The module of logarithmic p-forms of a locally free arrangement

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Abstract

For an essential, central hyperplane arrangement $\mathcal{A} \subseteq V \simeq k^{n+1}$ we show that $\Omega^1(\mathcal{A})$ (the module of logarithmic one forms with poles along $\mathcal{A}$) gives rise to a locally free sheaf on $\mathbf{P}^n$ if and only if for all $X \in L_{\mathcal{A}}$ with rank $X <$ dim V, the module $\Omega^1(\mathcal{A}_X)$ is free. Our main result says that in this case $\pi(\mathcal{A}, t)$ is essentially the Chern polynomial. The proof is based on a result of Solomon-Terao [16] and a formula we give for the Chern polynomial of a bundle $\mathcal{E}$ on $\mathbf{P}^n$ in terms of the Hilbert series of $\oplus_{m \in \mathbf{Z}} H^0(\mathbf{P}^n, \wedge^i \mathcal{E}(m))$. If $\Omega^1(\mathcal{A})$ has projective dimension one and is locally free, we give a minimal free resolution for Ω^p, and show that $\Lambda^p \Omega^1(\mathcal{A}) \simeq \Omega^p(\mathcal{A})$, generalizing results of Rose-Terao on generic arrangements.

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1 Introduction

If X is a complex manifold and D a divisor with normal crossings ($D = \sum D_i$, D_i smooth and meet transversely), then associated to D is the sheaf $\Omega^1(\log D)$ of meromorphic one forms with logarithmic poles on D. Deligne introduced this sheaf in [2] and shows (among other things) that it is locally free. Dolgachev and Kapranov seem to have been the first to examine in depth the case where D is a set of hyperplanes in general position in $\mathbf{P}^n$; one striking result they obtain is that if $D = \cup_{i=1}^d H_i$ and $d \geq 2n+3$, then the hyperplanes can be recovered from $\Omega^1(\log D)$ unless the H_i osculate a rational normal curve of degree n.

We also consider the case when the divisor is a set of hyperplanes in $\mathbf{P}^n$, but assume only that the hyperplanes are distinct. There are two main themes of this paper. In [16], Solomon and Terao give a formula for the Poincaré polynomial of an (essential, central) arrangement in terms of the Hilbert series of certain graded modules D^i associated to the arrangement. The formula generalizes Terao's famous freeness theorem [17]: If D^1 is a free module, then the Poincaré polynomial factors. This suggests a connection to Chern polynomials; motivated by the Solomon-Terao result we prove a formula relating the Chern polynomial of a bundle $\mathcal{E}$ on $\mathbf{P}^n$ to the Hilbert series of the modules $\oplus_{m \in \mathbf{Z}} H^0(\mathbf{P}^n, \wedge^i \mathcal{E}(m))$.

Since an arbitrary arrangement does not have normal crossings, $\Omega^1(\log D)$ is in general no longer locally free. Silvotti [15] studies this situation, and remedies the problem by blowing up the arrangement at the non-normal crossings; $\sigma^* D$ automatically has normal crossings on the blowup X, so yields a locally free sheaf $\Omega^1_X(\log \sigma^* D) \simeq \oplus \mathcal{F}_j$. Using a vanishing result of Esnault, Schechtman and Viehweg [6], Silvotti obtains a formula for the coefficients of the Poincaré polynomial in terms of $\chi(\Lambda^i \mathcal{F}_j)$. However, the computations can be quite complicated; in particular, Silvotti does not recover Terao's theorem.

The second point of this paper is that even when the hyperplanes are not in general position, there are situations where $\Omega^1(\log D)$ is a vector bundle on $\mathbf{P}^n$. We relate $\Omega^1(\log D)$ to the modules D^i mentioned above, and prove a criterion for the associated sheaves to be locally free. This class of arrangements was studied by Yuzvinsky in [20]; Yuzvinsky proves that for such arrangements the Hilbert polynomial of D^1 is a combinatorial invariant. We show that the Chern polynomial of the dual of D^1 is in fact the Poincaré polynomial of the arrangement (truncated by t^{n+1}). Hence, the Hilbert polynomial of D^1 may be obtained from the Poincaré polynomial via Hirzebruch-Riemann-Roch. We close with some results specific to the situation where Ω^1 or D^1 has projective dimension one. First, we review some facts about arrangements.

2 Hyperplane Arrangements

A hyperplane arrangement $\mathcal{A}$ is a finite collection of codimension one linear subspaces of a fixed vector space V. $\mathcal{A}$ is *central* if each hyperplane contains the origin $\mathbf{0}$ of V. A fundamental invariant of $\mathcal{A}$ is the Poincaré polynomial $\pi(\mathcal{A}, t)$. There are various ways of defining $\pi(\mathcal{A}, t)$; the simplest is from the intersection lattice $L_{\mathcal{A}}$ of $\mathcal{A}$. $L_{\mathcal{A}}$ consists of the intersections of the elements of $\mathcal{A}$, ordered by reverse inclusion. The rank function on $L_{\mathcal{A}}$ is given by the codimension in V. V is the lattice element $\hat{0}$; the rank one elements are the hyperplanes themselves. $\mathcal{A}$ is called *essential* if rank $L_{\mathcal{A}} = \dim V$. Henceforth, unless explicitly stated otherwise, all arrangements will be *essential* and *central*, and V will be k^{n+1}, with k an arbitrary field. We briefly review some fundamental definitions; for more information see Orlik and Terao ([11]).

Definition 2.1 *The Möbius function $\mu : L_{\mathcal{A}} \longrightarrow \mathbf{Z}$ is defined by*

$$\begin{aligned} \mu(\hat{0}) &= 1 \\ \mu(t) &= -\sum_{s<t} \mu(s), \text{ if } \hat{0} < t \end{aligned}$$

Definition 2.2 *The Poincaré polynomial $\pi(\mathcal{A},t)$ and characteristic polynomial $\chi(\mathcal{A},t)$ are defined by:*

$$\pi(\mathcal{A},t)=\sum_{X\in L_{\mathcal{A}}}\mu(X)\cdot(-t)^{rank(X)},\ \chi(\mathcal{A},t)=\sum_{X\in L_{\mathcal{A}}}\mu(X)\cdot t^{dim(X)}$$

The two polynomials are related via $\chi(\mathcal{A},t)=t^{n+1}\cdot\pi(\mathcal{A},-t^{-1})$. Let $S=\mathrm{Sym}(V^*)$, $\underline{m}$ the irrelevant maximal ideal, K the fraction field of S and suppose $\mathcal{A}$ consists of d distinct hyperplanes in V. For each hyperplane H_i of $\mathcal{A}$, fix l_i a nonzero linear form vanishing on H_i and put $Q=\prod_1^d l_i$. Denote the module of p differentials over k of S and K by Ω^p_S and Ω^p_K, respectively, and let $\mathrm{Der}_k S$ denote the module of k derivations of S.

