MULTI-VARIABLE POLYNOMIAL SOLUTIONS TO PELL'S EQUATION AND FUNDAMENTAL UNITS IN REAL QUADRATIC FIELDS

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Abstract. Solving Pell's equation is of relevance in finding fundamental units in real quadratic fields and for this reason polynomial solutions are of interest in that they can supply the fundamental units in infinite families of such fields.

In this paper an algorithm is described which allows one to construct, for each positive integer n, a finite collection, $\{F_i\}$, of multi-variable polynomials (with integral coefficients), each satisfying a multi-variable polynomial Pell's equation

$$C_i^2 - F_i H_i^2 = (-1)^{n-1},$$

where C_i and H_i are multi-variable polynomials with integral coefficients. Each positive integer whose square-root has a regular continued fraction expansion with period $n+1$ lies in the range of one of these polynomials. Moreover, the continued fraction expansion of these polynomials is given explicitly as is the fundamental solution to the above multi-variable polynomial Pell's equation.

Some implications for determining the fundamental unit in a wide class of real quadratic fields is considered.

1. Introduction

Solving Pell's equation is of relevance in finding fundamental units in real quadratic fields and for this reason polynomial solutions are interesting in that they can supply the fundamental units in infinite families of such fields.

There have been several papers written over the past thirty years which describe certain polynomials whose square roots have periodic continued fraction expansions which can be written down explicitly in terms of the coefficients and variables of the polynomials. See for example the papers of Bernstein [1], Levesque and Rhin [4], Madden [5], Van der Poorten [10] and Van der Poorten and Williams [11].

In this paper an algorithm is described which allows one to construct, for each positive integer n, a finite collection of multi-variable Fermat-Pell polynomials which have *all* positive integers whose square-roots have a continued fraction expansion of period $n+1$ in their range. If $F_i := F_i(t_0, t_1, \cdots, t_{\lfloor \frac{n+1}{2} \rfloor})$

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is any one of these polynomials, the fundamental polynomial solution to the equation

$$C_i^2 - F_i H_i^2 = (-1)^{n-1} \tag{1.1}$$

(where C_i and H_i are polynomials in the variables $t_0, t_1, \cdots, t_{\lfloor \frac{n+1}{2} \rfloor}$) can be found. Moreover, the continued fraction expansion of $\sqrt{F_i}$ can be written down when $t_1, \cdots, t_{\lfloor \frac{n+1}{2} \rfloor} \geq 0$ and $t_0 > g_i(t_1, \cdots, t_{\lfloor \frac{n+1}{2} \rfloor})$, a certain rational function of these variables. Some implications for single-variable Fermat-Pell polynomials are discussed as are the implications for writing down the fundamental units in a wide class of real quadratic number fields.

Definition: a multi-variable polynomial

$$F := F(t_0, t_1, \cdots, t_k) \ \in \mathbb{Z}[\, t_0, t_1, \cdots, t_k], \ k \geq 1$$

is called a *multi-variable Fermat-Pell polynomial* [1] if there exists polynomials

$$C := C(t_0, t_1, \cdots, t_k) \text{ and } H := H(t_0, t_1, \cdots, t_k) \ \in \mathbb{Z}[\, t_0, t_1, \cdots, t_k\,]$$

such that either

$$\begin{aligned} C^2 - F\,H^2 &= 1, \text{ for all } t_i, \ \ 0 \leq i \leq k, \ \ \text{or} \\ C^2 - F\,H^2 &= -1, \text{ for all } t_i, \ \ 0 \leq i \leq k. \end{aligned} \tag{1.2}$$

Such a triple of polynomials $\{C, H, F\}$ satisfying equation (1.2) constitute a *multi-variable polynomial solution* to Pell's equation.

