Deformations of chiral algebras and quantum cohomology of toric varieties

Fyodor Malikov[1] and Vadim Schechtman

Let X be a smooth complex variety. It was shown in [MSV] that the complex cohomology algebra $H^*(X)$ may be obtained as a cohomology of a certain vertex algebra $H^{ch}(X)$ canonically associated with X. By definition, $H^{ch}(X) = H^*(X; \Omega^{ch}_X)$, where Ω^{ch}_X is a sheaf of vertex superalgebras constructed in [MSV]. (If X is compact, then $H^{ch}(X)$ may be called the *chiral Hodge cohomology* algebra of X.) The algebra $H^{ch}(X)$ is equipped with a canonical odd derivation Q of square zero, and the cohomology of $H^{ch}(X)$ with respect to Q is equal to $H^*(X)$.

In the very interesting paper [B] Borisov defined for a toric complete intersection X a certain vertex superalgebra $V(X)$ equipped with an odd derivation of square zero so that $H^{ch}(X)$ equals the cohomology of $V(X)$ with respect to this derivation. It follows that $H^*(X)$ may also be represented as the cohomology of $V(X)$ with respect to another odd derivation d.

Let X be a smooth complete toric variety. In the present note we include Borisov's algebra $V(X)$ and its derivation d in a family $(V_q(X), d_q)$ of vertex superalgebras with derivation, parametrized by $q \in H^2(X)$, so that the cohomology of $V_q(X)$ with respect to d_q is equal to the *quantum cohomology* algebra of X.

In sect. 2.5 we present a simpler version of this construction in the case of $\mathbb{P}^N$ and apply the deformation technique to compute $H^*(\mathbb{P}^N; \Omega^{ch}_{\mathbb{P}^N})$

We also get similar (partial) results for Fano hypersurfaces in P^N.

§1. Borisov's construction

1.1. *Lattice vertex algebras.* Let L be a free abelian group on $2N$ generators $A^i, B^i,\ 1 \leq i \leq N$. Give L an integral lattice structure by defining a bilinear symmetric $\mathbb{Z}$-valued form

$$(.,.) : L \times L \to \mathbb{Z}$$

so that

$$(A^i, B^j) = \delta_{ij},\ (A^i, A^j) = (B^i, B^j) = 0.$$

Introduce the complexification of L:

$$\mathfrak{h}_L = L \otimes_{\mathbb{Z}} \mathbb{C}.$$

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The bilinear form $(.,.)$ carries over to $\mathfrak{h}_L$ by bilinearity. Let

$$\hat{\mathfrak{h}_L} = \mathfrak{h}_L \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$$

be a Lie algebra with bracket

$$[x \otimes t^i, y \otimes t^j] = i(x, y)\delta_{i+j}K,\ [x \otimes t^i, K] = 0.$$

Associated with L there is a group algebra $\mathbb{C}[L]$ with basis $e^\alpha, \alpha \in L$, and multipliciation

$$e^\alpha \cdot e^\beta = e^{\alpha+\beta},\ e^0 = 1,\ \alpha, \beta \in L.$$

Denote by $S_{\mathfrak{h}_L}$ the symmetric algebra of the space $\mathfrak{h}_L \otimes t^{-1}\mathbb{C}[t^{-1}]$. The space $S_{\mathfrak{h}_L} \otimes \mathbb{C}[L]$ carries the well-known vertex algebra structure, see for example [K]. Borisov proposes to enlarge this lattice vertex algebra by fermions as follows.

We tacitly assumed that $\mathfrak{h}_L$ is a purely even vector space: $\mathfrak{h}_L^{(0)} = \mathfrak{h}_L,\ \mathfrak{h}_L^{(1)} = 0$. Let $\Pi\mathfrak{h}_L$ satisfy the relations $\Pi\mathfrak{h}_L^{(1)} = \mathfrak{h}_L,\ \Pi\mathfrak{h}_L^{(0)} = 0$. Thus $\Pi\mathfrak{h}_L$ is a purely odd vector space with basis Ψ^i, Φ^i carrying the following odd bilinear form:

$$(.,.) : \Pi\mathfrak{h}_L \times \Pi\mathfrak{h}_L \to \mathbb{C},$$

$$(\Psi^i, \Phi^j) = \delta_{ij},\ (\Psi^i, \Psi^j) = (\Phi^i, \Phi^j) = 0.$$

Given all this, one defines the Clifford algebra, $Cl_{\mathfrak{h}_L}$, to be the vector superspace

$$Cl_{\mathfrak{h}_L} = \Pi\mathfrak{h}_L \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K',\ Cl_{\mathfrak{h}_L}^{(1)} = \Pi\mathfrak{h}_L \otimes \mathbb{C}[t, t^{-1}], Cl_{\mathfrak{h}_L}^{(0)} = \mathbb{C}K',$$

with (super)bracket $[x \otimes t^i, y \otimes t^j] = (x, y)\delta_{i+j}K'$.

Let $\Lambda_{\mathfrak{h}_L}$ be the symmetric algebra of the superspace

$$\oplus_{i=1}^N (\Phi^i \otimes \mathbb{C}[t^{-1}] \oplus \Psi^i \otimes t^{-1}\mathbb{C}[t^{-1}].$$

(If we had been allowed to forget about the parity, we would have equivalently defined $\Lambda_{\mathfrak{h}_L}$ to be the exterior algebra of the indicated space.) The space $\Lambda_{\mathfrak{h}_L}$ carries the well-known vertex algebra structure, see for example [K].

Finally let

$$V_L = \Lambda_{\mathfrak{h}_L} \otimes S_{\mathfrak{h}_L} \otimes \mathbb{C}[L].$$

Being a tensor product of vertex algebras, V_L is also a vertex algebra.

1.2. *Explicit description of the vertex algebra structure on V_L.* To simplify the notation, we identify $\mathbb{C}[L]$ with the subspace $1\otimes 1\otimes \mathbb{C}[L]$. As an $\hat{\mathfrak{h}_L}\oplus Cl_{\mathfrak{h}_L}$-module, V_L is a direct sum of irreducibles and there is one irreducible module, $V_L(\alpha)$, for each $\alpha\in L$. $V_L(\alpha)$ is freely generated by the supercommutative associative algebra $S_{\mathfrak{h}_L}\otimes\Lambda_{\mathfrak{h}_L}$ from the highest weight vector e^α. The words "highest weight vector" mean that the following relations hold:

$$A^i_n e^\alpha=\Psi^i_n e^\alpha=B^i_n e^\alpha=\Phi^i_{n+1}e^\alpha=0,\ n\geq 0,$$

$$Ke^\alpha=K'e^\alpha=e^\alpha,\ xe^\alpha=(x,\alpha)e^\alpha,\ x\in\mathfrak{h}_L.$$

Thus, $V_L(\alpha),\alpha\in L$, are different as $\hat{\mathfrak{h}_L}\oplus Cl_{\mathfrak{h}_L}$-modules, but isomorphic as $\hat{\mathfrak{h}_{L1}}\oplus Cl_{\mathfrak{h}_L}$-modules, where $\hat{\mathfrak{h}_{L1}}\subset\hat{\mathfrak{h}_L}$ is the subalgebra linearly spanned by $x\otimes t^i, i\neq 0, x\in\mathfrak{h}_L$. In fact, the multiplication by e^β provides an isomorphism of $\hat{\mathfrak{h}_{L1}}\oplus Cl_{\mathfrak{h}_L}$-modules:

$$e^\beta: V_L(\alpha)\longrightarrow V_L(\alpha+\beta),\ x\otimes e^\alpha\mapsto x\otimes e^{\alpha+\beta}.$$

Let us now define the state-field correspondence, that is, attach a field $x(z)\in\operatorname{End}(V_L)((z,z^{-1}))$ to each state $x\in V_L$. As has become customary, we shall write x_i for $x\otimes t^i$ ($x\in\mathfrak{h}_L$ or $\Pi\mathfrak{h}_L$). We have:

$$(x_{-n-1}e^0)(z)=\frac{1}{n!}x(z)^{(n)},\ x\in\mathfrak{h}_L,$$

where

$$x(z)=\sum_{j\in\mathbb{Z}}x_j z^{-j-1}.$$

In particular, $(x_{-1}e^0)(z)=x(z)$.

We continue in the same vein:

$$(\Phi^i_{-n}e^0)(z)=\frac{1}{n!}\Phi^i(z)^{(n)},$$

where

$$\Phi^i(z)=\sum_{j\in\mathbb{Z}}\Phi^i_j z^{-j};$$

$$(\Psi^i_{-n-1}e^0)(z)=\frac{1}{n!}\Psi^i(z)^{(n)},$$

where

$$\Psi^i(z)=\sum_{j\in\mathbb{Z}}\Psi^i_j z^{-j-1};$$

$$e^\alpha(z)=e^\alpha\cdot\exp\big(-\sum_{n<0}\frac{\alpha_n}{n}z^{-n}\big)\cdot\exp\big(-\sum_{n>0}\frac{\alpha_n}{n}z^{-n}\big)\cdot z^{\alpha_0}.$$

Finally,

$$x^{(1)}_{-n_1}\cdot x^{(2)}_{-n_2}\cdots x^{(k)}_{-n_k}\cdot e^{\alpha}(z)=:x^{(1)}_{-n_1}(z)x^{(2)}_{-n_2}(z)\cdots x^{(k)}_{-n_k}(z)e^{\alpha}(z):.$$

The vertex algebra structure on V_L is equivalently described by the following family of n-th products ($n\in\mathbb{Z}$):

$$_{(n)}:V_L\otimes V_L\to V_L,\ x\otimes y\mapsto x_{(n)}y\overset{\text{def}}{=}(\int x(z)z^n)(y),$$

where $\int x(z)z^n$ stands for the linear transformation of V_L equal to the coefficient of z^{-n-1} in the series $x(z)$.

1.3. *Degeneration of* V_L. Denote by L_A the subgroup of L generated by A^i, $i=1,...,N$. Any smooth toric variety X can be defined via a fan, Σ, that is, a collection of "cones" lying in L_A. Borisov uses such Σ to define a certain degeneration, V_L^{Σ}, of the vertex algebra structure on V_L. He further shows that the cohomology of V_L^{Σ} with respect to a certain differential $D^{\Sigma}:\ V_L^{\Sigma}\to V_L^{\Sigma}$ equals $H^*(X,\Omega_X^{ch})$, where Ω_X^{ch} is the chiral de Rham complex of [MSV]. Let us describe the outcome of this construction in the case when $X=\mathbb{P}^N$.

Consider the following set of $N+1$ elements of L_A: $\xi_1=A^1,\xi_2=A^2,...,\xi_N=A^N,\xi_{N+1}=-A^1-A^2-\cdots-A^N$. Define the cone $\Delta_i\subset L_A$ to be the set of all non-negative integral linear combinations of the elements $\xi_1,...,\xi_{i-1},\xi_{i+1},...,\xi_{N+1}$. It is easy to see that $L_A=\cup_i\Delta_i$ and the intersection $\Delta_i\cap\Delta_j$ is a face of both Δ_i and Δ_j. The fan Σ in this case is the set consisting of $\Delta_1,...,\Delta_{N+1}$ and their faces.

