SOME CYCLIC COVERS OF COMPLEMENTS OF ARRANGEMENTS

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Abstract. Motivated by the Milnor fiber of a central arrangement, we study the cohomology of a family of cyclic covers of the complement of an arbitrary arrangement. We give an explicit proof of the polynomial periodicity of the Betti numbers of the members of this family of cyclic covers.

1. Introduction

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^\ell$, with complement $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H$. The cohomology of $M(\mathcal{A})$ with coefficients in a local system arises in a number of applications, both outside and inside arrangement theory. Included among the former are the Aomoto-Gelfand theory of multivariable hypergeometric integrals [AK, Ge], and the representation theory of Lie algebras and quantum groups and solutions of the Knizhnik-Zamolodchikov differential equation in conformal field theory [Va]. This note is motivated by one of the latter applications, the cohomology of the Milnor fiber of a central arrangement.

Let $\mathcal{C}$ be a central arrangement of hyperplanes in $\mathbb{C}^{\ell+1}$, an arrangement for which each hyperplane $H \in \mathcal{C}$ contains the origin. For each such hyperplane, let α_H be a linear form with kernel H. Then $Q = Q(\mathcal{C}) = \prod_{H\in\mathcal{C}} \alpha_H$ is a defining polynomial for the arrangement $\mathcal{C}$, and is homogeneous of degree equal to the cardinality of $\mathcal{C}$. The complement $M(\mathcal{C}) = \mathbb{C}^{\ell+1} \setminus Q^{-1}(0)$ may be realized as the total space of the global Milnor fibration

$$F(\mathcal{C}) \longrightarrow M(\mathcal{C}) \xrightarrow{Q} \mathbb{C}^*,$$

where $F(\mathcal{C}) = Q^{-1}(1)$ is the Milnor fiber of Q, see [Mi]. We shall refer to $F(\mathcal{C})$ as the Milnor fiber of $\mathcal{C}$, and write $F = F(\mathcal{C})$ when the arrangement $\mathcal{C}$ is understood.

Suppose that the cardinality of $\mathcal{C}$ is n and let $(x_0, x_1, \ldots, x_\ell)$ be a choice of coordinates on $\mathbb{C}^{\ell+1}$. The geometric monodromy, $h : F \to F$ of the Milnor fibration is given by $h(x_0, \ldots, x_\ell) = (\xi_n x_0, \ldots, \xi_n x_\ell)$, where $\xi_n = \exp(2\pi\,\mathrm{i}/n)$. Since h has finite order n, the algebraic monodromy $h^* : H^q(F;\mathbb{C}) \to H^q(F;\mathbb{C})$ is diagonalizable and the eigenvalues of h^* belong to the set of n-th roots of unity. Denote the cohomology eigenspace of ξ_n^k by $H^q(F;\mathbb{C})_k$ and write $b_q(F)_k = \dim_{\mathbb{C}} H^q(F;\mathbb{C})_k$.

It is known [CS1] that these cohomology eigenspaces are isomorphic to the cohomology of the complement of a *decone* of $\mathcal{C}$ with coefficients in certain complex rank one local systems. See Section 2 for a summary of these results, and see [OT] as a general reference on arrangements. Let $\mathcal{A}$ be a decone of $\mathcal{C}$, an affine arrangement in $\mathbb{C}^\ell$, and denote the rank one local systems arising in the context of the Milnor fiber by $\mathcal{L}_k$, $1 \le k \le n$. Then $H^*(F(\mathcal{C});\mathbb{C})_k \simeq H^*(M(\mathcal{A});\mathcal{L}_k)$ for each k. Furthermore, the local systems $\mathcal{L}_k$ are *rational* in the sense of [CO]. The results of this work, in the context

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of the Milnor fiber problem, yield combinatorial bounds

$$\dim_{\mathbb{C}} H^q(A_{\mathbb{C}}(\mathcal{A}), a\wedge) \leq \dim_{\mathbb{C}} H^q(F;\mathbb{C})_k \leq \operatorname{rank}_{\mathbb{Z}_r} H^q(A_{\mathbb{Z}_r}(\mathcal{A}), \bar{a}\wedge),$$

where $r = n/(k,n)$ and $A_R(\mathcal{A})$ is the Orlik-Solomon algebra of $\mathcal{A}$ with coefficients in the ring R equipped with appropriate differential. See Section 5 for details. The lower bounds are well known. The upper bounds are new.

The local system $\mathcal{L}_n$ is trivial, and thus corresponds to the constant coefficient cohomology of $M(\mathcal{A})$. This is well understood in terms of the Orlik-Solomon algebra. While pursuing the remaining cases, we were led to a family of cyclic covers of $M(\mathcal{A})$, which includes the Milnor fiber $F(\mathcal{C})$. In this note, we show how a number of known results on the Milnor fiber extend naturally to all members of this family of covers, and give an explicit and elementary proof of the polynomial periodicity of the Betti numbers of the members of this family.

2. Milnor Fibration

Recall from the Introduction that $\mathcal{C}$ is a central arrangement of n hyperplanes in $\mathbb{C}^{\ell+1}$, with coordinates $(x_0, x_1, \ldots, x_\ell)$. Associated with $\mathcal{C}$, we have the defining polynomial $Q = Q(\mathcal{C})$, the complement $M(\mathcal{C}) = \mathbb{C}^{\ell+1} \setminus Q^{-1}(0)$, and the Milnor fiber $F = F(\mathcal{C}) = Q^{-1}(1)$. The geometric monodromy $h : F \to F$ of the Milnor fibration has order $n = |\mathcal{C}|$. The cyclic group $\mathbb{Z}_n$ generated by the geometric monodromy h acts freely on F. This free action gives rise to a regular n-fold covering $F \to F/(\mathbb{Z}_n)$.