Definition: The multi-variable Fermat-Pell polynomial F (as above) is said to have a *multi-variable polynomial continued fraction expansion* if there exists a positive integer n, a real constant T, a rational function $g(t_1, \cdots, t_k) \in \mathbb{Q}(t_1, \cdots, t_k)$ and polynomials $a_0 := a_0(t_0, t_1, \cdots, t_k) \in \mathbb{Z}[\, t_0, t_1, \cdots, t_k\,]$ and $a_j := a_j(t_1, \cdots, t_k) \in \mathbb{Z}[\, t_1, \cdots, t_k\,], \ 1 \leq j \leq n$, which take only positive integral values for integral $t_i \geq T, 1 \leq i \leq k$ and (possibly half-) integral $t_0 > g(t_1, \cdots, t_k)$ such that

$$\sqrt{F} = [\, a_0; \overline{a_1, \cdots, a_n, 2a_0}], \text{ for all } t_i\text{'s in the ranges stated, } 0 \leq i \leq k.$$

Remarks:

(1) From the point of view of simplicity it would be desirable to replace the condition $t_0 \geq g(t_1, \cdots, t_k)$ by $t_0 \geq T$ but it will be seen that for the polynomials examined here that the former condition is more natural and indeed cannot be replaced by the latter condition.

(2)The restriction that the $a_i(t_1, \cdots, t_k) \geq 0, \ 1 \leq i \leq n$ may also seem artificial to some since negative terms can easily be removed from a continued fraction expansion (see, for example [10]) but this changes the period of the continued fraction so is avoided here.

(3) It may also seem artificial to have a_0 depend on a variable t_0 while the

[1] These polynomials are called "Fermat-Pell polynomials" here to avoid confusion with "Pell Polynomials" and also because Fermat investigated the "Pell" equation.

other a_i's do not but this will also be seen to occur naturally.
(4)Finally, allowing t_0 to take half-integral values in some circumstances may also seem strange but this also will be seen to be natural and indeed necessary.

Definition: If, for all sets of integers $\{t'_0, t'_1, \cdots, t'_k\}$ satisfying $t'_0 \geq g(t'_1, \cdots, t'_k)$ and $t'_i \geq T$, $1 \leq i \leq k$,

$$X = C_i(t'_0, t'_1, \cdots, t'_k), \;\; Y = H_i(t'_0, t'_1, \cdots, t'_k)$$

constitutes the fundamental solution (in integers) to

$$X^2 - F_i(t'_0, t'_1, \cdots, t'_k)Y^2 = (-1)^{n-1}$$

then $(C_i(t_0, t_1, \cdots, t_k), H_i(t_0, t_1, \cdots, t_k))$ is termed the *fundamental polynomial solution* to equation (1.1).

Standard notations are used:

$$a_0 + \frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\,\cdots\frac{1}{a_N} := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots \cfrac{1}{a_N}}}}.$$

To save space this continued fraction is usually written $[a_0; a_1, \cdots, a_n]$. The infinite periodic continued fraction with initial non-periodic part a_0 and periodic part $a_1, ..., a_n, 2a_0$ is denoted by $[\,a_0; \overline{a_1, ..., a_n, 2a_0}]$. The i-th approximant of the continued fraction $[a_0; a_1, \cdots,]$ is denoted by P_i/Q_i.

Repeated use will be made of some basic facts about continued fractions, such as:

(1.3)
$$\begin{aligned} P_nQ_{n-1} - P_{n-1}Q_n &= (-1)^{n-1}, \\ P_{n+1} &= a_{n+1}P_n + P_{n-1}, \\ Q_{n+1} &= a_{n+1}Q_n + Q_{n-1}, \end{aligned}$$

each of these relations being valid for $n = 1, 2, 3 \cdots$.

Before coming to the main problem, it is necessary to first solve a related problem on symmetric strings of positive integers.

2. A problem concerning Symmetric Sequences

Question: For which symmetric sequences of positive integers $a_1, ..., a_n$ do there exist positive integers a_0 and D such that

(2.1)
$$\sqrt{D} = [\,a_0; \overline{a_1, ..., a_n, 2a_0}]?$$

Let P_i/Q_i denote the ith approximant of the continued fraction

(2.2)
$$0 + \frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\,\cdots\frac{1}{a_n}.$$

By the well known correspondence between convergents and matrices

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}a_1&1\\1&0\end{pmatrix}\cdots\cdots\begin{pmatrix}a_n&1\\1&0\end{pmatrix}=\begin{pmatrix}P_n&P_{n-1}\\Q_n&Q_{n-1}\end{pmatrix}$$

$$\Longrightarrow\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}a_1&1\\1&0\end{pmatrix}\cdots\cdots\begin{pmatrix}a_n&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}P_{n-1}&P_n\\Q_{n-1}&Q_n\end{pmatrix}.$$

