighted adjacencies too.

3 Circuit- and tree-like structures

The matrix (graph) spectral analysis can be carried out using some form of characteristic polynomial of matrix itself or some special matrixes (graphs) constructed from it. Thus it is valid known "theorem on coefficients for digraphs" [2].

Theorem. *Let*

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^N + a_1\lambda^{N-1} + \cdots + a_N$$

be characteristic polynomial of arbitrary digraph A with weighted adjacencies a_{ij}. Then

$$a_i = \sum_{L\in\mathcal{L}_i} (-1)^{p(L)}\pi_L , \quad i = 1, 2, \cdots, N , \tag{1}$$

where $\mathcal{L}_i$ is the set of all linear directed subgraphs L of graph A with exactly i vertices; $p(L)$ means the number of components (dicircuits) of L.

Coefficients in (1) are expressed in "circuit" structure of A, and this theorem is not more than rephrasing from the standard determinant notation $|\lambda\mathbf{I} - \mathbf{A}|$ in the form of matrix elements products sum with sign determined by substitution even (the number of dicircuits) into graph terms.

In terms of "tree" structure it is known the characteristic polynomial expression not for matrix $\mathbf{G}$ itself but for its Kirchhoff matrix $\mathbf{C}$ (or conductivity matrix) determined like

$$\mathbf{C} = \mathbf{D} - \mathbf{G} \ ,$$

where $\mathbf{D}$ is weighted powers matrix

$$\mathbf{D} \equiv \operatorname{diag}(\sum_{j=1}^{N} g_{1j}, \sum_{j=1}^{N} g_{2j}, \cdots, \sum_{j=1}^{N} g_{Nj}) \ .$$

Corresponding expression has a form [3]:

$$\det(\lambda\mathbf{I} - \mathbf{C}) = (-1)^N \sum_{k=0}^{N} (-\lambda)^k \Big[\sum_{F \in \mathcal{F}^k(G)} \pi_F \Big] \ , \tag{2}$$

$$\pi_F = \prod_{(i,j) \in \mathcal{A}F} f_{ij} = \prod_{(i,j) \in \mathcal{A}F} g_{ij} \ .$$

Here the set of forests containing directly k trees and being factors of G is designated by $\mathcal{F}^k(G)$. Note that as the sum of elements along every line of $\mathbf{C}$ is equal to zero, so its determinant is equal to zero too and the sum in (refein) one can lead from $k = 1$. In the following in clear cases we omit indication on graph.

4 The characteristic polynomial in tree-like structure terms

Knowing the characteristic polynomial expression of the admittance matrix $\mathbf{C}$ in terms of tree structure it is not hard to get analogous expression for characteristic polynomial of the matrix $\mathbf{G}$ itself. For this aim let us construct from G some new graph $G^\dagger$ by the next rule. Let $\mathcal{N}^\dagger = \mathcal{N} \cup \{\dagger\}$ be the set of vertices $\mathcal{N} = \{1, 2, \cdots, N\}$ of G which is supplemented by some new vertex designated $\dagger$. Let also add to G the vertex $\dagger$ and lay on out of every vertex $i \in \mathcal{N}$ the arc $(i, \dagger)$ with weight $g_{i\dagger} = -\sum_{j=1}^{N} g_{ij}$, and remove all loops (i, i). We denote the obtained graph by $G^\dagger$ ($G^\dagger : \mathcal{V}G^\dagger = \mathcal{N}^\dagger$, $\mathcal{A}G^\dagger = \{\mathcal{A}G \setminus \bigcup_{i \in \mathcal{N}}(i, i)\} \bigcup_{i \in \mathcal{N}}(i, \dagger)$, weights of arcs are equal g_{ij} , $i \in \mathcal{N}$, $j \in \mathcal{N}^\dagger$). Corresponding generalized adjacencies matrix $\mathbf{G}^\dagger$ of $G^\dagger$ has the following form

$$
\mathbf{G}^\dagger =
\begin{pmatrix}
0 & g_{12} & g_{13} & \cdots & g_{1N} & g_{1\dagger} \\
g_{21} & 0 & g_{23} & \cdots & g_{2N} & g_{2\dagger} \\
g_{31} & g_{32} & 0 & \cdots & g_{3N} & g_{3\dagger} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
g_{N1} & g_{N2} & g_{N3} & \cdots & 0 & g_{N\dagger} \\
0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix} .
$$