We now define V_L^{Σ} to be a vertex algebra equal to V_L as a vector space with n-th product $_{(n),\Sigma}$ as follows:

if $\{\sum_i n_iA^i,\sum_i n'_iA^i\}\subset\Delta_j$ for some j, then

$$(x\otimes e^{\sum_i m_iB^i+\sum_i n_iA^i})_{(n),\Sigma}(y\otimes e^{\sum_i m'_iB^i+\sum_i n'_iA^i})$$
$$=(x\otimes e^{\sum_i m_iB^i+\sum_i n_iA^i})_{(n)}(y\otimes e^{\sum_i m'_iB^i+\sum_i n'_iA^i});$$

otherwise

$$(x\otimes e^{\sum_i m_iB^i+\sum_i n_iA^i})_{(n),\Sigma}(y\otimes e^{\sum_i m'_iB^i+\sum_i n'_iA^i})=0,$$

where $_{(n)}$ stands for the n-th product on V_L. The fact that these new operations satisfy the Borcherds identities can be proved by including both V_L and V_L^{Σ} in a 1-parameter family of vertex algebras; this will be done in 2.1.

Let

$$D=\int\sum_{i=1}^{N}\Psi^i(z)(e^{A^i}-e^{-\sum_j A^j})(z).\tag{1.1}$$

It is obvious that $D\in\mathrm{End}(V_L^{\Sigma})$ and $D^2=0$; therefore, the cohomology $H_D(V_L^{\Sigma})$ arises.

Theorem 1.3. ([B])
$$H_D(V_L^\Sigma) = H^*(\mathbb{P}^N, \Omega^{ch}_{\mathbb{P}^N}).$$

§2. Deforming $H^*(\mathbb{P}^N)$

2.1. *The family $V_{L,q}$.*

Here we exhibit a family of vertex algebras, $V_{L,q}, q \in \mathbb{C}$, so that $V_{L,q}$ is isomorphic to V_L if $q \neq 0$ and $V_{L,0}$ is isomorphic to V_L^Σ; cf. the end of sect.8 [B].

Define the height function
$$ht:\ L_A \to \mathbb{Z}_>$$
as follows. It is easy to see that each $\alpha \in L_A$ is uniquely represented in the form
$$\alpha = \sum_{i=1}^{N+1} n_i \xi_i \tag{2.1}$$
so that all $n_i \geq 0$ and $\#\{i : n_i > 0\} \leq N$. Let
$$ht(\alpha) = \sum_i n_i,$$
where $n_1, ..., n_N$ are as in (2.1).

Define the linear automorphism
$$t_q : V_L \to V_L,\ q \in \mathbb{C} - \{0\}$$
by the formula
$$t_q(x \otimes e^{\sum_i m_i B^i + \sum_i n_i A^i}) = q^{ht(\sum_i n_i A^i)} x \otimes e^{\sum_i m_i B^i + \sum_i n_i A^i}.$$
Define $V_{L,q}$ to be the vertex algebra equal to V_L as a vector space with the following n-th product:
$$(x \otimes e^{\sum_i m_i B^i + \sum_i n_i A^i})_{(n),q}(y \otimes e^{\sum_i m'_i B^i + \sum_i n'_i A^i})$$
$$= t_q^{-1}(t_q(x \otimes e^{\sum_i m_i B^i + \sum_i n_i A^i})_{(n)} t_q(y \otimes e^{\sum_i m'_i B^i + \sum_i n'_i A^i})).$$
By definition,
$$t_q : V_{L,q} \to V_L,\ q \in \mathbb{C} - \{0\},$$
is a vertex algebra isomorphism. It is also easy to see that if $\sum_i n_i A^i$ and $\sum_i n'_i A^i$ belong to the same cone from Σ, then
$$(x \otimes e^{\sum_i m_i B^i + \sum_i n_i A^i})_{(n),q}(y \otimes e^{\sum_i m'_i B^i + \sum_i n'_i A^i})$$
$$= (x \otimes e^{\sum_i m_i B^i + \sum_i n_i A^i})_{(n)}(y \otimes e^{\sum_i m'_i B^i + \sum_i n'_i A^i});$$

otherwise
$$(x\otimes e^{\sum_i m_iB^i+\sum_i n_iA^i})_{(n),q}(y\otimes e^{\sum_i m_i'B^i+\sum_i n_i'A^i})$$
$$\in q\mathbb{C}[q](x\otimes e^{\sum_i m_iB^i+\sum_i n_iA^i})_{(n)}(y\otimes e^{\sum_i m_i'B^i+\sum_i n_i'A^i}).$$
Two things follow at once: first, the operations

$$_{(n),0}=\lim_{q\rightarrow 0}{}_{(n),q}, n\in\mathbb{Z}$$

are well defined and satisfy the Borcherds identities; second, the vertex algebra, $V_{L,0}$, obtained in this way is isomorphic to V_L^Σ. By the way, this remark proves that V_L^Σ is indeed a vertex algebra.

To get a better feel for this kind of deformation, and for the future use, let us consider the subspace $\mathbb{C}[L_A]\subset V_{L,q}$ with basis $e^\alpha, \alpha\in L_A$. The (-1)-st product makes this space a commutative algebra. The subspace $\mathbb{C}[\Delta_j]$ defined to be the linear span of $e^\alpha,\alpha\in\Delta_j$, is a polynomial ring on generators $e^{\xi_1},...,e^{\xi_{j-1}},e^{\xi_{j+1}},...,e^{\xi_{N+1}}$. For example, if we denote $x_i=e^{A^i}$, then $\mathbb{C}[\Delta_{N+1}]=\mathbb{C}[x_1,...,x_N]$ and this isomorphism identifies $e^{\sum_j n_jA^j}$ with the monomial $x_1^{n_1}\cdots x_N^{n_N}$.

The entire $\mathbb{C}[L_A]$ is not a polynomial ring. For example, as follows from the definition of the deformation, there is a relation

$$(e^{-A^1-\cdots-A^N})_{(-1)}(e^{A^1+\cdots+A^N})=q^{N+1}e^0,$$

because $ht(0)=0$, $ht(A^1+\cdots+A^N)=N$, $ht(-A^1-\cdots-A^N)=1$. If we let $T=e^{-A^1-\cdots-A^N}$, then the last equality rewrites as follows:

$$Tx_1x_2\cdots x_N=q^{N+1},$$

and a moment's thought shows that in fact

$$\mathbb{C}[L_A]=\mathbb{C}[x_1,...,x_N,T]/(Tx_1x_2\cdots x_N-q^{N+1}).$$

Being a group algebra, $\mathbb{C}[L_A]$ carries another algebra structure, a priori different from the one we just described and independent of q. We see that the two structures are isomorphic if $q\neq 0$; at $q=0$, however, the one we just described degenerates in an algebra with zero divizors.

2.2. *The algebra $H^*(\mathbb{P}^N)$.*

Let
$$Q(z)=A^i(z)\Phi^i(z)-\sum_j\Phi^j(z)',$$
$$G(z)=B^i(z)\Psi^i(z),$$
$$J(z)=:\Phi^i(z)\Psi^i(z):+\sum_j B^j(z)',$$
$$L(z)=:B^i(z)A^i(z):+:\Phi^i(z)'\Psi^i(z):,$$

where the summation with respect to repeated indices is assumed.

One checks that the Fourier components of these 4 fields satsify the commutation relations of the $N = 2$ algebra. It is also easy to see that the fields $G(z), L(z)$ commute with Borisov's differential D, see (1.1), and therefore define the fields, to be also denoted $G(z), L(z)$, acting on $H_D(V_L^\Sigma)$.

The fields $Q(z), J(z)$ do not commute with D, but their Fourier components $Q_0 = \int Q(z)$ and $J_0 = \int J(z)$ do:

$$[Q_0, D] = [J_0, D] = 0.$$

Thus we get 2 operators, to be also denoted Q_0, J_0, acting on $H_D(V_L^\Sigma)$. All this is summarized by saying that $H_D(V_L^\Sigma)$ is a topological vertex algebra.

A glance at the formulas on p. 17 of [B] shows that the isomorphism $H_D(V_L^\Sigma) = H^*(\mathbb{P}^N, \Omega^{ch}_{\mathbb{P}^N})$ (see Theorem 1.3) identifies these $G(z), L(z), Q_0, J_0$ with the fields (operators) constructed in [MSV] and denoted in the same way. One of the main results of [MSV] then gives

$$H^*(\mathbb{P}^N) = H_{Q_0}(H_D(V_L^\Sigma)). \tag{2.2}$$

Further, the algebra structure of $H^*(\mathbb{P}^N)$ is restored from the (-1)-st product on $H_D(V_L^\Sigma)$.

2.3. *Deformation of the algebra structure.*

It follows from the proof of Theorem 2.3 below that the cohomology (2.2) can be calculated in the reversed order:

$$H^*(\mathbb{P}^N) = H_D(H_{Q_0}(V_L^\Sigma)). \tag{2.3}$$

Note that D and Q_0 can also be regarded as well-defined operators acting on the deformed algebra:

$$D = \int \sum_{i=1}^{N} \Psi^i(z)(e^{A^i} - e^{-\sum_j A^j})(z), Q_0 = \int A^i(z)\Phi^i(z) : \ V_{L,q} \to V_{L,q}.$$

It is immediate to see that $D^2 = Q_0^2 = 0$ on $V_{L,q}$ as well. Moreover,

$$[D, Q_0] = 0. \tag{2.4}$$

Indeed, the formulas of 1.2 imply the following OPE:

$$\sum_{i=1}^{N} \Psi^i(z)(e^{A^i} - e^{-\sum_j A^j})(z) \cdot A^j(w)\Phi^j(w) = \frac{\sum_j e^{A^j}(w)' - e^{-\sum_j A^j}(w)'}{z-w}.$$

Therefore,

$$[D, Q_0] = \int \{\sum_j e^{A^j}(w) - e^{-\sum_j A^j}(w)\}' = 0.$$

Thus it is natural to take the space $H_D(H_{Q_0}(V_{L,q}))$ for a deformation of $H^*(\mathbb{P}^N)$.

Theorem 2.3.

$$H_D(H_{Q_0}(V_{L,q})) = \mathbb{C}[T]/(T^{N+1} - q^{N+1}).$$

Proof.