Definition 2.3 *$D^p(\mathcal{A})$ is the submodule of $\Lambda^p(\mathrm{Der}_k S)$ defined by*

$$D^p(\mathcal{A})=\{\theta\in\Lambda^p(\mathrm{Der}_k S)\mid\theta(Q,f_2,\ldots,f_p)\in(Q),\ \forall f_i\in S\}.$$

$\Omega^p(\mathcal{A})$ is the submodule of Ω^p_K defined by

$$\Omega^p(\mathcal{A})=\{\omega\in\Omega^p_K\mid Q\,\omega\in\Omega^p_S\ \text{and}\ Q\,d\omega\in\Omega^{p+1}_S\}.$$

$D^1(\mathcal{A})$ is usually called the module of $\mathcal{A}$ derivations, while $\Omega^p(\mathcal{A})$ is called the module of logarithmic p forms with poles along $\mathcal{A}$. When the arrangement is clear from the context, we will drop it from the notation. Note that we have $D^0(\mathcal{A})=\Omega^0(\mathcal{A})=S$ and we make the convention $D^p(\mathcal{A})=\Omega^p(\mathcal{A})=0$, for $p<0$.

If char $k\not| d$, then $D^1(\mathcal{A})$ has a direct sum decomposition as $D^1_0\oplus S(-1)$, where D^1_0 is the kernel of the Jacobian matrix of Q and $S(-1)$ corresponds to the Euler derivation. Correspondingly, we have a decomposition $\Omega^1\simeq\Omega^1_0\oplus S(1)$.

Since all these modules are graded, we may consider the corresponding sheaves on $\mathbf{P}^n$, written as usual as $\widetilde{\Omega^1}$ for the sheaf associated to Ω^1. An arrangement is called generic if for every $H_1,\ldots,H_m\in\mathcal{A}$, with $m\le n+1$, $\mathrm{rank}(H_1\cap\ldots\cap H_m)=m$. For a generic arrangement $\mathcal{A}$, the sheaves $\Omega^1(\log D)$ and $\widetilde{\Omega^1}$ are related as follows: after a change of coordinates, we may assume the first $n+1$ hyperplanes are the coordinate hyperplanes; for $i\in\{1,\ldots,d-n-1\}$ write $l_i=\sum_{j=0}^n a_{i,j}x_j$ for the remaining hyperplanes. In [21], Ziegler gives a free resolution for Ω^1 for a generic arrangement:

$$0\longrightarrow S^{d-n-1}\xrightarrow{\tau}S(1)^d\longrightarrow\Omega^1\longrightarrow 0,$$

where τ is given by

$$\begin{pmatrix} a_{1,0}x_0 & \cdots & \cdots & a_{d-n-1,0}x_0\\ \vdots & \ddots & \ddots & \vdots\\ a_{1,n}x_n & \cdots & \cdots & a_{d-n-1,n}x_n\\ -l_1 & 0 & \cdots & 0\\ 0 & -l_2 & \cdots & 0\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & -l_{d-n-1}\end{pmatrix}$$

In corollary 3.4 of [3], Dolgachev and Kapranov present $\Omega^1(\log D)$ as the cokernel of a map

$$V\otimes\mathcal{O}_{\mathbf{P}^n}(-1)\xrightarrow{\tau'}W\otimes\mathcal{O}_{\mathbf{P}^n},$$

where V is the subspace of k^d consisting of relations on the linear forms defining $\mathcal{A}$, W is the subspace of k^d orthogonal to $(1,1,\cdots,1)$, and $\tau':(a_1,\ldots,a_d)\longrightarrow(a_1l_1,\ldots,a_dl_d)$. Thus, the images of τ and τ' are isomorphic, and we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S^{d-n-1} & \xrightarrow{\tau'} & S^{d-1}(1) & \longrightarrow & \Omega^1(\log D)(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S^{d-n-1} & \xrightarrow{\tau} & S^{d}(1) & \longrightarrow & \Omega^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & S(1) & \longrightarrow & S(1) & \longrightarrow & 0
\end{array}$$

By the snake lemma, we have $\Omega^1(\log D)(1) \simeq \widetilde{\Omega}^1_0$. This also follows from the local description, but the above makes explicit the different gradings.

For every X in the intersection lattice $L_{\mathcal{A}}$, the subarrangement $\mathcal{A}_X$ of $\mathcal{A}$ is defined by

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subset H\}.$$

In general this is not an essential arrangement, but we can write it as $\mathcal{A}_X \simeq \mathcal{A}'_X \times \Phi$, where $\mathcal{A}'_X$ is essential and Φ is an empty arrangement.

The functors on the intersection lattice $X \longrightarrow D^p(\mathcal{A}_X)$ and $X \longrightarrow \Omega^p(\mathcal{A}_X)$ are local (see Orlik and Terao [11], Chapter 4.6). What we will use is the fact that if $\underline{q}$ is a prime ideal in S and $X = \cap_{\alpha_H \in \underline{q}} H$, then we have a canonical isomorphism:

$$D^p(\mathcal{A})_{\underline{q}} \simeq D^p(\mathcal{A}_X)_{\underline{q}}$$

and a similar one for Ω^p.

By definition, an arrangement $\mathcal{A}$ is free if $D^1(\mathcal{A})$ is a free S-module. Following Yuzvinsky [20], we will say that an (essential, central) arrangement $\mathcal{A}$ is locally free if for every $X \in L_{\mathcal{A}}$ with $\operatorname{rank} X < \dim V$, the arrangement $\mathcal{A}_X$ is free.

For a graded module M, let $P(M, X)$ be its Hilbert series. There is a beautiful relation between the modules $D^p(\mathcal{A})$ and the characteristic polynomial:

Theorem 2.4 *(Solomon and Terao,[16])*

$$\chi(\mathcal{A}, t) = (-1)^{n+1} lim_{X \to 1} \sum_{p \geq 0} P(D^p(\mathcal{A}); X)(t(X-1)-1)^p.$$

There is a dual version of this theorem, which replaces $D^p(\mathcal{A})$ with $\Omega^p(\mathcal{A})$. In certain situations, not all the modules $D^p(\mathcal{A})$ are needed to compute $\pi(\mathcal{A}, t)$; the paradigm for this is the case of free arrangements.

Theorem 2.5 *(Terao,[17]) If $D^1(\mathcal{A})$ is free, then $\pi(\mathcal{A}, t) = \prod(1 + a_i t)$, where the a_i are the degrees of the generators of $D^1(\mathcal{A})$.*

For arrangements on $\mathbf{P}^2$, $\widetilde{\Omega}^1$ is always locally free, and suffices to determine the Poincaré polynomial ([14]). For every coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n$, we denote by $H^0_*(\mathcal{F})$ the S-module $\oplus_{m \in \mathbf{Z}} H^0(\mathbf{P}^n, \mathcal{F}(m))$. Motivated by Theorem 2.4 we prove

Theorem *For every rank r bundle $\mathcal{E}$ on $\mathbf{P}^n$,*

$$c_t(\mathcal{E}) = lim_{X \to 1} (-1)^r t^r (1-X)^{n+1-r} \sum_{i=0}^{r} P(H^0_*(\wedge^i \mathcal{E}); X) \cdot (\frac{X-1}{t} - 1)^i.$$

As a consequence, we get the following generalization of Theorem 2.5:

Theorem *If $\mathcal{A}$ is an arrangement such that $\widetilde{\Omega^1}$ is locally free and $\overline{\pi}(\mathcal{A},t)$ is the class of $\pi(\mathcal{A},t)$ in $\mathbf{Z}[t]/(t^{n+1})$, then*

$$\overline{\pi}(\mathcal{A},t) = c_t(\widetilde{\Omega^1}).$$

We characterize those arrangements for which $\widetilde{\Omega}^p$ is a bundle:

Theorem *$\widetilde{\Omega}^p$ is a bundle iff for every $X \in L_{\mathcal{A}}$ with $\operatorname{rank} X < \dim V$, $\Omega^p(\mathcal{A}_X)$ is free.*

We will use freely results from commutative algebra for which our main reference is Eisenbud [5], as well as results about Chern classes of vector bundles on projective space, for which we refer to Fulton [7].