Note that, if $\mathbf{G} = \mathbf{P} - \mathbf{I}$, where $\mathbf{P}$ is probability matrix setting finite Markov's chain with killing, so the quantity $g_{i\dagger}$ is the probability of killing of the process if it is in state i. This has a sense of probability of outcoming to the bounder, so the additional vertex $\dagger$ could be interpreted in some sense as a bounder of the finite set $\mathcal{N} = \{1, 2, \cdots, N\}$.

It is easy to see, that $(N+1) \times (N+1)$ admittance matrix $\mathbf{C}^\dagger$ of graph $G^\dagger$ has the form

$$
\mathbf{C}^\dagger = -
\left(
\begin{array}{ccc|c}
& & & g_{1\dagger} \\
& \mathbf{G} & & g_{2\dagger} \\
& & & \vdots \\
& & & g_{N\dagger} \\
\hline
0 \ \ 0 & \cdots & 0 & 0
\end{array}
\right) ,
$$

so, using (2) we get the chain of equations

$$
\det(\lambda \mathbf{I} - \mathbf{G}) = \frac{1}{\lambda} \det(\lambda \mathbf{I} + \mathbf{C}^\dagger) = \frac{1}{\lambda} \sum_{k=1}^{N+1} \lambda^k \Big[\sum_{F \in \mathcal{F}^k(G^\dagger)} \pi_F \Big] =
$$

$$
= \sum_{k=0}^{N} \lambda^k \Big[\sum_{F \in \mathcal{F}^{k+1}(G^\dagger)} \pi_F \Big] .
$$

Note, that since the vertex $\dagger$ in graph $G^\dagger$ has zero outdegree and hence it is a root in every forest $F \in \mathcal{F}^l(G^\dagger)$, so the sets $\mathcal{F}^{k+1}(G^\dagger)$ and $\mathcal{F}_\dagger^k(G^\dagger)$ coincide. Thus it is valid

Theorem 1. *Characteristic polynomial of an arbitrary $N \times N$ matrix $\mathbf{G}$ can be expressed in the form*

$$
\det(\lambda \mathbf{I} - \mathbf{G}) = \sum_{k=0}^{N} \lambda^k \Big[\sum_{F \in \mathcal{F}_\dagger^k} \pi_F \Big] , \quad \pi_F = \prod_{(i,j) \in \mathcal{A}F} g_{ij} , \tag{3}
$$

where $\mathcal{F}_\dagger^k \equiv \mathcal{F}_\dagger^k(G^\dagger)$.

Under $\lambda = 0$ we obtain obvious

Consequence. *The determinant of $N \times N$ matrix $\mathbf{G}$ can be expressed in the form*

$$
\det \mathbf{G} = (-1)^N \sum_{F \in \mathcal{F}_\dagger} \pi_F , \tag{4}
$$

$$
\mathcal{F}_\dagger \equiv \mathcal{F}_\dagger(G^\dagger) .
$$

Let $\mathcal{R}$ be a subset of $\mathcal{N}$. Designate by $\mathbf{G}_{\mathcal{R}\mathcal{R}}$ matrix obtained from $\mathbf{G}$ by striking out the columns and lines with numbers $i \in \mathcal{R}$. So, $\mathbf{G}_{\mathcal{R}\mathcal{R}}$ is a diagonal minor of $\mathbf{G}$ of $(N - |\mathcal{R}|)$-th order. The corresponding to it digraph we denote by $G_{\mathcal{R}\mathcal{R}}$.

