

Optimal Redeeming Strategy of Stock Loans

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Abstract

A stock loan is a loan, secured by a stock, which gives the borrower the right to redeem the stock at any time before or on the loan maturity. The way of dividends distribution has a significant effect on the pricing of the stock loan and the optimal redeeming strategy adopted by the borrower. We present the pricing models subject to various ways of dividend distribution. Since closed-form price formulas are generally not available, we provide a thorough analysis to examine the optimal redeeming strategy. Numerical results are presented as well.

1 Introduction

Optimal stopping problems in finance have gained growing interests due to their close linkage with various optimal strategies. A typical and well-known example is the American vanilla option pricing model which has been extensively studied in the existing literature. Many researchers have also considered plenty of more sophisticated models. For example, Gerber and Shiu (1996) and Broadie and Detemple (1997) analyzed the optimal exercise strategy for American options on multi-assets, which was further addressed by Villeneuve (1999), Jiang (2002), Detemple et al. (2003), etc. Cheuk and Vorst (1997), Windcliff et al. (2001), and Dai et al. (2004) investigated the optimal shouting strategy for shout options. Dai and Kwok (2006) characterized the optimal exercise strategy for American-style Asian options and lookback options. Hu and Oksendal (1998) studied the optimal strategy of an investment problem. Other studies along this line include the pricing of game options, swing options and convertible bonds, and the multiple-stopping problems in finance, see Kifer (2000), Carmona and Touzi (2003), IBanez (2004), Meinshausen and Hambly (2004), Dai and Kwok (2005), Dai and Kwok (2008), etc.

In this paper, we take into consideration another optimal stopping problem arising from the pricing of a financial product: *stock loan*, which is a contract between a client (borrower) and a bank (lender). The borrower, who owns one share of a stock, obtains a loan, from the lender with the share as collateral. The borrower may redeem the stock at any time before or on the loan maturity by repaying the lender the principal and a predetermined load interest rate, or surrender the stock instead of repaying the loan. The accumulative dividends may be gained by the borrower or the lender, subject to the provision of the loan.

A natural pricing problem arises for both the borrower and the lender: given the principal K and the loan interest rate γ , what is the fair fee charged by the lender (referred to as the price of

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the stock loan)? Moreover, the borrower is facing another problem: what is the optimal redeeming strategy? Xia and Zhou (2007) initiated the study of the above problems under the Black-Scholes framework. Assuming that the loan is of infinite maturity and the dividends accrued are gained by the lender until the borrower redeems the stock, they revealed that the stock loan is essentially an American call option with a possibly negative interest rate. Moreover, they obtained a closed form price formula as well as the analytical optimal redeeming strategy. Zhang and Zhou (2009) further extended it to a regime switching market.

In this paper, we are concerned with finite maturity stock loans with various ways of dividend distribution. Except for special cases, closed form price formulas are no longer available. As in Xia and Zhou (2007), the pricing model of a stock loan resembles that of (finite maturity) American vanilla options if the dividends are gained by the lender before redemption. However, other ways of dividends distribution may significantly alter the pricing model as well as the optimal redeeming strategy. In particular, if the accumulative dividends are assumed to be returned to the borrower on redemption, the pricing model will get one path-dependent variable involved, which leads the study of the associated redeeming strategy to be challenging. We will provide an analytical method to analyze the optimal redeeming strategy.

Throughout the paper, we assume that the risk neutral stock price follows a geometric Brownian motion:

$$dS_t = (r - \delta) S_t dt + \sigma S_t dW_t,$$

where constants $r > 0$, $\delta \geq 0$ and $\sigma > 0$ are the riskless interest rate, continuous dividend yield¹ of and volatility of the stock, respectively, and $\{W_t; t > 0\}$ is a standard 1-dimension Brownian motion on a filtered probability space $(\mathbb{S}, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with $W_0 = 0$ almost surely. We denote by $K > 0$ the stock loan's principal, γ the (continuous compounding) loan interest rate, and $T > 0$ the maturity date. For later use, we let $C_E(\cdot, \cdot; r, \delta, X)$ (or $P_E(\cdot, \cdot; r, \delta, X)$) be the price of European vanilla call (or put) option with riskless rate r , dividend yield δ and strike price X .

The rest of the paper is organized as follows. In section 2, we consider a relatively simple case: the dividends are gained by the lender before redemption. In section 3, we assume that the dividends are reinvested in the stock and will be returned to the borrower on redemption. We will see that this reduces to a special case in section 2, and the resulting optimal strategy can be linked to those presented in the subsequent two sections. Section 4 is devoted to the scenario that the cash dividends are delivered to the borrower immediately, no matter whether the borrower redeems the stock. It is worth pointing out that the pricing models in Section 2-4 involve one state variable only and are then relatively easy to analyze. In section 5, we will take into consideration the most challenging and interesting case: the accumulative cash dividends are delivered to the borrower on redemption. The pricing model proves to be a two-dimensional parabolic variational inequality. Section 6 discusses some extensions with conclusive remarks.

¹All models can be trivially extended to the discrete dividend payment. On the other hand, the assumption of continuous dividend payments is true if the collateralized security is foreign currency.

2 Dividends gained by the lender before redemption

Assume that the dividends are gained by the lender before redemption. Let $V_1 = V_1(S, t)$ be the value of the stock loan at time t with stock price S . Then

$$V_1(S_t, t) = \sup_{v \in \mathcal{T}_{[t, T]}} \mathbb{E}_t \left[e^{-r(v-t)} (S_v - Ke^{\gamma v})^+ \right].$$

where \mathbb{E}_t is the risk neutral expectation conditionally on \mathcal{F}_t and $\mathcal{T}_{[t, T]}$ denotes the set of all $\{\mathcal{F}_s\}_{t \leq s \leq T}$ -stopping times with values in $[t, T]$. It turns out that V_1 satisfies the following variational inequality [cf. Karatzas and Shreve (1998)]:

$$\begin{cases} \min \left\{ -L_S^{r, \delta} V_1, V_1 - (S - Ke^{\gamma t}) \right\} = 0, \\ V_1(S, T) = (S - Ke^{\gamma T})^+, \end{cases} \quad (S, t) \in Q, \quad (2.1)$$

where $Q \equiv (0, \infty) \times [0, T]$, and

$$L_S^{r, \delta} = \frac{\partial}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - \delta) S \frac{\partial}{\partial S} - r.$$

Let us define the redemption region

$$E_1 \equiv \{(S, t) \in Q : V_1(S, t) = S - Ke^{\gamma t}\}.$$

The following proposition characterizes the properties of E_1 .

