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ON THE DERIVED FUNCTOR ANALOGY IN THE CUNTZ-QUILLEN FRAMEWORK FOR CYCLIC HOMOLOGY

by Guillermo Cortiñas

Affiliation: Departamento de Matemática, Facultad de Ciencias Exactas , Universidad de La Plata.

Abstract. Cuntz and Quillen have shown that for algebras over a field k with $char(k) = 0$, periodic cyclic homology may be regarded, in some sense, as the derived functor of (non-commutative) de Rham (co-)homology. The purpose of this paper is to formalize this derived functor analogy. We show that the localization $Def^{-1}\mathcal{PA}$ of the category $\mathcal{PA}$ of countable pro-algebras at the class of (infinitesimal) deformations exists (in any characteristic) (Theorem 3.2) and that, in characteristic zero, periodic cyclic homology is the derived functor of de Rham cohomology with respect to this localization (Corollary 5.4). We also compute the derived functor of rational K-theory for algebras over $\mathbb{Q}$, which we show is essentially the fiber of the Chern character to negative cyclic homology (Theorem 6.2).

0. Introduction.

In their paper [CQ2], Cuntz and Quillen show that, if $char(k) = 0$, then periodic cyclic homology may be regarded, in some sense, as the derived functor of (non-commutative) de Rham (co-)homology. The purpose of this paper is to formalize this derived functor analogy. We show that the localization $Def^{-1}\mathcal{PA}$ of the category $\mathcal{PA}$ of countable pro-algebras at the class of (infinitesimal) deformations exists (in any characteristic) (Theorem 3.2) and that, in characteristic zero, periodic cyclic homology is the derived functor of de Rham cohomology with respect to this localization (Corollary 5.4). We also compute the derived functor of rational K-theory for algebras over $\mathbb{Q}$, which we show is essentially the fiber of the Chern character to negative cyclic homology (Theorem 6.2). For the construction of $Def^{-1}\mathcal{PA}$, we equip $\mathcal{PA}$ with the analogy of a closed model category structure, where the analogy of cofibrant objects are the quasi-free pro-algebras and the analogy of trivial fibrations are the deformations. Further, we define notions of strong and weak nil-homotopy between pro-algebra homomorphisms such that –as is the case with "real" model categories ([Q])– $Def^{-1}\mathcal{PA}$ turns out to be isomorphic to the localization of $\mathcal{PA}$ at the class of weak nil-homotopy equivalences, and equivalent to the localization of the subcategory of quasi-free algebras (i.e. the cofibrant objects) at the class of strong nil-homotopy equivalences (cf. 3.2). Of

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course this result would be automatic if the structure we put on $\mathcal{PA}$ were a model category (cf [Q]), which we prove it is not (3.6). However the analogy we have is sufficient to prove those localization properties and to consider derived functors therefrom. Quillen proves (in [Q]) that a functor between model categories which maps weak equivalences between cofibrant objects into weak equivalences admits a derived functor. The analogy of this result also holds in our setting; it says roughly that if a functor $\mathcal{PA} \to \mathcal{C}$ remains invariant under pro-power series extensions of quasi-free pro-algebras (i.e. $F(A\{X\}/ < X >^\infty) \cong FA$), then its left derived functor exists (Theorem 5.2). Functors satisfying the latter condition are called Poincaré functors, as the condition that defines them is precisely a Poincaré lemma for (non commutative) power series. For example if F satisfies the stronger condition $FA = FA[t]$ then it is Poincaré; such is the case of de Rham cohomology in characteristic zero. Unless explicitly mentioned, all results in this paper hold over any characteristic.

The notion of nil-homotopy used here (although related to) is different from the usual notion of polynomial (or pol-) homotopy, as used for example in Karoubi-Villamayor K-theory (see Section 4 below). In fact, a typical homotopy equivalence under pol-homotopy is the inclusion into the polynomial pro-algebra $B \hookrightarrow B[t]$ which is not an equivalence under nil-homotopy. Instead, the inclusion into the power series pro-algebra $B \hookrightarrow B[t]/ < t >^\infty$ is a nil-homotopy equivalence. Under nil-homotopy, quasi-free pro-algebras are precisely those having the homotopy extension property; other properties of quasi-free pro-algebras proven in [CQ1] are shown here to have a natural interpretation in terms of homotopy (Theorem 2.1).

The rest of this paper is organized as follows. In section 1, the notion of (strong) nil-homotopy is introduced, and its first properties are proved. Section 2 is devoted to the interpretation of quasi-free pro-algebras as cofibrant objects with respect to the setting of the previous section (Theorem 2.1). The notion of weak nil-homotopy is introduced in section 3, where the existence of the localized category $Def^{-1}\mathcal{PA}$ is proved (Theorem 3.2). Section 2 is devoted to the comparison between our notion of nil-homotopy and the usual, polynomial homotopy. We prove that the localization at the union of the classes of nil-deformations and graded deformations exists and can be calculated as a homotopy category (Theorem 4.1). Section 5 deals with the formalization of the derived functor analogy of [CQ2]. We establish sufficient conditions for the existence of left derived functors (Theorem 5.2) and prove that, in characteristic zero, these conditions are met by the de Rham supercomplex functor $A \mapsto XA$ of Cuntz-Quillen (Corollary 5.4). In section 6 we compute the derived functor of the rational K-theory of rational pro-algebras, (Theorem 6.2) and of the negative cyclic homology of pro-algebras over any field (Corollary 6.9).

Note on Notation. We use most of the notations and notions established in [CQ 1,2,3]. However, some notations do differ: we write ∂_i $(i = 0, 1)$ for the natural inclusions $1 * 0$, $0 * 1 : A \to QA = A * A$, and $qa = \partial_0 a - \partial_1 a$. Thus our qa is twice Cuntz-Quillen's. Also our curvature is minus theirs; here $\omega(a, b) = \rho a \rho b - \rho(ab)$. In this paper, the superscript B^+ on a graded algebra B denotes the terms of positive degree, and not the even degree part as in *op. cit.*. The even and odd terms are indicated by B^{even} and B^{odd}. If A is a pro-algebra indexed by $\mathbb{N}$, then the map $A_{n+1} \to A_n$ is referred to as the structure map and is named σ or τ (subscripts are mostly omitted). Since for the most part we make no assumptions on charknone of the results of *op. cit.* which involve dividing by arbitrary integers holds. Such is the case of the isomorphism between QA and the de Rham algebra with Fedosov product ([CQ1]), – as it assumes $2 \neq 0$– which we do not use. We do use the fact that $qA^n/qA^{n+1} \cong \Omega^n A$ as A-bimodules, which does hold even if 2 is not invertible. On the other hand the isomorphism between the tensor algebra TA and the algebra of even differential forms holds in any characteristic with the same proof as in [CQ1].

1. A Closed Model Category Analogy.

1.0 We consider associative, non-necessarily unital algebras over a fixed ground field k. We write $\mathcal{A}$ and $\mathcal{V}$ for the categories of algebras and vector spaces and $\mathcal{PA}$ and $\mathcal{PV}$ for the corresponding pro-categories. As in [CQ3] we consider only countably indexed pro-objects. A map $f \in \mathcal{PA}(A, B)$ is called a *fibration* if it admits a right inverse as a map of pro-vector spaces, i.e. there exists $s \in \mathcal{PV}(B, A)$ such that $fs = 1$. Fibrations are denoted by a double headed arrow $\twoheadrightarrow$. By a (nil-) deformation ($\tilde{\twoheadrightarrow}$) of a pro-algebra A we mean a fibration onto A which is isomorphic to one of the form $P/K^\infty \tilde{\twoheadrightarrow} P/K$. Equivalently, $p : B \tilde{\twoheadrightarrow} A$ is a deformation iff it is a fibration and for $K = \mathrm{Ker}(p)$ we have $K^\infty = 0$. For example the map:

$$UA := TA/JA^\infty \overset{\pi^A}{\tilde{\twoheadrightarrow}} A$$

is a deformation, and is initial among all deformations with values in A. That is if $p : B \tilde{\twoheadrightarrow} A$ is a deformation then there exists a map $f : UA \longrightarrow B$ with $pf = \pi^A$. In particular if A is quasi-free in the sense of [CQ3] then p is split in $\mathcal{PA}$ (because π^A is). Deformations admitting a right inverse shall be called *deformation retractions*; thus A is quasi-free iff every deformation $B \tilde{\twoheadrightarrow} A$ is a retraction (or A is a retract of every deformation onto it). It follows that quasi-free pro-algebras are precisely those pro-algebras A such that the map $0 \rightarrowtail A$ has the left lifting property (LLP) with respect to deformations. Thus we have the analogy of closed model category ([Q]) where fibrations are as above, trivial fibrations are deformations, and cofibrant objects are quasi-free algebras. To pursue this analogy a step further, we define our weak nil-equivalences (or wne's) as follows. We say that a map $f \in \mathcal{PA}$ is a wne if any functor defined on $\mathcal{PA}$ and taking values in some category $\mathcal{C}$ which inverts (i.e. maps to isomorphisms) all nil-deformations also inverts f. Functors which invert wne's are called nil-invariant. We shall show that the localization of $\mathcal{PA}$ with respect to deformations exists, whence f is a wne iff it is inverted upon localizing. For completeness, we call a map f quasi-free if it has the LLP with respect to deformations. Thus quasi-free maps play the rôle of cofibrations. I hurry to point out that the above notions of fibration, cofibration, and weak equivalence DO NOT make $\mathcal{PA}$ into a closed model or even into a model category. Indeed, if the map $0 \longrightarrow A$ factors as a weak equivalence followed by a fibration then A is weak equivalent to 0 (3.5). As there are pro-algebras which are not equivalent to zero, axiom M2 for a model category ([Q]) does not hold. The latter problem would be solved if we allowed free maps of the form $A \longrightarrow A * TV$ to be weak equivalences; in fact any map $A \longrightarrow B$ factors as $A \longrightarrow A * TB$ followed by $a \mapsto f(a), \rho b \mapsto b$. This simply means that there are nil-invariant functors which do not invert free maps.

