A CHARACTERIZATION OF SEMIAMPLENESS AND CONTRACTIONS OF RELATIVE CURVES

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Revised version

ABSTRACT. I give a cohomological characterization of semiample line bundles. The result is a generalization of both the Fujita–Zariski Theorem on semiampleness and the Grothendieck–Serre Criterion for ampleness. As an application of the Fujita–Zariski Theorem I characterize contractible curves in 1-dimensional families.

INTRODUCTION

The Fujita–Zariski Theorem asserts that a line bundle $\mathcal{L}$ that is ample on its base locus is *semiample*. Semiampleness means that a multiple $\mathcal{L}^{\otimes n}$, $n > 0$ is globally generated. For discrete base locus the result goes back to Zariski ([17], Thm. 6.2), and the general form is due to Fujita ([3], Thm. 1.10). This note contains two applications of the Fujita–Zariski Theorem.

The first section contains a generalization of both the Fujita–Zariski Theorem and the cohomological criterion for ampleness due to Grothendieck–Serre. The result is the following characterization: A line bundle $\mathcal{L}$ is semiample if and only if the modules $H^1(X, \mathcal{I} \otimes \operatorname{Sym} \mathcal{L})$ are finitely generated over the ring $\Gamma(X, \operatorname{Sym} \mathcal{L})$ for every coherent ideal $\mathcal{I} \subset \mathcal{O}_B$. Here $B \subset X$ is the stable base locus of $\mathcal{L}$. This gives a positive answer to Fujita's question ([3], 1.16) whether it is possible to weaken the assumption in the Fujita–Zariski Theorem.

In the second section I generalize results of Piene [14] and Emsalem [2]. They used the Fujita–Zariski Theorem to obtain sufficient conditions for contractions in normal arithmetic surfaces. Our result is a characterization of contractible curves in 1-dimensional families over local noetherian rings in terms of complementary closed subsets. This also sheds some light on the noncontractible curve constructed by Bosch, Lütkebohmert, and Raynaud ([1], chap. 6.7). For proper normal algebraic surfaces, similar results appear in [15].

1. CHARACTERIZATION OF SEMIAMPLENESS

Throughout this section, R is a noetherian ring, X is a proper R-scheme, and $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module. According to the Grothendieck–Serre Criterion ([5], Prop. 2.6.1) $\mathcal{L}$ is ample if and only if for each coherent $\mathcal{O}_X$-module $\mathcal{F}$ there is an integer $n_0 > 0$ so that $H^1(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $n > n_0$. Let me reformulate this in terms of graded modules. For a coherent $\mathcal{O}_X$-module $\mathcal{F}$, set

$$H^p_*(\mathcal{F}, \mathcal{L}) = H^p(X, \mathcal{F} \otimes \operatorname{Sym} \mathcal{L}) = \bigoplus_{n \geq 0} H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}).$$

1991 *Mathematics Subject Classification.* 14A15, 14C20, 14H10, This is a graded module over the graded ring $\Gamma_*(\mathcal{L}) = \Gamma(X, \operatorname{Sym} \mathcal{L})$. The Grothendieck–Serre Criterion takes the form: $\mathcal{L}$ is ample if and only if the modules $H^1_*(\mathcal{F}, \mathcal{L})$ are finitely generated over the ring $\Gamma_0(\mathcal{L}) = \Gamma(\mathcal{O}_X)$ for all coherent $\mathcal{O}_X$-modules $\mathcal{F}$. In this form it generalizes to the semiample case. Following Fujita [3], we define the *stable base locus* $B \subset X$ of $\mathcal{L}$ to be the intersection of the base loci of $\mathcal{L}^{\otimes n}$ for all $n > 0$. We regard it as a closed subscheme with reduced scheme structure.