Proposition 2.1 *Assume that the dividends are gained by the lender before redemption.*

i) If $r \geq \gamma$ and $\delta = 0$, then $E_1 = \emptyset$. This indicates that early redemption should never happen. In this scenario the stock loan is equivalent to a European call option with strike price $Ke^{\gamma T}$, namely, $V_1(S, t) = C_E(S, t; r, 0, Ke^{\gamma T})$.

ii) If $r \geq \gamma$ and $\delta > 0$, or $r < \gamma$, then there is an optimal redeeming boundary $S_1^(t) : [0, T] \rightarrow (0, \infty)$ such that*

$$E_1 = \{(S, t) \in Q : S \geq S_1^*(t)\}.$$

In addition, $e^{-\gamma t} S_1^(t)$ is monotonically decreasing with t ,*

$$S_1^*(T) \equiv \lim_{t \rightarrow T^-} S_1^*(t) = \begin{cases} e^{\gamma T} \max \left(K, \frac{r-\gamma}{\delta} K \right) & \text{if } r \geq \gamma \text{ and } \delta > 0, \\ e^{\gamma T} K & \text{if } r < \gamma. \end{cases} \quad (2.2)$$

Proof: By similarity reduction

$$f_1(x, t) = e^{-\gamma t} V_1(S, t), \quad x = e^{-\gamma t} S,$$

we get

$$\begin{cases} \min \left\{ -L_x^{\bar{r}, \delta} f_1, f_1 - (x - K) \right\} = 0, \\ f_1(x, T) = (x - K)^+, \end{cases} \quad (x, t) \in Q, \quad (2.3)$$

where $\bar{r} = r - \gamma$. Then, $f_1(x, t)$ can be regarded as the value of an American call option with riskless rate \bar{r} that is likely to be non-positive. Hence, part i) is a well-known result when $\bar{r} > 0$ and $\delta = 0$.

When $\bar{r} \geq 0$ and $\delta > 0$, we know from (2.3) [cf. Jiang (2005)] that there is a decreasing function $x_1^*(t) : [0, T] \rightarrow (0, +\infty)$, such that

$$\{(x, t) \in Q : f_1(x, t) = x - K\} = \{(x, t) \in Q : x \geq x_1^*(t), t \in [0, T]\} \quad (2.4)$$

and

$$x_1^*(T) \equiv \lim_{t \rightarrow T^-} x_1^*(t) = \max\left(K, \frac{\bar{r}}{\delta}K\right). \quad (2.5)$$

Using a similar argument as in Jiang (2005), we are able to show that (2.4) and (2.5) are still true when $\bar{r} < 0$. Notice that $\max\left(K, \frac{\bar{r}}{\delta}K\right) = K$ for $\bar{r} < 0$, part ii) then follows. ■

Now we examine the asymptotic behavior of $S_1^*(t)$ as the time to maturity goes to infinity.

Proposition 2.2 *Assume $r \geq \gamma$ and $\delta > 0$, or $r < \gamma$. Let $S_1^*(t)$ be as given in part ii) of Proposition 2.1. Then*

$$S_{1,\infty}^* \equiv \lim_{T \rightarrow +\infty} S_1^*(t) = e^{\gamma t} x_{1,\infty}^*, \quad (2.6)$$

where

$$x_{1,\infty}^* = \begin{cases} \frac{\alpha_+}{\alpha_+ - 1}K, & \text{if } \delta > 0, \text{ or } \delta = 0 \text{ and } r < \gamma - \frac{1}{2}\sigma^2 \\ +\infty, & \text{if } \delta = 0 \text{ and } \gamma - \frac{1}{2}\sigma^2 \leq r < \gamma \end{cases} \quad (2.7)$$

and

$$\alpha_+ = \frac{-(r - \gamma - \delta - \frac{1}{2}\sigma^2) + \sqrt{(r - \gamma - \delta + \frac{1}{2}\sigma^2)^2 + 2\delta\sigma^2}}{\sigma^2}. \quad (2.8)$$

To prove the above proposition, we only need to consider a perpetual stock loan and $S_{1,\infty}^*$ is the corresponding optimal redeeming boundary. This has been done by Xia and Zhou (2007) in terms of a probabilistic approach. We would like to provide a simple PDE argument which is placed in Appendix. It is worth pointing out that the explicit solution of $x_{1,\infty}^*$ is nothing but the optimal exercise boundary of a perpetual American call option when $r \geq \gamma$ and $\delta > 0$.

3 Reinvested dividends returned to the borrower on redemption

Assume that the dividends are immediately re-invested in the stock and will be returned to the borrower on redemption. The intrinsic value (i.e. the redemption payoff) of the stock loan at time t becomes

$$\left(e^{\delta t} S_t - K e^{\gamma t}\right)^+, \quad t \in [0, T].$$

Let $V_2 = V_2(S, t)$ be the price function of the stock loan. Then

$$V_2(S_t, t) = \sup_{v \in \mathcal{T}_{[t, T]}} \mathbb{E}_t \left[e^{-r(v-t)} (e^{\delta v} S_v - K e^{\gamma v})^+ \right]. \quad (3.1)$$

Denote $\widehat{V}_2(\widehat{S}_t, t) = V_2(S_t, t)$, where

$$\widehat{S}_t \equiv e^{\delta t} S_t = S_0 \exp \left\{ \left(r - \frac{\sigma^2}{2} \right) t + \sigma W_t \right\}.$$

Then we can rewrite (3.1) as

$$\widehat{V}_2(\widehat{S}_t, t) = \sup_{v \in \mathcal{T}_{[t, T]}} \mathbb{E}_t \left[e^{-r(v-t)} (\widehat{S}_v - Ke^{\gamma v})^+ \right], \quad (3.2)$$

satisfying

$$\begin{cases} \min \left\{ -L_{\widehat{S}}^{r,0} \widehat{V}_2, \widehat{V}_2 - (\widehat{S} - Ke^{\gamma t}) \right\} = 0, \\ \widehat{V}_2(\widehat{S}, T) = (\widehat{S} - Ke^{\gamma T})^+, \end{cases} \quad (\widehat{S}, t) \in Q. \quad (3.3)$$

As a result, the stock loan can be regarded as the one written on the non-dividend paying stock \widehat{S} , which has been studied in Section 2.