The notion of weak equivalence defined above may be expressed as the weak homotopy relation associated to a notion of strong homotopy between pro-algebra homomorphisms. The definition of this strong homotopy is the subject of the next subsection.

Cylinders and nil-homotopy 1.1.

The *cylinder* of a pro-algebra A is the following pro-algebra:

$$Cyl(A) := QA/qA^{\infty} \tag{1}$$

Here $QA = A * A$ is the free product (or coproduct, or sum) and $qA = \operatorname{Ker}(QA \longrightarrow A)$ is the kernel of the folding map. We write $\partial_0 = 1 * 0$ and $\partial_1 = 0 * 1$ for the canonical inclusions $A \longrightarrow QA$, $\widetilde{\partial}_0 * \widetilde{\partial}_1 : QA \longrightarrow CylA$ for the completion map, and $p = p_A : CylA \overset{\sim}{\twoheadrightarrow} A$ for the the completion of the folding map $\mu : QA \longrightarrow A$. We have a commutative diagram:

$$\begin{array}{ccc} QA & & \\ {\scriptstyle\mu}\downarrow & \searrow^{\widetilde{\partial}_0 * \widetilde{\partial}_1} & \\ A & \xleftarrow{\sim} & CylA \end{array} \tag{2}$$

One checks that $\widetilde{\partial}_0 * \widetilde{\partial}_1$ is quasi-free if A is, whence $CylA$ is a cylinder object in the sense of [Q, 1.5. Def. 4]. Given homomorphisms $f, g : A \longrightarrow B$, we write $f \equiv g$ if there exists a map $h : CylA \longrightarrow B$ making the following diagram commute:

$$\begin{array}{ccc} QA & \xrightarrow{f*g} & B \\ {\scriptstyle\widetilde{\partial}_0 * \widetilde{\partial}_1}\downarrow & \nearrow_{h} & \\ CylA & & \end{array} \tag{3}$$

Note that as $QA \longrightarrow CylA$ is an epimorphism (although not a fibration), if a homotopy (i.e. a factorization through $CylA$) exists, it must be unique. For example if A and B are algebras, then $f \equiv g$ iff there exists n such that for all $a_1, \dots, a_n \in A$, we have

$$(f(a_1) - g(a_1)) \dots (f(a_n) - g(a_n)) = 0$$

and the homotopy is the map sending the class of qa to $f(a) - g(a)$. One checks that $\equiv$ is a reflexive and symmetric relation, and that it is compatible with composition on the left: $f_0 \equiv f_1 \Rightarrow f_2 f_0 \equiv f_2 f_1$ (whenever composition makes sense). It follows that the equivalence relation $\sim$ generated by $\equiv$ is compatible with composition on both sides. We say that f and g are (nil-) homotopic if $f \sim g$. We write $[\mathcal{PA}]$ for the category having the same objects as $\mathcal{PA}$ and as morphisms the sets of equivalence classes:

$$[A, B] := \mathcal{PA}(A, B)/$$

A map $f \in \mathcal{PA}$ is called a *strong* nil-homotopy equivalence if its class is an isomorphism in $[\mathcal{PA}]$.

Remark 1.2. The homotopy relation defined above may also be defined in terms of *n-fold cylinders*. Set $Cyl^1 A := CylA$, $\widetilde{\partial}_i^1 = \widetilde{\partial}_i$ and define the n-fold cylinder inductively by the pushout diagram:

$$\begin{array}{ccc} A & \xrightarrow{\widetilde{\partial}_0} & Cyl^1 A \\ {\scriptstyle \widetilde{\partial}_1^{n-1}}\downarrow & & \downarrow \\ Cyl^{n-1} A & \longrightarrow & Cyl^n A \end{array}$$

Define $\widetilde{\partial}_0^n$ as the composite map $A \xrightarrow{\widetilde{\partial}_0^{n-1}} Cyl^{n-1}A \to Cyl^n A$ and $\widetilde{\partial}_1^n$ as the composite $A \xrightarrow{\widetilde{\partial}_1^1} CylA \to Cyl^n A$. One checks that two maps $f, g : A \longrightarrow B$ are homotopic iff there exist n and $h : Cyl^n A \longrightarrow B$ such that the following diagram commutes:

$$\begin{array}{ccc} QA & \xrightarrow{f*g} & B \\ {\scriptstyle \widetilde{\partial}_0^n * \widetilde{\partial}_1^n}\downarrow & \nearrow_{h} & \\ Cyl^n A & & \end{array}$$

The map h in the diagram above will be called a homotopy between f and g.

The following lemma establishes a relation between the nil-homotopy equivalences just defined and the weak nil-equivalences of 1.0. above.

Lemma 1.3. *Let $f : A \tilde{\twoheadrightarrow} B$ be a deformation retraction. Then f is a strong nil-homotopy equivalence.*

Proof. We have to prove that $g = sf \sim 1$. Upon re-indexing, we can assume $f = \{f_n : A_n \longrightarrow B_n\}$, $s = \{s_n : B_n \longrightarrow A_n\}$ are inverse systems of maps commuting with the structure maps $\sigma = \sigma_n$, that $\sigma f_n s_n = \sigma$ and that for $K_n = \operatorname{Ker} f_n$ we have $K_n^n = 0$. Then for $a \in A_n$, we have $f(\sigma(g_n * 1)qa) = \sigma(fsfa - fa) = 0$, from which $\sigma(g_n * 1(qa)) \in K_{n-1}$. Thus $\sigma(g_n * 1)(qA_n)^n = 0$ whence $g * 1 : QA \longrightarrow B$ factors through $CylA$, and $g \equiv 1$. □

2. Quasi-free Algebras and the Homotopy Extension Property.

An interesting feature of nil-homotopy is that quasi-free algebras are precisely those having the homotopy extension property with respect to deformations. This fact is proven in Theorem 2.1 below. First we need:

Power pro-algebras, power spans and power deformations 2.0. By a *graded* pro-algebra we mean a non-negatively graded object in $\mathcal{PA}$, i.e. a pro-algebra B together with a direct sum decomposition of pro-vector spaces: $B = \bigoplus_{n=0}^{\infty} B^n$ such that the multiplication map $B \otimes B = \bigoplus B^n \otimes B^m \longrightarrow B$ maps $B^n \otimes B^m$

into B^{n+m}. Thus $B^+ = \bigoplus_{n=1}^{\infty} B^n$ is a two-sided ideal in B, in the sense that multiplication maps $B^+ \otimes B$ and $B \otimes B^+$ into B^+. It is straightforward to show that every graded pro-algebra is isomorphic–by a homogeneous isomorphism– to an inverse system of graded algebras and homogeneous maps. The *power* pro-algebra associated with B is the pro-algebra $\hat{B} := B/B^{+^{\infty}}$. Thus a power pro-algebra is a particular kind of graded algebra. For instance if A is an algebra then the power pro-algebra associated to the polynomials in a set X is the pro-algebra $\{A\{X\}/ < X >^n\}$, whose completion is the power series algebra in the non-commutative variables X. More generally, one considers the tensor algebra $T_{\tilde{A}}(M) = T_0(A) \bigoplus T^1(A) \bigoplus T^2(A) \cdots = A \oplus M \oplus M \otimes_{\tilde{A}} M \oplus \dots$ whose associated power algebra is $\hat{T}_{\tilde{A}}(M) = \{\bigoplus_{i=1}^{n} T^i(M) : n \in \mathbb{N}\}$ and when M is the free module on a set X one recovers the polynomial and power series algebras. These constructions can be copied for pro-algebras, pro-sets and pro-modules with the obvious definitions. However in general the free pro-module associated with a pro-set is not proyective, as it doesn't have the LLP with respect to all epimorphisms, but only with respect to fibrations (cf.[CQ3]). We use the following special notations. If V is a pro-vector space and $I \lhd A$ is an ideal in a pro-algebra, we write $P_A(V)$ for the power algebra associated with $T_{\tilde{A}}(\tilde{A} \otimes V \otimes \tilde{A})$ and $G_I(A)$ and $\hat{G}_I(A)$ for the graded pro-algebra $A \oplus I/I^2 \oplus I^2/I^3 \oplus \dots$ and its associated power algebra. If B is a graded pro-algebra and $u : A \longrightarrow B^0$ is a homomorphism, then by a *power span* of u we mean a k-linear map $T = \sum_{n=1}^{\infty} D_n : A \longrightarrow \hat{B}^+$ such that the following diagram commutes:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\text{multiplication}} & A \\ {\scriptstyle u\otimes T + T \otimes u - T \otimes T}\downarrow & & \downarrow{\scriptstyle T} \\ \hat{B} \otimes \hat{B} \oplus \hat{B} \otimes \hat{B} \oplus \hat{B} \otimes \hat{B} & \xrightarrow{\text{sum+multiplication}} & \hat{B}^+ \end{array} \tag{4}$$

Briefly, we write

$$T(xy) = uxTy + Txuy - TxTy \tag{4'}$$

to indicate the diagram above –even if A and B are not algebras. For example the ordinary Taylor span:

$$k[x] \longrightarrow k[x][[y]] = \{k[x,y]/ < y >^n\}, f(x) \mapsto \{\sum_{i=0}^{n} \frac{f^{(i)}(y)}{i!}\}$$

is a power span of the canonical inclusion. Note that the image of $f(x)$ in $k[x,y]/ < y >^n$ is just the class of $f(x) - f(y)$ and is therefore defined in any characteristic; if $f(x) = \sum_{i=0}^{n} a_i x^i$ then $\frac{f^{(i)}(y)}{i!}$ is just short for $\sum_{j=0}^{n-i} \binom{n}{j} a_{i+j}$ which is defined everywhere. Note also that any power span T induces a homomorphism $h : CylA \longrightarrow \hat{B}$ with $h\widetilde{\partial}_0 = u$, which is a homotopy between u and $h\widetilde{\partial}_1$. Conversely if h is a homotopy starting at u, then $T : A \xrightarrow{q} qA \longrightarrow qA/qA^{\infty} \xrightarrow{h} \hat{B}$ is a power span. Thus a power span is a special kind of homotopy where the target is a power algebra. By an *n-truncated span* we mean a linear map $T_n : A \longrightarrow B/B^{+^{n+1}}$ satisfying (4'). For example if T is a power span then $T_n : A \xrightarrow{T} B/B^{+^{\infty}} \longrightarrow B/B^{+^{n+1}}$ is an n-truncated power span. Finally, by a *power deformation retraction* we mean a deformation retraction of the form $\hat{B} \to B_0$ where B is a graded algebra.