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3 Locally Free Arrangements

We start with a general lemma about the depth of the modules Ω^p and D^p.

Lemma 3.1 *For every central arrangement $\mathcal{A}$ in V, with $\dim V = n+1 \geq 2$, we have $\operatorname{depth} \Omega^p \geq 2$ and $\operatorname{depth} D^p \geq 2$, for every p, $1 \leq p \leq n+1$.*

Proof. We consider the case of the modules Ω^p. Recall that if K is the quotient field of S, then

$$\Omega^p = \{\omega \in \wedge^p \Omega_S \otimes_S K \,|\, Q\omega \in \wedge^p \Omega_S, Q\, d\omega \in \wedge^{p+1}\Omega_S\},$$

where Q is the product of the linear forms defining the elements of $\mathcal{A}$. In particular, Ω^p is torsion-free and therefore $\operatorname{depth} \Omega^p \geq 1$.

We have to prove that if $0 \neq a \in S$ and $\omega \in \Omega^p$ are such that $\underline{m}\,\omega \subset a\,\Omega^p$, then $\omega \in a\,\Omega^p$. Note that since $\operatorname{depth} S \geq 2$ and the S-modules Ω^i_S are free, if for $\tau \in \Omega^i_K$ and $0 \neq b \in S$ we have $\underline{m}\,\tau \subset b\,\Omega^i_S$, then $\tau \in \Omega^i_S$.

By definition, we have $\underline{m}\, Q\omega \subset a\,\Omega^p_S$ and the above observation gives $Q\omega/a \in \Omega^p_S$. For every $f \in \underline{m}$ we have also $Q\, d(f\omega/a) \in \Omega^{p+1}_S$. We use

$$Q\, d(f\omega/a) = df \wedge Q\,\omega/a + Qf d(\omega/a).$$

Since we have already seen that $Q\,\omega/a \in \Omega^p_S$ we obtain $\underline{m}\, Qd(\omega/a) \subset \Omega^{p+1}_S$. One more application of the above observation gives $Q\, d(\omega/a) \in \Omega^{p+1}_S$ and therefore, $\omega/a \in \Omega^p$, which completes the proof.

The proof of the fact that $\operatorname{depth} D^p \geq 2$ is similar, using the definition of this module. □

The following Proposition is the generalization of Theorem 4.75 in Orlik and Terao [11] which is the case $p = 1$. Though the general result seems known to experts, we include the proof, as we could not find a reference in the literature.

Proposition 3.2 *For every central arrangement $\mathcal{A}$, each of the modules D^p and Ω^p is dual to the other.*

Proof. A standard generalization of the argument in Orlik and Terao [11], Proposition 4.74, gives a bilinear map of S-modules $\Omega^p \times D^p \longrightarrow S$, which induces morphisms $\alpha : \Omega^p \longrightarrow \operatorname{Hom}_S(D^p, S)$ and $\beta : D^p \longrightarrow$

$\mathrm{Hom}_S(\Omega^p, S)$. The proofs of the fact that α and β are isomorphisms are similar, so that we will give the proof only for α. We make induction on $n \geq 1$, the case $n = 1$ being straightforward.

Lemma 3.1 gives $\operatorname{depth} \Omega^p \geq 2$, while it is an easy exercise to see that since $n \geq 1$, for every graded S-module M, $\operatorname{depth} \mathrm{Hom}_S(M, S) \geq 2$. Since for a module N of depth at least two, $N \simeq \sum_t H^0(\tilde{N}(t))$, in order to prove that α is an isomorphism, it is enough to prove that it is an isomorphism at the sheaf level i.e. $\alpha \otimes S_{\underline{q}}$ is an isomorphism for every prime ideal $\underline{q} \neq \underline{m}$. Note that if the Proposition is true for an arrangement $\mathcal{A}$ and every p, then it is true also for $\mathcal{A} \times \Phi$, where Φ is the empty arrangement in k. Indeed, if $S' = \mathrm{Sym}(V^* \times k)$ is the polynomial ring corresponding to $\mathcal{A} \times \Phi$, then by Orlik and Terao [11], Proposition 4.84 and Solomon and Terao [16], Proposition 5.8, we have canonical isomorphisms

$$\Omega^p(\mathcal{A} \times \Phi) \simeq (\Omega^p(\mathcal{A}) \otimes_S S') \oplus (\Omega^{p-1}(\mathcal{A})(-1) \otimes_S S')$$

$$D^p(\mathcal{A} \times \Phi) \simeq (D^p(\mathcal{A}) \otimes_S S') \oplus (D^{p-1}(\mathcal{A})(1) \otimes_S S').$$

Therefore, it follows by induction that we may assume $\mathcal{A}$ to be essential.

For a prime ideal $\underline{q} \neq \underline{m}$, if we take $X = \cap_{\alpha_H \in \underline{q}} H$, then $X \in L_{\mathcal{A}}$, and $\operatorname{rank} X < \dim V$, since $\mathcal{A}$ is essential. But since $\Omega^p(-)$ and $D^p(-)$ are local functors, we have canonical isomorphisms $D^p_{\underline{q}} \simeq D^p(\mathcal{A}_X)_{\underline{q}}$ and $\Omega^p_{\underline{q}} \simeq \Omega^p(\mathcal{A}_X)_{\underline{q}}$. Since $\mathcal{A}_X$ is not essential, we have seen that it satisfies the conclusion of the Proposition, and therefore we get $\alpha \otimes S_{\underline{q}}$ isomorphism. □

Theorem 3.3 *For an (essential, central) arrangement $\mathcal{A}$ and every positive integer p, the following are equivalent:*

1. *$\widetilde{D^p}$ is locally free on $\mathbf{P}^n$.*

1'. *$\widetilde{\Omega^p}$ is locally free on $\mathbf{P}^n$.*

2. *For every $X \in L_{\mathcal{A}}$ with $\operatorname{rank} X < \dim V$, $D^p(\mathcal{A}_X)$ is free.*

2'. *For every $X \in L_{\mathcal{A}}$ with $\operatorname{rank} X < \dim V$, $\Omega^p(\mathcal{A}_X)$ is free.*

Proof. The equivalences $1 \Leftrightarrow 1'$ and $2 \Leftrightarrow 2'$ follow from Proposition 3.2.