Consider the Hopf bundle $\mathbb{C}^{\ell+1} \setminus \{0\} \to \mathbb{CP}^\ell$ with projection map $(x_0, x_1, \ldots, x_\ell) \mapsto (x_0 : x_1 : \cdots : x_\ell)$ and fiber $\mathbb{C}^*$. Let $p : M(\mathcal{C}) \to M^*$ denote the restriction of this projection to M, where $M^* = p(M)$. The restriction $p_F : F \to M^*$ of the Hopf bundle to the Milnor fiber is the orbit map of the free action of the geometric monodromy h on F and we therefore have $F/(\mathbb{Z}_n) \cong M^*$. These spaces and maps fit together with the Milnor fibration in the following diagram:

$$\begin{array}{ccccc}
\mathbb{Z}_n & \longrightarrow & F & \xrightarrow{p_F} & F/\mathbb{Z}_n \\
\downarrow & & \downarrow & & \downarrow \cong \\
\mathbb{C}^* & \longrightarrow & M(\mathcal{C}) & \xrightarrow{p} & M^* \\
\downarrow & & \downarrow Q & & \\
\mathbb{C}^*/\mathbb{Z}_n & \xrightarrow{\cong} & \mathbb{C}^* & &
\end{array}$$

Note that M^* is the complement of the projective hypersurface defined by the homogeneous polynomial Q. Thus it is the complement of the projective quotient of the arrangement $\mathcal{C}$. If we designate one of its hyperplanes, H_∞, the hyperplane at infinity, the remaining arrangement is called the decone of $\mathcal{C}$ with respect to H_∞. We call this ℓ-arrangement $\mathcal{A}$ here and observe that $M(\mathcal{A}) = M^*$ is independent of the choice of H_∞ and $|\mathcal{A}| = n - 1$. We shall assume that $\mathcal{A}$ contains ℓ linearly independent hyperplanes.

The cohomology groups of the Milnor fiber have been studied extensively, see for instance [OR, CS1, Ma, CS2, De]. We summarize some known results from [CS1]. Since h has finite order n, the algebraic monodromy $h^* : H^q(F;\mathbb{C}) \to H^q(F;\mathbb{C})$ is diagonalizable and the eigenvalues of h^* belong to the set of n-th roots of unity. Denote the cohomology eigenspace of $\xi_n^k = \exp(2\pi\,\mathrm{i}\,k/n)$ by $H^q(F;\mathbb{C})_k$, and denote the characteristic polynomial of $h^* : H^q(F;\mathbb{C}) \to H^q(F;\mathbb{C})$ by $\Delta_q(t) = \det(t \cdot h^* - \mathrm{id})$.

Proposition 2.1 ([CS1, 1.1])**.** *Let ξ_n^j and ξ_n^k be two n-th roots of unity which generate the same cyclic subgroup of $\mathbb{Z}_n = \langle \xi_n \rangle$. Then, for each q, the cohomology eigenspaces $H^q(F;\mathbb{C})_j$ and $H^q(F;\mathbb{C})_k$ are isomorphic.*

Corollary 2.2. *For each q, $0 \le q \le \ell$, there are nonnegative integers $d_{k,q}$ so that*

$$\Delta_q(t) = \prod_{k|n} \Phi_k(t)^{d_{k,q}},$$

where $\Phi_k(t)$ denotes the k-th cyclotomic polynomial.

Theorem 2.3 ([CS1, 1.6]). *Define the rank one local system $\mathcal{L}_k$ on $M(\mathcal{A})$ by the representation $\tau_k : \pi_1(M(\mathcal{A})) \to \mathbb{C}^*$ given by $\gamma_H \mapsto \xi_n^k$ for each meridian loop γ_H about the hyperplane $H \in \mathcal{A}$. Then, for each k, $1 \le k \le n$, we have*

$$H^*(F(\mathcal{C}); \mathbb{C})_k \simeq H^*(M(\mathcal{A}); \mathcal{L}_k).$$

In light of this result, it is natural to study the local system on $M(\mathcal{A})$ induced by the representation given by $\gamma_H \mapsto \xi_m^k$ for arbitrary m. In subsequent sections, we focus on the context in which these local systems arise.

3. Cyclic Covers

The realization of the Milnor fiber of a central arrangement as a cover of the complement of a decone in the previous section motivates the following construction.

Let $\mathcal{A}$ be an affine arrangement in $\mathbb{C}^\ell$, with coordinates $\mathbf{x} = (x_1, \dots, x_\ell)$. Associated with $\mathcal{A}$, we have the defining polynomial $f = Q(\mathcal{A})$, and the complement $M(\mathcal{A}) = \mathbb{C}^\ell \setminus f^{-1}(0)$. For each positive integer m, let $g_m : \mathbb{C}^* \to \mathbb{C}^*$ denote the cyclic m-fold covering defined by $g_m(z) = z^m$, and let $p_m : X_m(\mathcal{A}) \to M(\mathcal{A})$ denote the pullback of g_m along the map $f : M(\mathcal{A}) \to \mathbb{C}^*$, where

$$X_m(\mathcal{A}) = \{(\mathbf{x}, z) \in M(\mathcal{A}) \times \mathbb{C}^* \mid f(\mathbf{x}) = z^m\}.$$

This family of cyclic covers of $M(\mathcal{A})$ generalizes the Milnor fiber in a number of ways which we now pursue. First, we have the following.

Proposition 3.1. *Let $\mathcal{C}$ be a central arrangement of n hyperplanes in $\mathbb{C}^{\ell+1}$, with defining polynomial $Q = Q(\mathcal{C})$ and Milnor fiber $F(\mathcal{C})$. If $\mathcal{A}$ is a decone of $\mathcal{C}$, then the covering spaces $p_{{}_F} : F(\mathcal{C}) \to M(\mathcal{A})$ and $p_n : X_n(\mathcal{A}) \to M(\mathcal{A})$ are equivalent.*

Proof. The relation between the defining polynomials f of $\mathcal{A}$ and Q of $\mathcal{C}$ is given by $Q = x_0^n \cdot f(x_1/x_0, \dots, x_n/x_0)$, and $F(\mathcal{C}) = Q^{-1}(1)$. Using this, it is readily checked that the map $X_n(\mathcal{A}) \to F(\mathcal{C})$ defined by $(x_1, \dots, x_\ell, z) \mapsto (1/z, x_1/z, \dots, x_\ell/z)$ is a homeomorphism inducing an equivalence of covering spaces. □

The characteristic homomorphism $\Phi : \pi_1(M(\mathcal{A})) \to \mathbb{Z}_n$ of the covering $p_{{}_F} : F(\mathcal{C}) \to M(\mathcal{A})$ was identified in [CS1, 1.2]. It is given by $\Phi(\gamma_H) = g_n$, where γ_H is a meridian loop about the hyperplane $H \in \mathcal{A}$ and g_n is a fixed generator of $\mathbb{Z}_n$. A straightforward generalization of the proof of this fact from [CS1, 1.2] yields

Proposition 3.2. *The characteristic homomorphism $\Phi_m : \pi_1(M(\mathcal{A})) \to \mathbb{Z}_m$ of the covering $p_m : X_m(\mathcal{A}) \to M(\mathcal{A})$ is given by $\Phi_m(\gamma_H) = g_m$ for a fixed generator g_m of $\mathbb{Z}_m$ and meridian loops γ_H about the hyperplanes H of $\mathcal{A}$.*

The covering spaces $X_m(\mathcal{A})$ fit together nicely in the sense of the following.