For later use, we define the redemption region associated with \widehat{V}_2 as

$$\widehat{E}_2 = \left\{ (\widehat{S}, t) \in Q : \widehat{V}_2(\widehat{S}, t) = \widehat{S} - Ke^{\gamma t} \right\}. \quad (3.4)$$

Later we will see that \widehat{E}_2 has a close link with the redemption regions addressed in subsequent two sections. By Proposition 2.1, $\widehat{E}_2 = \emptyset$ when $r \geq \gamma$, and if $r < \gamma$, then there is a monotonically decreasing function $\widehat{S}_2^*(t) : [0, T] \rightarrow (0, \infty)$ such that

$$\widehat{E}_2 = \left\{ (\widehat{S}, t) \in Q : \widehat{S} \geq \widehat{S}_2^*(t), t \in [0, T] \right\}, \quad (3.5)$$

$\widehat{S}_2^*(T) \equiv \lim_{t \rightarrow T^-} \widehat{S}_2^*(t) = K$, and

$$\widehat{S}_{2,\infty}^* \equiv \lim_{T \rightarrow +\infty} \widehat{S}_2^*(t) \text{ is finite if } r < \gamma - \frac{1}{2}\sigma^2, \text{ and infinite if } \gamma - \frac{1}{2}\sigma^2 \leq r < \gamma. \quad (3.6)$$

4 Dividends always delivered to the borrower

Assume that the dividends are always delivered to the borrower during the lifetime of the stock loan. The intrinsic value of the stock loan becomes

$$(S_t - Ke^{\gamma t})^+ + \int_0^t \delta e^{r(t-u)} S_u du, \quad t \in [0, T]. \quad (4.1)$$

By introducing a path-dependent variable

$$I_t = \int_0^t \delta e^{r(t-u)} S_u du, \quad (4.2)$$

the value of the stock loan can be expressed as

$$\begin{aligned} V_3(S_t, I_t, t) &= \sup_{v \in \mathcal{T}_{[t, T]}} \mathbb{E}_t \left(e^{-r(v-t)} [(S_v - Ke^{\gamma v})^+ + I_v] \right) \\ &= I_t + \sup_{v \in \mathcal{T}_{[t, T]}} \mathbb{E}_t \left(e^{-r(v-t)} [(S_v - Ke^{\gamma v})^+] + \int_t^v \delta e^{-r(u-t)} S_u du \right). \end{aligned}$$

Observe that $V_3(S_t, I_t, t) - I_t$ is independent of I_t . Then we can write $H(S, t) \equiv V_3(S, I, t) - I$, which satisfies

$$\begin{cases} \min \left\{ -L_S^{r, \delta} H - \delta S, H - (S - Ke^{\gamma t}) \right\} = 0, \\ H(S, T) = (S - Ke^{\gamma T})^+, \end{cases} \quad (S, t) \in Q. \quad (4.3)$$

Here $H(S, t)$ can be regarded as the value of the stock loan excluding the accumulative dividends. Compared with (2.1) in Section 2, (4.3) has a source term δS due to the dividends delivered.

Now we define the redemption region

$$E_3 \equiv \{(S, t) \in Q : H(S, t) = S - Ke^{\gamma t}\}.$$

We proceed with a lemma.

Lemma 4.1 *Let $\widehat{V}_2(\cdot, \cdot)$ and \widehat{E}_2 be as given in (3.2) and (3.4) respectively. Then*

$$\widehat{V}_2(S, t) \leq H(S, t) \text{ and } E_3 \subset \widehat{E}_2.$$

Proof: First we prove $\widehat{V}_2(S, t) \leq H(S, t)$. Compared with (3.3), (4.3) can be rewritten as

$$\begin{cases} \min \left\{ -L_S^{r, 0} H - \delta S \left(1 - \frac{\partial H}{\partial S}\right), H - (S - Ke^{\gamma t}) \right\} = 0, \\ H(S, T) = (S - Ke^{\gamma T})^+, \end{cases} \quad (S, t) \in Q.$$

So, by the maximum principle [cf. Friedman (1988)], it suffices to show $\frac{\partial H}{\partial S} \leq 1$ or

$$\frac{\partial}{\partial S}(H - S) \leq 0. \quad (4.4)$$

Notice

$$\begin{cases} \min \left\{ -L_S^{r, \delta} (H - S), (H - S) + Ke^{\gamma t} \right\} = 0, \\ H(S, T) - S = \max(-S, -Ke^{\gamma T}), \end{cases} \quad (S, t) \in Q.$$

Again applying the maximum principle gives (4.4), then $\widehat{V}_2(S, t) \leq H(S, t)$ follows.

Now let us show $E_3 \subset \widehat{E}_2$. For any $(S, t) \in E_3$, we have

$$S - Ke^{\gamma t} \leq \widehat{V}_2(S, t) \leq H(S, t) = S - Ke^{\gamma t},$$

which implies $\widehat{V}_2(S, t) = S - Ke^{\gamma t}$, i.e., $(S, t) \in \widehat{E}_2$. This completes the proof. ■

The following proposition characterizes the shape of E_3 .