Theorem 1.1. *Let $B \subset X$ be the stable base locus of $\mathcal{L}$. Then the following are equivalent:*

- (i) *The invertible sheaf $\mathcal{L}$ is semiample.*
- (ii) *The modules $H^p_*(\mathcal{F}, \mathcal{L})$ are finitely generated over the ring $\Gamma_*(\mathcal{L})$ for each coherent $\mathcal{O}_X$-module $\mathcal{F}$ and all integers $p \geq 0$.*
- (iii) *The modules $H^1_*(\mathcal{I}, \mathcal{L})$ are finitely generated over the ring $\Gamma_*(\mathcal{L})$ for each coherent ideal $\mathcal{I} \subset \mathcal{O}_B$.*

Proof. The implication (i)⇒(ii) is well known, and (ii)⇒(iii) is trivial. To prove (iii)⇒(i) we assume that $\mathcal{L}$ is not semiample. According to the Fujita–Zariski Theorem the restriction $\mathcal{L}_B$ is not ample. By the Grothendieck–Serre Criterion there is a coherent ideal $\mathcal{I} \subset \mathcal{O}_B$ with $H^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n}) \neq 0$ for infinitely many $n > 0$. Thus $H^1_*(\mathcal{I}, \mathcal{L})$ is not finitely generated over $\Gamma_0(\mathcal{L})$. Since $B \subset X$ is the stable base locus, the maps $\Gamma(X, \mathcal{L}^{\otimes n}) \to \Gamma(B, \mathcal{L}_B^{\otimes n})$ vanish for all $n > 0$. Consequently, the irrelevant ideal $\Gamma_+(\mathcal{L}) \subset \Gamma_*(\mathcal{L})$ annihilates $H^1_*(\mathcal{I}, \mathcal{L})$, which is therefore not finitely generated over $\Gamma_*(\mathcal{L})$. □

Sommese [16] introduced a quantitative version of semiampleness: Let $k \geq 0$ be an integer; a semiample invertible sheaf $\mathcal{L}$ is called *k-ample* if the fibers of the canonical morphism $f : X \to \operatorname{Proj} \Gamma_*(\mathcal{L})$ have dimension $\leq k$. For example, 0-ampleness means ampleness.

Theorem 1.2. *Let $\mathcal{L}$ be a semiample invertible $\mathcal{O}_X$-module. Then $\mathcal{L}$ is k-ample if and only if the modules $H^{k+1}_*(\mathcal{F}, \mathcal{L})$ are finitely generated over the ground ring R for all coherent $\mathcal{O}_X$-modules $\mathcal{F}$.*

Proof. Set $Y = \operatorname{Proj} \Gamma_*(\mathcal{L})$ and let $f : X \to Y$ be the corresponding contraction. Suppose $\mathcal{L}$ is k-ample. Choose $n_0 > 0$ so that $\mathcal{L}^{\otimes n_0} = f^*(\mathcal{M})$ for some ample invertible $\mathcal{O}_Y$-module $\mathcal{M}$. Put $\mathcal{G} = \mathcal{F} \otimes (\mathcal{L} \oplus \mathcal{L}^{\otimes 2} \oplus \ldots \oplus \mathcal{L}^{\otimes n_0})$. Choose $m_0 > 0$ with $H^p(Y, R^q f_*(\mathcal{G}) \otimes \mathcal{M}^{\otimes m}) = 0$ for $p > 0$, $q \leq k+1$, and $m > m_0$. Consequently, the edge map $H^{k+1}(X, \mathcal{G} \otimes \mathcal{L}^{\otimes m n_0}) \to H^0(Y, R^{k+1} f_*(\mathcal{G}) \otimes \mathcal{M}^{\otimes m})$ in the spectral sequence

$$H^p(Y, R^q f_*(\mathcal{G}) \otimes \mathcal{M}^{\otimes m}) \Longrightarrow H^{p+q}(X, \mathcal{G} \otimes \mathcal{L}^{\otimes m n_0})$$

is injective for $m > m_0$. The fibers of $f : X \to Y$ are at most k-dimensional, so $R^{k+1} f_*(\mathcal{G}) = 0$. Thus $H^{k+1}(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$ for all $n > n_0 m_0$.

Conversely, assume that the condition holds. Seeking a contradiction we suppose that some fiber of $f : X \to Y$ has dimension $> k$. Using [13] we find a coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $R^{k+1} f_*(\mathcal{F}) \neq 0$. Replacing $\mathcal{L}$ by a suitable multiple, we have $\mathcal{L} = f^*(\mathcal{M})$ for some ample invertible $\mathcal{O}_Y$-module $\mathcal{M}$. Passing to a higher multiple if necessary, $H^p(Y, R^q f_*(\mathcal{F}) \otimes \mathcal{M}^{\otimes n}) = 0$ holds for $p > 0$, $q \leq k$, and $n > 0$. Then the edge map $H^{k+1}_*(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) \to H^0_*(Y, R^{k+1} f_*(\mathcal{F}) \otimes \mathcal{M}^{\otimes n})$ is surjective for $n > 0$. Choose a global section $s \in \Gamma(Y, \mathcal{M}^{\otimes n})$ for some $n > 0$ so that the open subset $Y_s \subset Y$ contains the set of associated points for $R^{k+1} f_*(\mathcal{F})$. Then $s \in \Gamma_*(\mathcal{M})$