For the proof of $1 \Rightarrow 2$, let $X \in L_{\mathcal{A}}$ be a nonzero linear subspace and $I_X \subset S$ its ideal. By making a linear change of variables we can assume that $I_X = (X_0, \ldots, X_{r-1})$, where $r = \operatorname{rank} X$. Since $\mathcal{A}_X \simeq \mathcal{A'}_X \times \Phi^{n+1-r}$, if $S_1 = k[X_0, \ldots, X_{r-1}]$, then

$$D(\mathcal{A}_X) \simeq (D(\mathcal{A'}_X) \otimes_{S_1} S) \oplus S^{n+1-r}$$

and more generally

$$D^p(\mathcal{A}_X) \simeq \bigoplus_{0 \leq i \leq p} (D^i(\mathcal{A'}_X) \otimes S^{\binom{n+1-r}{p-i}}).$$

Since $D^i(\mathcal{A'}_X)$ is a free S_1 module if and only if $D^i(\mathcal{A'}_X)_{I_X}$ is a free S_1-module, it follows that $D^p(\mathcal{A}_X)$ is free if and only if $D^p(\mathcal{A}_X)_{I_X}$ is free. But because $\widetilde{D^p}$ is locally free and $I_X \neq \underline{m}$, $D^p_{I_X}$ is free over S_{I_X}. On the other hand, since $D^p(-)$ is a local functor we have

$$D^p_{I_X} \simeq D^p(\mathcal{A}_X)_{I_X},$$

and therefore $D^p(\mathcal{A}_X)$ is free.

In order to prove $2 \Rightarrow 1$, let us consider a prime ideal $\underline{q}$ different from $\underline{m}$. If we take $X = \cap_{\alpha_H \in \underline{q}} H$, then $X \in L_{\mathcal{A}}$, $\operatorname{rank} X < \dim V$ and because $D^p(-)$ is a local functor we have $D^p_{\underline{q}} \simeq D^p(\mathcal{A}_X)_{\underline{q}}$, which is free over $S_{\underline{q}}$ by hypothesis. This concludes the proof of the theorem. □

By taking $p=1$ in the above theorem we obtain the following:

Corollary 3.4 *An arrangement $\mathcal{A}$ is locally free if and only if $\widetilde{D^1}$ is locally free on $\mathbf{P}^n$.*

Remark 3.5 A famous conjecture due to Terao asserts that the freeness of an arrangement depends only on the intersection lattice. This is equivalent to the fact that the local freeness of an arrangement depends only on the intersection lattice. Indeed, the fact that the second statement is a consequence of Terao's conjecture follows immediately by induction on rank. Conversely, if two arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ have isomorphic lattices and $\mathcal{A}_1$ is free, consider the product arrangements $\mathcal{A}'_1 = \mathcal{A}_1 \times B_1$ and $\mathcal{A}'_2 = \mathcal{A}_2 \times B_1$, where B_1 is the Boolean arrangement in $W = k$. Then we have $\mathcal{A}'_1$ free and in particular locally free, while $\mathcal{A}'_2$ is free if and only if it is locally free if and only if $\mathcal{A}_2$ is free.

Notice also that since free arrangements are always locally free, Terao's conjecture becomes a question on the splitting of vector bundles on $\mathbf{P}^n$.

Lemma 3.6 *For every arrangement $\mathcal{A}$, every $p \geq 1$ and every $X \in L_{\mathcal{A}}$, $D^p(\mathcal{A}_X)$ is free if and only if $D^i(\mathcal{A}'_X)$ is free, for all i with $p - \dim X \leq i \leq p$.*

Proof. We have seen in the proof of the above Theorem 3.3 that we can write

$$D^p(\mathcal{A}_X) = \bigoplus_{0 \leq i \leq p} D^i(\mathcal{A}'_X) \otimes_{S_1} S^{\binom{n+1-r}{p-i}},$$

where $S_1 = k[X_0, \ldots, X_{r-1}]$ and $r = n+1-\dim X$. To conclude it is enough to notice that $S^{\binom{n+1-r}{p-i}} \neq 0$ if and only if $p - \dim X \leq i \leq p$. $\square$

Corollary 3.7 *For every arrangement $\mathcal{A}$ and every $X \in L_{\mathcal{A}}$ such that $\dim X \geq p-1$, $D^p(\mathcal{A}_X)$ is free if and only if $\mathcal{A}_X$ is free. In particular, if $\widetilde{D^p}$ is locally free, then for every $X \in L_{\mathcal{A}}$ with $\dim X \geq \max\{1, p-1\}$, $\mathcal{A}_X$ is free.*

Proof. The first assertion follows from the above lemma, since we have $p - \dim X \leq 1$. The second assertion follows from the first one and Theorem 3.3. $\square$

Corollary 3.8 *For every arrangement $\mathcal{A}$, $\widetilde{D^1}$ is locally free if and only if $\widetilde{D^2}$ is locally free.*

Proof. The "if" part follows from the previous corollary in the case $p=2$, while the "only if" part is a consequence of the more general proposition below. $\square$

Proposition 3.9 *If $\widetilde{D^1}$ is locally free, then the natural map $\wedge^p(D^1) \longrightarrow D^p$ induces an isomorphism $\wedge^p(\widetilde{D^1}) \simeq \widetilde{D^p}$ for every $p \geq 1$. In particular, $\widetilde{D^p}$ is locally free. The similar assertion about the natural morphism $\wedge^p(\Omega^1) \longrightarrow \Omega^p$ is also true.*

Proof. We have to prove that for every $\underline{q} \in \operatorname{Spec}(S)$, $\underline{q} \neq \underline{m}$, the localized morphism:

$$\wedge^p(D^1_{\underline{q}}) \longrightarrow D^p_{\underline{q}}$$

is an isomorphism. Since $D^1(-)$ and $D^p(-)$ are local functors, if $X = \cap_{\alpha_H \in \underline{q}} H$, we have $D^1_{\underline{q}} = D^1(\mathcal{A}_X)_{\underline{q}}$ and $D^p_{\underline{q}} = D^p(\mathcal{A}_X)_{\underline{q}}$.

Because $\underline{q} \neq \underline{m}$, $X \neq (0)$ and by hypothesis $\mathcal{A}_X$ is free. But for free arrangements $D^p \simeq \wedge^p(D^1)$ (see Solomon-Terao [16], Prop.3.4), which concludes the proof of the first assertion.

The proof of the last statement is similar. □

When $\operatorname{char} k$ does not divide $|\mathcal{A}|$, it is possible to characterize the local freeness of $\mathcal{A}$ using the Ext modules of the Jacobian ideal.

Proposition 3.10 *For an arrangement $\mathcal{A}$ such that $\operatorname{char} k \not| |\mathcal{A}|$, $\mathcal{A}$ is locally free if and only if the modules $\operatorname{Ext}^i_S(S/J, S)$ are supported only at the maximal ideal $\underline{m}$, for all $i \geq 3$.*

Proof. Recall that we have the module D_0 defined by the exact sequence:

$$0 \longrightarrow D_0 \longrightarrow S^{n+1} \longrightarrow J(d-1) \longrightarrow 0,$$

where $d = |\mathcal{A}|$ and J is the Jacobian ideal. Since $\operatorname{char} k \not| d$, we have $D \simeq D_0 \oplus S(-1)$. Using Theorem 3.3 it follows that $\mathcal{A}$ is locally free if and only if $\widetilde{D_0}$ is locally free.

For a finitely generated S-module M it is known that the set

$$S(M) = \{\underline{q} \in \operatorname{Spec}(S) \,|\, M_{\underline{q}} \,\text{is not a free}\, S_{\underline{q}} - \text{module}\}$$

can be written as

$$S(M) = \bigcup_{i \geq 1} \operatorname{Supp} \operatorname{Ext}^i_S(M, S)$$

(see, Hartshorne [8], p. 238, exercise 6.6).