Consequence of consequence. *The determinant of minor* $\mathbf{G}_{\mathcal{R}\mathcal{R}}$ *can be expressed in the form*

$$\det \mathbf{G}_{\mathcal{R}\mathcal{R}} = (-1)^{N-|\mathcal{R}|} \sum_{F \in \mathcal{F}_{\{\dagger\} \cup \mathcal{R}}} \pi_F \ , \tag{5}$$

$$\mathcal{F}_{\{\dagger\} \cup \mathcal{R}} = \mathcal{F}_{\{\dagger\} \cup \mathcal{R}}(G^\dagger) \ .$$

Proof. By formula (4)

$$\det \mathbf{G}_{\mathcal{R}\mathcal{R}} = (-1)^{N-|\mathcal{R}|} \sum_{F \in \mathcal{F}_\dagger(G_{\mathcal{R}}^\dagger)} \pi_F \ .$$

Let us keep in $\mathbf{G}_{\mathcal{R}\mathcal{R}}$ the same numeration of elements as it is in matrix $\mathbf{G}$. So the elements $(g_{\mathcal{R}\mathcal{R}})_{il}$ of the corresponding matrix $\mathbf{G}_{\mathcal{R}\mathcal{R}}^\dagger$ (weights of arcs (i,l) of graph $G_{\mathcal{R}\mathcal{R}}^\dagger$) are equal:

$(g_{\mathcal{R}\mathcal{R}})_{il} = g_{il} \ , \ i,l \in \mathcal{N} \setminus \mathcal{R} \ ,$
$(g_{\mathcal{R}\mathcal{R}})_{il} = 0 \ \{i,l\} \cap \mathcal{R} \neq \emptyset,$
$(g_{\mathcal{R}\mathcal{R}})_{i\dagger} = \sum\limits_{m \in \{\dagger\} \cup \mathcal{R}} g_{im} \ , i \in \mathcal{N} \setminus \mathcal{R} \ .$

So productivity π_F of any forest $F \in \mathcal{F}_\dagger(G_{\mathcal{R}\mathcal{R}}^\dagger)$ represents the productivity $\pi_{F'}$ of some forest of the set $\mathcal{F}_{\{\dagger\} \cup \mathcal{R}}(G^\dagger)$. The sum of productivities π_F along all forests $F \in \mathcal{F}_\dagger(G_{\mathcal{R}\mathcal{R}}^\dagger)$ exhausts the set $\mathcal{F}_{\{\dagger\} \cup \mathcal{R}}(G^\dagger)$, which proves (5).

5 Formulas for components of eigenvectors

If it is known the eigenvalue λ of $\mathbf{G}$, so to calculate the components v_m of corresponding eigenvector $\vec{v}$ it is necessary to decide the standard system $(\lambda \mathbf{I} - \mathbf{G})\vec{v} = 0$. Let for certainty the n-th component of eigenvector $\vec{v}$ be not equal to zero. Without loss of generality one can accept it be equal to one: $v_n = 1$. Then we have for the rest of components following the Kramer rule

$$v_m = \frac{\det \Delta'_{nm}(\lambda)}{\det \Delta_{nn}(\lambda)} \ , \tag{6}$$

where

$$\Delta_{nn}(\lambda) = \lambda \mathbf{I} - \mathbf{G}_{nn} \ ,$$

$\mathbf{G}_{nn}$ is algebraic adjunct of the element g_{nn} of matrix $\mathbf{G}$, and $\Delta'_{nm}(\lambda)$ is a matrix obtained from $\Delta_{nn}(\lambda)$ by substitution of the m-th column of $\lambda \mathbf{I} - \mathbf{G}$ by n-th one with negative sign.