Theorem 2.1. (Compare [CQ1]). *The following conditions are equivalent for a pro-algebra A.*

(i) *(*LLP*) A is quasi-free.*

(ii) *(*Power Span Extension*) If B is a graded algebra and $u : A \longrightarrow B^0$ is a homomorphism then any truncated span $T_n : A \longrightarrow B/B^{+^{n+1}}$ lifts to a power span $T : A \longrightarrow \hat{B}$.*

(iii) *(*Tubular Neighborhood*) If $f : B \tilde{\twoheadrightarrow} A$ is a deformation with kernel I and B is quasi-free, then there is an isomorphism $\iota : B \overset{\cong}{\to} \hat{G}_I(B)$ such that $f\iota$ is the projection $\hat{G}_I(B) \tilde{\twoheadrightarrow} B/I = A$.*

(iv) *(*Even Forms*) There is a pro-algebra isomorphism $UA \cong \Omega^{even}A/\Omega^{even+^{\infty}}A$ which makes the following diagram commute:*

$$\begin{array}{ccc} UA & \longrightarrow & \widetilde{\Omega}^{even}A/\Omega^{even+^{\infty}}A \\ {\scriptstyle \pi^A \wr}\downarrow & \swarrow & \\ A & & \end{array}$$

(v) *(*de Rham Algebra*) There is a pro-algebra isomorphism $CylA \cong \Omega A/\Omega^{+^{\infty}}A$ which makes the following diagram commute:*

$$\begin{array}{ccc} CylA & \overset{\simeq}{\longrightarrow} & \Omega A/\Omega^{+^{\infty}}A \\ \downarrow & & \downarrow \\ A \oplus qA/qA^2 & \overset{\simeq}{\longrightarrow} & A \oplus \Omega^1 A \end{array}$$

Here the bottom arrow is the canonical isomorphism $aqb \mapsto adb$.

(vi) (Homotopy Extension) *Given any commutative solid arrow diagram:*

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle \tilde{\partial}_0}\downarrow & \nearrow & \downarrow{\scriptstyle \wr f} \\ CylA & \longrightarrow & C \end{array}$$

where f is a deformation, the dotted arrow exists and makes it commute.

Proof. (i)⇒(ii): Write $T_n = \sum_{i=1}^n D_i$ where D_i is the part of degree i; also let $D_0 = u$. Thus $u_n = u + T_n = \sum_{i=0}^n D_i$ is a homomorphism, from which the following identity follows:

$$-\delta D_i = \sum_{j=1}^{i} D_j \cup D_{i-j} \qquad (0 \leq i \leq n) \tag{5}$$

Here the maps D_i are regarded as 1-cochains with values in B, the cup product is the composite of $D_j \otimes D_{i-j}$ with the multiplication map $B \otimes B \longrightarrow B$ and δ is the

Hochschild co-boundary map—as defined by the appropriate arrow diagram. We must prove that a k-linear map $D_{n+1} : A \longrightarrow B^{n+1}$ exists so that

$$-\delta D_{n+1} = \sum_{i=1}^{n} D_i \cup D_{n+1-i} \tag{5'}$$

holds. It is straightforward to check that the right hand side of (5') is actually a cocycle, whence also a coboundary, as A is quasi-free. Explicitly, if $g : \Omega^2(A) \longrightarrow B_{n+1}$ is the bimodule homomorphism induced by the right hand side of (5') and if $f : A \longrightarrow \Omega^2(A)$ satisfies $-\delta f = d \cup d$, then we can take $D_{n+1} = gf$.

(i) $\iff$ (iii): If (iii) holds then $UA \tilde{\twoheadrightarrow} A$ is a retraction, whence A is quasi-free. Supose conversely that (i) holds. Because A is quasi-free, we have direct sum decompositions $B = A \oplus I$, and $B/I^2 = A \oplus I/I^2 = \hat{G}_I(B)/\hat{G}_I(B)^{+^2}$. Write $u : B \tilde{\twoheadrightarrow} A \hookrightarrow \hat{G}_I(B)$ for the composite map, and $p_1 : B \tilde{\twoheadrightarrow} \hat{G}_I(B)/\hat{G}_I(B)^{+^2}$ for the projection. Because B is quasifree, the truncated span $T_1 = p_1 - u : B \to \hat{G}_I(B)^+/\hat{G}_I(B)^{+^2}$ extends to a power span $T : A \to \hat{G}_I(B)^+/\hat{G}_I(B)^{+^\infty}$ (by (ii)). It is clear that T induces the identity on I/I^2; further, one checks –using (5)– that it also induces the identity on I^n/I^{n+1}. It follows that $p : u+T$ is an isomorphism.

(iii)$\Rightarrow$(iv): Applying (iii) to $\pi^A : UA \tilde{\twoheadrightarrow} A$, we get

$UA \cong \hat{G}_{JA/JA^\infty}(UA) = \Omega^{even}A/\Omega^{even+^\infty}A$.

(iv)$\Rightarrow$(i): Analogous to (iii)$\Rightarrow$(i).

(ii)$\Rightarrow$(v): By (ii), we can lift the de Rham derivation $d : A \longrightarrow \Omega^1 A$ to a power span $T : A \longrightarrow \Omega A/\Omega^{+^\infty}A$ of the identity map $A = \Omega^0 A$. By the discussion above, $1+T$ induces a homomorphism $h : CylA \longrightarrow \Omega A/\Omega^{+^\infty}A$ such that $hq = T$. In particular, h induces the canonical A-bimodule isomorphism $qA/qA^2 \cong \Omega^1 A$ mapping q to d. Thus we have $hq = d + D$, where $D(A) \subset \Omega^{\geq 2}/\Omega^{\geq 2^\infty}$. It follows that the composite $A^{\otimes n} \overset{hq^{\otimes n}}{\longrightarrow} \Omega^+/\Omega^{+^\infty} \longrightarrow \Omega^+/\Omega^{+^{n+1}}$ is just the cocycle $d^{\cup n}$, whence the induced bimodule homomorphism $qA^n/qA^{n+1} \cong \Omega^n A$ is the canonical isomorphism, and the proof ensues.

(v)$\Rightarrow$(i): By virtue of (5), if $T_2 = d + D_2 : A \longrightarrow \Omega^1 A \oplus \Omega^2 A$ is the 2-span induced by $\widetilde{\partial}_1$, then $-\delta D_2 = d \cup d$, whence A is quasi-free.

(vi)$\Rightarrow$(v): Since $\Omega A/\Omega^+ A^\infty \tilde{\twoheadrightarrow} \Omega^0 A \oplus \Omega^1 A$ is a deformation, there exists a homomotopy $h : CylA \longrightarrow \Omega A/\Omega^{+^\infty}A$ lifting the homotopy $1 \equiv 1 + d$. The same argument as in the proof of (ii)$\Rightarrow$(v) shows that h is an isomorphism.

(i)$\Rightarrow$(vi): As $0 \longrightarrow A$ is quasi-free, so are ∂_0 and $\widetilde{\partial}_0$. $\square$

Example 2.2. Let A be an algebra, and let $UA = TA/JA^\infty$ its universal quasi-free model. By the theorem above, we have $CylUA \cong \Omega UA/\Omega^+ UA^\infty$. We want to give an explicit isomorphism $CylUA \cong \Omega UA/\Omega^+ UA^\infty$ as well as to show that in this particular case, we also have an isomorphism

$$\Omega UA/\Omega^+ UA^\infty \cong P_{UA}(A)$$

First of all, we observe that given a vector space V, we have isomorphisms:

$$QTV \cong T(V \oplus V) \cong T(V \oplus qV) \cong T(V) * T(qV)$$

$$\cong T(\widetilde{TV} \otimes V \otimes \widetilde{TV}) \cong \Omega TV$$

Here $qV = \{(v, -v) : v \in V\}$ and the isomorphism $V \oplus V \cong V \oplus qV$ maps $(v, 0) = \partial_0 v$ to itself while $\partial_1 v \mapsto qv$. Thus the composite isomorphism $\alpha : QTV \overset{\to}{\cong} \Omega TV$ maps qv to dv and $\partial_0 x$ to x $(v \in V, x \in TV)$. In particular this holds when $V = A$; in this case α maps the ideal $< JA > \subset QTA$ generated by JA (which we identify with its image through $\widetilde{\partial}_0$) into the ideal $< JA > \subset \Omega TA$, and qTA into $\Omega^+ TA$. It follows that α induces an isomorphism $QTA/\mathcal{F}^\infty \cong \Omega TA/\mathcal{G}^\infty$, where $\mathcal{F}$ and $\mathcal{G}$ are respectively the $< JA > +qTA$ and $< JA > +\Omega^+ TA$-adic filtrations. On the other hand we have $CylUA = QTA/\mathcal{F}'^\infty$ and $\Omega UA/\Omega^+ UA^\infty = \Omega TA/\mathcal{G}'$ where $\mathcal{F}' =< JA^n > + < q(JA^n) > +(qTA)^n$ and $\mathcal{G}' =< JA^n > + < dJA^n > +(\Omega^+ TA)^n$. We have inclusions:

$\mathcal{F}^n \supset \mathcal{F}'^n \supset \mathcal{F}"^n =< JA^n > +(qTA)^n$

and

$\mathcal{G}^n \supset \mathcal{G}'^n \supset \mathcal{G}"^n =< JA^n > +(\Omega^+ TA)^n$

Lemma 2.3. below shows that for N sufficiently large, we also have inclusions $\mathcal{F}"^n \supset \mathcal{F}^N$ and $\mathcal{G}"^n \supset \mathcal{G}^N$. It follows that α induces the isomorphism $CylUA \overset{\to}{\cong} \Omega UA/\Omega^+ UA^\infty$ and that $\Omega UA/\Omega^+ UA^\infty = \Omega TA/G"^\infty = P_{UA}(A)$

Lemma 2.3. *Let $A \subset B$ be algebras and let $\epsilon : B \to A$ be a homomorphism such that $\epsilon a = a, (a \in A)$. Set $I = \operatorname{Ker} \epsilon$, and let $J \subset A$ be an ideal. Consider the following filtration in B:*

$$B \supset \mathcal{F}^n =< J^n > +I^n$$

Then there is an isomorphism:

$$B/\mathcal{F}^\infty \cong B/(< J > +I)^\infty$$

Proof. Let $\mathcal{G}^n =< J >^n +I^n$. It is straightforward to check that $(< J > +I)^{2n} \subset \mathcal{G}^n$, whence $B/(< J > +I)^\infty \cong B/\mathcal{G}^\infty$. Thus we must prove that $B/\mathcal{G}^\infty \cong B/\mathcal{F}^\infty$. It is clear that $\mathcal{G}^n \supset \mathcal{F}^n$. I claim that for $N = n^2 + n - 1$, we also have $\mathcal{G}^N \supset \mathcal{F}^n$. To prove the claim –and the lemma– it suffices to show that $< J >^N \subset \mathcal{F}^n$. Every element of $< J >^N$ is a sum of products of the form:

$$(j_1 + i_1) \dots (j_N + i_N) \qquad\qquad (j_r \in J, i_r \in I)$$

After fully expanding the product above, we get a large sum in which those terms not in I^n have at most n-1 i's and at least n^2 j's. Therefore, in each such term, at least n of the j's must appear side by side, forming a string. Hence the term in question lives in $< J^n >$. $\square$

Remark 2.4. The de Rham pro-algebra $\Omega A/\Omega^+ A^\infty = \{\bigoplus_{r=0}^n \Omega^r A_{n+1}\}$, of a pro-algebra $A = \{A_n\}$, together with the natural differentials b and d and the Karoubi operator κ, can be regarded as a pro-*truncated* mixed DGA in the sense of [Kar]. Indeed, the identity:

$$bd\omega + db\omega = \omega - \kappa\omega$$

holds in $\Omega^r(A_{n+1})$ for $r < n$ and in $\Omega^n_\natural A_{n+1} = \Omega^n A_{n+1}/[\Omega^0 A, \Omega^n A]$ for $r = n$. Thus:

$$\theta\Omega(A) \ldots ((\bigoplus^{n-1}\Omega^r A \ldots) \oplus \Omega^n A \ldots B + b)$$

is a pro-differential graded vector space, equipped with an even-odd gradation. This is the pro-complex of [CQ-2]; if $k \supset \mathbb{Q}$, it is homotopy equivalent to the (short) de Rham pro-complex:

$$XUA : \Omega^0 UA \underset{b}{\overset{d_\natural}{\rightleftarrows}} \Omega^1_\natural UA$$

In any characteristic, we still have $\theta\Omega UA \approx XUA$ for every algebra A and $\theta\Omega R \approx XR$ for every quasi-free algebra R. In particular $CylR$ carries all the relevant information for the cyclic homology of R.

3. The Homotopy Category.

Weak nil-homotopy 3.0. We write $[U\mathcal{PA}]$ for the category having the same objects as $\mathcal{PA}$ and where the set of maps from A to B is $[UA, UB]$. We have a functor $\gamma : \mathcal{PA} \to [U\mathcal{PA}]$, $A \mapsto A$, $f \mapsto [Uf]$. Two maps $f, g \in \mathcal{PA}(A, B)$ shall be called *weakly nil homotopic* if $\gamma f = \gamma g$; by a weak nil homotopy equivalence we shall mean a map $f \in \mathcal{PA}$ such that γf is an isomorphism. We show below that the class of weak nil homotopy equivalences is precisely the class of weak nil equivalences as defined in 1.0 above, and that γ is the localization of $\mathcal{PA}$ at this class. Further, we show that $[U\mathcal{PA}]$ is equivalent to the strong homotopy category $[\mathcal{PAQ}]$ of quasi-free algebras. First we need:

Lemma 3.1. *The functor $U : \mathcal{PA} \to \mathcal{PAQ}$ carries fibrations to fibrations and deformations to deformations.*

Proof. Let $f = \{f_n : A_n \twoheadrightarrow B_n\}$ be a fibration, and let $t = t_n : B_n \to A_n$ be a section of f in $\mathcal{PV}$. Upon re-indexing, we can assume that $ft\tau = \tau$ for the structure map of B. We want to construct a linear section $\hat{t}$ of Uf lifting t. Consider the following composite of linear maps:

$$s_n : \frac{TB_n}{JB_n^n} \overset{\sim}{\to} \bigoplus_{i=0}^{n-1} \Omega^{2i} B_n \hookrightarrow \bigoplus_{i=0}^{\infty} \Omega^{2i} B_n \cong TB_n$$

Note that s_n is a linear section of $TB_n \twoheadrightarrow TB_n/JB^n$. Consider the composite $\hat{t}_n : TB_n/JB^n \xrightarrow{s_n} TB_n \xrightarrow{Tt_n} TA_n \to TA_n/JA_n^n$; then $\hat{t}_n$ commutes with τ and

$$\begin{aligned}&\hat{t}_n(\rho b_0 \omega(b_1, b_2) \dots \omega(b_{2l-1}, b_{2l}) = \\ &= \rho t_n(b_0)(\omega(t_n b_1, t_n b_2) + \rho\omega_{t_n}(b_1, b_2)) \dots (\omega(t_n b_{2l-1}, t_n b_{2l}) + \rho\omega_{t_n}(b_{2l-1}, b_{2l}))\end{aligned}$$

for $0 \le l \le n-1$. Here $\rho : A \to TA$ is the canonical section, $\omega(a, b) = a \otimes b - ab$ is the curvature of ρ and ω_{t_n} is the curvature of t_n. Now since $ft\tau = \tau$, we have $\omega_{t_n}(b, b') \in \operatorname{Ker} \tau^B f_n$ $(b, b' \in B_n)$ and $\rho\omega_{t_n}(b, b') \in \operatorname{Ker} \tau^{TB} Tf_n$. It follows that $Uf_n \hat{t}_n \tau_n^{UB} = \tau_n^{UB}$, whence Uf is a fibration. Suppose further that f is also a deformation, and let $K = \operatorname{Ker} f$; we can assume $K_n^n = 0$. Let $L = \operatorname{Ker} Uf$; if $l \in L_n^n$ then $\pi_n^A l \in K_n^n = 0$, hence $L_n^n \subset JA_n/JA_n^n$, and $L_n^{n^2} = 0$. $\square$

Theorem 3.2. *(Compare [Qui, 1.13, Th.1])*

(i) *Strong nil-homotopy equivalences are precisely those maps which are inverted by every functor which inverts* deformation retractions*. Weak nil-homotopy equivalences are precisely those maps in* $\mathcal{PA}$ *that are inverted by every nil-invariant functor, i.e. every functor which inverts* all deformations.

(ii) *The functor* $\mathcal{PA} \to [\mathcal{PA}]$ *is the localization of* $\mathcal{PA}$ *at the class of* deformation retractions*, the functor* $\mathcal{PAQ} \to [\mathcal{PAQ}]$ *is the localization at the class of* power deformation retractions*, and the functor* $\gamma : \mathcal{PA} \to [U\mathcal{PA}]$ *is the localization at the class of* all deformations*. There is a category equivalence:* $[U\mathcal{PA}] \approx [\mathcal{PAQ}]$*.*

Proof. (i) Let se be the class of maps inverted by every functor which inverts deformation retractions and let se' be the class of strong homotopy equivalences. By virtue of Lemma 1.3, the functor $\mathcal{PA} \to [\mathcal{PA}]$ inverts deformation retractions, whence $se \subset se'$. Conversely, if F inverts deformation retractions then it inverts $CylA \tilde{\twoheadrightarrow} A$, and also $\widetilde{\partial}_i$, $i = 0, 1$. Thus F maps congruent maps to the same map; further, since $f \overset{F}{\sim} g \iff Ff = Fg$ is an equivalence relation, F also maps nil-homotopic maps to the same map, and strong nil-equivalences to isomorphisms. This proves the first assertion of (i). Next, write ω and ω' for the classes of weak nil-equivalences (as defined in 1.0 above) and weak nil-homotopy equivalences. We have to prove that $\omega = \omega'$. In view of Lemmas 1.3 and 3.1, the functor γ is nil-invariant, whence $\omega \subset \omega'$. Now let $F : \mathcal{PA} \to \mathcal{C}$ be a nil-invariant functor, and let $f \in \omega'(A, B)$. Because $F\pi^A$ and $F\pi^B$ are isomorphisms in $\mathcal{C}$, Ff will be an isomorphism iff FUf is. By definition, the fact that $f \in \omega'$ means that Uf is a strong equivalence, and therefore is inverted by F. Thus $\omega = \omega'$.