1) *Computation of* $H_{Q_0}(V_{L,q})$. By definition

$$Q_0 = \sum_{n\in\mathbb{Z}} A^i_{-n}\Phi^i_n \tag{2.5}$$

Therefore,

$$[Q_0, \Psi^j_0] = A^j_0,\ [Q_0, G_0] = L_0.$$

These relations imply that

$$H_{Q_0}(V_{L,q}) = H_{Q_0}(\cap_j KerA^j_0 \cap KerL_0). \tag{2.6}$$

It follows from 1.2 that the space $\cap_j KerA^j_0 \cap KerL_0$ is a linear span of elements of the form:

$$\Phi^{i_1}_0 \cdots \Phi^{i_m}_0 e^{\sum_i n_i A^i}.$$

Formula (2.5) shows that the restriction of Q_0 to this subspace is 0. Thus

$$H_{Q_0}(\cap_j KerA^j_0 \cap KerL_0) = \cap_j KerA^j_0 \cap KerL_0.$$

The (-1)-st product makes this subspace a supercommutative algebra. In the same way as in 2.1 we get an isomorphism

$$\cap_j KerA^j_0 \cap KerL_0 = \mathbb{C}[x_1, ..., x_N, T; \Phi_1, ..., \Phi_N]/(Tx_1 \cdots x_N - q^{N+1}),$$

where $\Phi_1, ..., \Phi_N$ are understood as grassman variables ($[x_i, \Phi_j] = [T, \Phi_j] = 0$, $\Phi_i\Phi_j + \Phi_j\Phi_i = 0$) and $(Tx_1 \cdots x_N - q^{N+1})$ stands for the ideal generated by $Tx_1 \cdots x_N - q^{N+1}$.

2) *Computation of* $H_D(H_{Q_0}(V_{L,q}))$. In view of Step 1), we have to restrict D to

$$\cap_j KerA^j_0 \cap KerL_0.$$

The isomorphism

$$\cap_j KerA^j_0 \cap KerL_0 = \mathbb{C}[x_1, ..., x_N, T; \Phi_1, ..., \Phi_N]/(Tx_1 \cdots x_N - q^{N+1}),$$

identifies D with $\sum_i (x_i - T)\partial/\partial(\Phi_i)$. Therefore, the complex

$$(\mathbb{C}[x_1, ..., x_N, T; \Phi_1, ..., \Phi_N]/(Tx_1 \cdots x_N - q^{N+1}), D)$$

is simply the Koszul resolution of the algebra

$$\{\mathbb{C}[x_1, ..., x_N, T]/(Tx_1 \cdots x_N - q^{N+1})\}/(x_1 - T, x_2 - T, ..., x_N - T)$$

associated with the sequence $x_1 - T, x_2 - T, ..., x_N - T$. This sequence is regular and we get at once

$$H_D(H_{Q_0}(V_{L,q})) = \mathbb{C}[T]/(T^{N+1} - q^{N+1}). \quad \square$$

It is easy to infer from Borisov's proof of Theorem 1.3 that the element $T = e^{-A^1 - \cdots - A^N} \in V_{L,q}$ is a cocycle representing the cohomology class proportional to that of a hyperplane in $\mathbb{P}^N$. This means that the deformation of $H^*(\mathbb{P}^N)$ we obtained coincides with the standard one, except that for some reason q happened to be raised to the power of N.

2.4. *Reduction to a single differential.*

Of course it would be nicer to get $H^*(\mathbb{P}^N)$, or its deformation, as the cohomology of this or that vertex algebra with respect to a single differential rather than to compute a repeated cohomology.

Theorem 2.4.

$$H_{D+Q_0}(V_{L,q})) = \mathbb{C}[T]/(T^{N+1} - q^{N+1}).$$

It is no wonder, in view of Theorem 2.3, that this assertion is a result of computation of a certain spectral sequence. We shall use several spectral sequences arising in the following situation, which is slightly different from the standard one. Let

$$W = \oplus_{n=-\infty}^{+\infty} W^n$$

be a graded vector space with two commuting differentials

$$d_1 : W^n \to W^{n+1}, d_2 : W^n \to W^{n-1}. \tag{2.7}$$

There arise the total differential $d = d_1 + d_2$ and the cohomology $H_{d_1+d_2}(W)$. Note that this cohomology group is not graded since d_1 and d_2 map in opposite directions. We can, however, introduce the filtration

$$W = \cup_n W^{\leq n}, \; W^{\leq n} = \oplus_{m=-\infty}^{n} W^m.$$

Then

$$(d_1 + d_2)(W^{\leq n}) \subseteq W^{\leq (n+1)}$$

and there arises a filtration $H_{d_1+d_2}(W)^{\leq n}$ on the cohomology and the graded object $GrH_{d_1+d_2}(W)$.

It is straightforward to define a spectral sequence

$$\{E(W)_r^n, d^{(r)} : E(W)_r^n \to E(W)_r^{n-r+1}\}, \; E(W)_{r+1}^n = H_{d^{(r)}}(E(W)_r^n), \tag{2.8}$$

the first three terms being as follows:

$$E(W)_0^n = W^n, \; E(W)_1^n = H_{d_1}(W^n), \; E(W)_2^n = H_{d_2}(H_{d_1}(W^n)), \tag{2.9}$$

where

$$H_{d_1}(W^n) = \frac{\text{Ker}\{d_1 : W^n \to W^{n+1}\}}{\text{Im}\{d_1 : W^{n-1} \to W^n\}},$$

$$H_{d_2}(H_{d_1}(W^n)) = \frac{\text{Ker}\{d_2 : H_{d_1}(W^n) \to H_{d_1}(W^{n-1})\}}{\text{Im}\{d_2 : H_{d_1}(W^{n+1}) \to H_{d_1}(W^n)\}}.$$

In the situation pertaining Theorem 2.4 we take $V_{L,q}$ for W, Q_0 for d_1, and D for d_2. The space $V_{L,q}$ is graded by fermionic charge; this grading is defined by letting the degree of Ψ^i_j be equal -1, the degree of Φ^i_j be equal 1, the degree of A^i_j, B^i_j, e^α be equal 0. By definition,

$$Q_0(V^n_{L,q}) \subseteq V^{n+1}_{L,q}, D(V^n_{L,q}) \subseteq V^{n-1}_{L,q}$$

and we get a spectral sequence $\{E(V_{L,q})^n_r, d^{(r)}\}$.

Observe that the grading by fermionic charge and the corresponding filtration are infinite in both directions. Therefore, the standard finiteness conditions that guarantee convergence of spectral sequences fail. Nevertheless the following lemma holds true.

Lemma 2.4. The spectral sequence $\{E(V_{L,q})^n_r, d^{(r)}\}$ converges to $H_{Q_0+D}(V_{L,q})$ and collapses:

$$H_D(H_{Q_0}(V_{L,q})) = H_{Q+D}(V_{L,q}).$$

Lemma 2.4 combined with Theorem 2.3 gives Theorem 2.4 at once and it remains to prove Lemma 2.4.

Proof of Lemma 2.4. Introduce yet another grading of the space $V^n_{L,q}$ as follows. Let $\alpha = (\alpha_1, ..., \alpha_{N+1})$ be an element of the group $\mathbb{Z}^{N+1}$. Let

$$V^n_{L,q}[\alpha] = (\cap^N_{i=1} Ker(A^i_0 - \alpha_i Id)) \cap Ker(L_0 - \alpha_{N+1} Id).$$

Of course

$$V^n_{L,q} = \oplus_{\alpha \in \mathbb{Z}^{N+1}} V^n_{L,q}[\alpha]$$

and both the differentials preserve this grading. Therefore all calculations can be carried out inside $V^n_{L,q}[\alpha]$ with a fixed α. Consider the following two cases.

1) $\alpha \neq 0$. In this case, as was observed in the beginnning of the proof of Theorem 2.3 (see e.g. (2.6)), $H_{Q_0}(V_{L,q}[\alpha]) = 0$ and, therefore, $E[\alpha]_1 = 0$. It remains to show that $H_{Q_0+D}(V_{L,q}[\alpha]) = 0$. Let $x \in V_{L,q}[\alpha]^{\leq n}$ be a cocycle. This means that there is a "chain" of elements $x_i \in V_{L,q}[\alpha]^{n-2i}, i = 0, 1, ..., k$ so that

$$x = \sum_{i=0}^{k} x_i,$$

and the following holds

$$Q_0(x_0) = 0, Q_0(x_{i+1}) + D(x_i) = 0, D(x_k) = 0, i = 0, ..., k-1. \tag{2.10}$$

We now repeatedly use the condition $H_{Q_0}(V_{L,q}[\alpha]) = 0$ and (2.10) to construct another chain $y_i \in V_{L,q}[\alpha]^{n-2i-1}$, $i \geq 0$, satisfying

$$Q_0(y_0) = x_0, Q_0(y_{i+1}) + D(y_i) = x_{i+1}. \tag{2.11}$$

Indeed, since $Q_0(x_0) = 0$, there is $y_0 \in V_{L,q}[\alpha]^{n-1}$ so that $Q_0(y_0) = x_0$.

Since

$$Q_0(-D(y_0) + x_1) = DQ_0(y_0) + Q_0(x_1) = D(x_0) + Q_0(x_1) = 0,$$

there is $y_1 \in V_{L,q}[\alpha]^{n-3}$ so that $Q_0(y_1) + D(y_0) = x_1$.

In general, having found $y_i \in V_{L,q}[\alpha]^{n-2i-1}, y_{i-1} \in V_{L,q}[\alpha]^{n-2i+1}$ so that $Q_0(y_i) + D(y_{i-1}) = x_i$, we calculate as follows:

$$0 = D(0) = D(Q_0(y_i) + D(y_{i-1}) - x_i) = DQ_0(y_i) - D(x_i).$$

Due to (2.10), the last expression rewrites as $DQ_0(y_i) + Q_0(x_{i+1})$ and we get

$$-Q_0D(y_i) + Q_0(x_{i+1}) = 0.$$

Therefore, $Q_0(D(y_i) - x_{i+1}) = 0$ and there is y_{i+1} so that $Q_0(y_{i+1}) = -D(y_i) + x_{i+1}$ as desired.

Formally, (2.11) means that

$$(D + Q_0)(\sum_{i=0}^{\infty} y_i) = x$$

and what does not allow us to conclude immediately that $x = \sum_{i=0}^{\infty} x_i$ is a coboundary is that the sum $\sum_{i=0}^{\infty} y_i$ looks infinite. To complete case 1) it remains to show that $y_i = 0$ for all sufficiently large i. This is achieved by the following dimensional argument. Note that by construction

$$y_i \in \oplus_{|m_j|<ki}(S_{\mathfrak{h}_L} \otimes \Lambda^{n-2i-1}_{\mathfrak{h}_L} \otimes e^{\sum_j m_j A^j + \sum_j \alpha_j B^j}), \tag{2.12}$$

where k is a number independent of i. Indeed, each application of D changes m_j by at most 1, Q_0 preserves m_j, and the linear estimate of m_j follows. On the other hand we have an explicit formula for L_0 (see the beginning of 2.2), and this formula implies that the smallest eigenvalue of L_0 restricted to $\Lambda^{n-2i-1}_{\mathfrak{h}_L}$ is nonnegative and grows faster than a polynomial of degree 2, say $q(i)$, as $i \to +\infty$. The same formula gives

$$L_0 e^{\sum_j m_j A^j + \sum_j \alpha_j B^j} = \sum_j m_j \alpha_j e^{\sum_j m_j A^j + \sum_j \alpha_j B^j}.$$

Therefore, if $y_i \neq 0$, then it is a sum of eigenvectors associated to eigenvalues of L_0 greater or equal $q(i) - (\alpha_1 + \cdots \alpha_n)ki$. Since this number tends to $+\infty$ as $i \to +\infty$, we arrive at contradiction with the assumption $L_0 y_i = \alpha_{N+1}$ if i is sufficiently large.