Therefore $\widetilde{D_0}$ is locally free if and only if $\operatorname{Supp} \operatorname{Ext}^i_S(D_0, S) \subset \{\underline{m}\}$ for every $i \geq 1$. Since $\operatorname{Ext}^i_S(S/J, S) \simeq \operatorname{Ext}^{i-2}_S(D_0, S)$ for every $i \geq 3$, the proof of the proposition is complete. □

4 Chern classes of vector bundles on $\mathbf{P}^n$

We consider a vector bundle $\mathcal{E}$ of rank r on $\mathbf{P}^n$. Motivated by the application in the context of arrangements which will be given in the next section, we introduce $R(\mathcal{E}; t, X) \in \mathbf{Z}[t]((X))/(t^{n+1})$, defined by

$$R(\mathcal{E}; t, X) = (-1)^r t^r (1-X)^{n+1-r} \sum_{i=0}^{r} P(H^0_*(\wedge^i \mathcal{E}); X) \cdot (\frac{X-1}{t} - 1)^i.$$

Here $H^0_*(\wedge^i \mathcal{E})$ is the finitely generated graded module $\oplus_{m \in \mathbf{Z}} H^0(\mathbf{P}^n, \wedge^i \mathcal{E}(m))$. Recall that for a finitely generated graded S-module M, $P(M; X)$ denotes the Hilbert series of M which is a Laurent series, but also a rational function in X. The main result of this section is that $R(\mathcal{E}; t, X)$ can be used to compute the Chern polynomial of $\mathcal{E}$. More precisely, we have the following

Theorem 4.1 *If $\mathcal{E}$ is a vector bundle on $\mathbf{P}^n$, then*

$$\lim_{X\to 1} R(\mathcal{E};t,X) = c_t(\mathcal{E}).$$

Remark 4.2 We will see in the proof that in order to compute the above limit, in the definition of $R(\mathcal{E};t,x)$ we may replace each $H^0_*(\wedge^i\mathcal{E})$ with a different finitely generated module M_i such that $\wedge^i\mathcal{E} \simeq \widetilde{M_i}$.

Before proving the theorem we give two lemmas.

Lemma 4.3 *The assertion of Theorem 4.1 is true in the case of a split vector bundle $\mathcal{E}$.*

Proof. Suppose that $\mathcal{E} \simeq \mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_r)$. In this case we have

$$R(\mathcal{E};t,X) = (-1)^r t^r (1-X)^{n+1-r} \sum_{i=0}^{r} P(H^0_*(\oplus_{1\leq k_1<\ldots<k_i\leq r}\mathcal{O}(a_{k_1}+\ldots+a_{k_r});X)\cdot(\frac{X-1}{t}-1)^i.$$

Since for every $a \in \mathbf{Z}$, $H^0_*(\mathcal{O}(a)) = S(a)$ and $P(S(a);X) = X^{-a}(1-X)^{-n-1}$, we get

$$R(\mathcal{E};t,X) = (-1)^r t^r (1-X)^{n+1-r} \sum_{i=0}^{r} \sum_{1\leq k_1<\ldots<k_i\leq r} X^{-a_{k_1}-\ldots-a_{k_i}} \cdot (1-X)^{-n-1} \cdot (\frac{X-1}{t}-1)^i$$

$$= (-1)^r t^r (1-X)^{-r} \prod_{i=1}^{r} (1 + X^{-a_i}(\frac{X-1}{t}-1)) = (-1)^r \prod_{i=1}^{r} (t\cdot\frac{1-X^{-a_i}}{1-X} - X^{-a_i}).$$

It follows from this that the limit exists and

$$\lim_{X\to 1} R(\mathcal{E};t,X) = (-1)^r \prod_{i=1}^{r} (-a_i t - 1) = \prod_{i=1}^{r} (1+a_i t) = c_t(\mathcal{E}).$$

□

Lemma 4.4 *If $r \geq n$ is fixed and $P \in \mathbf{Q}[X_1,\ldots,X_n]$ is a polynomial such that*

$$P(c_1(\mathcal{E}),\ldots,c_n(\mathcal{E})) = 0,$$

for every split vector bundle of rank r, $\mathcal{E} = \mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_r)$, then $P = 0$.

Proof. If s_i, $1 \leq i \leq n$ is the i^{th} symmetric polynomial in $a_1,\ldots,a_r$, then $c_i(\mathcal{E}) = s_i(a_1,\ldots,a_r)$, for every split vector bundle $\mathcal{E}$ as in the hypothesis. Therefore we get $P(s_1,\ldots,s_n) = 0$.

But the morphism

$$\phi\,:\,\mathbf{Q}[X_1,\ldots,X_n] \longrightarrow \mathbf{Q}[Y_1,\ldots,Y_r]$$

given by $\phi(X_i) = s_i(Y_1,\ldots,Y_r)$ is the restriction to $\mathbf{Q}[X_1,\ldots,X_n]$ of the monomorphism

$$\psi\,:\,\mathbf{Q}[X_1,\ldots,X_r] \longrightarrow \mathbf{Q}[Y_1,\ldots,Y_r],$$

where $\psi(X_i) = s_i(Y)$, for $1 \leq i \leq r$. Therefore we have $P = 0$. □

We are now ready to give the proof of the theorem.

Proof. It is easy to check that if $\mathcal{E}'=\mathcal{E}\oplus\mathcal{O}$, then $R(\mathcal{E}';t,X)=R(\mathcal{E};t,X)$. As we have also $c_t(\mathcal{E}')=c_t(\mathcal{E})$, it follows that by taking the direct sum with a large enough number of trivial bundles, we may suppose that $\operatorname{rank}\mathcal{E}=r\geq n$.

In this case, using the above two lemmas, we see that in order to prove the theorem it is enough to show that the existence of the limit and its value can be expressed in terms of some polynomial identities with rational coefficients in the Chern classes of $\mathcal{E}$. We have

$$R(\mathcal{E};t,X)=(-1)^r t^r(1-X)^{n+1-r}\sum_{i=0}^{r}P(H^0_*(\wedge^i\mathcal{E});X)\cdot\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}(\frac{X-1}{t})^j$$

$$=(-1)^r\sum_{j=0}^{r}t^{r-j}(1-X)^{n+1-r+j}\cdot\sum_{i=j}^{r}(-1)^i\binom{i}{j}P(H^0_*(\wedge^i\mathcal{E});X).$$

Since for $0\leq i\leq r$, $H^0_*(\wedge^i\mathcal{E})$ is a S-module of dimension $n+1$, we can write

$$P(H^0_*(\wedge^i(\mathcal{E}));X)=\frac{Q_i(X)}{(1-X)^{n+1}},$$

where $Q\in\mathbf{Z}[X,X^{-1}]$. Moreover, if we consider the Taylor expansion of $Q_i(X)$ around $X=1$:

$$Q_i(X)=\sum_{l\geq 0}e^{(i)}_l(X-1)^l,$$

then the first $n+1$ coefficients of this expansion can be recovered from the Hilbert polynomial of $H^0_*(\wedge^i\mathcal{E})$, which can be written as:

$$T_i(X)=\sum_{l=0}^{n}(-1)^{n-l}e^{(i)}_{n-l}\binom{X+l}{l}.$$

For these results, see for example Bruns and Herzog [1], Chapter 4.1.