Proposition 4.2 *Assume $\delta > 0$ and the dividends are always delivered to the borrower during the lifetime of the stock loan.*

i) If $r \geq \gamma$, then $E_3 = \emptyset$. That is, early redemption should never happen. In addition, we have

$$H(S, t) = C_E(S, t; r, \delta, Ke^{\gamma T}) + (1 - e^{-\delta(T-t)})S, \quad (4.5)$$

ii) If $r < \gamma$, then there is an optimal redeeming boundary $S_3^(t) : [0, T] \rightarrow (0, +\infty)$ such that*

$$E_3 = \{(S, t) \in Q : S \geq S_3^*(t)\}.$$

In addition, $e^{-\gamma t} S_3^*(t)$ is monotonically decreasing in t ,

$$S_3^*(T) \equiv \lim_{t \rightarrow T^-} S_3^*(t) = e^{\gamma T} K,$$

and

$$S_3^*(t) \geq \widehat{S}_2^*(t), \quad (4.6)$$

where $\widehat{S}_2^*(t)$ is as given in (3.5).

Proof: First, by an arbitrage argument, it is not hard to get

$$H(S, t) \geq C_E(S, t; r, \delta, K e^{\gamma T}) + (1 - e^{-\delta(T-t)})S, \quad (4.7)$$

where the right hand side is the price of the corresponding stock loan without early redemption right. Since $C_E(S, t; r, \delta, K e^{\gamma T}) > S e^{-\delta(T-t)} - K e^{\gamma T} e^{-r(T-t)}$ [see, for example, Hull (2003)], it follows

$$\begin{aligned} H(S, t) &\geq S - K e^{\gamma T} e^{-r(T-t)} \\ &> S - K e^{\gamma t}, \text{ for } r \geq \gamma \text{ and } t < T, \end{aligned}$$

which implies part i).

To show part ii), as before, we make a transformation: $f_3(x, t) = e^{-\gamma t} H(S, t)$, $x = e^{-\gamma t} S$. It follows

$$\begin{cases} \min \left\{ -L_x^{\bar{r}, \delta} f_3 - \delta x, f_3 - (x - K) \right\} = 0, \\ f_3(x, T) = (x - K)^+, \end{cases} \quad (x, t) \in Q. \quad (4.8)$$

By (4.4), we have $\frac{\partial}{\partial x}(f_3 - x) \leq 0$, from which we infer that there is a single-value function $x_3^*(t) : [0, T) \rightarrow (0, +\infty) \cup +\infty$ such that

$$\{(x, t) \in Q : f_3(x, t) = x - K\} = \{(x, t) \in Q : x \geq x_3^*(t)\}.$$

Since

$$-L_x^{\bar{r}, \delta}(x - K) - \delta x = -\bar{r}K > 0, \text{ for } r < \gamma,$$

using a similar analysis as in Brezis and Friedman (1976) or Dai et al. (2004), we can deduce $f_3(x, t) - (x - K)$ has a compact support for all t . This indicates $x^*(t) < +\infty$ for all t . So, we can take $S_3^*(t) = e^{\gamma t} x_3^*(t)$. (4.6) is a corollary of Lemma 4.1.

Apparently

$$\frac{\partial f_3}{\partial t} \leq 0, \quad (4.9)$$

which yields the monotonicity of $x_3^*(t)$. It remains to show $x_3^*(T) \equiv \lim_{t \rightarrow T^-} x_3^*(t) = K$. The argument is standard and is stated as follows. Obviously $x_3^*(t) \geq K$. If $x_3^*(T) > K$, we would have for $\tilde{x} \in (K, x_3^*(T))$,

$$\left. \frac{\partial f_3}{\partial t} \right|_{(\tilde{x}, T)} = \left[-\frac{1}{2} \sigma^2 x^2 \frac{\partial^2}{\partial x^2} - (\bar{r} - \delta) x \frac{\partial}{\partial x} + \bar{r} \right] (x - K) - \delta x \Big|_{x=\tilde{x}} = -\bar{r}K > 0,$$

which is in contradiction with (4.9). This completes the proof. ■

Now we analyze the asymptotic behavior of $S_3^*(t)$ as the time to maturity goes to infinity. By virtue of (3.6) and (4.6), we immediately obtain that $S_3^*(t) \rightarrow +\infty$ as $T \rightarrow +\infty$ in the case of $\gamma - \frac{1}{2}\sigma^2 \leq r < \gamma$. We will show that it is also true for $r < \gamma - \frac{\sigma^2}{2}$. Again, we need to study the corresponding perpetual stock loan. Denote $H_\infty(S) = \lim_{T \rightarrow +\infty} H(S, t)$. We assert

$$H_\infty(S) = S. \quad (4.10)$$

Indeed, from (4.7), we have

$$H(S, t) \geq S - Se^{-\delta(T-t)},$$

which yields $H_\infty(S) \geq S$ by letting $T \rightarrow \infty$. The converse inequality is apparent. So, we get (4.10), which implies that one should never redeem the perpetual stock loan. We summarize the above result as follows:

Proposition 4.3 *Assume $\delta > 0$ and the dividends are always delivered to the borrower during the lifetime of the stock loan. Then the perpetual stock loan is equivalent to the stock, i.e., $H_\infty(S) \equiv \lim_{T \rightarrow +\infty} H(S, t) = S$. In addition,*

$$S_{3,\infty}^* \equiv \lim_{T \rightarrow +\infty} S_3^*(t) = +\infty, \text{ for } r < \gamma,$$

where $S_3^*(t)$ is the optimal redeeming boundary in part ii) of Proposition 4.2.