Hence, $y_i = 0$ for all sufficiently large i, each cocycle is a coboundary, and case 1) is accomplished.

2) $\alpha = 0$. As we saw in the beginning of the proof of Theorem 2.3, the restriction of Q_0 to $V_{L,q}[0]$ is 0 and, by definition, the complex $(V_{L,q}[0], D + Q_0)$ is equal to $(E(V_{L,q})[0]_1, d^{(1)})$. □

2.5. *The vertex algebra $H_D(V_{L,q})$ and a computation of $H^*(\mathbb{P}^N, \Omega^{ch}_{\mathbb{P}^N})$.*

In this section we prove the following two theorems.

Theorem 2.5A *If $q \neq 0$, then $H_D(V_{L,q})$ equals the quantum cohomology of $\mathbb{P}^N$.*

Theorem 2.5B *The natural embedding of sheaves ([MSV], see also (2.20) below)*

$$\Omega^*_{\mathbb{P}^N} \hookrightarrow \Omega^{ch}_{\mathbb{P}^N}$$

provides an isomorphism

$$H^i(\mathbb{P}^N, \Omega^*_{\mathbb{P}^N}) \xrightarrow{\sim} H^i(\mathbb{P}^N, \Omega^{ch}_{\mathbb{P}^N}),\ 0 < i < N,$$

where $\Omega^*_{\mathbb{P}^N}$ is the sheaf of all differential forms.

Recall the previously known results on the cohomology of $\Omega^{ch}_{\mathbb{P}^N}$. $\Omega^{ch}_{\mathbb{P}^N}$ is a sheaf of $\widehat{sl}_{N+1}$-modules [MS1], see also 2.5.2. In particular, if $U_0 = \mathbb{C}^N \subset \mathbb{P}^N$ is a big cell, then $\Gamma(U_0, \Omega^{ch}_{\mathbb{P}^N})$ is a generalized Wakimoto module over $\widehat{sl}_{N+1}$ introduced in [FF]. We proved in [MS1] that

$$H^0(\mathbb{P}^N, \Omega^{ch}_{\mathbb{P}^N}) = \Gamma(U_0, \Omega^{ch}_{\mathbb{P}^N})^{int}, \tag{2.13}$$

where $\Gamma(U_0, \Omega^{ch}_{\mathbb{P}^N})^{int}$ stands for the maximal sl_{N+1}-integrable submodule of $\Gamma(U_0, \Omega^{ch}_{\mathbb{P}^N})$.

On the other hand, it follows from the chiral Serre duality [MS2] that

$$H^N(\mathbb{P}^N, \Omega^{ch}_{\mathbb{P}^N}) = H^0(\mathbb{P}^N, \Omega^{ch}_{\mathbb{P}^N})^d, \tag{2.14}$$

where ^{d} stands for the restricted dual.

Unfortunately, little is known about the structure of $\Gamma(U_0, \Omega^{ch}_{\mathbb{P}^N})$ and $\Gamma(U_0, \Omega^{ch}_{\mathbb{P}^N})^{int}$, if $N > 1$; see, however, [MS1] for the case of $N = 1$. Otherwise, Theorem 2.5 and (2.13-14) give a complete description of $H^*(\mathbb{P}^N, \Omega^{ch}_{\mathbb{P}^N})$.

The proofs of Theorems 2.5A and B are contained in 2.5.2. In 2.5.1 we collect some well-known material in order to place these results in the proper context and to formulate (2.18-19), two well-known assertions needed in 2.5.2.

2.5.1 A vertex algebra structure on a vector space V comprises a countable family of multiplications:

$${}_{(n)} : V \otimes V \to V,\ x \otimes y \mapsto x_{(n)}y,\ n \in \mathbb{Z},$$

a map

$$T : V \to V,$$

and a vacuum vector

$$\mathbf{1} \in V.$$

These data satisfy the Borcherds identities which imply, in particular, that T and $x_{(0)}, x \in V$, are derivations of the n-th product for all n. Thus,

$$[T, y_{(j)}] = (Ty)_{(j)},\ [x_{(0)}, y_{(j)}] = (x_{(0)}y)_{(j)}. \tag{2.15}$$

In the case of the vertex algebra $V_{L,q}$, the n-th multiplication was defined in the end of 1.2, $\mathbf{1}$ equals e^0, and T will be defined below.

Call V *commutative* (or holomorphic, see [K] 1.4) if $_{(n)} = 0$ for all $n \geq 0$. If V is commutative, then the (-1)-st multiplication gives it the structure of a commutative superalgebra with derivation T, and the functor arising in this way is an equivalence of the category of commutative vertex algebras and the category of commutative superalgebras with derivation, see again [K] 1.4.

If $d_x : V \to V$ is a differential ($d^2 = 0$), then the cohomology $H_{d_x}(V)$ arises. We assert that

$$d_x = x_{(0)} \text{ for some } x \in V \Rightarrow H_{d_x}(V) \text{ is a vertex algebra,} \tag{2.16}$$

since all products on V descend to $H_{d_X}(V)$ due to (2.15).

All vertex algebras we are concerned with are *conformal*. This means that there is a Virasoro field $L(z) = \sum_i L_i z^{-i-2}$, $L_i \in End(V)$, such that L_i satisfy the Virasoro commutation relations, $T = L_{-1}$, L_0 is diagonalizable, and $L(z)$ is the field attached to the state $L_{-2}\mathbf{1} \in V$. The formula at the beginning of 2.2 shows that $V_{L,q}$ is a conformal vertex algebra, the state $L_{-2}\mathbf{1}$ being equal to $\sum_i (B^i_{-1}A^i_{-1} + \Phi^i_{-1}\Psi^i_{-1})e^0$.

The eigenvalues of L_0 are called conformal weights. Hence a conformal vertex algebra V is graded by conformal weights, $V = \oplus_n V_n$, and in the case of $V = V_{L,q}$ this grading (but not the name) has already been used in the proofs of Theorems 2.3 and 2.4.

Returning to the cohomology vertex algebra $H_{d_x}(V)$ in the case when V is conformal and x is an eigenvector of L_0, we see that

$$L_{-2}\mathbf{1} \in Ker d_x \Rightarrow H_{d_x}(V) \text{ is conformal,} \tag{2.17}$$

$$L_{-2}\mathbf{1} \in Im d_x \Rightarrow H_{d_x}(V) \text{ is commutative.} \tag{2.18}$$

Indeed, if $L_{-2}\mathbf{1} \in Ker d_X$, then the operators $L_i \in End(V)$ descend to $H_{d_X}(V)$ due to (2.15). If, in addition, $L_{-2}\mathbf{1} = d_x(y)$, then all L_i's act on $H_{d_x}(V)$ trivially again due to (2.15). Hence L_0 acts on $H_{d_X}(V)$ trivially, each element of $H_{d_X}(V)$ is represented by a cocycle of conformal weight 0, and the n-th product on $H_{d_X}(V)$ vanishes unless $n = -1$.

If $x \in V_1$, then $d_x(V_n) \subset V_n$ for all n, and (2.18) can be sharpened as follows:

$$L_{-2}\mathbf{1} \in Im d_x \text{ and } x \in V_{-1} \Rightarrow H_{d_X}(V) = H_{d_X}(V_0). \tag{2.19}$$

In our previous work ([MSV], [MS1], [MS2]) we have dealt with conformal vertex algebras having the following properties: all conformal weights are nonnegative; the conformal weight 0 component is a finitely generated supercommutative ring and the corresponding multiplication coincides with the restriction of the (-1)st multiplication. For example, Ω^{ch}_X is a sheaf of such vertex algebras over a smooth manifold X: the conformal weight 0 component of $\Gamma(U,\Omega^{ch}_X)$ is the algebra of differential forms over $U\subset X$. In other words, there is a natural embedding

$$\Omega^*_X \stackrel{\sim}{\longrightarrow} \Omega^{ch}_{X,0}\subset\Omega^{ch}_X, \tag{2.20}$$

and it is this embedding that was invoked in Theorem 2.5B.

$H^*(X,\Omega^{ch}_{\mathbb{P}^N})$ is also a vertex algebra of this kind because its conformal weight 0 component equals the cohomology algebra $H^*(X)$. It is, therefore, natural to ask if there is a conformal vertex algebra with nonnegative conformal weights so that the (-1)-st multiplication identifies its conformal weight 0 component with the quantum cohomology of X.

The quantum cohomology itself is one such vertex algebra due to the equivalence of categories reviewed above. A more appealing possibility seems to be provided by $H_D(V_{L,q})$: it is a vertex algebra due to (2.16) because (1.1) is equivalent to

$$D=\sum_{i=1}^N(\Psi^i_{-1}(e^{A^i}-e^{-\sum_j A^j}))_{(0)}, \tag{2.21}$$

and it is conformal because, as one easily checks, $D(L_{-2}e^0)=0$.

Even though Theorem 2.5A says that in this way we do not get anything new either, it allows us to observe a curious phenomenon: $H_D(V_{L,q})$, $q\in\mathbb{C}$, is a family of vertex algebras over $\mathbb{C}$ with fiber that equals $H^*(\mathbb{P}^N)$ over any non-zero point and blows up to the non-commutative infinite dimensional vertex algebra $H^*(\mathbb{P}^N,\Omega^{ch}_{\mathbb{P}^N})$ over $0\in\mathbb{C}$.

Rather unexpectedly, Theorem 2.5B turns out to be a by-product of the proof of Theorem 2.5A.

