

An Application of the Whitehouse Module to Riffle Shuffles Followed by a Cut

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Abstract

Using representation theoretic work on the Whitehouse module, a formula is obtained for the cycle structure of a riffle shuffle followed by a cut. This result will be merged with the paper [F6].

1 Introduction

In an effort to study the way real people shuffle cards, Bayer and Diaconis [BaD] performed a definitive analysis of the Gilbert-Shannon-Reeds model of riffle shuffling. For an integer $k \geq 1$, a k -shuffle can be described as follows. Given a deck of n cards, one cuts it into k piles with probability of pile sizes j_1, \dots, j_k given by $\frac{\binom{n}{j_1, \dots, j_k}}{k^n}$. Then cards are dropped from the packets with probability proportional to the pile size at a given time (thus if the current pile sizes are A_1, \dots, A_k , the next card is dropped from pile i with probability $\frac{A_i}{A_1 + \dots + A_k}$). It was proved in [BaD] that $\frac{3}{2} \log_2 n$ shuffles are necessary and suffice for a 2-shuffle to achieve randomness (the paper [A] had established this result asymptotically in n). It was proved in [DMP] that if $k = q$ is a prime power, then the chance that a permutation distributed as a q -shuffle has n_i i -cycles is equal to the probability that a uniformly chosen monic degree n polynomial over the field F_q factors into n_i irreducible polynomials of degree i .

A very natural question is to study the effects of cuts on the results of the previous paragraph. For example, it is shown in [F6] that performing the process of “a riffle shuffle followed by a cut at a uniform position” also gets random in $\frac{3}{2} \log_2 n$ steps. This can be contrasted with a result of Diaconis [D], who proves that although shuffling by doing random transpositions gets random in $\frac{1}{2} n \log(n)$ steps, the use of cuts at each stage drops the convergence time to $\frac{3}{8} n \log(n)$ steps. The main result of this note is that if $k = q$ is a prime power, then the chance that a permutation distributed as a q -shuffle has n_i i -cycles is equal to the probability that a uniformly chosen monic degree n polynomial over the field F_q with non-zero constant term factors into n_i irreducible polynomials of degree i . The result is proved by showing it to be completely equivalent to representation theoretic results of Whitehouse [W].

Before jumping into the proof, we remark that the theory of riffle shuffling appears in numerous parts of mathematics. Among these are:

1. Cyclic and Hochschild homology [H],[GerS],[L]
2. Hopf algebras, Poincaré-Birkhoff-Witt theorem (Chapter 3.8 of [SSt])
3. Representation theory of the symmetric group [Sta]
4. Dynamical systems [BaD],[La1],[La2],[F5]
5. Free Lie algebras [Ga]
6. Random matrices [Sta]
7. Algebraic number theory (speculative) [F3],[F5]
8. Potential theory [F7]

A survey paper describing these connections, to be titled “Riffle shuffling: a unifying theme” is in preparation.

In the past few years interesting combinatorial generalizations of riffle shuffling have emerged. Roughly, they can be classified as

1. Biased riffle shuffles [DFP],[F1],[Sta]
2. Other Coxeter groups [BaD], [BeBe], [F2]
3. Hyperplane arrangements [BiHR], [F2]
4. Affine shuffles [Ce1],[Ce2],[F5],[F6]
5. Riffle shuffles with cuts [BaD], [Ce3], [F6]

The point is that since riffle shuffles are related to so many parts of mathematics, these generalizations should be interesting too. In particular, as is clear from [DMP] and the papers of the author just cited (see also [Sta] for connections with quasi-symmetric functions and extensions to infinite support) for these generalizations the induced distribution on conjugacy classes seems to have lovely properties. This note gives further support to that philosophy.

We remark that the Whitehouse module also appears in interesting mathematical contexts ([HS],[RW], [LeSo]). Richard Stanley's MIT website contains transparencies from an illuminating talk about the Whitehouse module.

The structure of this note is as follows: Section 2 gives the main result, and Section 3 suggests two open problems.

2 Main Result

To begin some notation is necessary. Recall that an element w of S_n is said to have a descent at position i (with $1 \leq i \leq n-1$) if $w(i) > w(i+1)$, and a cyclic descent at position n if $w(n) > w(1)$. One lets $d(w)$ be the number of descents of w and defines $cd(w)$ to be $d(w)$ if w has no cyclic descent at n , and to be $d(w) + 1$ if w has a cyclic descent at n . Thus $cd(w)$ can be thought of as the total number of descents of w , viewed cyclically.