(ii) The first assertion of (ii) is immediate from the proof of the first assertion of (i). The second assertion follows similarly, in view of 2.1-iii). Now let F be a nil invariant functor as above. We have to show that F factors as $F = \tilde{F}\gamma$ for some $\tilde{F} : [U\mathcal{PA}] \to \mathcal{C}$, and that such $\tilde{F}$ is unique. We put $\tilde{F}(A) = F(A)$ and for $[f] \in [U\mathcal{PA}](A, B)$, we set $\tilde{F}[f] = F\pi^B Ff(F\pi^A)^{-1}$. It is clear that $\tilde{F}$ is well-defined and that $F = \tilde{F}\gamma$. Now suppose G is another functor with the same property as $\tilde{F}$. Then $GA = A$ on objects and if $f \in \mathcal{PA}(A, B)$ then G must map $[Uf]$ onto $FUf = \tilde{F}[Uf]$. Since any map $[g] \in [U\mathcal{PA}](UA, UB)$ factors as $[g] = [\pi^{UB}][Ug][\pi^{UA}]^{-1}$, it suffices to prove that $[\pi^{UA}] = [U\pi^A]$. But both π^{UA} and $U\pi^A$ are left inverse to the same map $\iota : UA \to U^2A$ induced by $T\rho : TA \to T^2A$, whence (by Lemma 1.3) $[\pi^{UA}] = [\iota]^{-1} = [U\pi^A]$. This proves the third assertion. By the proof of (i), the functor $\gamma : \mathcal{PAQ} \to [U\mathcal{PA}]$ induces a functor $\overline{\gamma} : [\mathcal{PAQ}] \to [U\mathcal{PA}]$. Let $\gamma' : [U\mathcal{PA}] \to [\mathcal{PAQ}]$, $A \mapsto UA$, $[f] \mapsto [f]$. Then $[\pi^R] : \gamma'\overline{\gamma}(R) = UR \to R$ and $[\pi^{UA}] : \overline{\gamma}\gamma'(A) = UA \to A$ are natural isomorphisms $\overline{\gamma}\gamma' \overset{\cong}{\to} 1$ and $\gamma'\overline{\gamma} \overset{\cong}{\to} 1$. This concludes the proof. □

Corollary 3.3. *Let* $f, g : A \to B$ *be pro-algebra homomorphisms. We have:*

(i) *Strong* $\Rightarrow$ *Weak: If* f *is a strong equivalence then it is also a weak equivalence. If* f *and* g *are strongly nil-homotopic then they are also weakly homotopic.*

(ii) *Weak* $\Rightarrow$ *Strong: The converse of (i) holds if* A *and* B *are quasi-free.*

Proof. As $CylA \to A$ is a deformation, any nil invariant functor maps strong equivalences into isomorphisms and homotopic maps to the same map. In particular, this happens with the localization functor γ, proving (i). Part (ii) follows from the identities: $[A,B] = [\mathcal{PAQ}](A,B) = [U\mathcal{PA}](A,B) = [UA, UB]$. □

By defintion, the class *Def* of deformations sits into the intersection of the class *we* of weak equivalences and the class *Fib* of fibrations. The proposition below shows that in fact $Def = we \cap Fib$. In particular this proves that quasi-free maps are precisely those having the LLP with respect to those fibrations which are weak equivalences.

Proposition 3.4. *A fibration is a deformation iff it is a weak equivalence.*

Proof. If f is deformation then it is a weak equivalence by definition of the latter. Suppose now $f : A \twoheadrightarrow B$ is a fibration and a weak equivalence, and write $K = \operatorname{Ker} f$. Upon re-indexing, we can assume f is an inverse system of epimorphisms $\{f_n : A_n \twoheadrightarrow B_n\}$ commuting with structure maps. We must prove $K^\infty = 0$. I claim it suffices to check this for the particular case when f is a strong equivalence. For if f is a weak equivalence and a fibration then Uf is both a strong equivalence (by 3.3) and a fibration (by 3.1). Whence, if we know the proposition for strong equivalences, we have $\operatorname{Ker} Uf^\infty = 0$. Now a little diagram chasing shows that $\operatorname{Ker} Uf_n \twoheadrightarrow K_n$ is an epimorphism ($n \geq 1$), whence also $K^\infty = 0$, proving the claim. Assume then that there exists $g \in \mathcal{PA}(B,A)$ with $\beta := gf \sim 1$, and that $g = \{g_n : B_n \to A_n\}$ is an inverse system of homomorphisms commuting with the structure maps. By definition of homotopy, there exist $r \geq 1$ and $\alpha_i \in \mathcal{PA}(A,A)$ with $1 = \alpha^0 \equiv \alpha^1 \equiv \cdots \equiv \alpha^r = \beta$. Because $\alpha := \alpha^1 \equiv 1$, for every $n \in \mathbb{N}$ there exists $m_0 \geq n$ such that for $m \geq m_0$, $\tau_{mn}(\alpha * 1)$ factors as follows:

$$\begin{array}{ccc} QA_m & \xrightarrow{\alpha * 1} & A_m \\ \downarrow & & \downarrow \tau_{mn} \\ \frac{QA_m}{qA_m^m} & \xrightarrow{h} & A_n \end{array}$$

Therefore, given $a_1, \dots, a_m \in A_m$, we have:

$$\begin{aligned} 0 = \tau(\alpha * 1)(qa_1 \dots qa_m) \\ = \tau((\alpha a_1 - a_1) \dots (\alpha a_m - a_m)) \\ \equiv (-1)^m \tau(a_1 \dots a_m) \mod < \tau\alpha a_1, \dots \alpha a_m > \end{aligned}$$

Thus if $a_1, \dots, a_m \in \operatorname{Ker}(\tau\alpha)$, we have $\tau(a_1 \dots a_m) = 0$. We have proven the following statement:

$$(6) \quad (\forall n \geq 1)(\exists m_0 \geq n) \text{ and for each } m \geq m_0 \\ \text{an } N = N_m \geq m \text{ such that } (\operatorname{Ker} \tau_{mn}\alpha_n)^N = 0$$

We are going to show next that if α satisfies (6) and $\gamma \equiv \alpha$, then γ satisfies (6) too. It will follow that β, and then also f, satisfies (6), whence $K^\infty = 0$ as we had to prove. So assume (6) holds for α and let $\gamma : A \to A$ with $\gamma \equiv \alpha$. Proceeding as above, we can find, for each n, an $m_1 \geq m_0 \geq n$ such that if $m \geq m_1$, then

$$0 \equiv (-1)^m \tau\alpha(a_1 \dots a_m) \mod < \tau\gamma a_1, \dots \gamma a_m >$$

In particular $\tau_{mn}(\operatorname{Ker}\gamma_m)^m \subset \operatorname{Ker}\tau_{mn}\alpha$ whence for N as in (6) we have $(\operatorname{Ker}\tau\gamma_m)^{mN} = 0$. $\square$

Corollary 3.5. *A pro-algebra A is weak equivalent to zero iff $A^\infty = 0$.*

Proof. If $A \sim 0$ then $UA \overset{\sim}{\twoheadrightarrow} 0$ is a deformation by 3.2-i) and 3.4. Therefore $UA^\infty = 0$, whence $A^\infty = 0$. The converse is trivial. $\square$

Remark 3.6. We can now see how far $\mathcal{PA}$ is from being a closed model category. Indeed: by 3.5 above, if $0 \to A$ factors as a weak equivalence followed by a fibration, then $A \sim 0$. On the other hand, if TV is a tensor algebra then clearly $TV^\infty \neq 0$, despite the fact that the map $0 \rightarrowtail TV$ has the LLP with respect to all fibrations.

4. Nil-homotopy v. Polinomial homotopy.

4.0. We want to compare our nil-homotopy relation with the more usual notion of homotopy defined via polynomial homotopies, as used for example to define Karoubi-Villamayor K-theory ([KV]). Given two homomorphisms $f, g \in \mathcal{PA}(A, B)$, we shall write $f \overset{pol}{\equiv} g$ if there exists a homomorphism $h : A \to B[t]$, with values in the polynomial ring on the commuting variable t, such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{h} & B[t] \\ {\scriptstyle (f,g)}\downarrow & \swarrow {\scriptstyle (\epsilon_0,\epsilon_1)} & \\ B \times B & & \end{array}$$

Here ϵ_i stands for "evaluation at i" ($i = 0, 1$). Note ϵ_1 is defined even if B is not unital, in which case $t \notin B[t]$; we set $\epsilon_1(\sum_{i=0}^n a_i t^i) = \sum_{i=0}^n a_i$. Also note that the map (ϵ_0, ϵ_1) is a fibration; a natural linear section is given by $(b_0, b_1) \to b_0 + b_1 t$. We observe that $\overset{pol}{\equiv}$ is a reflexive and symmetric relation, and that if $f \overset{pol}{\equiv} g$ then $fh \overset{pol}{\equiv} gh$ (whenever the composition makes sense). It follows that the equivalence relation $\overset{pol}{\sim}$ generated by $\overset{pol}{\equiv}$ is preserved by composition on both sides. Thus $B[t]$ plays the rôle the free path space of a topological space plays in ordinary topological homotopy. We showed in 1.2 above that nil-homotopy can be described in terms of higher fold cylinders. Analogously, polynomial homotopy (or simply pol-homotopy) can be defined in terms of higher free path spaces. Set $B^I = B[t]$, and define B^{I^n} inductively by the pull-back square:

$$\begin{array}{ccc} B^{I^n} & \longrightarrow & B^{I^{n-1}} \\ \downarrow & & \downarrow {\scriptstyle \epsilon_0^{n-1}} \end{array}$$

We write ϵ_0^n and ϵ_1^n for the composite maps $B^{I^n} \to B^I \overset{\epsilon_0}{\to} B$ and $B^{I^n} \to B^{I^{n-1}} \overset{\epsilon_1^{n-1}}{\to} B$. Thus $(\epsilon_0^n, \epsilon_1^n) : B^{I^n} \to B \times B$ is a fibration, and two maps $f_0, f_1 : A \to B$ are $\overset{pol}{\sim}$ iff there exist n and $h : A \to B^{I^n}$ such that $h\epsilon_i^n = f_i$.