5 Dividends returned to the borrower on redemption

In this section, we assume that the accumulative dividends will be returned to the borrower on redemption. In contrast to (4.1), the intrinsic value of the stock loan is now

$$\left(S_t - Ke^{\gamma t} + \int_0^t \delta e^{r(t-u)} S_u du \right)^+, \quad t \in [0, T].$$

Again we introduce the path-dependent variable I_t as given in (4.2), then the value of the stock loan can be expressed as

$$V_4(S_t, I_t, t) = \sup_{v \in \mathcal{T}_{[t, T]}} \mathbb{E}_t \left[e^{-r(v-t)} (S_v - Ke^{\gamma v} + I_v)^+ \right].$$

Note that

$$dI_t = (\delta S_t + rI_t) dt.$$

It follows that $V_4(S, I, t)$ satisfies

$$\begin{cases} \min \left\{ -L_S^{r, \delta} V_4 - (\delta S + rI) \frac{\partial V_4}{\partial T}, V_4 - (S + I - Ke^{\gamma t}) \right\} = 0, \\ V_4(S, I, T) = (S + I - Ke^{\gamma T})^+, \end{cases} \quad (S, I, t) \in \Omega, \quad (5.1)$$

where $\Omega = (0, +\infty) \times (0, +\infty) \times [0, T]$. (5.1) is analogous to the pricing model for Asian options and the existence of (strong) solution can be proved using a similar argument as in Pascucci (2008).

We emphasize that (5.1) is indeed a two-dimensional time-dependent problem and does not permit dimension reduction.

As before, we define the redemption region by

$$E_4 = \{(S, I, t) \in \Omega : V_4(S, I, t) = S + I - Ke^{\gamma t}\}.$$

Apparently $V_4(S, I, t) \leq V_3(S, I, t) = H(S, t) + I$, from which we immediately get

$$\{(S, I, t) \in \Omega : (S, t) \in E_3\} \subset E_4.$$

The following lemma presents a stronger result which plays an important role in analysis of the shape of E_4 :

Lemma 5.1 *Let $\widehat{V}_2(\cdot, \cdot)$ and \widehat{E}_2 be as defined in (3.2) and (3.4) respectively. Then we have*

$$V_4(S, I, t) \leq \widehat{V}_2(S + I, t) \tag{5.2}$$

and thus

$$\{(S, I, t) \in \Omega : (S + I, t) \in \widehat{E}_2\} \subset E_4. \tag{5.3}$$

Proof: We only need to prove $V_4(S, I, t) \leq \widehat{V}_2(S + I, t)$. Let us adopt $y \equiv S + I$ as a new state variable in place of S , and denote

$$U(y, I, t) = V_4(S, I, t),$$

which satisfies

$$\begin{cases} \min \{-\mathcal{L}U, U - (y - Ke^{\gamma t})\} = 0, \\ U(y, I, T) = (y - Ke^{\gamma T})^+, \end{cases} \quad (y, I, t) \in \Omega_y,$$

where $\Omega_y \equiv \{(y, I, t) : 0 < I < y < +\infty, 0 \leq t < T\}$,

$$\mathcal{L} = \frac{\partial}{\partial t} + (\delta y + (r - \delta)I) \frac{\partial}{\partial I} + \frac{1}{2}\sigma^2(y - I)^2 \frac{\partial^2}{\partial y^2} + ry \frac{\partial}{\partial y} - r.$$

Note that (3.3) can be rewritten as

$$\begin{cases} \min \left\{ -\mathcal{L}\widehat{V}_2 - \frac{1}{2}\sigma^2 [y^2 - (y - I)^2] \frac{\partial^2 \widehat{V}_2}{\partial y^2}, \widehat{V}_2 - (y - Ke^{\gamma t}) \right\} = 0, \\ \widehat{V}_2(y, T) = (y - Ke^{\gamma T})^+, \end{cases} \quad (y, I, t) \in \Omega_y.$$

Due to the convexity of $(y - Ke^{\gamma t})^+$ in y , we can deduce $\frac{\partial^2 \widehat{V}_2}{\partial y^2} \geq 0$, then

$$\frac{1}{2}\sigma^2 [y^2 - (y - I)^2] \frac{\partial^2 \widehat{V}_2}{\partial y^2} \geq 0, \text{ for any } y > I > 0, t \in [0, T].$$

Applying the maximum principle then gives

$$U(y, I, t) \leq \widehat{V}_2(y, t),$$

which is desired. ■

We would like to give a financial interpretation to (5.2). $\widehat{V}_2(\cdot, \cdot)$ and $V_4(\cdot, \cdot, \cdot)$ represent the prices for the stock loans respectively with reinvested dividends and with cash dividends, both of which will be returned to the borrowers on redemption. In the risk-neutral world, the return rate of reinvested dividends is the same as that of cash dividends. Since we can regard the combination of the dividends and the stock as an imaginary underlying asset (i.e. y), the cash dividends essentially decrease the volatility of the underlying asset, which leads to a lower price of the corresponding stock loan.

5.1 The case of $r \geq \gamma$

Now let us investigate the case of $r \geq \gamma$.

Proposition 5.2 *Assume $\delta > 0$ and the dividends are gained by the borrower on redemption.*

- i) *If $r > \gamma$, then $E_3 = \emptyset$. That is, early redemption should never happen.*
- ii) *If $r = \gamma$, then it is optimal to hold the option before expiry.*

Proof: We adopt an arbitrage argument. If the borrower redeems the loan at time $t < T$, then he or she will get

$$\text{one stock } S_t + \text{cash } I_t - Ke^{\gamma t},$$

which amounts to at expiry

$$\begin{aligned} S_T + I_T - Ke^{\gamma t} e^{r(T-t)} &< S_T + I_T - Ke^{\gamma T} \\ &\leq (S_T + I_T - Ke^{\gamma T})^+, \end{aligned} \quad (5.4)$$

This indicates that early redemption is not optimal. Part i) then follows.

To prove part ii), the argument is similar and the unique difference lies in that “<” in (5.4) should be replaced by “=”. ■

It is worth distinguishing two statements of part i) and ii) in Proposition 5.2. For the latter case, early redemption may be optimal on some occasions. For example, when $I \geq Ke^{rT}$ and $r = \gamma$, it is not hard to show that $V_4(S, I, t) = S + I - Ke^{rT}$, which implies that the redemption at any time is optimal.