2.5.2 *Proof Theorems 2.5A and B.*

By definition, the complex $(V_{L,q},D)$ is the constant vector space V_L with differential D polynomially depending on $q\in\mathbb{C}$. To make this more precise, observe that V_L is graded by the function ht defined in 2.1:

$$V_L=\oplus_{n\geq 0}V_L^n, \tag{2.22}$$

where V_L^n is a linear span of $x\otimes e^{\sum_i m_iB^i+\sum_i n_iA^i}$ with $ht(\sum_i n_iA^i)=n$. The differential D then breaks in a sum

$$D=d_+ + q^N d_-, \tag{2.23a}$$

so that

$$d_+(V_L^n)\subset V_L^{n+1}, \tag{2.23b}$$

$$d_-(V_L^n)\subset V_L^{n-N}, \qquad (2.23c)$$

and

$$(d_+)^2=(d_-)^2=[d_+,d_-]=0. \qquad (2.23d)$$

Again by definition, the complex (V_L,d_+) coincides with Borisov's complex (V_L^{Σ},D). It follows from formulas (2.23a-d) and Theorem 1.3 that there is a spectral sequence of the same type as (2.8), the 1st term and the 1st differential being as follows

$$E_1=H^*(\mathbb{P}^N,\Omega^{ch}_{\mathbb{P}^N}) \qquad (2.24)$$

$$d_1=q^Nd_-:H^*(\mathbb{P}^N,\Omega^{ch}_{\mathbb{P}^N})\rightarrow H^*(\mathbb{P}^N,\Omega^{ch}_{\mathbb{P}^N}),$$
$$d_-(H^n(\mathbb{P}^N,\Omega^{ch}_{\mathbb{P}^N}))\subset H^{n-N}(\mathbb{P}^N,\Omega^{ch}_{\mathbb{P}^N}). \qquad (2.25)$$

Simply because $\dim\mathbb{P}^N=N$, the 2nd term equals

$$\frac{H^0(\mathbb{P}^N,\Omega^{ch}_{\mathbb{P}^N})}{\mathrm{Im}\{d_-:H^N(\mathbb{P}^N,\Omega^{ch}_{\mathbb{P}^N})\rightarrow H^0(\mathbb{P}^N,\Omega^{ch}_{\mathbb{P}^N})\}}$$

$$\oplus\mathrm{Ker}\{d_-:H^N(\mathbb{P}^N,\Omega^{ch}_{\mathbb{P}^N})\rightarrow H^0(\mathbb{P}^N,\Omega^{ch}_{\mathbb{P}^N})\}\oplus\oplus_{i=1}^{N-1}H^i(\mathbb{P}^N,\Omega^{ch}_{\mathbb{P}^N}),$$

and all higher differentials vanish. An argument similar to (and simpler than) the one used in the proof of Lemma 2.4 shows that this spectral sequence converges to $H_D(V_{L,q})$. Therefore

$$H_D(V_{L,q})$$
$$=\frac{H^0(\mathbb{P}^N,\Omega^{ch}_{\mathbb{P}^N})}{\mathrm{Im}\{d_-:H^N(\mathbb{P}^N,\Omega^{ch}_{\mathbb{P}^N})\rightarrow H^0(\mathbb{P}^N,\Omega^{ch}_{\mathbb{P}^N})\}}$$
$$\oplus\mathrm{Ker}\{d_-:H^N(\mathbb{P}^N,\Omega^{ch}_{\mathbb{P}^N})\rightarrow H^0(\mathbb{P}^N,\Omega^{ch}_{\mathbb{P}^N})\}\oplus\oplus_{i=1}^{N-1}H^i(\mathbb{P}^N,\Omega^{ch}_{\mathbb{P}^N}). \qquad (2.26)$$

Lemma 2.6. *There is $y\in H^N(\mathbb{P}^N,\Omega^{ch}_{\mathbb{P}^N})$ such that $d_-(y)\in H^0(\mathbb{P}^N,\Omega^{ch}_{\mathbb{P}^N})$ equals the Virasoro element $L_{-2}e^0$.*

This lemma allows us to complete the proof of Theorems 2.5A and B instantaneously. Our differentials come from elements of V_L of conformal weight 1, see (2.21); hence, due to Lemma 2.6, (2.18) and (2.19) apply: $H_D(V_{L,q})$ equals $H_D((V_{L,q})_0)$, which is known (Theorem 2.4) to be equal to the quantum cohomology. In particular, as follows from (2.26),

$$H^i(\mathbb{P}^N,\Omega^{ch}_{\mathbb{P}^N})=H^i(\mathbb{P}^N,(\Omega^{ch}_{\mathbb{P}^N})_0),\ 0<i<N,$$

the latter space being canonically isomorphic to $H^i(\mathbb{P}^N,\Omega^*_{\mathbb{P}^N})$ due to (2.20). Thus it remains to prove Lemma 2.6.

Proof of Lemma 2.6 To find an appropriate $y\in H^N(\mathbb{P}^N,\Omega^{ch}_{\mathbb{P}^N})$ and calculate $d_-(y)$ we need to take a plunge in [MSV,B].

Let $x^0:x^1:\cdots:x^N$ be homogeneous coordinates on $\mathbb{P}^N$ and $b^i=x^i/x^0$. Consider the N-dimensional torus $\mathbb{T}^N=\mathrm{Spec}\mathbb{C}[(b^1)^{\pm1},...,(b^N)^{\pm1}]\subset\mathbb{P}^N$.

We shall need the following facts about the sheaf $\Omega^{ch}_{\mathbb{P}^N}$.

First,

$$\Gamma(\mathbb{T}^N,\Omega^{ch}_{\mathbb{P}^N})=\mathbb{C}[(b^i_0)^{\pm 1},b^i_{j-1},a^i_{j-1};\phi^i_j,\psi^i_{j-1};1\leq i\leq N,j\leq 0], \tag{2.27}$$

where b^i_j, a^i_{j-1} are even, ϕ^i_j,ψ^i_{j-1} odd.

By letting $\deg x^i_j=-j$, $x=b,a,\phi$ or ψ, we recover the grading by conformal weight. By letting $\deg b^i_j=\deg a^i_j=0$, $\deg\phi^i_j=1$, $\deg\psi^i_j=-1$ we get another grading, that by *fermionic charge*. Therefore, $\Gamma(\mathbb{T}^N,\Omega^{ch}_{\mathbb{P}^N})$ is bigraded and this bigrading extends to the entire sheaf:

$$\Omega^{ch}_{\mathbb{P}^N}=\oplus_{m=-\infty}^{+\infty}\oplus_{n=0}^{+\infty}\Omega^{ch,m}_{\mathbb{P}^N,n}. \tag{2.28}$$

Next, we discuss "tensor" properties of $\Omega^{ch}_{\mathbb{P}^N}$. We identify $\Gamma(\mathbb{T}^N,\Omega^{*}_{\mathbb{P}^N})$ with $\Gamma(\mathbb{T}^N,\Omega^{ch}_{\mathbb{P}^N})_0$ by identifying b^i with b^i_0 and db^i with ϕ^i_0. This identification extends to the isomorphism (2.20).

The structure of higher conformal weight components is more complicated, but here is what we can say about the component of conformal weight 1. Consider the following elements of $\Gamma(\mathbb{T}^N,\Omega^{ch,0}_{\mathbb{P}^N,1})$:

$$e_{ij}=b^{i-1}_0a^{j-1}_{-1}+\phi^{i-1}_0\psi^{j-1}_{-1},\ i,j\neq 1, \tag{2.29a}$$

$$e_{1j}=a^{j-1}_{-1},\ j\neq 1 \tag{2.29b}$$

$$e_{i1}=-\sum_{l=1}^N b^{i-1}_0b^l_0a^l_{-1}-\sum_{l=1}^N b^{i-1}_0\phi^l_0\psi^l_{-1}$$
$$-\sum_{l=1}^N b^l_0\phi^{i-1}_0\psi^l_{-1},\ i\neq 1. \tag{2.29c}$$

It was checked in [MS1] III that these elements come from $H^0(\mathbb{P}^N,\Omega^{ch,0}_{\mathbb{P}^N,1})\subset\Gamma(\mathbb{T}^N,\Omega^{ch,0}_{\mathbb{P}^N,1})$ and that the Fourier components of the corresponding fields span a Lie subalgebra of $\mathrm{End}(\Omega^{ch}_{\mathbb{P}^N})$ isomorphic to the loop algebra $Lsl_{N+1}=sl_{N+1}\otimes\mathbb{C}[t,t^{-1}]$. Therefore,

$$Lsl_{N+1}\hookrightarrow\mathrm{End}(\Omega^{ch}_{\mathbb{P}^N}), \tag{2.30a}$$

so that

$$sl_{N+1}\hookrightarrow H^0(\mathbb{P}^N,\Omega^{ch,0}_{\mathbb{P}^N,1}),E_{ij}\mapsto e_{ij}. \tag{2.30b}$$

is a morphism of sl_{N+1}-modules, where E_{ij}, $i\neq j$, $1\leq i,j\leq N+1$ are the standard generators of sl_{N+1}, and sl_{N+1} operates on $H^0(\mathbb{P}^N,\Omega^{ch,0}_{\mathbb{P}^N,1})$ by means of the composite map $sl_{N+1}\xrightarrow{\sim}sl_{N+1}\otimes 1\subset\widehat{sl}_{N+1}$

Elements (2.29a-c) have fermionic charge 0. For the fermionic charge $N+1$ component there is an isomorphism:

$$\Omega^1_{\mathbb{P}^N}\otimes\Omega^N_{\mathbb{P}^N}\xrightarrow{\sim}\Omega^{ch,N+1}_{\mathbb{P}^N,1}. \tag{2.31}$$

Over $\mathbb{T}^N$ it is defined by the assignment

$$f_i(b^1,...,b^N)db^i \otimes (db^1 \wedge db^2 \wedge \cdots \wedge db^N) \mapsto f_i(b_0^1,...,b_0^N)\phi_{-1}^i\phi_0^1\phi_0^2\cdots\phi_0^N,$$

$$f_i(b^1,...,b^N) \subset \mathbb{C}[(b^1)^{\pm 1},...,(b^N)^{\pm 1}].$$

Isomorphism (2.31) induces the isomorphism

$$H^N(\mathbb{P}^N, \Omega^1_{\mathbb{P}^N} \otimes \Omega^N_{\mathbb{P}^N}) \stackrel{\sim}{\longrightarrow} H^N(\mathbb{P}^N, \Omega^{ch,N+1}_{\mathbb{P}^N,1}). \tag{2.32}$$

By the Serre duality,

$$H^N(\mathbb{P}^N, \Omega^1_{\mathbb{P}^N} \otimes \Omega^N_{\mathbb{P}^N}) \stackrel{\sim}{\longrightarrow} H^0(\mathbb{P}^N, \mathcal{T})^*, \tag{2.33}$$

where $\mathcal{T}$ is the tangent sheaf. The Lie algebra sl_{N+1} operates on $\mathbb{P}^N$, therefore there arises the map $sl_{N+1} \to H^0(\mathbb{P}^N, \mathcal{T})^*$, which is well known to be an isomorphism. Hence, (2.33) combined with (2.32) rewrites as follows

$$H^N(\mathbb{P}^N, \Omega^{ch,N+1}_{\mathbb{P}^N,1}) \stackrel{\sim}{\longrightarrow} sl_{N+1}. \tag{2.34}$$

This map is an isomorphism of sl_{N+1}-modules, and it is not hard to find a Cech cochain representing a highest weight vector of $H^N(\mathbb{P}^N, \Omega^{ch,N+1}_{\mathbb{P}^N})_1$, that is, a non-zero vector v satisfying

$$E_{ij}v = 0,\ i < j. \tag{2.35}$$

If we denote by U_i the open subset of $\mathbb{P}^N$ satisfying $x_i \neq 0$, then $\{U_0, ..., U_N\}$ is an affine cover of $\mathbb{P}^N$, so that $\mathbb{T}^N = U_0 \cap U_1 \cap ... \cap U_N$. The N-th term of the Cech complex equals, therefore, $\Gamma(\mathbb{T}^N, \Omega^{ch}_{\mathbb{P}^N})$, and it is an exercise to check that

$$(b_0^1)^{-1}(b_0^2)^{-1}\cdots(b_0^{N-1})^{-1}(b_0^N)^{-3}\phi_{-1}^i\phi_0^1\phi_0^2\cdots\phi_0^N \tag{2.36}$$

represents a highest weight vector of $H^N(\mathbb{P}^N, \Omega^{ch,N+1}_{\mathbb{P}^N,1})$.