Since the left side in the second equation is a symmetric sequence of symmetric matrices it follows that

$$P_n = Q_{n-1}. \tag{2.3}$$

Suppose $\sqrt{D} = [\,a_0; \overline{a_1, ..., a_n, 2a_0}] = a_0+\beta$, where $\beta =[0; \overline{a_1, ..., a_n, 2a_0}]$ so that

$$\begin{aligned}
&\beta = [\,0; a_1, ..., a_n, 2a_0 + \beta],\\
&\Longrightarrow \beta = \frac{(2a_0+\beta)P_n + P_{n-1}}{(2a_0+\beta)Q_n + Q_{n-1}} = \frac{\beta P_n + (2a_0P_n + P_{n-1})}{\beta Q_n + (2a_0Q_n + Q_{n-1})},\\
&\Longrightarrow \beta^2 Q_n + (2a_0Q_n + Q_{n-1} - P_n)\beta - (2a_0P_n + P_{n-1}) = 0,\\
&\Longrightarrow \beta^2 Q_n + (2a_0Q_n)\beta - (2a_0P_n + P_{n-1}) = 0, (\text{by } (2.3))\\
&\Longrightarrow \sqrt{D} = a_0 + \beta = \sqrt{a_0^2 + \frac{2a_0P_n + P_{n-1}}{Q_n}}
\end{aligned}$$

The problem now becomes one of determining for which symmetric sequences of positive integers $a_1, ..., a_n$ does there exist positive integers a_0 such that $(2a_0P_n + P_{n-1})/Q_n$ is an integer.

Theorem 1. *There exists a positive integer a_0 such that $(2a_0P_n+P_{n-1})/Q_n$ is an integer if and only if $P_{n-1}Q_{n-1}$ is even.*

Proof. $\Longleftarrow$ Suppose first of all that $P_{n-1}Q_{n-1}$ is even. By equation (1.3)

$$P_nQ_{n-1} + (-1)^n = P_{n-1}Q_n$$

(i) Suppose n is even. Then $P_nQ_{n-1}P_{n-1} + P_{n-1} = P_{n-1}^2Q_n$. Choose t to be any integer or half-integer such that tQ_n is an integer and $a_0 := Q_{n-1}P_{n-1}/2 + tQ_n > 0$. Then

$$\frac{2a_0P_n + P_{n-1}}{Q_n} = \frac{Q_{n-1}P_{n-1}P_n + 2tP_nQ_n + P_{n-1}}{Q_n} = 2tP_n + P_{n-1}^2$$

(ii)Similarly, in the case n is odd, $-P_nQ_{n-1}P_{n-1} + P_{n-1} = -P_{n-1}^2Q_n$. Choose t to be any integer or half-integer such that tQ_n is an integer and $a_0 := -Q_{n-1}P_{n-1}/2 + tQ_n > 0$. In this case

$$\frac{2a_0P_n + P_{n-1}}{Q_n} = 2tP_n - P_{n-1}^2$$

$\Longrightarrow$ Suppose next that P_{n-1} and Q_{n-1} are both odd and that there exists a positive integer a_0 such that $(2a_0P_n+P_{n-1})/Q_n$ is a positive integer, m, say.

Using (1.3) and (2.3) it follows that Q_n is even. Then $2a_0P_n + P_{n-1} = mQ_n$ implies P_{n-1} is even - a contradiction. □

Remarks:
(i)Note that this process gives all a_0 such that $(2a_0P_n + P_{n-1})/Q_n$ is an integer. Indeed,

$$\begin{aligned}
&(2a_0P_n + P_{n-1})/Q_n = k, \text{ an integer} \\
&\Longleftrightarrow 2a_0P_nQ_{n-1} = -P_{n-1}Q_{n-1} + kQ_nQ_{n-1}, \\
&\Longleftrightarrow 2a_0(-1)^{n-1} = 2a_0(P_nQ_{n-1} - P_{n-1}Q_n), \text{ (by (1.3))} \\
&\qquad\qquad\qquad = -P_{n-1}Q_{n-1} + Q_n(kQ_{n-1} - 2a_0P_{n-1}), \\
&\Longleftrightarrow a_0 = (-1)^{n-1}\left(\frac{-P_{n-1}Q_{n-1}}{2} + Q_n\frac{kQ_{n-1} - 2a_0P_{n-1}}{2}\right).
\end{aligned}$$

Notice also that if there is one such a_0 that there are infinitely many of them.