Now we use representation theory to obtain a formula for the cycle structure of a riffle shuffle followed by a cut. To begin we recall the following result which gives a formula for the chance of a permutation w after a k -riffle shuffle followed by a cut.

Theorem 1 ([F6]) *The chance of obtaining a permutation w after a k -riffle shuffle followed by a cut is*

$$\frac{1}{nk^{n-1}} \binom{n+k-cd(w^{-1})-1}{n-1}.$$

It is useful to recall the notion of a cycle index associated to a character of the symmetric group. Letting $n_i(w)$ be the number of i -cycles of a permutation w and N be a subgroup of S_n , one defines $Z_N(\chi)$ as

$$Z_N(\chi) = \frac{1}{|N|} \sum_{w \in N} \chi(w) \prod_i a_i^{n_i(w)}.$$

The cycle index stores complete information about the character χ . For a proof of the following attractive property of cycle indices, see [Fe].

Lemma 1 *Let N be a subgroup of S_n and χ a class function on N . Then*

$$Z_{S_n}(Ind_N^{S_n}(\chi)) = Z_N(\chi).$$

Next, recall that an idempotent e of the group algebra of a finite group G defines a character χ for the action of G on the left ideal KG_e of the group algebra of G over a field K of characteristic zero. For a proof of Lemma 2, which will serve as a bridge between representation theory and computing measures over conjugacy classes, see [H]. For its statement, let $e \langle w \rangle$ be the coefficient of w in the idempotent e .

Lemma 2 *Let C be a conjugacy class of the finite group G , and let χ be the character associated to the idempotent e . Then*

$$\frac{1}{|G|} \sum_{w \in C} \chi(w) = \sum_{w \in C} e \langle w \rangle .$$

It is also convenient to define

$$Z_{S_n}(e) = \sum_{w \in S_n} e \langle w \rangle \prod_i a_i^{n_i(w)},$$

which makes sense for any element e of the group algebra. Note that one does not divide by the order of the group. When e is idempotent and χ is the associated character, Lemma 2 can be rephrased as

$$Z_{S_n}(\chi) = Z_{S_n}(e).$$

To proceed recall the Eulerian idempotents e_n^j , $j = 1, \dots, n$ in the group algebra QS_n of the symmetric group over the rationals. These can be defined [GerS] as follows. Let $s_{i,n-i} = \sum w$ where the sum is over all $\binom{n}{i}$ permutations w such that $w(1) < \dots < w(i)$, $w(i+1) < \dots < w(n)$ and let $s_n = \sum_{i=1}^{n-1} s_{i,n-i}$. Letting $\mu_j = 2^j - 2$, the e_n^j are defined as

$$e_n^j = \prod_{i \neq j} \frac{s_n - \mu_i}{(\mu_j - \mu_i)}.$$

They are orthogonal idempotents which sum to the identity.

The following result, which we shall need, is due to Hanlon. The symbol μ denotes the Moebius function of elementary number theory.

Theorem 2 ([H])

$$1 + \sum_{n=1}^{\infty} \sum_{i=1}^n k^i Z_{S_n}(e_n^i) = \prod_{i \geq 1} (1 - a_i)^{-(1/i) \sum_{d|i} \mu(d) k^{i/d}}.$$

Theorem 3 ([Ga])

$$\sum_{i=1}^n k^i e_n^i = \sum_{w \in S_n} \binom{n+k-d(w)-1}{n} w.$$

Remark: Combining Lemma 2 and Theorem 3, one sees that the formula for the cycle structure of a riffle shuffle [DMP] and Theorem 2 imply each other. It is interesting that both proofs used a bijection of Gessel and Reutenauer [GesR].