Observe that another copy of sl_{N+1} we have discovered earlier has e_{1N+1} for its highest weight vector, see (2.29b,2.30). The assertion crucial for our proof is that d_- sends one highest weight vector to another:

$$d_-((b_0^1)^{-1}b_0^2)^{-1}\cdots b_0^{N-1})^{-1}b_0^N)^{-3}\phi_{-1}^i\phi_0^1\phi_0^2\cdots\phi_0^N) = e_{1N+1} \in H^0(\mathbb{P}^N, \Omega^{ch,0}_{\mathbb{P}^N,1}). \tag{2.37}$$

Lemma 2.6 follows from (2.37) easily. To explain this implication we have to digress on elementary representation theory of Lsl_{N+1}.

Consider the decomposition

$$Lsl_{N+1} = L_-sl_{N+1} \oplus sl_{N+1} \oplus L_+sl_{N+1},$$

where

$$L_\pm sl_{N+1} = t^{\pm 1}\mathbb{C}[t^{\pm 1}].$$

Let $L_\geq sl_{N+1} = sl_{N+1} \oplus L_+sl_{N+1}$. Any sl_{N+1}-module becomes an $L_\geq sl_{N+1}$-module if the action of sl_{N+1} is extended to the entire $L_\geq sl_{N+1}$ by the requirement $L_+sl_{N+1} \mapsto$

0. Therefore for any sl_{N+1}-module U there arises the *Weyl module*, denoted $\mathbb{W}_U$ and defined as follows:

$$\mathbb{W}_U = \mathrm{Ind}_{L_{\geq} sl_{N+1}}^{Lsl_{N+1}} U.$$

The Weyl module induced from the trivial representation, $\mathbb{W}_{\mathbb{C}}$, is well-known to be a conformal vertex algebra due to [FZ], see also [K] 4.7. Therefore, it has vacuum vector, $\mathbf{1}$, and Virasoro element, $L_{-2}^{aff}\mathbf{1}$. Other Weyl modules are modules over $\mathbb{W}_{\mathbb{C}}$. This means, in particular, that Fourier components L_i^{aff} act on Weyl modules. The action of L_0^{aff} is diagonalizable and defines a grading on each Weyl module also called the grading by conformal weight. The aim of this digression was to formulate the following well-known (and easily derived from the Kac-Kazhdan equations) assertion:

$$I \subset \mathbb{W}_{sl_{N+1}} \text{ is a } \textit{proper } Lsl_{N+1}\text{-submodule } \Rightarrow I \cap (\mathbb{W}_{sl_{N+1}})_2 = \{0\}, \qquad (2.38)$$

where $\mathbb{W}_{sl_{N+1}}$ stands for the Weyl module induced from the adjoint representation, and $(\mathbb{W}_{sl_{N+1}})_2$ is its conformal weight 2 component.

Return to the proof of Lemma 2.6. Due to (2.30a), $H^i(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^{ch,m})$ is an Lsl_{N+1}-module for all i and m. The component $H^N(\mathbb{P}^N, \Omega_{\mathbb{P}^N,1}^{ch,N+1})$ is an $L_{\geq} sl_{N+1}$-module isomorphic to sl_{N+1}, see (2.34), on which $L_+ sl_{N+1}$ acts trivially because $H^N(\mathbb{P}^N, \Omega_{\mathbb{P}^N,m}^{ch,N+1}) = 0$ for all $m < 1$. By the universality property of induced modules, $\mathbb{W}_{sl_{N+1}}$ maps onto the Lsl_{N+1}-submodule of $H^N(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^{ch,N+1})$ generated by $H^N(\mathbb{P}^N, \Omega_{\mathbb{P}^N,1}^{ch,N+1})$. Denote this submodule $\widehat{\mathbb{W}}_{sl_{N+1}}$.

Similarly, $H^0(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^{ch,0})$ is an Lsl_{N+1}-module, and the Lsl_{N+1}-submodule generated by $\mathbf{1}$ is a quotient of $\mathbb{W}_{\mathbb{C}}$. This quotient contains yet another submodule, the one generated by $sl_{N+1} \hookrightarrow H^0(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^{ch})_1$, see (2.30b), to be denoted $\widehat{\widehat{\mathbb{W}}}_{sl_{N+1}}$. This submodule, again for the same reason, is a quotient of $\mathbb{W}_{sl_{N+1}}$. By definition, the above mentioned Virasoro element $L_{-2}^{aff}\mathbf{1}$ belongs to $(\widehat{\widehat{\mathbb{W}}}_{sl_{N+1}})_2$

We are practically done. It is easy to derive from [B] that

$$d_- : H^N(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^{ch,N+1}) \to H^0(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^{ch,0})$$

is an Lsl_{N+1}-morphism. Equality (2.37) then means that $d_-(\widehat{\mathbb{W}}_{sl_{N+1}}) \subset \widehat{\widehat{\mathbb{W}}}_{sl_{N+1}}$ is non-zero, and is therefore a quotient of $\mathbb{W}_{sl_{N+1}}$ by a proper submodule. Due to (2.38)

$$(d_-(\widehat{\mathbb{W}}_{sl_{N+1}}))_2 = (\mathbb{W}_{sl_{N+1}})_2 = (\widehat{\widehat{\mathbb{W}}}_{sl_{N+1}})_2.$$

Hence $L_{-2}^{aff}\mathbf{1} \in d_-(\widehat{\mathbb{W}}_{sl_{N+1}})$. To complete the proof of Lemma 2.6 it remains to check that the affine Virasoro element, $L_{-2}^{aff}\mathbf{1}$, coincides with $L_{-2}\mathbf{1}$ and this is easy.

Finally we have to prove (2.37). The difficulty with computation of

$$d_-((b_0^1)^{-1}(b_0^2)^{-1}\cdots(b_0^{N-1})^{-1}(b_0^N)^{-3}\phi_{-1}^i\phi_0^1\phi_0^2\cdots\phi_0^N)$$

lies in that the operator d_- is defined in terms of the vertex algebra V_L, while

$$(b_0^1)^{-1}(b_0^2)^{-1}\cdots(b_0^{N-1})^{-1}(b_0^N)^{-3}\phi_{-1}^i\phi_0^1\phi_0^2\cdots\phi_0^N$$

is an element of $\Gamma(\mathbb{T}^N,\Omega^{ch}_{\mathbb{P}^N})$. The vertex algebra embedding

$$\Gamma(\mathbb{T}^N,\Omega^{ch}_{\mathbb{P}^N})\hookrightarrow V_L,$$

an important ingredient of Borisov's proof of Theorem 1.3, is determined by the rules

$$(b_0^i)^{\pm}\mapsto e^{\pm B^i},\phi_0^i\mapsto\Phi_0^i e^{B^i},\psi_{-1}^i\mapsto\Psi_{-1}^i e^{-B^i}, \tag{2.39a}$$

$$a_{-1}^i\mapsto A_{-1}^i e^{-B^i}-\Phi_0^i\Psi_{-1}^i e^{-B^i}, \tag{2.39b}$$

$$x\mapsto X\ \Rightarrow L_{-1}x\mapsto L_{-1}X, \tag{2.39c}$$

$$x\mapsto X, y\mapsto Y\ \Rightarrow x_{(-1)}y\mapsto X_{(-1)}Y. \tag{2.39d}$$

These rules imply

$$(b_0^1)^{-1}(b_0^2)^{-1}\cdots(b_0^{N-1})^{-1}(b_0^N)^{-3}\phi_{-1}^i\phi_0^1\phi_0^2\cdots\phi_0^N\mapsto e^{-B^N}\Phi_{-1}^N\Phi_0^1\Phi_0^2\cdots\Phi_0^N.$$

It follows from Borisov's proof of Theorem 1.3 that an element of V_L representing the class of

$$(b_0^1)^{-1}(b_0^2)^{-1}\cdots(b_0^{N-1})^{-1}(b_0^N)^{-3}\phi_{-1}^i\phi_0^1\phi_0^2\cdots\phi_0^N$$

can be chosen to be equal to

$$(\Psi_{-1}^N e^{A^N})_{(0)}(\Psi_{-1}^{N-1}e^{A^{N-1}})_{(0)}\cdots(\Psi_{-1}^1 e^{A^1})_{(0)}e^{-B^N}\Phi_{-1}^N\Phi_0^1\Phi_0^2\cdots\Phi_0^N.$$

The formulas of 1.1-2 imply that $(\Psi_{-1}^i e^{A^i})_{(0)}$, $1\le i\le N-1$, simply erases Φ_0^i. Hence

$$(\Psi_{-1}^{N-1}e^{A^N})_{(0)}(\Psi_{-1}^{N-2}e^{A^{N-1}})_{(0)}\cdots(\Psi_{-1}^1 e^{A^1})_{(0)}e^{-B^N}\Phi_{-1}^N\Phi_0^1\Phi_0^2\cdots\Phi_0^N$$

$$=\Phi_{-1}^N\Phi_0^N e^{-B^N+A^1+A^2+\cdots A^{N-1}}.$$

The calculation of the last operation is a little more tedious, but also straightforward; the result is this:

$$(\Psi_{-1}^N e^{A^N})_{(0)}(\Phi_{-1}^N\Phi_0^N e^{-B^N+A^1+A^2+\cdots A^{N-1}})$$

$$(\Psi_{-1}^N\Phi_{-1}^N\Phi_0^N-\Phi_{-1}^N A_{-1}^N+\frac{1}{2}\Phi_0^N A_{-2}^N+\frac{1}{2}\Phi_0^N(A_{-1}^N)^2)e^{-B^N+A^1+A^2+\cdots A^N}. \tag{2.40}$$

To complete our calculation we have to apply d_- to this element. Observe that this element comes from the interior of the cone spanned by $A^1,...,A^N$ and has height N. It follows from the definition of the spectral sequence and (1.1) or (2.21) that on this element d_- equals

$$-((\Psi_{-1}^1+\Psi_{-1}^2+\cdots+\Psi_{-1}^N)e^{-A^1-A^2-\cdots-A^N})_{(0)}.$$

Indeed, it is precisely the component of Borisov's differential (1.1) that decreases the height of the element (2.40). (By the way, it decreases it by N, which explains the assertion (2.23c).) Another calculation similar to those performed shows that

$$-((\Psi^1_{-1}+\Psi^2_{-1}+\cdots+\Psi^N_{-1})e^{-A^1-A^2-\cdots-A^N})_{(0)}.$$

sends the element (2.40) to

$$A^N_{-1}e^{-B^N}-\Phi^N_0\Psi^N_{-1}e^{-B^N}.$$

According to (2.39b), the latter element corresponds to a^N_{-1} and hence to e_{1N+1}, see (2.29b), as desired. □

§3. Deforming cohomology algebras of hypersurfaces in projective spaces

Let $\mathcal{L}\to\mathbb{P}^N$ be a degree $-n<0$ line bundle, $\mathcal{L}^*\to\mathbb{P}^N$ its dual, $s\in\Gamma(\mathbb{P}^N,\mathcal{L}^*)$ a global section so that its zero locus $Z(s)\subset\mathbb{P}^N$ is a smooth hypersurface. The way Borisov calculates the cohomology of the chiral de Rham complex over $Z(s)$ is as follows.