is not a zero divisor for $H^0_*(R^{k+1}f_*(\mathcal{F}),\mathcal{M})$. It follows that $H^0_*(R^{k+1}f_*(\mathcal{F}),\mathcal{M})$ is nonzero for infinitely many degrees. Consequently, the same holds for $H^{k+1}_*(\mathcal{F},\mathcal{L})$, which is therefore not finitely generated over R. □

Remark 1.3. For a *vector bundle* $\mathcal{E}$, it might happen that $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is semiample, whereas $\operatorname{Sym}^n(\mathcal{E})$ fails to be globally generated for all $n>0$. For example, let k be an algebraically closed field of characteristic $p>0$, and X be a smooth proper curve of genus $g>p-1$ so that the absolute Frobenius $\operatorname{Fr}_X : H^1(\mathcal{O}_X)\to H^1(\mathcal{O}_X)$ is zero. For an example see [11], p. 348, ex. 2.14. Let $D\subset X$ be a divisor of degree 1. According to the commutative diagram

$$\begin{array}{ccccccc}
H^0(\mathcal{O}_X) & \longrightarrow & H^0(\mathcal{O}_D) & \longrightarrow & H^1(\mathcal{O}_X(-D)) & \longrightarrow & H^1(\mathcal{O}_X) \\
\downarrow{\scriptstyle \operatorname{Fr}_X^*} & & \downarrow{\scriptstyle \operatorname{Fr}_X^*} & & \downarrow{\scriptstyle \operatorname{Fr}_X^*} & & \downarrow{\scriptstyle \operatorname{Fr}_X^*=0} \\
H^0(\mathcal{O}_X) & \longrightarrow & H^0(\mathcal{O}_{pD}) & \longrightarrow & H^1(\mathcal{O}_X(-pD)) & \longrightarrow & H^1(\mathcal{O}_X),
\end{array}$$

the p-linear map $\operatorname{Fr}_X^* : H^1(\mathcal{O}_X(-D))\to H^1(\mathcal{O}_X(-pD))$ is not injective. Hence there is a nontrivial extension

$$0\longrightarrow \mathcal{O}_X\longrightarrow \mathcal{E}\longrightarrow \mathcal{O}_X(D)\longrightarrow 0$$

whose Frobenius pull back $\operatorname{Fr}_X^*(\mathcal{E})$ splits. The surjection $\mathcal{E}\to\mathcal{O}_X(D)$ gives a section $A\subset\mathbb{P}(\mathcal{E})$ representing $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ with $A^2=1$ ([11], Prop. 2.6, p. 371). The Fujita–Zariski Theorem implies that $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is semiample, and we obtain a birational contraction $\mathbb{P}(\mathcal{E})\to Y$. It is easy to see that the exceptional set is an integral curve $R\subset\mathbb{P}(\mathcal{E})$ which has degree p on the ruling. Hence $\mathbb{P}(\mathcal{E})\to Y$ does not restrict to closed embeddings on the fibers of $\mathbb{P}(\mathcal{E})\to X$. Consequently, $\operatorname{Sym}^n(\mathcal{E})$ is not globally generated at any point $x\in X$.

2. Contractions of relative curves

Throughout this section, R is a local noetherian ring, and X is a proper R-scheme with 1-dimensional closed fiber $X_0\subset X$. Then all fibers of the structure morphism $X\to\operatorname{Spec}(R)$ are at most 1-dimensional. For example, X could be a flat family of curves.

A *Stein factor* of X is a proper R-scheme Y together with a proper morphism $f:X\to Y$ so that $\mathcal{O}_Y\to f_*(\mathcal{O}_X)$ is bijective (compare [12], sec. 5). Our objective is to describe the set of all Stein factors for a given X.