Proposition 3.3. *If $m = k \cdot r$, then the map $\mathbb{C}^\ell \times \mathbb{C}^* \to \mathbb{C}^\ell \times \mathbb{C}^*$ defined by $(\mathbf{x}, z) \mapsto (\mathbf{x}, z^r)$ induces a cyclic r-fold covering $p_{m,k} : X_m(\mathcal{A}) \to X_k(\mathcal{A})$.*

Proof. Let $X_{k,r}(\mathcal{A}) \to X_k(\mathcal{A})$ denote the pullback of $g_r : \mathbb{C}^* \to \mathbb{C}^*$ along the map $X_k(\mathcal{A}) \to \mathbb{C}^*$ defined by $(x_1, \dots, x_\ell, z) \mapsto z$, with

$$X_{k,r}(\mathcal{A}) = \{(\mathbf{x}, z, w) \in M(\mathcal{A}) \times \mathbb{C}^* \times \mathbb{C}^* \mid f(\mathbf{x}) = z^k \text{ and } z = w^r\}.$$

It is then readily checked that the map $X_m(\mathcal{A}) \to X_{k,r}(\mathcal{A})$ defined by $(\mathbf{x}, z) \mapsto (\mathbf{x}, z^r, z)$ is a homeomorphism compatible with the projection maps. □

Remark 3.4. The space $X_m(\mathcal{A})$ admits a self-map $h_m : X_m(\mathcal{A}) \to X_m(\mathcal{A})$ defined by $h_m(\mathbf{x}, z) = (\mathbf{x}, \xi_m^{-1} \cdot z)$, where $\xi_m = \exp(2\pi\,\mathrm{i}/m)$. In the case $m = n$ of the Milnor fiber, the map $h_n : X_n(\mathcal{A}) \to X_n(\mathcal{A})$ corresponds to the geometric monodromy $h : F(\mathcal{C}) \to F(\mathcal{C})$ under the equivalence of covering spaces exhibited in the proof of Proposition 3.1. For arbitrary m, the "monodromy" map h_m generates a cyclic group of order m, which acts freely on $X_m(\mathcal{A})$. The resulting regular m-fold covering $X_m(\mathcal{A}) \to X_m(\mathcal{A})/\langle h_m \rangle$ clearly coincides with $p_m : X_m(\mathcal{A}) \to M(\mathcal{A})$. More generally, for $m = k \cdot r$ composite, the map h_m^k generates a cyclic group of order r, which also acts freely on $X_m(\mathcal{A})$, and the covers $X_m(\mathcal{A}) \to X_m(\mathcal{A})/\langle h_m^k \rangle$ and $p_{m,k} : X_m(\mathcal{A}) \to X_k(\mathcal{A})$ coincide.

4. Cohomology

We now study the cohomology of the covering spaces $X_m(\mathcal{A})$. Fix a basepoint $\mathbf{x}_0 \in M(\mathcal{A})$. From the Leray-Serre spectral sequence of the fibration $p_m : X_m(\mathcal{A}) \to M(\mathcal{A})$, we obtain $H^*(X_m(\mathcal{A}); \mathbb{C}) = H^*(M(\mathcal{A}); \mathcal{L}^m)$, the cohomology of $X_m(\mathcal{A})$ with trivial $\mathbb{C}$-coefficients is isomorphic to the cohomology of the base $M(\mathcal{A})$ with coefficients in the rank m local system $\mathcal{L}^m$ with stalk $\mathcal{L}^m_{\mathbf{x}} = H^0(p_m^{-1}(\mathbf{x}); \mathbb{C}) \simeq \mathbb{C}^m$. The results presented in this section are natural generalizations in the context of arrangements of those of [CS1, Section 1].

Proposition 4.1 (cf. [CS1, 1.3–1.5])**.** *Let $T \in \mathrm{GL}(m, \mathbb{C})$ be the cyclic permutation matrix of order m defined by $T(\vec{e}_i) = \vec{e}_{i+1}$ for $1 \le i \le n-1$ and $T(\vec{e}_n) = \vec{e}_1$, where $\{\vec{e}_i\}$ is the standard basis for $\mathbb{C}^m$. Note that T is diagonalizable with eigenvalues ξ_m^k, $1 \le k \le m$.*

1. *The local system $\mathcal{L}^m$ is induced by the representation $\tau^m : \pi_1(M(\mathcal{A}), \mathbf{x}_0) \to \mathrm{GL}(m, \mathbb{C})$ given by $\tau^m(\gamma_H) = T$ for each meridian γ_H.*
2. *The local system $\mathcal{L}^m$ decomposes into a direct sum, $\mathcal{L}^m = \bigoplus_{k=1}^m \mathcal{L}^m_k$ of rank one local systems. For each k, the local system $\mathcal{L}^m_k$ is induced by the representation $\tau^m_k : \pi_1(M(\mathcal{A}), \mathbf{x}_0) \to \mathbb{C}^*$ defined by $\tau^m_k(\gamma_H) = \xi_m^k$.*
3. *We have $H^*(X_m(\mathcal{A}); \mathbb{C}) = \bigoplus_{k=1}^m H^*(M(\mathcal{A}); \mathcal{L}^m_k)$.*

The above result provides one decomposition of the cohomology $H^*(X_m(\mathcal{A}); \mathbb{C})$. Another is given by the monodromy maps $h_m : X_m(\mathcal{A}) \to X_m(\mathcal{A})$ of Remark 3.4. Since h_m has finite order m, the induced map $h_m^* : H^q(X_m(\mathcal{A}); \mathbb{C}) \to H^q(X_m(\mathcal{A}); \mathbb{C})$ is diagonalizable, with eigenvalues among the m-th roots of unity. Denote the cohomology eigenspace of ξ_m^k by $H^q(X_m(\mathcal{A}); \mathbb{C})_k$, and let $\Delta_q^{(m)}(t) = \det(t \cdot h_m^* - \mathrm{id})$ denote the characteristic polynomial of $h_m^* : H^q(X_m(\mathcal{A}); \mathbb{C}) \to H^q(X_m(\mathcal{A}); \mathbb{C})$. We then have the following generalizations of Theorem 2.3, Proposition 2.1, and Corollary 2.2.