Therefore we have $e^{(i)}_k=(-1)^k\Delta^{n-k}T_i(0)$. For every graded S-module M, its Hilbert polynomial is given by the formula $T(m)=\chi(\widetilde{M}(m))$, for every $m\in\mathbf{Z}$. Using the short exact sequences corresponding to successive hyperplane sections we get

$$e^{(i)}_k=(-1)^k\chi(\wedge^i\mathcal{E}|_{H_k}),$$

where $H_k\subset\mathbf{P}^n$ is a linear subspace of dimension k, for $0\leq k\leq n$. We deduce

$$R(\mathcal{E};t,X)=(-1)^r\sum_{j=r-n}^{r}t^{r-j}(1-X)^{n+1-r+j}\cdot\sum_{i=j}^{r}(-1)^i\binom{i}{j}\cdot\frac{\sum_{k\geq 0}(-1)^k e^{(i)}_k(1-X)^k}{(1-X)^{n+1}}.$$

$$R(\mathcal{E};t,X)=(-1)^r\sum_{j=r-n}^{r}t^{r-j}\cdot\sum_{i=j}^{r}(-1)^i\binom{i}{j}\sum_{k\geq 0}(-1)^k\frac{e^{(i)}_k}{(1-X)^{r-j-k}}.$$

By considering the coefficient of t^{r-j}, for $r-n\leq j\leq r$, the fact that $\lim_{X\to 1}R(\mathcal{E};t,X)$ exists and is equal to $c_t(\mathcal{E})$ is equivalent to:

$$(1)\ \sum_{i=j}^{r}(-1)^i\binom{i}{j}e_k^{(i)}=0,\ \text{for}\ 0\le k\le r-j-1,$$

$$(2)\ \sum_{i=j}^{r}(-1)^i\binom{i}{j}e_{r-j}^{(i)}=(-1)^r c_{r-j}(\mathcal{E}),$$

for every j with $r-n\le j\le r$.

Using the formulas we have for $e_k^{(i)}$, these relations become:

$$(1')\ \sum_{i=j}^{r}(-1)^i\binom{i}{j}\chi(\wedge^i\mathcal{E}|_{H_k})=0,\ \text{for}\ 0\le k\le r-j-1$$

$$(2')\ \sum_{i=j}^{r}(-1)^i\binom{i}{j}\chi(\wedge^i\mathcal{E}|_{H_{r-j}})=(-1)^r c_{r-j}(\mathcal{E}),$$

for every j, with $r-n\le j\le r$.

For future reference, notice that for $j=r-n$, $(2')$ becomes

$$(2'')\ \sum_{i=r-n}^{r}(-1)^i\binom{i}{r-n}\chi(\wedge^i\mathcal{E})=(-1)^r c_n(\mathcal{E}).$$

In order to finish the proof of the theorem, it is enough to notice that using Hirzebruch-Riemann-Roch theorem (see Fulton [7] Corollary 15.2.1) all the Euler-Poincaré characteristics can be expressed as polynomials with rational coefficients in the Chern classes of the exterior powers $\wedge^i\mathcal{E}$. Indeed, since the Chern classes of these exterior powers can be computed as polynomials in the Chern classes of $\mathcal{E}$, we can apply Lemma 4.4 and Lemma 4.3 to conclude the proof of the theorem. □

Corollary 4.5 *If $\mathcal{E}$ is a vector bundle on $\mathbf{P}^n$ with $\operatorname{rank}\mathcal{E}=r\ge n$, then we have the following formula for the top Chern class:*

$$\sum_{i=r-n}^{r}(-1)^i\binom{i}{r-n}\chi(\wedge^i\mathcal{E})=(-1)^r c_n(\mathcal{E}).$$

In particular, if $r=n$ we have

$$\sum_{i=0}^{n}(-1)^i\chi(\wedge^i\mathcal{E})=(-1)^n c_n(\mathcal{E}).$$

Proof. This is just the identity $(2'')$ in the proof of Theorem 4.1. □

Remark 4.6 If $\mathcal{E}$ is a vector bundle on $\mathbf{P}^n$ such that $\mathcal{E}^*$ has $r-n+1$ sections $\sigma_1,\ldots,\sigma_{r-n+1}$ such that the degeneration locus Γ is zero dimensional and the degeneration locus Γ' of $\sigma_1,\ldots,\sigma_{r-n}$ is empty, then the formula in Corollary 4.5 is equivalent to the formula giving the degree of Γ.

Indeed, $\sigma_1,\ldots,\sigma_{r-n+1}$ define a morphism $\mathcal{E}\longrightarrow\mathcal{F}=\mathcal{O}_{\mathbf{P}^n}^{r-n+1}$ which gives an Eagon-Northcott type complex

$$0\longrightarrow\wedge^r\mathcal{E}\otimes(S_n\mathcal{F})^*\longrightarrow\ldots\longrightarrow\wedge^{r-n+1}\mathcal{E}\otimes\mathcal{F}^*\longrightarrow\wedge^{r-n}\mathcal{E}.$$

It follows from Eisenbud [5], Theorems A.2.10 and A.2.14 that since $\dim \Gamma = 0$ and $\Gamma' = \emptyset$, this complex is exact and moreover, it gives a resolution of $\mathcal{O}_\Gamma$. Therefore we obtain

$$\sum_{i=r-n}^{r} (-1)^i \chi(\wedge^i \mathcal{E}) = (-1)^{r-n} \chi(\mathcal{O}_\Gamma) = (-1)^{r-n} \deg \Gamma.$$

On the other hand, the Thom-Porteous formula (see Fulton [7], Theorem 14.4) says that under our hypothesis $\deg \Gamma = (-1)^n c_n(\mathcal{E})$ and we get the formula in Corollary 4.5.

Let $\mathcal{E}$ be a vector bundle on $\mathbf{P}^n$, with $\operatorname{rank} \mathcal{E} = n$. For every i, we denote by $Q(\wedge^i \mathcal{E}; X)$ the Hilbert polynomial of $\wedge^i \mathcal{E}$.

Corollary 4.7 *With the above notation, we have*

$$\sum_{i=0}^{n} (-1)^i Q(\wedge^i(\mathcal{E}); iX) = (-1)^n \sum_{i=0}^{n} c_i(\mathcal{E}) X^{n-i}.$$

Proof. We prove that the two polynomials take the same value for every $a \in \mathbf{Z}$. Indeed, if in Corollary 4.5 we replace $\mathcal{E}$ by $\mathcal{E}(a)$, then we have

$$\chi(\wedge^i(\mathcal{E}(a))) = \chi(\wedge^i \mathcal{E})(ai) = Q(\wedge^i \mathcal{E}; ai),$$

while $c_n(\mathcal{E}(a)) = \sum_{i=0}^{n} c_i(\mathcal{E}) a^{n-i}$ (see Fulton [7], Remark 3.2.3). $\square$

5 The Characteristic Polynomial

In this section we will apply the general results we have obtained so far to the case of the vector bundle associated to the module of $\mathcal{A}$ derivations of a locally free arrangement $\mathcal{A}$. Recall that $\pi(\mathcal{A}, t)$ is a polynomial of degree $n+1$. We will denote by $\overline{\pi}(\mathcal{A}, t)$ its class in $\mathbf{Z}[t]/(t^{n+1})$. The main result is the following.