Let C_i, $i\in I$ be the finite collection of all 1-dimensional integral components of the closed fiber X_0. A subset $J\subset I$ yields a subcurve $C=\bigcup_{i\in J}C_i$. We call such a curve $C\subset X$ *contractible* if there is a Stein factor $f:X\to Y$ so that $f(C_i)$ is a closed point if and only if $i\in J$. According to [5], Theorem 5.4.1, a Stein factor is determined up to isomorphism by its restriction $f_0:X_0\to Y_0$. The task now is to determine the contractible curves $C\subset X$. It follows from [14] and [2] that all curves $C\subset X$ are contractible provided that the ground ring R is henselian. In particular this holds if R is complete. On the other hand, a noncontractible curve is discussed in [1], chapter 6.7.

We seek to describe contractible curves $C\subset X$ in terms of complementary closed subsets $D\subset X$. We need a definition: Suppose $D\subset X$ is a closed subset of codimension ≤ 1. Let $R\subset R^\wedge$ be the completion with respect to the maximal ideal, X' the normalization of $X\otimes_R R^\wedge$, and $C_i', C', D'\subset X'$ the preimages of

$C_i, C, D \subset X$, respectively. Let $h : X' \to Z'$ be the contraction of all $C'_i \subset X'_0$ disjoint from C'. We call D *persistent* if $h(D') \subset Z'$ has codimension ≤ 1.

Example 2.1. Suppose R is a discrete valuation ring with residue field k and fraction field K. Let X be the proper R-scheme obtained from $X' = \mathbb{P}^1_R$ by identifying the closed points $0, \infty \in \mathbb{P}^1_k$. Then the closure $D \subset X$ of the point $0 \in \mathbb{P}^1_K$ is not persistent.

Theorem 2.2. *Suppose $J \subset I$ is a subset so that the curve $C = \bigcup_{i\in J} C_i$ is connected. Then $C \subset X_0$ is contractible if and only if there is a persistent closed subset $D \subset X$ of codimension ≤ 1 disjoint from C and intersecting each irreducible component $C_i \subset X_0$ with $i \notin J$.*

Proof. Assume that C is contractible. The corresponding contraction $f : X \to Y$ maps C to a single point. Let $V \subset Y$ be an affine open neighborhood of $f(C)$. Set $U = f^{-1}(V)$ and $D = X - U$. Clearly $D \cap C = \emptyset$. Furthermore, $D \cap C_i \neq \emptyset$ for $i \notin J$; otherwise $f(C_i)$ would be a proper curve contained in the affine scheme V, which is absurd. Let X', Y' be the normalizations of $X \otimes_R R^\wedge, Y \otimes_R R^\wedge$, respectively. The induced morphism $f' : X' \to Y'$ is the contraction of the preimage $C' \subset X'$ of C. The preimage $V' \subset Y'$ of V is affine, so $Y - V$ is of codimension ≤ 1 ([10] II, 2.2.6). Hence the preimage $D' \subset X'$ of D is of codimension ≤ 1. Obviously, the same holds if we contract the preimages $C'_i \subset X'$ of C_i disjoint from C'. Thus $D \subset X$ is of codimension ≤ 1 and persistent.

Conversely, assume the existence of such a subset $D \subset X$. Set $U = X - D$. We claim that the affine hull $U^{\text{aff}} = \operatorname{Spec}\Gamma(U, \mathcal{O}_X)$ is of finite type over R and that the canonical morphism $U \to U^{\text{aff}}$ is proper.

Suppose this for a moment. Then $U \to U^{\text{aff}}$ contracts C and is a local isomorphism near each $x \in U_0 - C$. Choose for each $x \in X_0 - C$ an affine open neighborhood $U_x \subset X$ of x disjoint to the exceptional set of $U \to U^{\text{aff}}$. Then $U_x \cap U \to U^{\text{aff}}$ is an open embedding. It is easy to see that the schemes $U_x \bigcup_{U_x \cap U} U^{\text{aff}}$, $x \in X_0 - C$ and U^{aff} form an open cover of a proper R-scheme Y. The induced morphism $f : X \to Y$ is the desired contraction.

It remains to verify the claim. Let $R \subset R^\wedge$ be the completion. According to [9], VIII Corollary 3.4, the scheme U^{aff} is of finite type if and only if $U^{\text{aff}} \otimes_R R^\wedge$ is of finite type. Furthermore, $U \to U^{\text{aff}}$ is proper if and only if if is proper after tensoring with $R^\wedge$ ([9], VIII Cor. 4.8). Since $U^{\text{aff}} \otimes_R R^\wedge = (U \otimes_R R^\wedge)^{\text{aff}}$ by [8], Proposition 21.12.2, it suffices to prove the claim under the additional assumption that R is complete.