To continue, we let $\overline{e_n^j}$ denote the idempotent obtained by multiplying the coefficient of w in e_n^j by $\text{sgn}(w)$. Let λ_{n+1} be the $n+1$ cycle $(1\ 2 \dots n+1)$ and $\Lambda_{n+1} = \frac{1}{n+1} \sum_{i=0}^n (\text{sgn} \lambda_{n+1}^i) \lambda_{n+1}^i$. Viewing $\overline{e_n^j}$ as in the group algebra of S_{n+1} , Whitehouse [W] proves that for $j = 1, \dots, n$ the element $\Lambda_{n+1} \overline{e_n^j}$ is an idempotent in the group algebra QS_{n+1} , which we denote by f_{n+1}^j . Whitehouse's main result is the following:

Theorem 4 ([W]) Let $F_{n+1}^j, \overline{E_n^j}$ be the irreducible modules corresponding to the idempotents f_{n+1}^j and $\overline{e_n^j}$. Then

$$F_{n+1}^j \oplus \bigoplus_{i=1}^j \overline{E_{n+1}^i} = \bigoplus_{i=1}^j \text{Ind}_{S_n}^{S_{n+1}} \overline{E_n^i}.$$

As final preparation for the main result of this section, we link the idempotent $\Lambda_{n+1} \overline{e_n^j}$ with riffle shuffles followed by a cut.

Lemma 3 The coefficient of w in $\sum_{j=1}^n k^j \Lambda_{n+1} \overline{e_n^j}$ is $\text{sgn}(w) \frac{1}{n+1} \binom{k+n-cd(w)}{n}$.

PROOF: Given Theorem 3, this is an elementary combinatorial verification. \square

Theorem 5 now derives the cycle structure of a permutation distributed as a shuffle followed by a cut. So as to simplify the generating functions, recall that $\sum_{d|i} \mu(d)$ vanishes unless $i = 1$.

Theorem 5

$$\begin{aligned} & 1 + \sum_{n \geq 1} \sum_{w \in S_{n+1}} \frac{1}{(n+1)k^{n+1}} \binom{n+k-cd(w)}{n} \prod_i a_i^{n_i(w)} \\ &= 1 - \frac{1}{k-1} - \frac{a_1}{k} + \frac{1}{k-1} \prod_{i \geq 1} \left(1 - \frac{a_i}{k^i}\right)^{-1/i \sum_{d|i} \mu(d)(k^{i/d}-1)}. \end{aligned}$$

If $k = q$ is the size of a finite field, this says that the cycle type of a permutation distributed as a shuffle followed by the cut has the same law as the factorization type of a monic degree n polynomial over F_q with non-vanishing constant term.

PROOF: Replacing a_i by $a_i k^i (-1)^{i+1}$, it is enough to show that

$$\begin{aligned} & 1 + \sum_{n \geq 1} \sum_{w \in S_{n+1}} \text{sgn}(w) \frac{1}{(n+1)} \binom{n+k-cd(w)}{n} \prod_i a_i^{n_i(w)} \\ &= 1 - \frac{1}{k-1} - a_1 + \frac{1}{k-1} \prod_{i \geq 1} \left(1 - (-1)^{i+1} a_i\right)^{-1/i \sum_{d|i} \mu(d)(k^{i/d}-1)}. \end{aligned}$$

Using Lemmas 1, 2, 3 and Theorem 4, one sees that

$$\begin{aligned} & 1 + \sum_{n \geq 1} \sum_{w \in S_{n+1}} \text{sgn}(w) \frac{1}{(n+1)} \binom{n+k-cd(w)}{n} \prod_i a_i^{n_i(w)} \\ &= 1 + \sum_{n=1}^{\infty} \sum_{j=1}^n k^j Z_{S_{n+1}}(f_n^j) \\ &= 1 + \sum_{n=1}^{\infty} \sum_{j=1}^n k^j \sum_{i=1}^j Z_{S_{n+1}}(\text{Ind}_{S_n}^{S_{n+1}}(\overline{e_n^i})) - \sum_{n=1}^{\infty} \sum_{j=1}^n k^j \sum_{i=1}^j Z_{S_{n+1}}(\overline{e_{n+1}^i}) \\ &= 1 + a_1 \sum_{n=1}^{\infty} \sum_{i=1}^n Z_{S_n}(\overline{e_n^i}) \left(\frac{k^{n+1} - k^i}{k-1}\right) - \sum_{n=1}^{\infty} \sum_{i=1}^n Z_{S_{n+1}}(\overline{e_{n+1}^i}) \left(\frac{k^{n+1} - k^i}{k-1}\right) \\ &= 1 + a_1 k Z_{S_1}(\overline{e_1}) + \frac{a_1 k - 1}{k-1} \sum_{n=2}^{\infty} k^n \sum_{i=1}^n Z_{S_n}(\overline{e_n^i}) + \frac{1 - a_1}{k-1} \sum_{n=2}^{\infty} \sum_{i=1}^n k^i Z_{S_n}(\overline{e_n^i}). \end{aligned}$$