We write $[\mathcal{PA}]^{pol}$ for the (strong) polynomial homotopy category, and call a map $f \in \mathcal{PA}(A, B)$ a polynomial equivalence if its class $[f]^{pol}$ is an isomorphism in $[\mathcal{PA}]^{pol}$. A typical polynomial equivalence is the projection $B = \bigoplus_{n=0}^{\infty} B_n \twoheadrightarrow B_0$ of a graded algebra or pro-algebra onto the part of degree zero, which is homotopy inverse to the inclusion $B_0 \hookrightarrow B$. A homotopy between the composite $B \twoheadrightarrow B_0 \hookrightarrow B$ and the identity map is given by $h : B \to B[t]$, $h(b) = bt^{deg(b)}$. Projections of the form $\bigoplus_{n=0}^{\infty} B_n \overset{pol}{\tilde{\twoheadrightarrow}} B_0$ shall be called *graded deformations*. For example power deformations are graded, because power algebras are. We also consider the category $[U\mathcal{PA}]^{pol}$ having as objects those of $\mathcal{PA}$ and as homomorphisms from A to B the homotopy classes $[UA, UB]^{pol}$. The relation between nil-homotopy and polynomial homotopy is established by the following:

Theorem 4.1.

(i) *The functor $U : \mathcal{PA} \to \mathcal{PAQ}$ carries pol-homotopic maps to pol-homotopic maps.*

(ii) *If $f, g : A \to B$ are nil-homotopic and if A is quasi-free, then they are also pol-homotopic.*

(iii) *The functor $\mathcal{PA} \to [\mathcal{PA}]^{pol}$ is the localization at the class of graded deformations, and the functor $\gamma' : \mathcal{PA} \to [U\mathcal{PA}]^{pol}$ is the localization at the union of the classes of nil-deformations and graded-deformations. There is a category equivalence $[\mathcal{PAQ}]^{pol} \approx [U\mathcal{PA}]^{pol}$.*

Proof. (i) It suffices to show that if $f, g : A \to B \in \mathcal{PA}$ satisfy $f \overset{pol}{\equiv} g$, then $Uf \overset{pol}{\equiv} Ug$. Let $H : A \to B[t]$ be a homotopy from f to g. Then $H' = H\pi^A : UA \to B[t]$ is a homotopy from $f\pi^A$ to $g\pi^A$ and Uf, Ug are liftings of $f\pi^A, g\pi^A$ to UB. Hence by [CQ-2, Lemma 9.1], we have $Uf \overset{pol}{\equiv} Ug$.

(ii)By Theorem 2.1, the map $CylA \to A$ is a power deformation retraction, hence a graded deformation. It follows that $\tilde{\partial}_0 \overset{pol}{\equiv} \tilde{\partial}_1$, and then $f \overset{pol}{\sim} g$.

(iii) The proof of the first assertion is analogous to the proof of the first assertion of Theorem 3.2-ii). Next, we must show that γ' inverts both nil and graded deformations and is initial among functors with such property. That γ' inverts graded deformations follows from (i), and that it inverts nil-deformations from (ii) and 3.2. If $F : \mathcal{PA} \to \mathcal{C}$ inverts both types of deformation, then $\tilde{F} : [U\mathcal{PA}]^{pol} \to \mathcal{C}$, $A \mapsto FA$, $[U\mathcal{PA}](A, B) \ni [f] \mapsto F\pi^B Ff(F\pi^A)^{-1}$ stisfies $\tilde{F}\gamma' = F$ and is the only such functor. □

5. Derived Functors.

Notations 5.0. Recall from [Q] that if $F : \mathcal{M} \to \mathcal{M}'$ is a functor between model categories, then the total (left) derived functor LF : $Ho\mathcal{M} \to Ho\mathcal{M}'$ is the (left) derived functor of the composite $\Gamma' F : \mathcal{C} \to Ho\mathcal{M}'$ with respect to the localization and a functor $F : \mathcal{PA} \to C$, we may (and do) consider the total left and right derived functors of F with respect to Γ and to $\gamma : \mathcal{PA} \to [U\mathcal{PA}]$ and $\gamma' : \mathcal{PA} \to [U\mathcal{PA}]^{pol}$.

Motivation 5.1. The following proposition generalizes a common procedure for deriving functors. As a motivation, recall the way crystalline (or infinitesimal) cohomology is defined for commutative algebras of finite type over a field of characteristic zero. Given an algebra A one chooses a smooth k-algebra R and an epimorphism $p : R \twoheadrightarrow A$ and defines $H^*_{cris}A$ as the cohomology of the (commutative) de Rham pro-complex Ω_{R/I^∞} where $I = \operatorname{Ker} p$ (cf. [H], [I]). The essential step in proving that H^*_{cris} is well-defined is the observation that if A above is quasi-free, then $\hat{R}_I$ is an algebra of power series over A, and that (continuous) H^*_{dR} satisfies the Poincaré Lemma: $H_{dR}(A) \cong H^*_{dR}A[[t]]$ ([H]). Here $H^*_{dR}A[[t]] \overset{def}{=} H^*(\varprojlim \Omega^*(A[t]/ < t^n >)$. Actually Poincaré Lemma is derived from the stronger fact that $\Omega^*(A) \xrightarrow{\sim} \Omega^* A[t]$ is a homotopy equivalence of pro-complexes ([H]). A non-commutative analogue of this construction was given by Cuntz and Quillen in [CQ2]. They showed that the non-commutative de Rham pro-complex XUA associated to an associative algebra A has the homotopy type of the periodic cyclic complex $\theta\Omega(A)$. In the framework of this paper, we interpret these results as saying that crystalline and periodic (co)-homology are respectively the derived functors of commutative and of non-commutative de Rham cohomology (see 5.4 below). The next proposition gives sufficient and necessary conditions so that when the construction above is applied to an arbitrary functor F, the result represents the left derived functor LF. We call this condition the Poincaré condition because it resembles the Poincaré Lemma quoted above. In both the commutative and non-commutative cases, one uses the fact that, in characteristic zero, de Rham cohomology is invariant under polynomial equivalence. Thus the Poincaré condition is automatic (see 5.3). However there are Poincaré functors which are not pol-homotopy invariant. For instance the Grothendieck group K_0 is nil-invariant (and therefore represents its derived functor) despite the fact that in general, $K_0(A[t]) \neq K_0(A)$.

Theorem-Definition 5.2. (Poincaré Functors) *Let $F : \mathcal{PA} \to \mathcal{C}$ and $\Gamma : \mathcal{C} \to \mathcal{C}'$ be functors. The following are equivalent:*

(i) *FU represents the derived functor of F with respect to Γ and to $\gamma : \mathcal{PA} \to [U\mathcal{PA}]$.*

(ii) *ΓFU is nil-invariant.*

(iii) *Given any commutative diagram:*

$$\begin{array}{ccc} R_0 & \xrightarrow{f} & R_1 \\ {\scriptstyle p_0}\downarrow & \swarrow{\scriptstyle p_1} & \\ A & & \end{array}$$

where p_i is a nil-deformation and R_i is quasi-free ($i = 0, 1$), the map $\Gamma F f$ is an isomorphism in $\mathcal{C}'$.

(iv) *Given any pro-vector space V and any quasi-free pro-algebra R, the map $\Gamma F(R \hookrightarrow P_R(V))$ is an isomorphism in $\mathcal{C}'$.*

(v) *Condition (iv) holds for $V = A$ and $R = UA$ ($A \in \mathcal{PA}$).*

We call F a Poincaré *functor if it satisfies the equivalent conditions above.*

Proof. We mimic the proof of the fact that a functor between model categories which preserves homotopy equivalences between cofibrant objects admits a derived functor ([Qui 1.4.1]).

(i) $\iff$ (ii) That (i)$\Rightarrow$(ii) is clear. Assume now ΓFU is nil-invariant, and let $\hat{F} : [U\mathcal{PA}] \to \mathcal{C}'$ be the induced functor. We have to prove that $\hat{F} = \mathrm{LF}$, i.e. that $\Gamma FU = \hat{F}\gamma$ is equipped with a natural map $\alpha : \Gamma FU \to \Gamma F$ such that if $\hat{G} : [U\mathcal{PA}] \to \mathcal{C}'$ is another functor and $\beta : G := \hat{G}\gamma \to F$ is a natural map then β factors uniquely through α. Let $\alpha = \Gamma F(\pi^A) : \Gamma FUA \to \Gamma FA$ and set $\overline{\beta} = (\beta U)(G\pi^A)^{-1} : GA \to \Gamma FUA$. Then $\overline{\beta}$ satisfies $\beta = \alpha\overline{\beta}$ and is the only such map.

(ii)$\Rightarrow$(iii) The map f is a strong equivalence because each p_i is a deformation and R_i is quasi-free. Therefore ΓFUf is an isomorphism. On the other hand we have $\pi^{R_1}Uf = f\pi^{R_0}$ where each π^{R_i} is a deformation retraction; thus it is enough to show that each $\Gamma F\pi^{R_i}$ is an isomorphism. But if $\iota_i : R_i \to UR_i$ is a right inverse for π^{R_i}, then $\iota_i\pi^{R_i} : UR_i \to UR_i$ is a nil-equivalence, whence the proof reduces to showing that if $g : UB \to UB$ is a strong equivalence, then ΓFg is an isomorphism. We know by hypothesis that ΓFUg is an isomorphism, and we have $\pi^{UB}FUg = g\pi^{UB}$. But $\Gamma F\pi^{UB}$ must be an isomorphism, because $\Gamma FU\pi^B$ is, and both π^{UB} and $U\pi^B$ have a right inverse in common; namely the map induced by $T\rho : TB \to T^2B$.

(iii)$\Rightarrow$(iv) Let $r : P_R(V) \tilde{\twoheadrightarrow} R$ be the projection map. Then r is a deformation and is a retraction of the canonical inclusion. Thus (iv) is a particular case of (iii), with $R_0 = A = R$ and $R_1 = P_R(V)$.

(v) is logically weaker than (iv).