Proposition 4.2 (cf. [CS1, 1.6])**.** *For each k, $1 \le k \le m$, we have*

$$H^*(X_m(\mathcal{A}); \mathbb{C})_k \simeq H^*(M(\mathcal{A}); \mathcal{L}^m_k)$$

Proposition 4.3 (cf. [CS1, 1.1])**.** *Let ξ_m^j and ξ_m^k be two m-th roots of unity which generate the same cyclic subgroup of $\mathbb{Z}_m = \langle \xi_m \rangle$. Then the cohomology eigenspaces $H^*(X_m(\mathcal{A}); \mathbb{C})_j$ and $H^*(X_m(\mathcal{A}); \mathbb{C})_k$ are isomorphic.*

Corollary 4.4. *If ξ_m^j and ξ_m^k are m-th roots of unity which generate the same cyclic subgroup of $\mathbb{Z}_m = \langle \xi_m \rangle$, then $H^*(M(\mathcal{A}); \mathcal{L}^m_j) \simeq H^*(M(\mathcal{A}); \mathcal{L}^m_k)$.*

Corollary 4.5. *For each q, $0 \le q \le \ell$, there are nonnegative integers $d_{k,q}^{(m)}$ so that*

$$\Delta_q^{(m)}(t) = \prod_{k|n} \Phi_k(t)^{d_{k,q}^{(m)}},$$

where $\Phi_k(t)$ denotes the k-th cyclotomic polynomial.

We relate the cohomology of the spaces $X_m(\mathcal{A})$ for various m using these results.

Theorem 4.6. *If k divides m, then the cohomology $H^*(X_k(\mathcal{A});\mathbb{C})$ is a direct summand of $H^*(X_m(\mathcal{A});\mathbb{C})$.*

Proof. From Proposition 4.1, we have $H^*(X_m(\mathcal{A});\mathbb{C}) = \bigoplus_{q=1}^{m} H^*(M(\mathcal{A});\mathcal{L}_q^m)$, where the local system $\mathcal{L}_q^m$ is induced by the representation τ_q^m given by $\gamma_H \mapsto \xi_m^q$. Writing $m = k \cdot r$, we see that the representations τ_p^k, $1 \le p \le k$, are among the m representations τ_q^m, $1 \le q \le m$. In other words, $\bigoplus_{p=1}^{k} H^*(M(\mathcal{A});\mathcal{L}_p^k) = H^*(X_k(\mathcal{A});\mathbb{C})$ is a direct summand of $H^*(X_m(\mathcal{A});\mathbb{C})$. □

Remark 4.7. This result may be interpreted in terms of the intermediate coverings $p_{m,k} : X_m(\mathcal{A}) \to X_k(\mathcal{A})$ of Proposition 3.3 as follows. One can check that the projection $p_{m,k}$ commutes with the monodromy maps, $p_{m,k} \circ h_m = h_k \circ p_{m,k}$. Consequently, the induced map $p_{m,k}^* : H^*(X_k(\mathcal{A});\mathbb{C}) \to H^*(X_m(\mathcal{A});\mathbb{C})$ preserves the eigenspaces of h_k^*. Using Proposition 4.2 and the above theorem, one can show that $p_{m,k}^*$ maps $H^*(X_k(\mathcal{A});\mathbb{C}) = \bigoplus_{p=1}^{k} H^*(M(\mathcal{A});\mathcal{L}_p^k)$ isomorphically to the summand $\bigoplus_{p=1}^{k} H^*(M(\mathcal{A});\mathcal{L}_{pr}^{kr})$ of $H^*(X_m(\mathcal{A});\mathbb{C})$.

These results also show that, to determine the cohomology of $X_m(\mathcal{A})$, it suffices to compute $H^*(M(\mathcal{A});\mathcal{L}_1^k)$ for divisors k of m. Proposition 4.2 and Corollary 4.4 yield

Theorem 4.8. *The Betti numbers of the space $X_m(\mathcal{A})$ are given by*

$$b_q(X_m(\mathcal{A})) = \dim_{\mathbb{C}} H^q(X_m(\mathcal{A});\mathbb{C}) = \sum_{k|m} \phi(k) \cdot b_q(\mathcal{L}_1^k),$$

where ϕ is the Euler phi function and $b_q(\mathcal{L}_1^k) = \dim_{\mathbb{C}} H^q(M(\mathcal{A});\mathcal{L}_1^k)$.

The summand $H^*(M(\mathcal{A});\mathcal{L}_1^1)$ of $H^*(X_m(\mathcal{A});\mathbb{C})$ corresponds to the constant coefficient cohomology of $M(\mathcal{A})$. This is well understood in terms of the Orlik-Solomon algebra defined next.