The expression for $\Delta_{nn}(\lambda)$ it is easy to get using already obtained formula (5) concerning diagonal minors $\mathbf{G}_{\mathcal{R}\mathcal{R}}$ of matrix $\mathbf{G}$. Since the coefficient at λ^k at the expression of characteristic polynomial of algebraic adjunct $\mathbf{G}_{nn}$ is itself a sum of determinants of

diagonal minors of matrix $\mathbf{G}$ of $(N-k-1)$-th order with sign $(-1)^{N-k-1}$ and not including n-th column and n-th line so one can write

$$\det(\lambda\mathbf{I}-\mathbf{G}_{nn}) = \sum_{k=0}^{N-1}\lambda^k\Big[\sum_{\substack{\mathcal{R}\in\mathcal{N},\ n\in\mathcal{R}\\|\mathcal{R}|=k+1}}\det(-\mathbf{G}_{\mathcal{R}\mathcal{R}})\Big] =$$

$$= \sum_{k=0}^{N-1}\lambda^k\Big[(-1)^{N-k-1}\sum_{\substack{\mathcal{R}\in\mathcal{N},\ n\in\mathcal{R}\\|\mathcal{R}|=k+1}}\det\mathbf{G}_{\mathcal{R}\mathcal{R}}\Big] =$$

$$= \sum_{k=0}^{N-1}\lambda^k\Big[\sum_{\substack{\mathcal{R}\in\mathcal{N},\ n\in\mathcal{R}\\|\mathcal{R}|=k+1}}\Big\{\sum_{F\in\mathcal{F}_{\{\dagger\}\cup\mathcal{R}}(G^\dagger)}\pi_F\Big\}\Big] .$$

The sum at square brackets in the last expression obviously is equal to the sum of productivities π_F of all forests belonging to the unification

$$\bigcup_{\substack{\mathcal{R}\in\mathcal{N},\ n\in\mathcal{R}\\|\mathcal{R}|=k+1}}\mathcal{F}_{\{\dagger\}\cup\mathcal{R}}(G^\dagger) ,$$

which in its turn is equal to $\mathcal{F}^k_{\{\dagger,n\}}(G^\dagger)$. So, the denominator at (6) has the form

$$\det\Delta_{nn}(\lambda) = \det(\lambda\mathbf{I}-\mathbf{G}_{nn}) = \sum_{k=0}^{N-1}\lambda^k\Big[\sum_{F\in\mathcal{F}^k_{\{\dagger,n\}}}\pi_F\Big] . \qquad (7)$$

$$\mathcal{F}^k_{\{\dagger,n\}} \equiv \mathcal{F}^k_{\{\dagger,n\}}(G^\dagger) ,$$

It is more difficult to obtain the formula for numerator at (6) having the form

$$\det\Delta'_{nm}(\lambda) = \sum_{k=0}^{N-1}\lambda^k\Big[\sum_{F\in\mathcal{F}^k_{\{\dagger,n\}}(m\cdot n)}\pi_F\Big] , \qquad (8)$$

where by $\mathcal{F}^k_{\{\dagger,n\}}(m\cdot n)$ the subset of $\mathcal{F}^k_{\{\dagger,n\}}(G^\dagger)$ having $m\cdot n$-walk is denoted. The summing over k one can lead till $N-2$, because the set $\mathcal{F}^{N-1}_{\{\dagger,n\}}(m\cdot n)$ is empty.

It is sufficiently to establish (8) under $\lambda=0$, because the coefficients at powers of λ are the diagonal minors of the same matrix and are utterly of analogical to each other form.

Let $\Delta'_{nm}\equiv\Delta'_{nm}(0)$. It is necessary to prove that

$$\det\Delta'_{nm} = \sum_{F\in\mathcal{F}_{\{\dagger,n\}}(m\cdot n)}\pi_F . \qquad (9)$$

In passing let us remark that using (9) it is easy to obtain the expression for algebraic adjunct G_{nm} of the element g_{nm}. Actually, matrices $-\Delta'_{nm}$ and G_{nm} differ only in such a way. In $-\Delta'_{nm}$ (if for example $n<m$) n-th column of matrix G_{nm} is at the $(m-1)$-th place and is of opposite sign. Hence

$$G_{nm} = (-1)^{N+m-n-1} \sum_{F \in \mathcal{F}_{\{\dagger,n\}}} \pi_F \ .$$

To prove (9) let us exchange in $\mathbf{G}$ places of n-th and m-th columns. Obtained auxiliary matrix we denote by $\mathbf{H}$. Nondiagonal elements h_{ij} of $\mathbf{H}$ and corresponding expanded matrix $\mathbf{H}^\dagger$ are equal

i) $h_{ij} = g_{ij}$, $j \neq n, m$,

ii) $h_{in} = g_{im}$, $h_{im} = g_{in}$.