(v)$\Rightarrow$(ii) By virtue of Example 2.2, if (v) holds, then ΓF sends homotopy equivalences $UA \to UB$ to isomorphisms, whence ΓFU sends weak nil-equivalences to isomorphisms. □

Corollary 5.3. *If F preserves either nil-deformation retractions or graded deformations, then it is Poincaré. In the latter case FU represents the left derived functor with respect to both $\gamma : \mathcal{PA} \to [U\mathcal{PA}]$ and to $\gamma' : \mathcal{PA} \to [U\mathcal{PA}]^{pol}$.*

Proof. That F is Poincaré means that its restriction to $\mathcal{PAQ}$ preserves nil homotopy (cf. 3.2). Such is the case if F preserves either nil-homotopy or, by 4.1-ii), pol-homotopy of arbitrary pro-algebras. The same argument as in the proof of the theorem shows that, in the latter case, FU also represents the derived functor with respect to γ'. □

Corollary 5.4. *Let $X : \mathcal{PA} \to \mathcal{PS} :=$((Pro-Supercomplexes)) be the functor which assigns to every pro-algebra A the de Rham pro-super complex XA of 2.4 above. Let $\Gamma : \mathcal{PS} \to Ho\mathcal{PS}$ be the localization at the class of homotopy equivalences and let γ and γ' be as above. If the ground field k has $char(k) = 0$ then the functor X is Poincaré (relative to Γ and to γ), and its left derived functor with respect to both γ and γ' is represented by the periodic cyclic pro-complex $\theta\Omega$ of 2.4 above.*

Proof. In characteristic zero, the functor X preserves polynomial homotopy (e.g. by [Kas], or by [CQ2&3]), whence it is Poincaré and XU represents LX (by 5.3).

On the other hand, in any characteristic, XUA is homotopy equivalent to $\theta\Omega UA$, because UA is quasi-free (e.g. by [P]). In characteristic zero, by virtue of Goodwillie's theorem ([G1], [CQ2]), $\theta\Omega UA$ has the homotopy type of $\theta\Omega A$. Summing up, if $char(k) = 0$ then $FU \approx \theta\Omega$ represents LX. □

Remark 5.5. In characteristic $p > 0$, the lemma above fails to hold. Indeed, if X were Poincaré then –by 5.2–the homology of the periodic cyclic complex $CP(P_0(k)) = Hom(Xk, XP_0(k))$ should be zero, which –as a straightforward calculation shows– it is not. See also Lemma 6.6 below.

6. The derived functors of rational K-theory and Cyclic Homology.

The purpose of this section is to show that the functor which assigns to every $\mathbb{Q}$-pro-algebra its rational K-theory space is (almost) a Poincaré functor, and that its left derived functor is essentially the fiber of the Chern character with values in negative cyclic homology. See Theorem 6.2 below for a precise statement. The proof of Theorem 6.2 has two main ingredients. The first ingredient is Goodwillie's isomorphism

$$K^{\mathbb{Q}}_*(A, I) \cong HN_*(A, I) \tag{7}$$

between the relative rational K-group of a nilpotent ideal and its analogue in negative cyclic homology [G2]. Actually Goodwillie's result is stated and proven for unital algebras; we shall use an adaptation of this that holds for arbitrary pro-algebras, which is obtained in 6.1 below. This adaptation says that the relative K-group of an infinitesimal deformation is isomorphic to the corresponding negative cyclic homology group, and essentially reduces the question of the Poincaréness of K to that of HN. The second ingredient is the calculation of relative HN for a power deformation. This calculation is carried out without any hypothesis on the characteristic of k (Proposition 6.8).

6.0. The derived functor of rational K-theory.

We use the following model for the rational K-theory of a unital algebra or ring:

$$K^{\mathbb{Q}}(A) := \mathbb{Q}_\infty BGlA$$

Here Gl is the general linear group, and B denotes the simplicial set associated to the category of Gl. Thus for us $K^{\mathbb{Q}}(A)$ is a simplicial set; note that its homotopy groups are precisely Quillen's rational K-groups. For general, non-necessarily unital algebras over the ground field k we set:

$$K^{\mathbb{Q}}(A) := \text{fiber}(K^{\mathbb{Q}}(\tilde{A}) \to K^{\mathbb{Q}}(k))$$

Thus in general $K^{\mathbb{Q}}(A)$ depends on k, and coincides with the usual rational K-group if A is unital or more generally if it is excisive for $K^{\mathbb{Q}}$. Now we extend this definition to the case of pro-algebras, by taking homotopy inverse limits, as follows. If $A = \{A_\lambda : \lambda \in \Lambda\}$ we put:

$$K^{\mathbb{Q}}(A) := \underset{\overleftarrow{\Lambda}}{\mathrm{holim}}\ K^{\mathbb{Q}}(A_\lambda)$$

Next we generalize Goodwillie's isomorphism to the pro-algebra case; we assume throughout that $char k = 0$. Recall from [G2] that the isomorphism (7) is induced by a natural Chern character $K_*^{\mathbb{Q}}(A) \longrightarrow HN_*(A) := HN_*(A/k)$ which is defined for every unital algebra A. By [W] this character may be realized as a simplicial map $ch : K^{\mathbb{Q}}(A) \longrightarrow SN(A)$, where SN is constructed as follows. First truncate the total chain complex for negative cyclic homology to obtain a complex CN^t such that $H_n(CN^t) = HN_n(A)$ $(n \geq 1)$ and $H_n(CN^t) = 0$ if $n \leq 0$. Next define SN as the result of applying the Dold-Kan correspondence to CN^t. Hence SN is a connected, fibrant simplicial set with $\pi_n SN(A) = HN_n A$ $(n \geq 1)$, and the isomorphism (7) says that the map between fibers $K^{\mathbb{Q}}(A, I) \longrightarrow SN(A, I)$ is a weak equivalence. If now A is any –non necessarily unital– algebra, and $I \lhd A$ is a nilpotent ideal, then we have weak equivalences:

$$K^{\mathbb{Q}}(A, I) \cong K^{\mathbb{Q}}(\tilde{A}, I) \overset{\overset{ch}{\sim}}{\longrightarrow} SN(\tilde{A}, I) \cong SN(A, I) \tag{8}$$

He have thus extended (7) to non-unital algebras. If now $A = \{A_\lambda : \lambda \in \Lambda\}$ is a pro-algebra, we set $SN(A) = \underset{\overleftarrow{\Lambda}}{\mathrm{holim}}\ SN(A_\lambda)$, and write $ch : K^{\mathbb{Q}}(A/k) \longrightarrow SN(A)$ for the map induced by passage to $\overleftarrow{\mathrm{holim}}$. As $\overleftarrow{\mathrm{holim}}$ preserves fibers, fibrations and weak equivalences of fibrant s. sets, (cf. [BK]) it follows that the weak equivalences (8) hold for arbitrary deformations and pro-algebras. We have proven:

Lemma 6.1. *With the notations and definitions of 6.0 above, there is a natural map of fibrant simplicial sets $ch : K^{\mathbb{Q}}(A) \longrightarrow SN(A)$ which is defined for all pro-algebras A, and coincides with Goodwillie's character in the case of unital algebras. If $f : A \tilde{\twoheadrightarrow} B$ is a deformation, then the induced map $K^{\mathbb{Q}}(f) \approx SN(f)$ is a weak equivalence.*

Proof. See the discussion above. □

6.1.1. In particular the lemma above holds if f is a power deformation of quasi-free pro-algebras, whence –by Theorem 5.2-iv)– $K^{\mathbb{Q}}$ will be Poincaré iff SN is. In the next subsection we compute the homotopy groups of $SN(f)$ for power deformations of quasi-free pro-algebras and show that these are all zero except for π_1, which is nonzero. Thus the simplicial set SN' obtained from the complex CN by truncating in degree 2, so that $\pi_n(SN') = HN_n$ if $n \geq 2$ and zero otherwise is a Poincaré functor; further, its derived functor is null-homotopic, cf. 6.9 below. It follows that the K-theory space obtained by the same process as above using the elementary group instead of the general linear group is a Poincaré functor. Explicitly, the functor:

$$KE^{\mathbb{Q}}(A) := \underset{\overleftarrow{\Lambda}}{\mathrm{holim}}\ \ \mathrm{fiber}(\mathbb{Q}_\infty(E\tilde{A} \longrightarrow \mathbb{Q}_\infty Ek) \tag{9}$$

is Poincaré.

Theorem 6.2. *(*The derived functor of K-theory*)*

The functor $A \mapsto K^{\mathbb{Q}}(A)$ is not Poincaré. However, the functor $A \mapsto KE^{\mathbb{Q}}(A)$ of (9) above is, and therefore it has a left derived functor $LKE^{\mathbb{Q}}$. Set $LK_n^{\mathbb{Q}}(A) := \pi_n LKE^{\mathbb{Q}}$; then there is an exact sequence:

$$\begin{gathered}\ldots \to HN_{n+1}A \to LK_n^{\mathbb{Q}}(A) \to K_n^{\mathbb{Q}}(A) \to H_n(A) \to \\ \ldots \to HN_3(A) \to LK_2^{\mathbb{Q}}(A) \to K_2^{\mathbb{Q}}(A) \to HN_2(A)\end{gathered}$$

Proof. The first two assertions follow from the discussion above and 6.9 below. To prove the third assertion consider the exact sequence of K-groups associated with the universal deformation $\pi^A : UA \tilde{\twoheadrightarrow} A$. Then $LK_n^{\mathbb{Q}}(A) = K_n^{\mathbb{Q}}(UA)$ $(n \geq 2)$ (by 5.2) and $K_n(\pi^A) \cong HN_n(\pi^A)$ $(n \geq 1)$ (by 6.1). Because UA is quasi-free, $HN_n(UA) = 0$ for $n \geq 2$, and therefore $HN_n(\pi^A) \cong HN_{n+1}(A)$, for $n \geq 2$. This proves that the sequence is exact at $LK_2^{\mathbb{Q}}(A)$ and to the left. By the same argument, the natural map $HN_2(A) \hookrightarrow HN_1(\pi^A)$ is injective, whence $K_2^{\mathbb{Q}}(A) \to K_1^{\mathbb{Q}}(\pi^A)$ factors through ch_2. It follows that the sequence is exact also at $K_2^{\mathbb{Q}}(A)$, completing the proof. □