5. Orlik-Solomon Algebra

Let $A = A(\mathcal{A})$ be the Orlik-Solomon algebra of $\mathcal{A}$ generated by the 1-dimensional classes a_H, $H \in \mathcal{A}$. It is the quotient of the exterior algebra generated by these classes by a homogeneous ideal, hence a finite dimensional graded $\mathbb{C}$-algebra. There is an isomorphism of graded algebras $H^*(M(\mathcal{A});\mathbb{C}) \simeq A(\mathcal{A})$. In particular, $\dim A^q(\mathcal{A}) = b_q(\mathcal{A})$ where $b_q(\mathcal{A}) = \dim H^q(M(\mathcal{A});\mathbb{C})$ denotes the q-th Betti number of $M(\mathcal{A})$ with trivial local coefficients $\mathbb{C}$. The absolute value of the Euler characteristic of the complement is a combinatorial invariant:

$$\beta(\mathcal{A}) = (-1)^{\ell} \sum_{q=0}^{\ell} (-1)^q b_q(\mathcal{A}) = |\chi(M(\mathcal{A}))|. \tag{1}$$

Let $\boldsymbol{\lambda} = \{\lambda_H \mid H \in \mathcal{A}\}$ be a collection of complex weights. Define a differential $A^q \to A^{q+1}$ by multiplication by $a_{\boldsymbol{\lambda}} = \sum_{H\in\mathcal{A}} \lambda_H\, a_H$. This provides a complex $(A^\bullet, a_{\boldsymbol{\lambda}}\wedge)$. Associated to $\boldsymbol{\lambda}$, we have a rank one representation $\rho : \pi_1(M(\mathcal{A})) \to \mathbb{C}^*$ given by $\gamma_H \mapsto t_H = \exp(2\pi\,\mathrm{i}\,\lambda_H)$ for any meridian loop γ_H about the hyperplane $H \in \mathcal{A}$, and a corresponding rank one local system $\mathcal{L}$ on $M(\mathcal{A})$. Note that ρ and $\mathcal{L}$ are unchanged if we replace the weights $\boldsymbol{\lambda}$ with $\boldsymbol{\lambda} + \mathbf{m}$, where $\mathbf{m} = \{m_H \mid H \in \mathcal{A}\}$ is a collection of integers. The following inequalities are well known, see [CO].

Proposition 5.1. *For all $\boldsymbol{\lambda}$ and all q we have*

$$\sup_{\mathbf{m}\in\mathbb{Z}^{|\mathcal{A}|}} \dim_{\mathbb{C}} H^q(A^\bullet, a_{\boldsymbol{\lambda}+\mathbf{m}}\wedge) \le \dim_{\mathbb{C}} H^q(M(\mathcal{A});\mathcal{L}) \le \dim_{\mathbb{C}} H^q(M(\mathcal{A});\mathbb{C}).$$

For the local systems arising in the context of the covers $X_m(\mathcal{A})$, the weights are rational. Suppose $\lambda_H = k_H/N$ for all H, with integers k_H and N, and assume without loss that the g.c.d. of the k_H is prime to N. In this case there are better upper bounds. The Orlik-Solomon ideal is defined by integral linear combinations of the generators, hence the algebra may be defined over any commutative ring R, denoted $A_R(\mathcal{A})$. Write $A_{\mathbb{Q}} = A_{\mathbb{Q}}(\mathcal{A})$. Left-multiplication by the element $a_{\boldsymbol{\lambda}} = \sum \lambda_H a_H \in A^1_{\mathbb{Q}}$ induces a differential on the Orlik-Solomon algebra, and we denote the resulting complex by $(A^{\bullet}_{\mathbb{Q}}, a_{\boldsymbol{\lambda}}\wedge)$. Similarly, associated to the element $a_{\mathbf{k}} = Na_{\boldsymbol{\lambda}} = \sum k_H a_H$, we have the complex $(A^{\bullet}_{\mathbb{Q}}, a_{\mathbf{k}}\wedge)$. We showed in [CO] that the complexes $(A^{\bullet}_{\mathbb{Q}}, a_{\boldsymbol{\lambda}}\wedge)$ and $(A^{\bullet}_{\mathbb{Q}}, a_{\mathbf{k}}\wedge)$ are chain equivalent. The coefficients of $a_{\mathbf{k}}$ are integers, so we consider the Orlik-Solomon algebra with integer coefficients and the associated complex $(A^{\bullet}_{\mathbb{Z}}, a_{\mathbf{k}}\wedge)$. Let $(A^{\bullet}_N, \bar{a}_{\mathbf{k}}\wedge)$ be the reduction of $(A^{\bullet}_{\mathbb{Z}}, a_{\mathbf{k}}\wedge) \mod N$, where $A_N = A_{\mathbb{Z}_N}$ denotes the Orlik-Solomon algebra with coefficients in the ring $\mathbb{Z}_N$ and $\bar{a}_{\mathbf{k}} = a_{\mathbf{k}} \mod N$.

Theorem 5.2 ([CO, 4.5])**.** *Let $\boldsymbol{\lambda} = \mathbf{k}/N$ be a system of rational weights, and let $\mathcal{L}$ be the associated rational local system on the complement M of $\mathcal{A}$. Then, for each q,*

$$\dim_{\mathbb{C}} H^q(M(\mathcal{A}); \mathcal{L}) \leq \operatorname{rank}_{\mathbb{Z}_N} H^q(A^{\bullet}_N, \bar{a}_{\mathbf{k}}\wedge).$$

There are examples in [CO] which show that the inequality can be strict.

6. Polynomial Periodicity

We continue the study of the cohomology of the spaces $X_m(\mathcal{A})$. We first investigate the implications of a well known vanishing theorem of Schechtman, Terao, and Varchenko in this context. We then establish the polynomial periodicity of the Betti numbers of this family of spaces.

An edge of $\mathcal{A}$ is a nonempty intersection of hyperplanes and $L(\mathcal{A})$ is the set of edges. Given $Y \in L(\mathcal{A})$, let $\mathcal{A}_Y = \{H \in \mathcal{A} \mid Y \subset H\}$. Define the weight of Y by $\lambda_Y = \sum_{H \in \mathcal{A}_Y} \lambda_H$. Call $Y \in L(\mathcal{A})$ *dense* if $\beta((\mathcal{A}_Y)_0) > 0$ where $(\mathcal{A}_Y)_0$ is a decone of $\mathcal{A}_Y$. The projective closure, $\mathcal{A}_\infty$, of $\mathcal{A}$ adds the infinite hyperplane, H_∞, with weight $-\sum_{H \in \mathcal{A}} \lambda_H$. Recall that $\mathcal{A} \subset \mathbb{C}^\ell$ contains ℓ linearly independent hyperplanes.