Now each curve in X_0 is contractible. Observe that the contraction of C does not change U^{aff}, so we can as well assume that C is empty. Now our goal is to prove that U is affine. Since R is complete, hence universally japanese, the normalization $X' \to X$ is finite. Using Chevalley's Theorem ([4], Thm. 6.7.1), we reduce the problem to the case that X is normal. Now the irreducible components of X are the connected components. Treating them separately we may assume that X is connected. Contracting the curves C_i contained in D we can assume that D_0 is finite and intersects each C_i. If $D = X$ or $D = \emptyset$ there is nothing to prove. Assume that $D \subset X$ is of codimension 1, in other words a Weil divisor. The problem is that it might not be Cartier. To overcome this, consider the graded quasicoherent $\mathcal{O}_X$-algebra $\mathcal{R} = \bigoplus_{n\geq 0} \mathcal{O}_X(nD)$. The graded subalgebra $\mathcal{R}' \subset \mathcal{R}$ generated by $\mathcal{R}_1 = \mathcal{O}_X(D)$ is of finite type over $\mathcal{O}_X$. Set $X' = \mathcal{P}roj\,(\mathcal{R}')$ and let $g : X' \to X$

be the structure morphism. Then g is projective and $\mathcal{O}_{X'}(1)$ is a g-very ample invertible $\mathcal{O}_{X'}$-module. The canonical maps $D : \mathcal{O}_X(nD) \to \mathcal{O}_X((n+1)D)$ induce a homomorphism $\mathcal{R}' \to \mathcal{R}'$ of degree one, hence a section $s : \mathcal{O}_{X'} \to \mathcal{O}_{X'}(1)$. It follows from the definition of homogeneous spectra that s is bijective over U and vanishes on $g^{-1}(D)$. Thus the corresponding Cartier divisor $D' \subset X'$ representing $\mathcal{O}_{X'}(1)$ has support $g^{-1}(D)$.

Let $A \subset X'_0$ be a closed integral subscheme of dimension $n > 0$. If $g(A) \subset X_0$ is a curve, then A is not contained in D' but intersects D'. Hence $D' \cdot A > 0$. If $g(A) \subset X$ is a point, then $\mathcal{O}_A(1)$ is ample, so $(D')^n \cdot A > 0$. By the Nakai criterion for ampleness we conclude that $\mathcal{O}_{X'}(1)$ is ample on its base locus. Now the Fujita–Zariski Theorem tells us that $\mathcal{O}_{X'}(1)$ is semiample. It follows that $U \simeq X' - D'$ is affine. This finishes the proof. ☐

Let us consider the special case that the total space X is a normal surface. Replacing R by $\Gamma(X, \mathcal{O}_X)$, we are in the following situation: Either R is a discrete valuation ring, such that $X \to \operatorname{Spec}(R)$ is a flat deformation of X_0. Or R is a local normal 2-dimensional ring, hence $X \to \operatorname{Spec}(R)$ is the birational contraction of X_0. In either case we call a Weil divisor $H \in Z^1(X)$ *horizontal* if it is a sum of prime divisors not supported by X_0.

Suppose $J \subset I$ is a subset with $C = \bigcup_{i \in J} C_i$ connected. Let $V \subset X_0$ be the union of all C_i disjoint from C.

Corollary 2.3. *Notation as above. Then $C \subset X_0$ is contractible if and only if there is a horizontal Weil divisor $H \subset X$ disjoint from C with the following property: For each C_i, $i \notin J$, either H intersects C_i, or H intersects a connected component $V' \subset V$ with $V' \cap C_i \neq \emptyset$.*

Proof. Suppose $C \subset X_0$ is contractible. Let $D \subset X$ be a persistent Weil divisor as in Theorem 2.2. Then its horizontal part $H \subset D$ satisfies the above conditions. Conversely, assume there is a horizontal Weil divisor $H \subset X$ as above. It follows that $D = H + V$ is a persistent Weil divisor disjoint from C intersecting each C_i with $i \notin J$. Thus $C \subset X_0$ is contractible. ☐

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