It is easy to see that algebraic adjunct $\mathbf{H}_{nn}$ of the element h_{nn} of $\mathbf{H}$ differs from matrix $(-\Delta'_{nm})$ by only the sign of m-th column. So taking into account (7)

$$\det \Delta'_{nm} = -(-1)^{N-1} \det \mathbf{H}_{nn} = - \sum_{F \in \mathcal{F}_{\{\dagger,n\}}(H)} \pi_F \ . \tag{10}$$

Now in (10) it is necessary to cross from the sum of productivities π_F of factor forests of $H^\dagger$ to the sum of productivities π'_F of corresponding subgraphs of graph $G^\dagger$. Notice that in $H^\dagger$ weight of arc (m, n) is equal to minus sum of weights of arcs (m, i) going out of the vertex m in graph $G^\dagger$:

$$h_{mn} = g_{mm} = - \sum_{i \in \mathcal{N}^\dagger \setminus \{m\}} g_{mi} \ . \tag{11}$$

That is why we divide the set of forests $\mathcal{F}_{\{\dagger,n\}}(H^\dagger)$ in to two nonintersecting subsets: the set $\mathcal{F}_{\{\dagger,n\}}(H^\dagger; (m, n))$ of forests including the arc (m, n) and the set $\mathcal{F}_{\{\dagger,n\}}(H^\dagger;](m, n))$ of forests without this arc. Then

$$\sum_{F \in \mathcal{F}_{\{\dagger,n\}}(H^\dagger)} \pi_F = \sum_{F \in \mathcal{F}_{\{\dagger,n\}}(H^\dagger;(m,n))} \pi_F + \sum_{F \in \mathcal{F}_{\{\dagger,n\}}(H^\dagger;](m,n))} \pi_F \ . \tag{12}$$

The set of arcs $\mathcal{A}F$ of the forest $F \in \mathcal{F}_{\{\dagger,n\}}(H^\dagger; (m, n))$ is representable in a form $\mathcal{A}F = (m, n) \cup \mathcal{A}P$, where P is some graph belonging to the set $\mathcal{F}_{\{\dagger,n,m\}}(H^\dagger)$. By it as $h_{ij} = g_{ij}, j \neq m, n$ the sets $\mathcal{F}_{\{\dagger,n,m\}}(H^\dagger)$ and $\mathcal{F}_{\{\dagger,n,m\}}(G^\dagger)$ coincide. Hence, taking into account (11)

$$\sum_{F \in \mathcal{F}_{\{\dagger,n\}}(H^\dagger;(m,n))} \pi_F = - \sum_{i \in \mathcal{N}^\dagger \setminus \{m\}} g_{mi} \left[\sum_{F \in \mathcal{F}_{\{\dagger,n,m\}}(G^\dagger)} \pi_F \right] \ .$$

Note that any forest $F \in \mathcal{F}_{\{\dagger,n,m\}}(G^\dagger)$ consists of exactly of three trees $T_\dagger$, T_n and T_m with roots correspondingly $\dagger$, n and m. Therefore the addition to such forest the arc (m, i) with weight g_{mi} leads either to graph belonging to $\mathcal{F}_{\{\dagger,n\}}(G^\dagger)$ (if $i \in \mathcal{V}T_\dagger \cup \mathcal{V}T_n$), or (if $i \in \mathcal{V}T_m$) to graph, in which outdegree of every vertex is equal to one with the exception of $\dagger$ and n: $d^+(j) = 1$ $j \neq \dagger, n$, $d^+(\dagger) = d^+(n) = 0$, and there is exactly one circuit in graph and this circuit contains the vertex m. We denote the set of such graphs by $\mathcal{O}_m(G^\dagger)$. So

$$\sum_{F \in \mathcal{F}_{\{\dagger,n\}}(H^\dagger;(m,n))} \pi_F = - \sum_{F \in \mathcal{F}_{\{\dagger,n,\}}(G^\dagger)} \pi_F - \sum_{F \in \mathcal{O}_m(G^\dagger)} \pi_F \ . \tag{13}$$