6.3. The derived functor of negative cyclic homology.

The purpose of this subsection is to compute the homotopy type of the relative space $SN(P_A(V) \to A)$ associated with a power deformation retraction of a quasi-free pro-algebra A over a field. We do not make any assumptions with regards to *chark*. The calculation uses two lemmas (6.4 and 6.6) which show the patologies that appear in characteristic $p > 0$. In particular, 6.6 gives a different proof of the fact that the de Rham pro-complex X is Poincaré iff $chark = 0$. In Lemma 6.4 we give a formula for the homotopy type of the X pro-complex of a free product. Recall that if A and B are algebras, then there is an isomorphism of vector spaces:

$$A * B = A \oplus B \oplus T(A \otimes B) \oplus T(B \otimes A) \oplus T(A \otimes B) \otimes A \oplus T(B \otimes A) \otimes B$$

In particular, the natural inclusion $T(A \otimes B) \hookrightarrow A * B$ is an algebra homomorphism. Putting this map together with the natural inclusions $A \hookrightarrow A * B$ and $B \hookrightarrow A * B$, we get map of super complexes:

$$XA \oplus XB \oplus XT(A \otimes B) \overset{\iota}{\hookrightarrow} X(A * B)$$

As all the maps in the above discussion are natural, all of this generalizes immediately to the case of pro-algebras. The following lemma may be regarded as a particular, easy case of [FT, 3.2.1]. We give an independent proof in this particular case.

Lemma 6.4. *(*Compare [FT, 3.2.1]*) Let A, B be pro-algebras. There exist a natural map of pro-mixed complexes: $\pi : X(A*B) \to XA \oplus XB \oplus XT(A\otimes B)$ such that $\pi\iota = 1$ and a natural homotopy $h : 1 \sim \iota\pi$.*

Proof. By naturality, we may assume A and B are algebras. The map $A*B \to A\times B$, $a \mapsto (a,0)$, $b \mapsto (0,b)$ induces a retraction $XA*B \to XA \oplus XB$. Write $XA*B = XA\oplus XB \oplus Y$. Thus $Y_0 = U \oplus V := T(A\otimes B)\oplus T(B\otimes A)\oplus T(A\otimes B)\otimes A \oplus T(B\otimes A)\otimes B$. where U is the sum of the first two terms and V is the sum of the last two. Further, one checks that:

$$Y_1 \cong T(A\otimes B)dA \oplus T(B\otimes A)dB \oplus T\widetilde{(A\otimes B)}\otimes AdB \oplus T\widetilde{(B\otimes A)}\otimes BdA \cong Y_0$$

Consider the maps: $\alpha : U \to U$, $x_0y_0\dots x_ny_n \mapsto y_nx_0\dots y_{n-1}x_n$, and $\mu : V \to U$, $x_0y_0\dots x_ny_nx \mapsto xx_0y_0\dots x_ny_n$. Under the identifications above, the map ι_1 sends $x \in T(A\otimes B) \cong \Omega^1T(A\otimes B)_\natural$ onto $x + \alpha x \in U$. Define a mixed complex map $\pi : Y \to XT(A\otimes B)$, $\pi_0(u_0,u_1,v_0,v_1) = x_0 + \mu y_0 + \alpha x_1 + \alpha\mu y_1$, $\pi_1(u_0,u_1,v_0,v_1) = u_0$, $u_i \in U$, $v_i \in V$; 0 denotes the alphabetical order, and 1 denotes the inverse order. One checks that $\pi\iota = 1$. Further the map $h : Y_0 \to Y_1$, $h(x_0,x_1,y_0,y_1) = (0, x_1 + \mu y_1, y_0, y_1)$ verifies $\iota_1\pi_1 = hb$ and $\iota_0\pi_0 = bh$. □

Corollary 6.5. *(*Compare [CQ-3, 7.3]*) If $char k = 0$, then there is a homotopy equivalence of supercomplexes: $X(A*B) \approx XA \oplus XB$.*

Proof. Immediate from the well-known calculation of the cyclic homology of a tensor algebra (e.g. [FT, 2.3.1]). □

Lemma 6.6. *Let A be an algebra, V a vector space and $P_A(V)$ the power pro-algebra. Give TV and $T(A\otimes TV)$ a gradation by setting $deg(a) = 0$ and $deg(v) = 1$ ($a \in A$, $v \in V$). Then there exists a natural homotopy equivalence of pro-mixed complexes*

$$XP_A(V) \approx XA \oplus \{X^{deg\le n}TV \oplus X^{deg\le n}T(A\otimes TV) : n \ge 1\}$$

Proof. By definition the power pro-algebra $P_A(V)$ is graded, and the gradation is given by the prescription of the lemma. This gradation is reflected by the X-complex; we have a degree decomposition:

$$C := X(P_A(V)) = \{\oplus_{i=0}^{2n} X^{deg=i}(P_A(V)_{n+1})\}$$

We observe that for $i \le n$ the direct summand subcomplexes corresponding to degree i in the X complex of $P_A(V)_n = A*TV/<V>^n$ and of $A*TV$ are isomorphic. Further the pro-complex $D := \{\oplus_{i\ge n+1}^{2n} C_{n+1}^{deg=i}\}$ is the zero pro-complex, as the structure maps $\tau_{n,2n}^D$ are all zero. Therefore C is isomorphic to the pro-complex $\{X^{deg\le n}(A*TV)\}$. Now the lemma is immediate from 6.4. □

Remark 6.7. As the homotopy equivalence in the lemma above is natural, it extends automatically to pro-algebras. Since on the other hand the Hochschild, cyclic and related homology groups of a tensor algebra are well known, one could conceivably write down explicitly all the relative pro-homology groups for the projection $P_A(V) \longrightarrow A$ in any characteristic. In the next proposition we calculate the negative cyclic group for the particular case when A is an algebra and V is a vector space. Since in characteristic zero $HH_0(TV) \cong HH_1(TV)$, our calculation can also be derived from [G2] in this particular case.

Proposition 6.8. *Let k be a field of characteristic $p \geq 0$, and let A be a quasi-free k-pro-algebra. If V is a pro-vector space and $f : P_A(V) \longrightarrow A$ is the natural projection, then $SN(f)$ is an Eilenberg-Maclane space $E(\Upsilon(A,V),1)$, where $\Upsilon(A,V)$ is an abelian group which depends functorially on A and V. Explicitly if A is a quasi-free algebra and V is a vector space, then $\Upsilon(A,V) = \Pi_{n=0}^{\infty}(C_n \oplus \bigoplus_{r\geq 0} D_{n,r})$ is the infinite product of the following co-invariant spaces:*

$$C_n = (T^nV)_{\mathbb{Z}/n} \qquad \textit{and}$$

$$D_{n,r} = (\bigoplus_{i_1+\dots+i_r=n} A \otimes T^{i_1}V \otimes \cdots \otimes A \otimes T^{i_r}V)_{\mathbb{Z}/r}$$

Here $\mathbb{Z}/n$ and $\mathbb{Z}/r$ act by $v_1 \otimes \cdots \otimes v_n \mapsto v_n \otimes v_1 \otimes \cdots \otimes v_{n-1}$ and by $a_1 \otimes x_1 \otimes \cdots \otimes a_r \otimes x_r \mapsto a_r \otimes x_r \otimes a_1 \otimes x_1 \otimes \cdots \otimes a_{r-1} \otimes x_{r-1}$

Proof. By the cofinality theorem for $\operatorname{holim}_{\leftarrow}$ ([BK]), we may assume A and V are indexed by $\mathbb{N}$. Thus for $n \geq 1$ we have an exact sequence:

$$0 \longrightarrow \varprojlim{}^1 HN_n(f_i) \longrightarrow \pi_n(SN(f)) \longrightarrow \varprojlim H_n(f_i) \longrightarrow 0 \tag{10}$$

Since $P_A(V)$ is quasi-free, the inverse system $\{HN_n(f_i) : i \in \mathbb{N}\}$ is isomorphic to the inverse system $\{HN_n(X(f)) : i \in \mathbb{N}\}$ (here X is regarded as a mixed complex). Thus both ends in the exact sequence above are zero for $n \geq 2$. Furthermore $SN(f)$ is connected by definition; this concludes the proof of the first assertion. Assume now A is a quasi-free algebra and V is a vector space. It follows form 6.6 that we have an isomorphism of pro-vector spaces

$$\begin{aligned}\{HN_1(f_n) : n \in \mathbb{N}\} \cong \{&\bigoplus_{i=0}^{n} T^iV_{\mathbb{Z}_i}\}\oplus \\ &\bigoplus_{i=0}^{n}\bigoplus_{r\geq 0}\bigoplus_{j_1+\dots j_r=i}(A \otimes T^{i_1}V \otimes \cdots \otimes A \otimes V^{i_r})_{\mathbb{Z}/r} : n \in \mathbb{N}\}\end{aligned} \tag{11}$$

As every map in the pro-vector space of the right hand of (11) is a surjection, the $\varprojlim{}^1$ term in (10) is zero, and the second assertion of the proposition follows. $\square$

Corollary 6.9. *The functor $A \mapsto SN(A)$ is not Poincaré, regardless of the characteristic of k. The functor $A \mapsto SN'(A)$ of 6.1.1 above is Poincaré (in any characteristic) and its left derived functor is null homotopic.* $\square$

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Guillermo Cortiñas, Departamento de Matemática, Facultad de Ciencias Exactas, Calles 50 y 115, (1900) La Plata, Argentina.

E-mail address: `willie@mate.unlp.edu.ar`