Theorem 5.1 *If $\mathcal{A}$ is a locally free arrangement, then*

$$\overline{\pi}(\mathcal{A}, t) = c_t(\widetilde{\Omega^1}).$$

Proof. From the basic relation between the Poincaré and the characteristic polynomial we get

$$\pi(\mathcal{A}, t) = (-t)^{n+1} \chi(\mathcal{A}, -t^{-1}).$$

Combining this with Theorem 2.4 we obtain

$$\pi(\mathcal{A}, t) = \lim_{X \to 1} t^{n+1} \sum_{i \geq 0} P(D^i; X)(-\frac{1}{t}(X-1) - 1)^i.$$

By Proposition 3.9 we have $\widetilde{D^i} = \wedge^i \widetilde{D^1}$; so from Remark 4.2 it follows that

$$\lim_{X \to 1} R(\widetilde{D^1}; -t, X) = \lim_{X \to 1} \sum_{i=0}^{n} P(D^i; X) \cdot (-\frac{1}{t}(X-1) - 1).$$

By Theorem 4.1 this limit is equal to

$$c_{-t}(\widetilde{D^1}) = c_t(\widetilde{\Omega^1}).$$

We therefore obtain

$$\overline{\pi}(\mathcal{A}, t) = c_t(\widetilde{\Omega^1}).$$

□

Remark 5.2 Since it is known that $\pi(\mathcal{A}; -1) = 0$ (see Orlik and Terao [11], Proposition 2.5.1), in order to know $\pi(\mathcal{A}, t)$ it is enough to know $\overline{\pi}(\mathcal{A}, t)$.

In fact, when $\operatorname{char} k$ does not divide $|\mathcal{A}|$, then from Theorem 5.1 we can deduce a formula for $\pi(\mathcal{A}, t)$ involving the vector bundle $\widetilde{\Omega^1_0}$.

Corollary 5.3 *If $\mathcal{A}$ is a locally free arrangement such that* $\operatorname{char} k$ *does not divide $|\mathcal{A}|$, then we have*

$$\pi(\mathcal{A}, t) = (1 + t)c_t(\widetilde{\Omega^1_0}).$$

Proof. Since $\pi(\mathcal{A}, t)/(1 + t)$ is a polynomial of degree n, it follows that it is enough to prove that its class in $\mathbf{Z}[t]/(t^{n+1})$ is equal to $c_t(\widetilde{\Omega^1_0})$. But by Theorem 5.1 it follows that this class is equal to

$$\overline{\pi}(\mathcal{A}, t)/(1 + t) = c_t(\widetilde{\Omega^1})/(1 + t) = c_t(\widetilde{\Omega^1_0}),$$

since $\widetilde{\Omega^1} \simeq \widetilde{\Omega^1_0} \oplus \mathcal{O}(1)$. □

In [20], Yuzvinsky proves that for a locally free arrangement, the Hilbert polynomial of the module $D^1(\mathcal{A})$ depends only on the lattice. The following corollary makes this more precise.

Corollary 5.4 *If $\mathcal{A}$ is a locally free arrangement, then giving the Hilbert polynomial of $D^1(\mathcal{A})$ is equivalent to giving the Poincaré polynomial $\pi(\mathcal{A}, t)$ of the arrangement.*

Proof. The statement follows from Theorem 5.1 and the Hirzebruch-Riemann-Roch theorem (see Fulton [7], Corollary 15.2.1). □

Example 5.5 Let $\mathcal{A}$ be the arrangement in $\mathbf{P}^3$ defined by the vanishing of the fifteen linear forms $a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3$, where a_i is either zero or one. This example was constructed by Edelman-Reiner ([4]) as a counterexample to a conjecture of Orlik; $\pi(\mathcal{A}, t) = 1 + 15t + 80t^2 + 170t^3 + 104t^4$. There are 45 rank three elements of $L_{\mathcal{A}}$; we consider the corresponding subarrangements as essential arrangements in $\mathbf{P}^2$. The subarrangements are of three distinct types, described below ($\mu(L_2)$ is the Möbius function of the rank two elements of $L_{\mathcal{A}_X}$) :

20	subarrangements on 3 hyperplanes,	$\mu(L_2) = (1, 1, 1)$
15	subarrangements on 5 hyperplanes,	$\mu(L_2) = (2, 2, 1, 1, 1, 1)$
10	subarrangements on 7 hyperplanes,	$\mu(L_2) = (2, 2, 2, 2, 2, 2, 1, 1, 1)$

It is easy to check that all of these subarrangements are free, so $\mathcal{A}$ is locally free. We illustrate Corollary 5.4 for this example. For a rank three bundle $\mathcal{E}$ on $\mathbf{P}^3$ we have:

$$\int \mathrm{ch}(\mathcal{E}(m)) \cdot \mathrm{td}(\mathcal{T}_{\mathbf{P}^3}) = \frac{1}{2}m^3 + (3 + \frac{c_1}{2})m^2 + (\frac{11}{2} + 2c_1 + \frac{c_1^2}{2} - c_2)m + 3 + \frac{11c_1}{6} + c_1^2 - 2c_2 + \frac{c_1^3}{3} - \frac{c_1c_2}{2} + \frac{c_3}{2}.$$

Since we know $\pi(\mathcal{A}, t)$, we may apply Corollary 5.3 to obtain the Chern classes c_i, and from Hirzebruch-Riemann-Roch we obtain the Hilbert polynomial:

$$\chi(D_0^1(m)) = \frac{1}{2}m^3 - 4m^2 + \frac{57}{6}m - 6.$$

The point is that for a locally free arrangement, knowing the combinatorial data (i.e. Poincaré polynomial) means knowing the Hilbert polynomial, which can make the computation of the free resolution much faster; for this example the free resolution is:

$$0 \longrightarrow S(-6) \longrightarrow S^4(-5) \longrightarrow D_0^1 \longrightarrow 0.$$

Thus, we see that $\widetilde{\Omega_0^1} \simeq \Omega_{\mathbf{P}^3}(6)$, and $c_t(\widetilde{\Omega_0^1}) = \frac{(1+5t)^4}{(1+6t)} \bmod t^4 = 1 + 14t + 66t^2 + 104t^3$, as expected.

Remark In [3], Dolgachev and Kapranov point out that for a generic arrangement, $\Omega^1(\log D)$ is a Steiner bundle (hence stable); in the previous example $\widetilde{D_0^1}$ is Steiner. In $\mathbf{P}^2$ there are many examples of non-generic arrangements for which $\widetilde{\Omega_0^1}$ is indecomposable but not semistable.

6 Minimal free resolutions for the modules of logarithmic forms

In this section we give a minimal free resolution for the modules $\Omega^p(\mathcal{A})$ of logarithmic forms in the case of a locally free arrangement $\mathcal{A}$ with $\operatorname{pdim} \Omega^1(\mathcal{A}) = 1$. This generalizes the results of Rose and Terao [12] in the case of generic arrangements. The same idea can be used to give a minimal free resolution for the modules $D^p(\mathcal{A})$ when $\operatorname{pdim} D^1(\mathcal{A}) = 1$.

We first recall the definition of syzygy modules:

Definition 6.1 *A module M is a k^{th} syzygy if there exists an exact sequence*

$$0 \longrightarrow M \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \ldots \longrightarrow F_k,$$

with F_i free.