Let us now consider the last sum at the right part of (12). The productivity π_F of any forest $F \in \mathcal{F}_{\{\dagger,n,\}}(H^\dagger; \rceil(m,n))$ by force of i) ii) one can represent as $\pi_F = \pi_E$, where E is some subgraph of $G^\dagger$, to obtain which from the forest F one must exchange the arcs ending in m into arcs ending in n and back. The forest $F \in \mathcal{F}_{\{\dagger,n,\}}(H^\dagger; \rceil(m,n))$ consists of two trees $T_\dagger$ and T_n with the roots $\dagger$ and n correspondingly. By this if $m \in \mathcal{V}T_\dagger$ so the graph E is still a forest and besides the sequence of arcs from m does not lead to n (but leads to $\dagger$). In the case of $m \in T_n$ in F such an exchanging of arcs results in E be graph containing the only circuit and the vertex m belongs to this circuit, i.e. $E \in \mathcal{O}_m(G^\dagger)$. Thus

$$\sum_{F\in\mathcal{F}_{\{\dagger,n\}}(H^\dagger;\rceil(m,n))} \pi_F = \sum_{F\in\mathcal{F}_{\{\dagger,n\}}(G^\dagger;\rceil m\cdot n)} \pi_F + \sum_{F\in\mathcal{O}_m(G^\dagger)} \pi_F \ . \tag{14}$$

Here we denote by $\mathcal{F}_{\{\dagger,n\}}(G^\dagger; \rceil m \cdot n)$ the subset of $\mathcal{F}_{\{\dagger,n\}}(G^\dagger)$ not containing $m\cdot n$-way. Combining together (10),(12),(13) and (14) we get, that products π_F along graphs containing circuits reduce and

$$\det \Delta'_{nm} = \sum_{F\in\mathcal{F}_{\{\dagger,n\}}(G^\dagger)} \pi_F - \sum_{F\in\mathcal{F}_{\{\dagger,n\}}(G^\dagger;\rceil m\cdot n)} \pi_F = \sum_{F\in\mathcal{F}_{\{\dagger,n\}}(G^\dagger;m\cdot n)} \pi_F \ ,$$

which is the same as (9). Thus it is proved

Theorem 2. *Let λ be the eigenvalue of order 1 of matrix $\mathbf{G}$, $\vec{v}$ — corresponding eigenvector and its n-th component is not equal to zero, then accurate to constant factor the components of $\vec{v}$ are following*

$$v_n = 1 \ , \quad v_m = \frac{\sum\limits_{k=0}^{N-2} \lambda^k \left[\sum\limits_{F\in\mathcal{F}^k_{\{n,\dagger\}}(m\cdot n)} \pi_F \right]}{\sum\limits_{k=0}^{N-1} \lambda^k \left[\sum\limits_{F\in\mathcal{F}^k_{\{n,\dagger\}}} \pi_F \right]} \ . \tag{15}$$

where $\mathcal{F}^k_{\{n,\dagger\}}(m \cdot n)$ is a subset of set $\mathcal{F}^k_{\{n,\dagger\}}$, consisting of forests having $m \cdot n$-way.

Remark. To obtain the expression for m-th component of eigenvector of the transform matrix $\mathbf{G}^\top$ it is necessary only to exchange indices m, n with each other at the right part of (15).