To simplify things further, recall that $\sum_{i=1}^n Z_{S_n}(\overline{e_n^i})$ is a_1^n since the $\overline{e_n^i}$'s sum to the identity. The above then becomes

$$1 - \frac{1}{k-1} - a_1 + \frac{1-a_1}{k-1} \left(1 + \sum_{n=1}^{\infty} \sum_{i=1}^n k^i Z_{S_n}(\overline{e_n^i})\right),$$

so the sought result follows from Theorem 2. \square

Before continuing, we observe that a combinatorial proof of Theorem 5 (which must exist) would give a new proof of Theorem 4, by reversing the steps.

Upon hearing about Theorem 5, Persi Diaconis immediately asked for the expected number of fixed points after a k -riffle shuffle followed by a cut, suggesting that it should be smaller than for a k riffle shuffle. Using the methods of Section 5 of [DMP], one can readily derive analogs of all of the results there. As an illustrative example, Corollary 1 shows that the expected number of fixed points after a k -riffle shuffle followed by a cut is the same as for a uniform permutation, namely 1 (the answer for k -riffle shuffles is $1 + 1/k + \dots + 1/k^{n-1}$). Two other examples are worth mentioning and will be treated in Corollary 2.

Corollary 1 *The expected number of fixed points after a k -riffle shuffle followed by a cut is 1.*

PROOF: The case $n = 1$ is obvious. Multiplying a_i by u in the statement of Theorem 5 shows that

$$\begin{aligned} & 1 + \sum_{n \geq 1} \sum_{w \in S_{n+1}} u^{n+1} \frac{1}{(n+1)k^{n+1}} \binom{n+k-cd(w)}{n} \prod_i a_i^{n_i(w)} \\ &= 1 - \frac{1}{k-1} - \frac{ua_1}{k} + \frac{1}{k-1} \prod_{i \geq 1} \left(1 - \frac{u^i a_i}{k^i}\right)^{-1/i \sum_{d|i} \mu(d)(k^{i/d}-1)}. \end{aligned}$$

To get the generating function in u (for $n \neq 1$) for the expected number of fixed points in a riffle shuffle followed by a cut, one multiplies the right hand side by k , sets $a_2 = a_3 = \dots = 1$, differentiates with respect to a_1 , and then sets $a_1 = 1$. Doing this yields the generating function

$$-u + u \prod_{i \geq 1} \left(1 - \frac{u^i}{k^i}\right)^{-1/i \sum_{d|i} \mu(d)k^{i/d}}.$$

The result now follows from the identity

$$\prod_{i \geq 1} \left(1 - \frac{u^i}{k^i}\right)^{-1/i \sum_{d|i} \mu(d)k^{i/d}} = \frac{1}{1-u},$$

which is equivalent to the assertion that a monic degree n polynomial over F_q has a unique factorization into irreducibles, since $1/i \sum_{d|i} \mu(d)k^{i/d}$ is the number of irreducible polynomials of degree i over the field F_k . \square

Corollary 2 *Fix u with $0 < u < 1$. Let N be chosen from $\{0, 1, 2, \dots\}$ according to the rule that $N = 0$ with probability $\frac{1-u}{1-u/k}$ and $N = n \geq 1$ with probability $\frac{(k-1)(1-u)u^n}{k-u}$. Given N , let w be the result of a random k shuffle followed by a cut. Let N_i be the number of cycles of w of length i . Then the N_i are independent and N_i has a negative binomial distribution with parameters $1/i \sum_{d|i} \mu(d)(k^{i/d}-1)$ and $(u/k)^i$. Consequently, for fixed k as $n \rightarrow \infty$, the joint distribution of the number of i cycles after a k -shuffle followed by a cut converges to independent negative binomials with parameters $1/i \sum_{d|i} \mu(d)(k^{i/d}-1)$ and $(1/k)^i$.*

PROOF: Theorem 5 and straightforward manipulations give that

$$\begin{aligned} & 1 + \frac{k-1}{k} \sum_{n \geq 1} \sum_{w \in S_n} \frac{u^n}{nk^{n-1}} \binom{n+k-cd(w)-1}{n-1} \prod_i a_i^{n_i(w)} \\ &= \prod_{i \geq 1} \left(1 - \frac{a_i u^i}{k^i}\right)^{-1/i \sum_{d|i} \mu(d)(k^{i/d}-1)}. \end{aligned}$$