Lemma 6.2 *If $\operatorname{pdim} M = 1$ and $Ext^1(M, S)_{\underline{q}} = 0$ for every prime ideal $\underline{q} \neq \underline{m}$, then M is an $(n-1)^{st}$ syzygy.*

Proof.

From the short exact sequence:

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

we obtain an exact sequence:

$$0 \longrightarrow M^* \longrightarrow F_0^* \longrightarrow F_1^* \longrightarrow Ext^1(M,S) \longrightarrow 0.$$

Since $Ext^1(M,S)$ is supported only at m, $Ext^1(M,S)$ is a module of finite length, so has a free resolution of length $n+1$; hence M^* has projective dimension $n-1$. Dualizing once more and using the fact that $Ext^i(Ext^1(M,S),S)$ is zero for $i \neq n+1$, we find that M is an $(n-1)^{st}$ syzygy. □

Suppose now that M is a finitely generated S-module. Note that if $\widetilde{M}$ is locally free, then the second condition in Lemma 6.2 is automatically satisfied. In [9] Lebelt shows that if $\operatorname{pdim} M = i$ and if M is an $i(p-1)^{st}$ syzygy, then one can obtain a free resolution for $\Lambda^p M$ from a free resolution of M. In the situation considered above, i.e. $i=1$ and M is an $(n-1)^{st}$ syzygy, we obtain a minimal free resolution of $\Lambda^p(M)$, for $p \leq n-1$, which is an Eagon-Northcott type complex. Namely, if

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

is a minimal free resolution of M, then a minimal free resolution of $\wedge^p M$ is given by:

$$(F_\bullet^{(p)})(M): \quad 0 \longrightarrow D_p F_1 \longrightarrow D_{p-1}F_1 \otimes F_0 \longrightarrow D_{p-2}F_1 \otimes \Lambda^2 F_0 \longrightarrow \ldots \longrightarrow \Lambda^p F_0 \longrightarrow \Lambda^p M \longrightarrow 0,$$

where $D_i F_1 = (S_i(F_1^*))^*$ denotes the i^{th} divided power of F_1. If $\operatorname{char} k = 0$, then $D_i F_1 \simeq S_i F_1$ is the usual symmetric power of F_1.

We can now give the main result of this section:

Theorem 6.3 *If $\mathcal{A}$ is a locally free arrangement and* $\operatorname{pdim} \Omega^1 = 1$*, then the natural morphism*

$$\wedge^p \Omega^1 \longrightarrow \Omega^p$$

is an isomorphism, and $(F_\bullet^{(p)}(\Omega^1))$ gives a minimal free resolution of Ω^p, for every p, $p \leq n-1$.

Proof. From Lemma 6.2 and Lebelt's result cited above, it follows that for every p, $p \leq n-1$, $F_\bullet^{(p)}(\Omega^1)$ is a (minimal) free resolution of $\wedge^p \Omega^1$. In particular, we have $\operatorname{pdim} \wedge^p \Omega^1 = p$. By the Auslander-Buchsbaum formula we obtain $\operatorname{depth} \wedge^p \Omega^1 = n+1-p \geq 2$.

We consider the commutative diagram:

$$\begin{array}{ccc} \wedge^p \Omega^1 & \xrightarrow{\alpha} & \Omega^p \\ \downarrow & & \downarrow \\ H^0_*(\wedge^p \widetilde{\Omega^1}) & \xrightarrow{\beta} & H^0_*(\widetilde{\Omega^p}) \end{array}$$

Proposition 3.9 implies that β is an isomorphism. By Lemma 3.1, $\operatorname{depth} \Omega^p \geq 2$ and we also have $\operatorname{depth} \wedge^p \Omega^1 \geq 2$ and therefore both the vertical maps in the diagram are isomorphisms. We conclude that α is an isomorphism. □

Corollary 6.4 *If $\mathcal{A}$ is a locally free arrangement with* $\operatorname{pdim} \Omega^1 = 1$*, then we have*

$$\operatorname{pdim} \Omega^p = p, \text{ for } p \leq n-1,$$

$$\operatorname{pdim} \Omega^{n+1} = 0.$$

Moreover, if $\operatorname{char} k$ *does not divide $|\mathcal{A}|$, then we have also*

$$\operatorname{pdim} \Omega^n = n-1,$$

and therefore $\operatorname{pdim} D^1 = n-1$.

Proof. The first assertion follows from Theorem 6.3, while the second one is true for an arbitrary arrangement (see Orlik and Terao [11], Proposition 4.68).

If $\mathrm{char}\, k \not| |\mathcal{A}|$, then we have $\Omega^1 = \Omega^1_0 \oplus S(1)$. Therefore,

$$\wedge^p \Omega^1 = \wedge^p \Omega^1_0 \oplus \wedge^{p-1} \Omega^1_0(1),$$

for every p. It follows immediately that $\mathrm{pdim} \wedge^p \Omega_0 = p$, for $p \leq n-1$. In particular, from the Auslander-Buchsbaum formula we get depth $\wedge^{n-1} \Omega^1_0(1) = 2$.

From this and the decomposition $\wedge^n \Omega^1 = \wedge^n \Omega^1_0 \oplus \wedge^{n-1} \Omega^1_0(1)$ it follows that $\wedge^{n-1} \Omega^1_0(1)$ is a direct summand of $H^0_*(\widetilde{\wedge^n \Omega})$.

By Lemma 3.1 we have $\mathrm{depth}\, \Omega^n \geq 2$. This implies that $\Omega^n \simeq H^0_*(\widetilde{\Omega^n}) \simeq H^0_*(\wedge^n \widetilde{\Omega}^1)$. From the fact that this module has a summand of depth two and has depth at least two, we conclude that the depth is exactly two, and the result follows from one more application of the Auslander-Buchsbaum formula.

The fact that $\mathrm{pdim}\, D^1 = n-1$ is a consequence of the general fact that $\Omega^n \simeq D^1(d)$, where $d = |\mathcal{A}|$ (see Rose and Terao [12], Lemma 4.4.1). □

A generic arrangement $\mathcal{A}$ is locally free since for every $X \in L_{\mathcal{A}}$, with $\mathrm{rank}\, X < \dim V$, $\mathcal{A}_X$ is isomorphic to the product between a Boolean arrangement and an empty arrangement. On the other hand, Ziegler [21], Corollary 7.7 shows that in this case $\mathrm{pdim}\, \Omega^1 = 1$ and therefore Theorem 6.3 and Corollary 6.4 apply in this case and give the results in Rose and Terao [12].

Analogous results with similar proofs hold if we replace the modules Ω^p with the modules D^p. Namely, we have

Theorem 6.5 *If $\mathcal{A}$ is a locally free arrangement and* $\mathrm{pdim}\, D^1 = 1$, *then the natural morphism*

$$\wedge^p D^1 \longrightarrow D^p$$

is an isomorphism and $(F^{(p)}_\bullet(D^1))$ is a minimal free resolution of D^p, for every p, $p \leq n-1$. We have

$$\mathrm{pdim}\, D^p = p, \text{ for } p \leq n-1,$$

$$\mathrm{pdim}\, D^{n+1} = 0,$$

and if $\mathrm{char}\, k$ *does not divide* $|\mathcal{A}|$, *then*

$$\mathrm{pdim}\, D^n = n-1.$$

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