Owing to Proposition 5.2, (5.1) is reduced to a linear problem:

$$\begin{cases} -L_S^{r, \delta} V_4 - (\delta S + rI) \frac{\partial V_4}{\partial I} = 0, \\ V_4(S, I, T) = (S + I - Ke^{\gamma T})^+, \end{cases} \quad (S, I, t) \in \Omega.$$

5.2 The case of $r < \gamma$

By Lemma 5.1 and the properties of \widehat{E}_2 , we know that E_4 is always non-empty when $r < \gamma$. Furthermore, we have

Proposition 5.3 *Assume $\delta > 0$ and the dividends are gained by the borrower on redemption. If $r < \gamma$, then $\{(S, I, t) \in \Omega : I \geq Ke^{\gamma t}\} \subset E_4$.*

Proof: Given $(S_t, I_t, t) \in \Omega$ with $I_t \geq Ke^{\gamma t}$, we claim that the loan should be redeemed immediately at time t . Indeed, if it is redeemed at time t , then we have a payoff: stock S_t + non-negative cash $I_t - Ke^{\gamma t}$, which becomes at a later time $t' \in (t, T]$

$$\text{stock } S_{t'} + \text{cash } I_{t'} - Ke^{\gamma t} e^{r(t'-t)} > (S_{t'} + I_{t'} - Ke^{\gamma t'})^+ \text{ if } r < \gamma.$$

This implies the conclusion. ■

We stress that Proposition 5.2 and Lemma 5.3 only rely on the no-arbitrage principle, and therefore are independent of the geometric Brownian motion assumption of stock price. Such a remark also applies to some of previous results.

Denote $\widehat{\Omega} = \{(S, I, t) \in \Omega : I < Ke^{\gamma t}\}$. Due to (5.3), we only need to study

$$\widehat{E}_4 \equiv E_4 \cap \widehat{\Omega}.$$

Proposition 5.4 *Assume that the dividends are gained by the borrower on redemption and $r < \gamma$. Then, there is an optimal redeeming boundary $S_4^*(I, t) : (0, Ke^{\gamma t}) \times [0, T) \rightarrow (0, \infty)$ such that*

$$\widehat{E}_4 = \{(S, I, t) \in \widehat{\Omega} : S \geq S_4^*(I, t)\}.$$

Moreover, $e^{-\gamma t} S_4^*(I, t)$ is monotonically decreasing in I and t ,

$$S_4^*(I, T) \equiv \lim_{t \rightarrow T} S_4^*(I, t) = e^{\gamma T} K - I, \text{ for all } I,$$

and

$$S_4^*(I, t) \leq \widehat{S}_2^*(t) - I, \text{ for all } t, \quad (5.5)$$

where $\widehat{S}_2^*(t)$ is as given in (3.5). In particular, $S_4^*(0, t) \equiv \lim_{I \rightarrow 0} S_4^*(I, t) \leq \widehat{S}_2^*(t)$, for all t .

Proof: Since $(S + I - Ke^{\gamma t})^+$ is convex in S , we infer that $V_4(S, I, t)$ is also convex in S , namely

$$V_4(aS_1 + (1-a)S_2, I, t) \leq aV_4(S_1, I, t) + (1-a)V_4(S_2, I, t).$$

which implies the convexity of \widehat{E}_4 in S . On other hand, due to Lemma 5.1, we have $(S, I, t) \in \widehat{E}_4$ for $S > \widehat{S}_2^*(t) - I$. This indicates the existence of $S_4^*(I, t)$ and $S_4^*(I, t) \leq \widehat{S}_2^*(t) - I$.

Using the similarity reduction $V_4(S, I, t) = e^{-\gamma t} f_4(x, A, t)$, $x = e^{-\gamma t} S$ and $A = e^{-\gamma t} I$, we get

$$\begin{cases} \min \left\{ -L_x^{\bar{r}, \delta} f_4 - (\delta x + \bar{r}A) \frac{\partial f_4}{\partial A}, f_4 - (x + A - K) \right\} = 0, \\ f_4(x, A, T) = (x + A - K)^+, \quad (x, A, t) \in \Omega. \end{cases} \quad (5.6)$$

Let $x_4^*(A, t) \equiv e^{-\gamma t} S_4^*(I, t)$. It suffices to show that $x_4^*(A, t)$ is monotonically decreasing in A and t , and

$$x_4^*(A, T) \equiv \lim_{t \rightarrow T} x_4^*(A, t) = K - A, \text{ for all } A. \quad (5.7)$$

By the maximum principle, it is not hard to get

$$\frac{\partial f_4}{\partial t} \leq 0,$$

and

$$\frac{\partial}{\partial A} [f_4 - (x + A - K)] = \frac{\partial f_4}{\partial A} - 1 \leq 0,$$

which implies the monotonicity of $x_4^*(A, t)$ in A and t . Using a similar argument as in Proposition 4.2, we are able to obtain (5.7). ■

6 Numerical examples

In this section we present numerical results to verify our theoretical results. Let us first look at the pricing models in Section 2-4 which belong to standard one-dimensional parabolic variational inequalities. They can be numerically solved using many sophisticated numerical methods such as the projected SOR [cf. Wilmott et al. (1993)], the recursive integration method [cf. Huang et al. (1996)] and the penalty approach [cf. Forsyth and Vetzal (2002)]. We simply make use of the binomial tree method [cf. Hull (2003)] which is easy to implement. The default data used are $r - \gamma = -0.04$, $\delta = 0.03$, $\gamma = 0.1$ and $K = 0.7$. Figure 1 and Figure 2 plot the optimal redeeming boundaries against the time to maturity $\tau = T - t$ with $\sigma = 0.4$ and with $\sigma = 0.15$, respectively. Here $x_1^*(\cdot) = e^{-\gamma t} S_1^*(t)$, $x_2^*(\cdot) = e^{-\gamma t} \hat{S}_2^*(t)$ and $x_3^*(\cdot) = e^{-\gamma t} S_3^*(t)$. Observe that $x_1^* \leq x_2^* \leq x_3^*$, and these boundaries are monotonically increasing in τ (so, decreasing in t). Since $r < \gamma$, we can see that all boundaries go to $x = K \equiv 0.7$ at $\tau \rightarrow 0$. Theoretical results indicate that $x_1^*(\cdot)$ has an asymptotic line ($x = 1.97$ for Figure 1 and $x = 0.82$ for Figure 2) while $x_3^*(\cdot)$ does not as $\tau \rightarrow +\infty$, which can be seen from the figures. In addition, $x_2^*(\cdot)$ has an asymptotic line ($x = 0.97$) as $\tau \rightarrow +\infty$ in Figure 2 due to $r < \gamma - \frac{\sigma^2}{2}$, whereas not in Figure 1 ($\gamma - \frac{\sigma^2}{2} \leq r < \gamma$). This is also consistent with our theoretical analysis.