Theorem 6.1 ([STV, 4.3])**.** *Call the local system $\mathcal{L}$ nonresonant if $\lambda_Y \notin \mathbb{Z}_{\geq 0}$ for every dense edge $Y \in L(\mathcal{A}_\infty)$. In this case*

$$H^q(M(\mathcal{A}); \mathcal{L}) = 0 \text{ for } q \neq \ell, \quad \text{and} \quad \dim_{\mathbb{C}} H^\ell(M(\mathcal{A}); \mathcal{L}) = \beta(\mathcal{A}).$$

Recall the decomposition $H^*(X_m(\mathcal{A}); \mathbb{C}) = \bigoplus_{q=1}^m H^*(M(\mathcal{A}); \mathcal{L}_q^m)$ of the cohomology of $X_m(\mathcal{A})$ from Proposition 4.1. By Theorem 4.8, it suffices to consider the case $q = 1$. In the notation of the previous section, the local system $\mathcal{L}_1^m$ arises from the rational and equal weights $\lambda_H = 1/m$ for all $H \in \mathcal{A}$. The projective closure, $\mathcal{A}_\infty$, of $\mathcal{A}$ is the projective quotient of the cone $\mathcal{C}$ of $\mathcal{A}$. Assign the weight $-|\mathcal{A}|/m$ to H_∞.

Proposition 6.2. *If either* (i) $m > |\mathcal{A}|$, *or* (ii) $|(\mathcal{A}_\infty)_Y|$ *is relatively prime to m for every dense edge $Y \in L(\mathcal{A}_\infty)$, then the local system $\mathcal{L}_1^m$ is nonresonant.*

Proof. If $Y \subset H_\infty$, then $\lambda_Y < 0$. Otherwise, we have $\lambda_Y = |\mathcal{A}_Y|/m$, which cannot be a positive integer in either case (i) or (ii). ∎

Call a positive integer k nonresonant if k satisfies either of the hypotheses of Proposition 6.2. Recall the factorization, $\Delta_q^{(m)}(t) = \prod_{k|m} \Phi_k(t)^{d_{k,q}^{(m)}}$, of the characteristic polynomial of the monodromy $h_m^* : H^q(X_m(\mathcal{A}); \mathbb{C}) \to H^q(X_m(\mathcal{A}); \mathbb{C})$ provided by Corollary 4.5. The results of Section 4 and this section provide the following information concerning the exponents $d_{k,q}^{(m)}$ arising in this factorization.

Proposition 6.3. *For every m, we have $d_{1,q}^{(m)} = b_q(\mathcal{A})$ for all q. If k is nonresonant, then $d_{k,q}^{(m)} = 0$ if $q < \ell$ and $d_{k,\ell}^{(m)} = \beta(\mathcal{A})$.*

Proof. The first statement follows from Proposition 4.2 and the fact that $\mathcal{L}_1^1$ is the trivial local system. The second statement follows from Proposition 6.2. □

Proposition 6.2 also facilitates an elementary and explicit proof of the polynomial periodicity of the Betti numbers of the family of covering spaces $X_m(\mathcal{A})$. We refer to Sarnak and Adams [SA, Ad] for results along these lines in greater generality, and to Hironaka [H1, H2] and Sakuma [Sk] for related results on branched covers of surfaces and links. A sequence $\{a_m\}_{m\in\mathbb{N}}$ is said to be *polynomial periodic* if there are polynomials $p_1(x), \dots, p_N(x) \in \mathbb{Z}[x]$ so that $a_m = p_i(m)$ whenever $m \equiv i \mod N$.

Theorem 6.4. *For each q, $0 \le q \le \ell$, the sequence, $\{b_q(X_m(\mathcal{A}))\}_{m\in\mathbb{N}}$, of Betti numbers of the cyclic covers $X_m(\mathcal{A})$ of the complement $M(\mathcal{A})$ is polynomial periodic.*

Proof. First note that $b_0(X_m(\mathcal{A})) = 1$ for all m.

Let $N = \prod_{p \text{ prime}} p^e$ be the product of all prime powers p^e for which the exponent e is maximal so that $p^e \le |\mathcal{A}|$. Evidently, N is the smallest positive integer for which $k|N$ for all $k \le |\mathcal{A}|$. Note also that if $m \equiv i \mod N$, then

$$\{k \in \mathbb{N} \mid 1 \le k \le |\mathcal{A}| \text{ and } k|m\} = \{k \in \mathbb{N} \mid 1 \le k \le |\mathcal{A}| \text{ and } k|i\}. \tag{2}$$

For $1 \le q \le \ell - 1$ and $1 \le i \le N$, define constant polynomials $p_{q,i} = b_q(X_i(\mathcal{A}))$. From Theorem 4.8, we have $p_{q,i} = \sum_{k|i} \phi(k) \cdot b_q(\mathcal{L}_1^k)$. Similarly, if $m \equiv i \mod N$, then $b_q(X_m(\mathcal{A})) = \sum_{k|m} \phi(k) \cdot b_q(\mathcal{L}_1^k)$, and by Proposition 6.2, the sum is over all $k \le |\mathcal{A}|$. Thus, polynomial periodicity of the Betti numbers $b_q(X_m(\mathcal{A}))$ for $1 \le q \le \ell - 1$ follows from the relation between such divisors of m and i noted in (2) above.

The polynomial periodicity of the top Betti number $b_\ell(X_m(\mathcal{A}))$ may be established by an Euler characteristic argument as follows. We have $\chi(X_m(\mathcal{A})) = m \cdot \chi(M(\mathcal{A}))$. This, together with (1), yields

$$b_\ell(X_m(\mathcal{A})) = m \cdot \beta(\mathcal{A}) + (-1)^{\ell+1}\left[1 + \sum_{q=1}^{\ell-1} (-1)^q b_q(X_m(\mathcal{A}))\right].$$

Defining linear polynomials $p_{\ell,i}(x) = \beta(\mathcal{A}) \cdot x + (-1)^{\ell+1}\left[1 + \sum_{q=1}^{\ell-1}(-1)^q p_{q,i}\right]$ for each i, $1 \le i \le N$, we have $b_\ell(X_m(\mathcal{A})) = p_{\ell,i}(m)$ if $m \equiv i \mod N$. □

Remark 6.5. The polynomial periodicity of the Betti numbers of more general classes of covers of a finite CW-complex are established in [SA, Ad]. Noteworthy in the above proof are the explicit identifications of the "period" N and the polynomials $p_{q,i}(x)$ for the cyclic covers $X_m(\mathcal{A})$, see the concluding remarks in [H1].