(ii)Notice that, with P_n, P_{n-1}, Q_n and Q_{n-1} as defined above, if there exists a positive integer D satisfying (2.1) then $D = p(t_0)$, for some allowed t_0, where

$$p(t) = \left(\frac{Q_{n-1}P_{n-1}}{2} + tQ_n\right)^2 + 2tP_n + P_{n-1}^2,\ t > \frac{-Q_{n-1}P_{n-1}}{2Q_n}, \quad (n \text{ even}),$$
$$p(t) = \left(\frac{-Q_{n-1}P_{n-1}}{2} + tQ_n\right)^2 + 2tP_n - P_{n-1}^2,\ t > \frac{Q_{n-1}P_{n-1}}{2Q_n}, \quad (n \text{ odd}).$$

The above theorem suggests a simple algorithm for deciding if, for a given symmetric sequence of positive integers $a_1, \cdots, a_n$, there exist positive integers a_0 and D such that (2.1) holds. Notice that all that matters is the parity of the a_i so all calculations can be done in $\mathbb{Z}_2$. First of all define the following matrices:

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ K = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Convert the sequence $a_1, a_2, \cdots, a_n$ to a sequence of J- and K-matrices, according to whether each a_i is odd (replace by a K) or even (replace by a J). Prefix a J-matrix (to account for the initial 0 in the continued fraction (2.2)). Multiply this sequence together (modulo 2) using the facts that $J^2 = K^3 = I$, and $JK = K^2J$.

The final matrix $\equiv \begin{pmatrix} * & 1 \\ * & 1 \end{pmatrix}$ mod 2 $\Longleftrightarrow$ there do not exist positive integers a_0 and D such that (2.1) holds.

Example 1. *Do there exist positive integers a_0 and D such that*

$$\sqrt{D} = [\,a_0; \overline{22, 34, 97, 32, 15, 17, 17, 15, 32, 97, 34, 22, 2a_0}]?$$

As described above convert the sequence $22, 34, 97, 32, 15, 17, 17, 15, 32, 97, 34, 22$ to a sequence of J- and K-matrices, prefix a J-matrix and multiply the sequence together:

$$\underbrace{JJ}JKJ\underbrace{KKK}KJK\underbrace{JJ} = JK(JK)JK = J\underbrace{K(K^2}\underbrace{J)J}K$$
$$= JK = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Therefore there do exist positive integers a_0 and D such that

$$\sqrt{D} = [\,a_0; \overline{22, 34, 97, 32, 15, 17, 17, 15, 32, 97, 34, 22, 2a_0}].$$

3. Multi-variable Fermat-Pell Polynomials

Definition: If $\{\,a_1, \cdots, a_n\}$ is a symmetric zero-one sequence such that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \prod_{i=1}^{n} \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix} \not\equiv \begin{pmatrix} * & 1 \\ * & 1 \end{pmatrix} \mod 2$$

then the sequence $\{\,a_1, \cdots, a_n\}$ is termed a *permissible* sequence. Let $r(n)$ denote the number of permissible sequences of length n.

Note: It is not difficult to show that $r(2m) = ((-1)^m + 2^{m+1})/3$ and that $r(2m+1) = ((-1)^m + 5 \times 2^m)/3$.

If D is a positive integer such that $\sqrt{D} = [\,a_0; \overline{a_1, ..., a_n, 2a_0}]$ then $\{\,a_1, \cdots, a_n\} \mod 2$ must equal one of the above permissible sequences and D is said to be *associated* with this permissible sequence . The collection of all positive integers associated with a particular permissible sequence is termed the *parity class* of this permissible sequence. Sometimes, if there is no danger of ambiguity, these collections of positive integers will be referred to simply as *parity classes*.