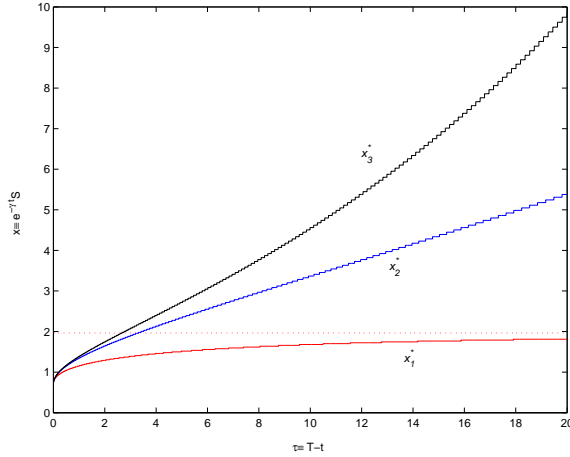


Figure 1: Optimal redeeming boundaries (Parameters: $r - \gamma = -0.04$, $\delta = 0.03$, $\sigma = 0.4$ and $K = 0.7$.)

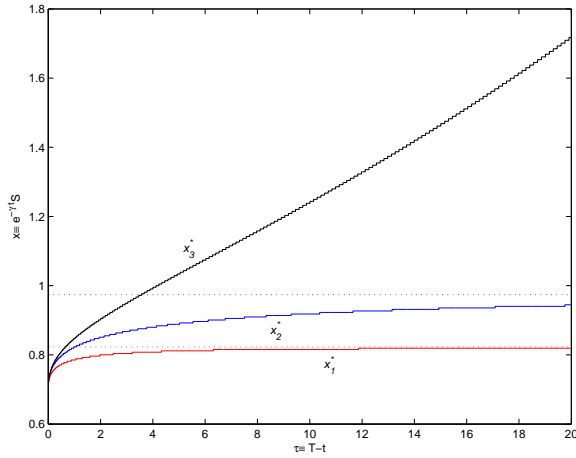


Figure 2: Optimal redeeming boundaries (Parameters: $r - \gamma = 0.06$, $\delta = 0.03$, $\sigma = 0.15$ and $K = 0.7$.)

Now let us move on to the case in Section 5, where the pricing model, resembling that of American-style Asian options, is a degenerate parabolic variational inequality. We then employ the forward shooting grid method [cf. Barraquand and Pudet (1996)], which is widely used to deal with this type of degenerate problems. Figure 3 shows the optimal redeeming boundary $x_4^*(\cdot, \cdot)$ in x - A - τ plane, where $x = e^{-\gamma t}S$, $A = e^{-\gamma t}I$ and $\tau = T - t$. The data used are $r - \gamma = -0.04$, $\delta = 0.03$, $\sigma = 0.4$ and $K = 0.7$. Time snapshots of the boundary are depicted in Figure 4. It can be seen that the boundary, as a function of A and τ , is decreasing in A , increasing in τ (i.e. decreasing in t). At maturity the boundary is a straight line $x + A = K \equiv 0.7$. Nevertheless, theoretical analysis indicates that $x_4^*(\cdot, \tau) \leq x_2^*(\tau)$. Figure 1 shows that approximately x_2^* equals 1.35 when $\tau = 1$, and 1.9 when $\tau = 3$. It can be observed from Figure 4 that x_4^* is indeed bounded from above by $x_2^*(\cdot)$. All these verify Proposition 5.4.

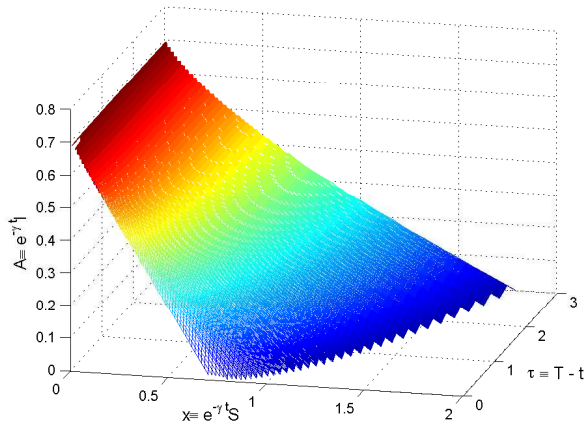


Figure 3: Optimal redeeming boundary $x_4^*(A, \tau) \equiv e^{-\gamma t}S_4^*(I, t)$ (Parameters: $r - \gamma = -0.04$, $\delta = 0.03$, $\sigma = 0.4$ and $K = 0.7$)

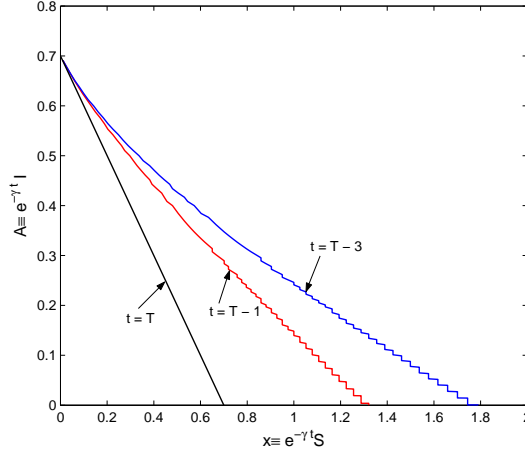


Figure 4: Time snapshots of $x_4^*(\cdot, \cdot)$ (Parameters: $r - \gamma = -0.04$, $\delta = 0.03$, $\sigma = 0.4$ and $K = 0.7$)

7 Conclusion and extensions

In the Black-Schole framework, we formulate the pricing of stock loans as an optimal stopping problem, or equivalently, a variational inequality. Since closed-form price formulas are generally not available, we provide an analytic approach to analyze the optimal redeeming strategy. It turns out that the way of dividends distribution significantly alters the pricing model and the optimal redeeming strategy. Numerical results are presented as well.