A generating function for the Betti numbers $b_q(X_m(\mathcal{A}))$ is given by the following *zeta function*, suggested by A. Adem. For each q, $0 \le q \le \ell$, define

$$\zeta_{\mathcal{A},q}(s) = \sum_{m=1}^{\infty} \frac{b_q(X_m(\mathcal{A}))}{m^s}. \tag{3}$$

Theorem 6.6. *We have*

$$\zeta_{\mathcal{A},q}(s) = \zeta(s) \cdot \left[\sum_{k \le |\mathcal{A}|} \frac{\phi(k) \cdot b_q(\mathcal{L}_1^k)}{k^s} + \delta_{q,\ell} \cdot \beta(\mathcal{A}) \sum_{k > |\mathcal{A}|} \frac{\phi(k)}{k^s}\right]$$

where $\zeta(s)$ is the classical Riemann zeta function and $\delta_{q,\ell}$ is the Kronecker delta.

Proof. From Theorem 4.8, we have $b_q(X_m(\mathcal{A})) = \sum_{k|m} \phi(k) \cdot b_q(\mathcal{L}_1^k)$. If $k > |\mathcal{A}|$, we have $b_q(\mathcal{L}_1^k) = 0$ for $q < \ell$ and $b_\ell(\mathcal{L}_1^k) = \beta(\mathcal{A})$ by Proposition 6.2. A calculation using these observations yields the result. □

7. Bounds and Examples

The results of Sections 4 and 6 show that, to determine the cohomology of $X_m(\mathcal{A})$, it suffices to compute $H^*(M(\mathcal{A});\mathcal{L}_1^k)$ for those divisors k of m for which $k < |\mathcal{A}|$ and hypothesis (ii) of Proposition 6.2 fails. Combining the results of Proposition 5.1 and Theorem 5.2, we have combinatorial bounds on the local system Betti numbers,

$$\sup_{\mathbf{m}\in\mathbb{Z}^{|\mathcal{A}|}} \dim_{\mathbb{C}} H^q(A^\bullet, a_{\boldsymbol{\lambda}+\mathbf{m}}\wedge) \le b_q(\mathcal{L}_1^k) \le \operatorname{rank}_{\mathbb{Z}_k} H^q(A^\bullet_{\mathbb{Z}_k}, \bar{a}_{\mathbf{1}}\wedge), \tag{4}$$

where $k\cdot\boldsymbol{\lambda} = \mathbf{1}$ and $a_{\mathbf{1}} = \sum_{H\in\mathcal{A}} a_H$. Evidently, if the two extreme non-negative integers in the above inequalities are equal, then $b_q(\mathcal{L}_1^k)$ is determined. In particular, if $\operatorname{rank}_{\mathbb{Z}_k} H^q(A^\bullet_{\mathbb{Z}_k}, \bar{a}_{\mathbf{1}}\wedge) = 0$, we have $b_q(\mathcal{L}_1^k) = 0$ as well. We conclude with several examples which illustrate the utility of these bounds.

Example 7.1. Let $\mathcal{C}$ be a realization of the MacLane (8_3) configuration, with defining polynomial $Q(\mathcal{C}) = xy(y-x)z(z-x-\xi_3^2y)(z+\xi_3y)(z-x)(z+\xi_3^2x+\xi_3y)$, and let $\mathcal{A}$ be a decone of $\mathcal{C}$. The Poincaré polynomial of $\mathcal{A}$ is $P(\mathcal{A},t) = 1+7t+13t^2$, and $\beta(\mathcal{A}) = 7$. A calculation in the Orlik-Solomon algebra of $\mathcal{A}$ reveals that $H^q(A^\bullet_{\mathbb{Z}_k}, \bar{a}_{\mathbf{1}}\wedge) = 0$ for $q \neq 2$ and all $k > 1$. Thus, $P(X_m(\mathcal{A}),t) = 1+7t+(6+7m)t^2$ for all m. In particular, for $m = 8 = |\mathcal{C}|$, the Poincaré polynomial of the Milnor fiber of the MacLane arrangement is $P(F(\mathcal{C}),t) = 1+7t+62t^2$.

Example 7.2. Let $\mathcal{A}$ be the Selberg arrangement in $\mathbb{C}^2$, with defining polynomial $Q(\mathcal{A}) = xy(x-y)(x-1)(y-1)$. The Poincaré polynomial of $\mathcal{A}$ is given by $P(\mathcal{A},t) = \sum_{q\ge 0} b_q(\mathcal{A})t^q = 1+5t+6t^2$, and we have $\beta(\mathcal{A}) = 2$, see (1). The dense edges of $\mathcal{A}_\infty$ all have cardinality 3, so by Proposition 6.2, if $k > 5$ or k is prime to 3, the local system $\mathcal{L}_1^k$ on $M(\mathcal{A})$ is nonresonant. Thus if $(m,3) = 1$, the Poincaré polynomial of the cover $X_m(\mathcal{A})$ is $P(X_m(\mathcal{A}),t) = 1+5t+(4+2m)t^2$.

If $k = 3$, one can check that $\dim_{\mathbb{C}} H^q(A^\bullet, a_{\boldsymbol{\lambda}+\mathbf{m}}\wedge) = \operatorname{rank}_{\mathbb{Z}_3} H^q(A^\bullet_{\mathbb{Z}_3}, \bar{a}_{\mathbf{1}}\wedge) = 1$, where $3\boldsymbol{\lambda} = \mathbf{1}$ and $\mathbf{m} = (\dots m_H \dots) \in \mathbb{Z}^5$ satisfies $m_H = -1$ if $H = \{x-y=0\}$ and $m_H = 0$ otherwise. Consequently, $b_1(\mathcal{L}_1^3) = 1$ as well, and if 3 divides m, we have $P(X_m(\mathcal{A}),t) = 1+7t+(6+2m)t^2$. It follows that the zeta function $\zeta_{\mathcal{A},1}(s)$ of (3) is given by $\zeta_{\mathcal{A},1}(s) = \zeta(s)\cdot[5+2\cdot 3^{-s}]$.