6   Example

Let us consider an easy example demonstrating the application of singless form technique. Let $\mathbf{M} = \mathbf{P} - \mathbf{I}$, where $\mathbf{P}$ is a probability matrix, setting the finite Markov's chain with killing (that is the sum of elements along any line may be less than one, corresponding residual is just a killing probability), and also let the transition probabilities $M_{ij} = P_{ij}$, $i \neq j$, and the killing ones $M_{i\dagger} = -\sum\limits_{j=1}^{N} M_{ij} = 1 - \sum\limits_{j=1}^{N} P_{ij}$ be exponentially small. Namely, let $N = 3$ and $M_{ij} = m_{ij}e^{-V_{ij}/\varepsilon}$ and $V_{12}=V_{13} = 4$, $V_{1\dagger} = 5$, $V_{21} = 3$, $V_{23} = 2$, $V_{2\dagger} = 5$, $V_{32} = 1$, $V_{3\dagger} = 4$, $V_{31} = 3$; ε is a small parameter.

Thus if instead of diagonal elements M_{ii} we use their expression through the quantities $M_{i\dagger}$ $(M_{ii} = -M_{i\dagger} - \sum\limits_{j\neq i}^{N} M_{ij})$ the matrix $\mathbf{M}$ gets the form

$$\mathbf{M} = \begin{pmatrix} -M_{12} - M_{13} - M_{1\dagger} & M_{12} & M_{13} \\ M_{21} & -M_{21} - M_{23} - M_{2\dagger} & M_{23} \\ M_{31} & M_{32} & -M_{32} - M_{3\dagger} - M_{31} \end{pmatrix} .$$

By virtue of lack of sign and non-negativity of transition (nondiagonal elements of $\mathbf{M}$) and killing ($M_{i\dagger}$) probabilities that are used at tree-like structure formulas, it is ought to keep at the asymptotic of characteristic polynomial coefficients (3) only terms reaching the maximal order on small parameter. Thus the eigenvalues asymptotic one can extract from the equation

$$\lambda^3 + \lambda^2 m_{32}e^{-1/\varepsilon} + \lambda m_{32}m_{21}e^{-5/\varepsilon} + m_{32}m_{21}m_{1\dagger}e^{-10/\varepsilon} = 0 .$$

The exponential orders $V_k = -\lim\limits_{\varepsilon\to 0}\varepsilon\ln a_k$ of the coefficients a_k of characteristic polynomial $\sum\lambda^k a_k$ satisfy convex nonequalities system analogical to [6]: $V_{k-1}-V_k \geq V_k-V_{k+1}$, and the eigenvalues asymptotic one can look in the form $\lambda_k^\varepsilon \underset{\varepsilon\to 0}{\sim} \Lambda_k\ e^{\frac{V_k-V_{k-1}}{\varepsilon}}$ [7]. Substituting model λ in such a form we get the following eigenvalues asymptotics:

$$\lambda_1 \underset{\varepsilon\to 0}{\sim} - m_{1\dagger}e^{-5/\varepsilon} , \quad \lambda_2 \underset{\varepsilon\to 0}{\sim} - m_{21}e^{-3/\varepsilon} , \quad \lambda_3 \underset{\varepsilon\to 0}{\sim} - m_{32}e^{-1/\varepsilon} .$$

The asymptotic of eigenvectors of the matrix $\mathbf{M}$ and the transform matrix $\mathbf{M}^*$ one can find from (15). Denoting by $\mathbf{C}$ (by $\mathbf{C}'$) the matrix, composed of ultimate values of vector-lines of $\mathbf{M}$ (vector-columns of $\mathbf{M}^*$) one gets

$$\mathbf{C} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} , \quad \mathbf{C}' = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} , \quad \mathbf{C}\mathbf{C}' = \mathbf{I} .$$

Notice also that the matrix

$$\tilde{\mathbf{M}} = \begin{pmatrix} -m_{1\dagger}e^{-\frac{5}{\varepsilon}} & 0 & 0 \\ m_{21}e^{-\frac{3}{\varepsilon}} & -m_{21}e^{-\frac{3}{\varepsilon}} & 0 \\ 0 & m_{32}e^{-\frac{1}{\varepsilon}} & -m_{32}e^{-\frac{1}{\varepsilon}} \end{pmatrix}$$

demonstrates the same in leading order eigenvalues asymptotic and the same ultimate values of eigenvectors' components.