Extend the lattice $(L,(.,.))$ introduced in 1.1 to the lattice $(\hat{L},(.,.))$ so that

$$\hat{L}=L\oplus\mathbb{Z}A^u\oplus\mathbb{Z}B^u,\ (A^u,B^u)=1,(A^u,L)=0,(B^u,L)=0.$$

There arises the corresponding lattice vertex algebra $V_{\hat{L}}$. Observe that any subset $L'\subset\hat{L}$ closed under addition gives rise to the vertex subalgebra $V_{L'}\subset V_L$ generated by $\mathfrak{h}_{\hat{L}}$ and $Cl_{\hat{L}}$ from the highest weight vectors $e^\beta,\beta\in L'$; see 1.1-1.2. In our geometric situation let $\hat{L}_n$ be the span of B^i $(i=1,...,N)$, B^u with arbitrary integral coefficients and A^i $(i=1,...,N)$, A^u, $nA^u-A^1-\cdots-A^N$ with nonnegative integral coefficients.

The vertex algebra $V_{\hat{L}_n}$ affords a degeneration, $V^\Sigma_{\hat{L}_n}$, and includes in a family, $V_{\hat{L}_n,q}$, $q\in\mathbb{C}$, in the same way the algebra V_L did, see 1.3, 2.1. To construct $V^\Sigma_{\hat{L}_n}$, consider the following $N+1$ elements of $\hat{L}:\ \xi_1=A^1,\xi_2=A^2,...,\xi_N=A^N,\xi_{N+1}=nA^u-A^1-A^2-\cdots-A^N$. Define the cone Δ_i to be the set of all non-negative integral linear combinations of the elements $\xi_1,...,\xi_{i-1},\xi_{i+1},...,\xi_{N+1},A^u$ and let $\Sigma=\{\Delta_1,...,\Delta_{N+1}\}$. The vertex algebra $V^\Sigma_{\hat{L}_n}$ is now defined by repeating word for word the definition of V^Σ_L in 1.3.

Similarly, the family $V_{\hat{L}_n,q}$, $q\neq 0$, is defined by repeating word for word the definition of $V_{L,q}$ in 2.1. This family extends "analytically" to $q=0$ if $n\leq N+1$ and we again obtain an isomorphism

$$V_{L,0}=V^\Sigma_{\hat{L}_n}\text{ if }n<N+1. \tag{3.1}$$

(The condition $n<N+1$ will be clarified below.)

Borisov's differential is as follows:

$$D=\int\{\sum_{i=1}^{N}\Psi^i(z)(e^{A^i}-e^{nA^u-\sum_j A^j})(z)+\Psi^u(z)(ne^{nA^u-\sum_j A^j}-e^{A^u})(z)\}. \quad (3.2a)$$

(For the future use let us note that the right hand side of this equality can be rewritten as a sum over lattice points:

$$D=\int\{\sum_{i=1}^{N+1}\Psi^{\xi_i}(z)e^{\xi_i}(z)-\Psi^u(z)e^{A^u}(z)\}, \quad (3.2b)$$

where $\Psi^{\xi_i}=\Psi^i$ $(i\leq N)$ and $\Psi^{\xi_{N+1}}=n\Psi^u-\sum_j\Psi^j$.)

It is obvious that $D\in\mathrm{End}(V_{\hat{L}_n,q})$ and $D^2=0$; therefore there arise the cohomology groups $H_D(V_{\hat{L}_n,q})$ and $H_D(V^{\Sigma}_{\hat{L}_n})=H_D(V_{\hat{L}_n,0})$.

Theorem 3.1. ([B])

$$H_D(V^{\Sigma}_{\hat{L}_n})=H^*(\mathcal{L},\Omega^{ch}_{\mathcal{L}}).$$

Borisov proposes to calculate the chiral de Rham complex over the hypersurface $Z(s)\subset\mathbb{P}^N$ by means of a certain Koszul-type resolution of the complex $\Omega^{ch}_{\mathcal{L}}$. The combinatorial data that determine $s\in\Gamma(\mathbb{P}^N,\mathcal{L}^*)$ consists of the finite set

$$\Delta^*=\{\beta=B^u+\sum_{j=1}^{N}n_jB^j\text{ s.t. }(\beta,\xi_i)\geq 0,\ i=1,2,...,N+1\}, \quad (3.3)$$

and a function

$$g:\Delta^*\to\mathbb{Z}_{\geq}. \quad (3.4)$$

Define

$$K_g=\sum_{\beta\in\Delta^*}\int g(\beta)\Phi^{\beta}(z)e^{\beta}(z), \quad (3.5)$$

where $\Phi^{\beta}=\Phi^u+\sum_j n_j\Phi^j$ provided $\beta=B^u+\sum_j n_jB^j$. It is easy to see that

$$K_g\in\mathrm{End}(V_{\hat{L}_n,q}),\ K_g^2=0,[K_g,D]=0.$$

Therefore, there arise the cohomology groups $H_{D+K_g}(V_{\hat{L}_n,q})$ and $H_{D+K_g}(V^{\Sigma}_{\hat{L}_n})=H_{D+K_g}(V_{\hat{L}_n,0})$.

Theorem 3.2. ([B])

$$H_{D+K_g}(V^{\Sigma}_{\hat{L}_n})=H^*(Z(s),\Omega^{ch}_{Z(s)}).$$

All the vertex algebras in sight being topological (see the beginning of 2.2), Theorem 3.2 and the main result of [MSV] give

$$H^*(Z(s))=H_{Q_0}(H_{D+K_g}(V^{\Sigma}_{\hat{L}_n})), \quad (3.6a)$$

or, equivalently,

$$H^*(Z(s)) = H_{D+K_g}(V^{\Sigma}_{\hat{L}_n})_0, \tag{3.6b}$$

where $H_{D+K_g}(V^{\Sigma}_{\hat{L}_n})_0$ stands for the kernel of L_0.

This prompts the following

Conjecture 3.3. *If $n < N+1$, then the algebra $H_{D+K_g}(V_{\hat{L}_n,q})_0$ is isomorphic to the quantum cohomology algebra of $Z(s)$.*

Unfortunately we do not have a proof of this conjecture; we cannot even prove that $H_{D+K_g}(V_{\hat{L}_n,q})_0$ is a deformation of $H^*(Z(s))$. What we know is collected in the following

Proposition 3.4. *(i) The element $e^{nA^u-\sum_j A^j}$ satisfies*

$$(D+K_g)(e^{nA^u-\sum_j A^j}) = 0,$$

and, therefore, determines an element of $H_{D+K_g}(V_{\hat{L}_n,q})_0$ for all q. If $q=0$, then this element, considered as an element of $H^*(Z(s))$ (see (3.6b)), is proportional to the cohomology class of a hyperplane section.

(ii) Due to (i), $e^{nA^u-\sum_j A^j}$ generates a subalgebra of $H_{D+K_g}(V_{\hat{L}_n,q})_0$ to be denoted $\mathcal{A}_q$. This subalgebra is a deformation of $\mathcal{A}_0$.

(iii) If $Z(s)$ is a hyperplane (i.e. $n=1$), then Conjecture 3.3 is correct.

(iv) If $Z(s)$ is a non-degenerate quadric in $\mathbb{P}^3$, then $H_{D+K_g}(V_{\hat{L}_n,q})_0$ is isomorphic to $\mathbb{C}[x,y]/(x^2-1,y^2-1)$.. Hence Conjecture 3.3 is true in this case.

Since these results are by no means complete, we shall confine ourselves to sketching a proof of Proposition 3.4. The first part of assertion (i) is a result of the obvious calculation using the formulas of 1.1-1.2. The fact that at $q=0$ the element $e^{nA^u-\sum_j A^j}$ is proportional to the cohomology class of a hyperplane section follows from Borisov's proof of Theorem 3.2; this observation is completely analogous to the one made in the end of 2.3.

To prove (ii) observe that we have a constant family of vector spaces $V_{\hat{L}_n,q}$, $q \in \mathbb{C}$, with differential $D+K_g$ depending on q. At $q=0$ the complex $(V_{\hat{L}_n,q}, D+K_g)$ degenerates in Borisov's complex $(V^{\Sigma}_{\hat{L}_n}, D+K_g)$. As it always happens in situations of this kind, the differential $d = D+K_g$ breaks in a sum $d = d_-(q) + d_+$ so that $[d_-(q), d_+] = 0$, $d_-(0) = 0$, and d_+ equals Borisov's differential on $V^{\Sigma}_{\hat{L}_n}$. There arises a spectral sequence converging to $H_{D+K_g}(V_{\hat{L}_n,q})$ with 1-st term equal to $H^*(Z(s), \Omega^{ch}_{Z(s)})$. The 2-nd term equals the cohomology of the complex $(H^*(Z(s), \Omega^{ch}_{Z(s)}), d^{(1)})$ with $d^{(1)} = d_-(q))$. It remains to show that

$$\mathcal{A}_0 \subset \mathrm{Ker} d^{(r)},\ \mathcal{A}_0 \cap \mathrm{Im} d^{(r)} = 0,\ r \geq 1. \tag{3.7}$$

All these spaces are subquotients of the subalgebra of $V_{\hat{L}_n,0}$ generated by e^0, e^{A^i} $(i = 1, ..., N)$, e^{A^u}, and Φ^i_0 $(i = 1, ..., N)$, Φ^u_0, the product being equal to ${}_{(-1)}$. This is a supercommutative algebra isomorphic to

$$\mathbb{C}[x_1, ..., x_N, T, u; \Phi_1, ..., \Phi_N, \Phi_u]/(x_1 x_2 \cdots x_N T),$$

where we let $x_i = e^{A^i}$, $u = e^{A^u}$, $T = e^{nA^u - \sum_j A^j}$, Φ_i, Φ_u being the corresponding grassman variables. (All this is completely analogous to our discussion in the end of 2.1.) Formula (3.2a) says that when restricted to this space Borisov's differential D coincides with the Koszul differential associated with the sequence $x_i - T, u - nT$ $(i = 1, ..., N)$ and our space quickly shrinks to

$$\mathbb{C}[T]/(T^{N+1}),$$

on which (3.7) is obviously true at least when $r = 1$. If $r \geq 2$, then the first part of (3.7) is obviously true and the second follows from a simple dimensional argument.