Since the Selberg arrangement is a decone of the braid arrangement $\mathcal{B}$ of rank three, the cover $X_6(\mathcal{A})$ is homeomorphic to the Milnor fiber $F(\mathcal{B})$, and $P(F(\mathcal{B}),t) = 1+7t+18t^2$, as is well known. For further calculations along these lines, see [CS2, De].

Example 7.3. Let $\mathcal{C}$ be the Hessian configuration, with defining polynomial $Q(\mathcal{C}) = x_1x_2x_3\prod_{i,j=0,1,2}(x_1+\xi_3^ix_2+\xi_3^jx_3)$, and let $\mathcal{A}$ be a decone of $\mathcal{C}$ with Orlik-Solomon algebra A. The Poincaré polynomial of $\mathcal{A}$ is $P(\mathcal{A},t) = 1+11t+28t^2$, and $\beta(\mathcal{A}) = 18$. One can check that $H^q(A^\bullet_{\mathbb{Z}_k}, \bar{a}_{\mathbf{1}}\wedge) = 0$ for $q \neq 2$ if $k \neq 2,4$, and that, if $k = 2,4$,

$$\operatorname{rank}_{\mathbb{Z}_k} H^q(A^\bullet_{\mathbb{Z}_k}, \bar{a}_{\mathbf{1}}\wedge) = \begin{cases} 2 & \text{if } q = 1,\\ 20 & \text{if } q = 2,\\ 0 & \text{otherwise.}\end{cases}$$

So $b_1(\mathcal{L}_1^k) = 0$ and $b_2(\mathcal{L}_1^k) = 18$ if $k \neq 2,4$, while $b_1(\mathcal{L}_1^k) \le 2$ and $b_2(\mathcal{L}_1^k) \le 20$ if $k = 2,4$.

Concerning the lower bound of (4), it is known that the resonance variety $\mathcal{R}_1(\mathcal{C})$ of the Hessian arrangement has a non-local three-dimensional component $S \subset \mathbb{C}^{12}$, see [CS3, 5.8], [Li, 3.3]. For $\boldsymbol{\lambda} \in S$, we have $\dim_{\mathbb{C}} H^1(A^\bullet(\mathcal{C}), a_{\boldsymbol{\lambda}}\wedge) = 2$, see [Fa, 3.12]. For an appropriate ordering of the hyperplanes of $\mathcal{C}$, this component has basis

$$\vec{e}_5+\vec{e}_7+\vec{e}_{12}-\vec{e}_1-\vec{e}_2-\vec{e}_3, \vec{e}_4+\vec{e}_9+\vec{e}_{11}-\vec{e}_1-\vec{e}_2-\vec{e}_3, \vec{e}_6+\vec{e}_8+\vec{e}_{10}-\vec{e}_1-\vec{e}_2-\vec{e}_3.$$

Using this basis, one can show that, for $a_{\boldsymbol{\lambda}} = \frac{1}{k}a_{\mathbf{1}} \in \mathbb{C}^{12}$, there exists $\mathbf{m} \in \mathbb{Z}^{12}$ so that $\boldsymbol{\lambda}+\mathbf{m} \in S$ if and only if $k = 2,4$. Thus, $\sup_{\mathbf{m}\in\mathbb{Z}^{12}} \dim_{\mathbb{C}} H^1(A^\bullet(\mathcal{C}), a_{\boldsymbol{\lambda}+\mathbf{m}}\wedge) = 2$ for these k. A standard argument then shows that $\sup_{\mathbf{m}\in\mathbb{Z}^{11}} \dim_{\mathbb{C}} H^1(A^\bullet(\mathcal{A}), a_{\boldsymbol{\lambda}+\mathbf{m}}\wedge) = 2$ for $a_{\boldsymbol{\lambda}} = \frac{1}{k}a_{\mathbf{1}} \in \mathbb{C}^{11}$ as well. Consequently, the inequalities in the upper bounds on $b_q(\mathcal{L}_1^k)$ noted above are in fact equalities. It follows that the zeta function $\zeta_{\mathcal{A},1}(s)$ is given by $\zeta_{\mathcal{A},1}(s) = \zeta(s)\cdot[11 + 2\cdot 2^{-s} + 4\cdot 4^{-s}]$.

These calculations determine the Betti numbers of the cover $X_m(\mathcal{A})$ for any m, as well as the dimensions of the eigenspaces of the maps $h_m^* : H^*(X_m(\mathcal{A});\mathbb{C}) \to H^*(X_m(\mathcal{A});\mathbb{C})$. In particular, for $m = 12 = |\mathcal{C}|$, the Poincaré polynomial of the Milnor fiber of the Hessian arrangement is $P(F(\mathcal{C}),t) = 1 + 17t + 232t^2$, and the characteristic polynomials $\Delta_q(t) = \Delta_q^{(12)}$ of the algebraic monodromy are

$$\Delta_0(t) = t-1,\ \Delta_1(t) = (t-1)^9(t^4-1)^2,\ \text{and}\ \Delta_2(t) = (t-1)^8(t^4-1)^2(t^{12}-1)^{18}.$$

Example 7.4. Let $\mathcal{A}$ be the Ceva(3) arrangement in $\mathbb{C}^3$, with defining polynomial $Q(\mathcal{A}) = (x^3-y^3)(x^3-z^3)(y^3-z^3)$. It is known that $\dim_{\mathbb{C}} H^1(A^\bullet, a_{\boldsymbol{\lambda}}\wedge) \leq 1$ for all $\boldsymbol{\lambda}$. In the case $k=3$, is is also known that $b_1(\mathcal{L}_1^3) = \operatorname{rank}_{\mathbb{Z}_3} H^1(A^\bullet_{\mathbb{Z}_3}, \bar{a}_{\mathbf{1}}\wedge) = 2$. This example is discussed in detail in [Fa, 4.5], [Li, 3.3], [CS3, 6.2], and [CO, 3.5 and 4.7].

As illustrated by this arrangement, the lower bound of (4) may be strict. On the other hand, we know of no example where the upper bound of (4) is strict.

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