Setting all $a_i = 1$ gives the equation

$$1 + \frac{(k-1)u}{k(1-u)} = \prod_{i \geq 1} \left(1 - \frac{u^i}{k^i}\right)^{-1/i \sum_{d|i} \mu(d)(k^{i/d}-1)}.$$

Taking reciprocals and multiplying by the first equation gives

$$\begin{aligned} & \left(\frac{1-u}{1-u/k}\right) + \frac{(k-1)(1-u)}{k-u} \sum_{n \geq 1} \sum_{w \in S_n} \frac{u^n}{nk^{n-1}} \binom{n+k-cd(w)-1}{n-1} \prod_i a_i^{n_i(w)} \\ &= \prod_{i \geq 1} \left(\frac{1 - \frac{u^i}{k^i}}{1 - \frac{a_i u^i}{k^i}}\right)^{1/i \sum_{d|i} \mu(d)(k^{i/d}-1)}, \end{aligned}$$

proving the first assertion of the corollary.

For the second assertion there is a technique simpler than that in [DMP]. Rearranging the last equation gives that

$$\begin{aligned} & \left(\frac{1-u}{1-1/k}\right) + \sum_{n \geq 1} \sum_{w \in S_n} \frac{(1-u)u^n}{nk^{n-1}} \binom{n+k-cd(w)-1}{n-1} \prod_i a_i^{n_i(w)} \\ &= \frac{1-u/k}{1-1/k} \prod_{i \geq 1} \left(\frac{1 - \frac{u^i}{k^i}}{1 - \frac{a_i u^i}{k^i}}\right)^{1/i \sum_{d|i} \mu(d)(k^{i/d}-1)}. \end{aligned}$$

Letting $g(u)$ be a generating function with a convergent Taylor series, the limit coefficient of u^n in $\frac{g(u)}{1-u}$ is simply $g(1)$. This proves the second assertion. \square

3 Open problems

To finish the paper, we mention two open problems. The most interesting is to prove the conjecture from [F5] that the cycle structure of an affine q -shuffle is given by the factorization type of a monic degree n polynomial over F_q with constant term 1. For the identity conjugacy class, it amounts to the $m = 0$ case of the following observation.

Corollary (loc. cit.): For any positive integers x, y , the number of ways (disregarding order and allowing repetition) of writing $m \pmod{y}$ as the sum of x integers of the set $0, 1, \dots, y-1$ is equal to the number of ways (disregarding order and allowing repetition) of writing $m \pmod{x}$ as the sum of y integers of the set $0, 1, \dots, x-1$.

More generally, let $f_{n,k,d}$ be the coefficient of z^n in $(\frac{z^k-1}{z-1})^d$ and let μ be the Moebius function. Let $n_i(w)$ be the number of i -cycles in a permutation w . Then (loc. cit.) the conjecture is equivalent to the truly bizarre assertion (which we intentionally do not simplify) that for all n, k ,

$$\begin{aligned}
& \sum_{m=0 \pmod n} \text{Coef. of } q^m u^n t^k \text{ in } \sum_{n=0}^{\infty} \frac{u^n}{(1-tq) \cdots (1-tq^n)} \sum_{w \in S_n} t^{cd(w)} q^{\text{maj}(w)} \prod x_i^{n_i(w)} \\
= & \sum_{m=0 \pmod{k-1}} \text{Coef. of } q^m u^n t^k \text{ in } \sum_{k=0}^{\infty} t^k \prod_{i=1}^{\infty} \prod_{m=0}^{\infty} \left(\frac{1}{1-q^m x_i u^i} \right)^{1/i} \sum_{d|i} \mu(d) f_{m,k,i/d}.
\end{aligned}$$

This assertion is bizarre because on one side the summation is taken mod n , and on the other side it is taken mod $k - 1$!

A second problem, considered in [F6] is to determine whether or not the following statement is true. Recall that the major index of a permutation is the sum of the positions of its descents.

Statement: For $n \geq 1$, let t be the largest divisor of n such that $\gcd(cd - 1, t) = 1$. Then for every conjugacy class C of S_n , the set of permutations in C with cd cyclic descents has its major index equidistributed mod t .

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