To conclude the paper, we briefly discuss some possible extensions of the pricing models. Without loss of generality, we assume that the dividends are gained by the lender and let $V = V(S, t)$ be the price function of the stock loan.

7.1 Amortized loan

Assume that the loan is amortized. Let C be the amortized rate defined as follows:

$$\int_0^T C e^{\gamma(T-t)} dt = K e^{\gamma T},$$

which yields $C = \frac{\gamma}{1 - e^{-\gamma T}} K$. At time t , the amount to be repaid is

$$K e^{\gamma t} - \int_0^t C e^{\gamma(t-u)} du = \frac{C}{\gamma} \left(1 - e^{-\gamma(T-t)} \right).$$

So, the early redemption payoff is $S_t - \frac{C}{\gamma} (1 - e^{-\gamma(T-t)})$, $t \in [0, T)$. Then the pricing model becomes

$$\begin{cases} \min \left\{ -L_S^{r, \delta} V + C, V - \left[S - \frac{C}{\gamma} (1 - e^{-\gamma(T-t)}) \right] \right\} = 0, \\ V(S, T) = S, \quad (S, t) \in Q. \end{cases}$$

7.2 Withdrawal feature

The withdrawal feature aims to protect the lender from the extremely downside risk of the collateralized stock. It allows the lender to withdrawal the loan at any time by requesting the borrower to redeem the stock with the price L , where $L < K$. This implies

$$V(S, t) \leq L.$$

Like the pricing of game options, the model is described as a double obstacle problem [cf. Kiefr (2000), Dai and Kwok (2005)] :

$$\begin{cases} \max \left\{ \min \left\{ -L_S^{r, \delta} V, V - (S - Ke^{\gamma t}) \right\}, V - L \right\} = 0, \\ V(S, T) = \min \left\{ (S - Ke^{\gamma T})^+, L \right\}, \end{cases} \quad (S, t) \in Q.$$

7.3 Renewal feature

If we assume that the borrower has right to renew the loan, this will lead to multiple stopping problems. See, for example, Carmona and Touzi (2003), IBanez (2004), Meinshausen and Hambly (2004) and Dai and Kwok (2008). From the point of view of PDEs, the pricing model is described by a series of variational inequalities [cf. Dai and Kwok (2008)]. We refer interested readers to the references.

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Appendix: The proof of Proposition 2.2

Applying the maximum principle, it is easy to see that the solution to problem (2.3) possesses the following properties:

$$0 \leq f(x, t) \leq 1 \text{ and } 0 \leq \frac{\partial f}{\partial x} \leq 1. \quad (\text{A-1})$$

Denote $f_\infty(x) = \lim_{(T-t) \rightarrow +\infty} f(x, t)$. Then $f_\infty(x)$ satisfies the stationary counterpart of (2.3), which is equivalent to a free boundary problem:

$$-\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f_\infty}{\partial x^2} - (\bar{r} - \delta) x \frac{\partial f_\infty}{\partial x} + \bar{r} f_\infty = 0, \quad x \leq x_\infty^* \quad (\text{A-2})$$

$$f_\infty(x_\infty^*) = x - K, \text{ and } f'_\infty(x_\infty^*) = 1. \quad (\text{A-3})$$

Here x_∞^* is the free boundary to be determined. Due to (A-1), we are only concerned with the solution to (A-2)-(A-3) under the restrictions

$$0 \leq f_\infty(x) \leq x \text{ and } 0 \leq f'_\infty(x) \leq 1. \quad (\text{A-4})$$

As is well-known, the general solution to equation (A-2) is

$$f_\infty(x) = C_1 x^{\alpha_+} + C_2 x^{\alpha_-},$$

where C_1 and C_2 are to be determined, and α_+ (as given in (2.8)) and $\alpha_- = 1 - \frac{2(r-\gamma-\delta)}{\sigma^2} - \alpha_+$ are two roots of the algebraic equation $\frac{\sigma^2}{2}\alpha^2 + (\bar{r} - \delta - \frac{\sigma^2}{2})\alpha - \bar{r} = 0$. Let us first assume $\delta > 0$. It is easy to check that $\alpha_- < 1 < \alpha_+$, which yields $C_2 = 0$ or

$$f_\infty(x) = C_1 x^{\alpha_+}. \quad (\text{A-5})$$

Otherwise we would have $f'_\infty(x) = C_1 \alpha_1 x^{\alpha_+ - 1} + C_2 \alpha_2 x^{\alpha_- - 1} \rightarrow \infty$ as $x \rightarrow 0$, which contradicts (A-4).

From (A-5), we can make use of (A-3) to get

$$C_1 = \frac{1}{\alpha_+} \left(\frac{\alpha_+ - 1}{\alpha_+ K} \right)^{\alpha_+ - 1} \text{ and } x_\infty^* = \frac{\alpha_+}{\alpha_+ - 1} K \quad (\text{A-6})$$

for $\delta > 0$. To deal with the case of $\delta = 0$, we let δ go to 0 in (A-6). It is easy to see that if $r + \frac{1}{2}\sigma^2 \geq 0$ and $\delta \rightarrow 0$, then $\alpha_+ \rightarrow 1$, which yields $x_\infty^* = +\infty$. If $\bar{r} + \frac{1}{2}\sigma^2 < 0$ and $\delta \rightarrow 0$, then $\alpha_+ > 1$ and the expression of x_∞^* is the same as in (A-6). The proof is complete. ■