Before turning to (iii) let us note that a quantum version of this argument gives:

$$\mathcal{A}_q \text{ is a quotient of } \mathbb{C}[T]/(T^{N+1} - q^{N+1-n} n^n T^n). \tag{3.8}$$

Indeed, again by definition (as in the end of 2.1), the subalgebra of $V_{\hat{L}_n,q}$ generated by e^{A^i} $(i = 1, ..., N)$, e^{A^u}, and Φ_0^i $(i = 1, ..., N)$, Φ_0^u is isomorphic to

$$\mathbb{C}[x_1, ..., x_N, T, u; \Phi_1, ..., \Phi_N, \Phi_u]/(x_1 x_2 \cdots x_N T - q^{N+1-n} u^n),$$

and the restriction of D to this supercommutative algebra coincides with the Koszul differential associated with the regular sequence $x_i - T, u - nT$ $(i = 1, ..., N)$. The relation (3.8) follows at once. By the way, the appearance of $N + 1 - n$ as a power of q in (3.8) explains why the condition $n < N + 1$ was imposed in (3.1).

Return to the proof of (iii). In this case the quantum cohomology algebra is isomorphic to the algebra of functions on an N-point set. Because of (ii), $\mathcal{A}_q$ is isomorphic to $\mathbb{C}[T]/p(T)$, $\deg p(T) = N$, and, because of (3.8), $p(T)$ divides $T^{N+1} - q^N T$. The latter has no multiple roots. Hence $\mathcal{A}_q$ is also the algebra of functions on an N-point set.

(iv) follows from the same spectral sequence that was used for the proof of (ii): due to (3.6b) $H^*(Z(s), \Omega^{ch}_{Z(s)})_0 = \mathbb{C}[x, y]/(x^2, y^2)$ and the elements x, y are annihilated by all higher differentials because on the one hand $x, y \in H^1(Z(s), \Omega^{ch}_{Z(s)})$ and on the other hand it is true in general that all $d^{(r)}$, $r \geq 1$, send $H^1(Z(s), \Omega^{ch}_{Z(s)})$ to 0. The rest follows from (3.8), which in this case reads as follows:

$$T^4 - 4q^2 T^2 = 0.$$

□

Remarks. (i) By Corollary 9.3 of [G], the cohomology class p of a hyperplane section satsifies in the quantum cohomology of $Z(s)$ the relation

$$p^N = q n^n p^{n-1}.$$

The amusing similarity between this equality and (3.8) suggests that $\mathcal{A}_q$ might be equal to $\mathbb{C}[T]/(T^N - q^{N+1-n} n^n T^{n-1})$.

(ii) Borisov's suggestion to treat the mirror symmetry as a flip interchanging A's and B's seems to be working in our "quantized" situation as well. Compare (3.5) with (3.2b) to note that D and K_g are sums over two sets of lattice points defined by self-dual condition (3.3). Hence the $A-B$ flip changes D to a similar differential to be associated with the mirror partner of $Z(s)$ lying in another toric manifold, see the next section. Of course the vertex algebra $V_{\hat{L}_n}$ bears a certain asymmetry, since not all elements of the type $e^{\sum_j n_j A^j + n_u A^u}$ are allowed, but Borisov's "transition to the whole lattice" (see Theorem 8.3 in [B]) and the above spectral sequence seem to straighten things out.

§4. Quantum cohomology of toric varieties

Let us briefly explain how the constructions and results of section 2 carry over to an arbitrary smooth compact toric variety of dimension N. Each such variety is determined by a *complete regular* fan in L_A. This and other relevant concepts can be defined as follows (see [D, Bat] for details).

4.1 Let $I \subset L_A$. The cone generated by I is said to be the set of all non-negative integral combinations of elements of I and is denoted Δ_I.

A cone generated by part of a basis of L_A is called *regular*.

A *complete regular fan* Σ is defined to be a collection of regular cones $\{\sigma_1, ..., \sigma_s\}$ so that the following conditions hold:

(i) If σ' is a face of $\sigma \in \Sigma$, then $\sigma' \in \Sigma$;

(ii) If $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma'$ is a face of σ;

(iii) (the completeness condition) $L_A = \sigma_1 \cup ... \cup \sigma_s$.

We skip the construction of the smooth compact toric manifold X_Σ attached to a regular complete fan Σ referring the reader to [D], but formulate Batyrev's result on $H^2(X_\Sigma, \mathbb{R})$, see [Bat].

A function $\phi : L_A \to \mathbb{R}$ is called *piecewise linear* if its restriction to any cone in Σ is a morphism of abelian groups. Denote by $PL(\Sigma)$ the space of all piecewise linear functions.

Let $G(\Sigma) = \{\xi_1, ..., \xi_n\}$ be the set of the generators of all 1-dimensional cones in Σ. Since each piecewise linear function is determined by its values on ξ_i ($i = 1, ..., n$), $PL(\Sigma)$ is an n-dimensional real vector space. It contains the N-dimensional subspace of globally linear functions; the latter is naturally isomorphic to $L_B \otimes_{\mathbb{Z}} \mathbb{R}$.

Theorem 4.1 ([Bat])

$$H^2(X_\Sigma, \mathbb{R}) = PL(\Sigma)/L_B \otimes_{\mathbb{Z}} \mathbb{R}.$$

4.2 Let us return to the vertex algebra V_L. Having fixed an arbitrary $\mathbb{R}$-valued function ϕ on L_A, we proceed in much the same way as in 2.1.

Define the linear automorphism

$$t_\phi : V_L \to V_L$$

by the formula

$$t_\phi(x \otimes e^{\sum_i m_i B^i + \sum_i n_i A^i}) = e^{-\phi(\sum_i n_i A^i)} x \otimes e^{\sum_i m_i B^i + \sum_i n_i A^i}. \tag{4.1}$$

Define $V_{L,\phi}$ to be the vertex algebra equal to V_L as a vector space with the following n-th product:

$$(x \otimes e^{\sum_i m_i B^i + \sum_i n_i A^i})_{(n),\phi}(y \otimes e^{\sum_i m'_i B^i + \sum_i n'_i A^i})$$

$$= t_\phi^{-1}(t_\phi(x \otimes e^{\sum_i m_i B^i + \sum_i n_i A^i})_{(n)} t_\phi(y \otimes e^{\sum_i m'_i B^i + \sum_i n'_i A^i})). \qquad (4.2)$$

By definition,

$$t_\phi : V_{L,\phi} \to V_L$$

is a vertex algebra isomorphism. This provides us with a constant family of vertex algebras parametrized by ϕ and we would like to study the behavior of this family as ϕ tends to ∞. For this we have to impose certain restrictions on ϕ.

Following [Bat], call a piecewise linear function ϕ *convex* if

$$\phi(x) + \phi(y) \geq \phi(x+y) \text{ all } x, y \in L_A. \qquad (4.3)$$

The cone of all convex piecewise linear functions descends to the cone in $H^2(X_\Sigma, \mathbb{R}) = PL(\Sigma)/L_B \otimes_{\mathbb{Z}} \mathbb{R}$, see Theorem 4.1. Denote this cone by $K(\Sigma)$ and its interior by $K^0(\Sigma)$. $K^0(\Sigma)$ consists of classes of all *strictly convex* piecewise linear functions, that is, of all those functions ϕ for which equality in (4.1) is achieved if and only if x and y belong to the same cone in Σ.

We see immediately that

(i) if ϕ is convex piecewise linear, then the operations

$$_{(n),\infty\phi} = \lim_{\tau \to +\infty} {}_{(n),\tau\phi}, \ n \in \mathbb{Z}$$

are well defined and satisfy the Borcherds identities; denote the vertex algebra arising in this way by $V_{L,\infty\phi}$;

(ii) if ϕ is strictly convex piecewise linear, then $V_{L,\infty\phi}$ is isomorphic to Borisov's algebra V_L^Σ.

These assertions mean that the family $V_{L,\phi}$ produces a deformation of V_L^Σ with base equal to the cone of strictly convex piecewise linear functions. It is also immediate to see that if $\phi - \phi'$ is a linear function, then the two deformations $V_{L,\tau\phi}$ and $V_{L,\tau\phi'}$, $\tau \geq 0$, are equivalent. Therefore we have obtained the family of vertex algebras $V_{L,\phi}$, $\phi \in K^0(\Sigma)$, which is a deformation of V_L^Σ with base $K^0(\Sigma)$.

Denote by $QH^*_\phi(X_\Sigma, \mathbb{R})$ the quantum cohomology of X_Σ as defined in section 5 of [Bat]. Borisov's differential is as follows

$$D = \int \sum_{i=1}^{N} \Psi^i(z)(\sum_{j=1}^{n}(B^i, \xi_j)e^{\xi_j}(z)), \qquad (4.4)$$

where $\{\xi_1, ..., \xi_n\}$ is the set of generators of all 1-dimensional cones in Σ.

Theorem 4.2

$$H_{Q_0+D}(V_{L,\phi}) = QH^*_\phi(X_\Sigma, \mathbb{R}).$$

Sketch of Proof. First of all,

$$(Q_0)^2 = 0,\ D^2 = 0,\ [Q_0, D] = 0.$$

(The first two of these assertions are obvious, the last one is obtained in the same way as (2.4).) Hence there arises a spectral sequence completely analogous to the one used in 2.4. It converges and collapses:

$$H_{Q_0+D}(V_{L,\phi}) = H_D(H_{Q_0}(V_{L,\phi}));$$

this is done in exactly the same way as in the proof of Theorem 2.4.

In part 1) of the proof of Theorem 2.3 the space $H_{Q_0}(V_{L,\phi})$ was shown to be equal to the group algebra $\mathbb{R}[L_A]$ extended by grassman variables Φ_0^i $(i = 1, ..., N)$. Thus $H_{Q_0}(V_{L,\phi})$ is a Koszul complex, and the restriction of D to this space equals the Koszul differential associated with the sequence

$$\sum_{j=1}^{n}(B^i, \xi_j)e^{\xi_j},\ i = 1, ..., N. \tag{4.5}$$

Therefore, $H_D(H_{Q_0}(V_{L,\phi}))$ is the corresponding "Koszul cohomology".

On the other hand, Batyrev defines $QH^*_\phi(X_\Sigma, \mathbb{R})$ to be the polynomial ring $\mathbb{R}[z_1, ..., z_n]$ modulo the sum of two ideals denoted $P(\Sigma)$ and $Q_\phi(\Sigma)$. It follows from the proof of either Theorem 9.5 or Theorem 8.4 in [Bat] that

$$\mathbb{R}[L_A] = \mathbb{R}[z_1, ..., z_n]/Q_\phi(\Sigma).$$

Under this identification, the image of the ideal $P(\Sigma)$ in $\mathbb{R}[L_A]$ is generated by the elements (4.5) as follows from the comparison of (4.5) above and Definition 3.7 in [Bat]. The ring $\mathbb{R}[L_A]$ is Cohen-Macaulay; hence the sequence (4.5) is regular. □

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F.M.: Department of Mathematics, University of Southern California, Los Angeles, CA 90089, USA; fmalikov@mathj.usc.edu

V.S.: IHES, 35 Route de Chartres, 91440 Bures-sur-Yvette, France; vadik@ihes.fr