Although matrices $\mathbf{M}$ and $\tilde{\mathbf{M}}$ outwardly are rather different, however the solutions of the evolution equation

$$\frac{d\vec{p}}{dt} = \mathbf{L}\vec{p}$$

with guiding matrix $\mathbf{L}$ equal to $\mathbf{M}^*$ or to $\tilde{\mathbf{M}}^*$ are the same under exponentially large times $t = \tau e^{V/\varepsilon}$, $V > 0$, and in "slow" time" τ with exponential scale V the corresponding evolution $\vec{p}_V(\tau) = \lim\limits_{\varepsilon\to 0}\vec{p}(\tau e^{V/\varepsilon})$ has the form

$$\vec{p}_V(\tau) = \mathbf{C}' \left[\lim_{\varepsilon \to 0} \mathrm{diag} \left\{ e^{\Lambda_1 \tau \exp(\frac{V-5}{\varepsilon})}, e^{\Lambda_2 \tau \exp(\frac{V-3}{\varepsilon})}, e^{\Lambda_1 \tau \exp(\frac{V-1}{\varepsilon})} \right\} \right] \mathbf{C} \vec{p}_V(0) \ .$$

Under exponential scale $V = 1$ part of the initial distribution concentrated at the third state crosses to the second one. Such crossing rapidness at "slow" time τ is determined by the quantity $\Lambda_3 = -m_{32}$. Under exponential scale $V = 3$ if $\tau \to \infty$ entire distribution turns out to be at the first state, corresponding rapidness is determined by the quantity $\Lambda_2 = -m_{21}$. Finally at the scale $V = 5$ killing of the process takes place and it is governed by the quantity $\Lambda_1 = -m_{1\dagger}$. Note, that in spite of the killing probability $M_{3\dagger} = m_{3\dagger} e^{-4/\varepsilon}$ at state 3 is greater than the killing probability $M_{1\dagger} = m_{1\dagger} e^{-5/\varepsilon}$ at state 1, but the part of distribution concentrated at state 3 crosses to the second state (and then to state 1) before killing at state 3 could take place.

Considered example shows from one side small matrix elements are not negligible comparatively with the large ones. From another side nevertheless one can neglect by some of the elements, may be not small (here these are elements M_{12}, $M_{2\dagger}$, M_{23}, M_{31}, $M_{3\dagger}$). Which of elements do not affect on the spectrum and on the evolution is determined by signless formulas (3) and (15).

7 Discussion

Obtained expressions for characteristic polynomial (3) and components of eigenvectors (15) at the situation of arbitrary matrices give few substantial for practical computation in comparison with usual methods. Moreover, the formula (1) being only reformulation of standard characteristic polynomial expression is more preferable than (3), because the volume of calculations needed for determination of coefficients at powers of λ using (1) is less than using (3). The reason is that G has less subgraphs being circuits than $G^\dagger$ has forests. In such case signless formulas obtained are not of interest for practical calculations.

However, the situation strongly changes in the case of large matrices with non-negative elements which coefficients depend on small (large) parameter and may possess on it different orders (such matrices, as it was noted before, often appear in singular and stochastic problems). Usual methods in such a situation are practically unsuitable due to large amount of calculations and low accuracy. On the contrary, formulas (3) and (15) due to sign absence at sums become extremely effective and allow to keep an eye on only higher order in asymptotic. This circumstance permits not only to carry out practical calculations, but also (which is highly essentially from theoretical point oh view) to determine singular limits and analyze structure spectrum properties. Here it is necessary to note that though for matrix $\mathbf{G}$ with non-negative elements introducing quantities $g_{i\dagger} = -\sum_{j=1}^{N} g_{ij}$ are not positive however at the expense of spectrum shift it is always possible to make them non-negative ones. The problem of picking out at characteristic polynomial terms having highly order is reducing to determination forests F with maximal productivity π_F and is a separate difficult one. This problem is solved, the effective technique for determination of extreme forests is elaborated [9]. Analogical technique can be used for analysis of large very sparse matrices.

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