Theorem 2. *(i)For each positive integer n there exists a finite collection of multi-variable Fermat-Pell polynomials $F_j(t_0, t_1, \cdots, t_{\lfloor \frac{n+1}{2} \rfloor}), 1 \leq j \leq r(n)$, such that each positive integer whose square root has a continued fraction expansion with period $n+1$ lies in the range of exactly one of these polynomials. Moreover, these polynomials can be constructed;*

(ii) These polynomials have a polynomial continued fraction expansion which can be explicitly determined;

(iii) The fundamental polynomial solution $C = C_j(t_0, t_1, \cdots, t_{\lfloor \frac{n+1}{2} \rfloor})$, $H = H_j(t_0, t_1, \cdots, t_{\lfloor \frac{n+1}{2} \rfloor})$ to

$$C^2 - F_j(t_0, t_1, \cdots, t_{\lfloor \frac{n+1}{2} \rfloor})H^2 = (-1)^{n-1} \tag{3.1}$$

exists and can be explicitly determined.

Proof. (i)The proof will be by construction.

Step 1: Find all permissible sequences. This will involve checking $2^{\lfloor \frac{n+1}{2} \rfloor}$ zero-one sequences in a way similar to the example (1) above.

Step 2: For each permissible sequence $\{a_1, \cdots, a_n\}$ create a new symmetric polynomial sequence $\{a_1(t_1), a_2(t_2), \cdots, a_{n-1}(t_2), a_n(t_1)\}$ by replacing each a_i and its partner a_{n+1-i} in the symmetric sequence by $a_i(t_i) = a_{n+1-i}(t_i) = 2t_i + 1$ if $a_i = 1$ and by $a_i(t_i) = a_{n+1-i}(t_i) = 2t_i + 2$ if $a_i = 0$. This new sequence will sometimes be referred to as the sequence $\{a_1, \cdots, a_n\}$, if there is no danger of ambiguity. Each of the integer variables t_i (in the polynomial being constructed) will be allowed to vary independently over the range $0 \le t_i < \infty$ and each of the new a_i's will keep the same parity and stay positive.

Step 3 As in (2.2), form the continued fraction

$$0 + \frac{1}{a_1(t_1)+}\,\frac{1}{a_2(t_2)+}, \ldots, \frac{1}{a_{n-1}(t_2)+}\,\frac{1}{a_n(t_1)}$$

and calculate P_n, Q_n, P_{n-1} and Q_{n-1} for this polynomial continued fraction, where these expressions are now polynomials in the t_i's.

Step 4 Construct $F_j := F_j(t_0, t_1, \cdots, t_{\lfloor \frac{n+1}{2} \rfloor})$, the multi-variable Fermat-Pell polynomial corresponding to the particular parity sequence under consideration. This is simply done by defining

$$F_j := \begin{cases} \left(\frac{Q_{n-1}P_{n-1}}{2} + t_0 Q_n\right)^2 + 2t_0 P_n + P_{n-1}^2, \ (n \text{ even}) \\ \left(\frac{-Q_{n-1}P_{n-1}}{2} + t_0 Q_n\right)^2 + 2t_0 P_n - P_{n-1}^2, \ (n \text{ odd}) \end{cases} \tag{3.2}$$

where $(-1)^{n+1} Q_{n-1}P_{n-1}/(2Q_n) < t_0 < \infty$ and t_0 can take half-integral values if Q_n is even and otherwise takes integral values.

Every positive integer whose square root has a continued fraction expansion with period $n+1$ lies in the range of exactly one of these polynomials. That these polynomials are multi-variable Fermat-Pell polynomials follows from equation (3.4) below.

(ii) With t_0 in the range given, then

$$\sqrt{F_j} = [\,a_0(t_0, t_1, \cdots, t_{\lfloor \frac{n+1}{2} \rfloor}); \overline{a_1(t_1), \cdots, a_n(t_1), 2a_0(t_0, t_1, \cdots, t_{\lfloor \frac{n+1}{2} \rfloor})}],$$

for all $t_i \ge 0$. Here

$$a_0 = a_0(t_0, t_1, \cdots, t_{\lfloor \frac{n+1}{2} \rfloor}) := \begin{cases} \frac{Q_{n-1}P_{n-1}}{2} + t_0 Q_n, \ (n \text{ even}) \\ \frac{-Q_{n-1}P_{n-1}}{2} + t_0 Q_n, \ (n \text{ odd}). \end{cases} \tag{3.3}$$

(iii) Notice (using (1.3) and (2.3)) that

$$(a_0 Q_n + P_n)^2 - (a_0^2 + (2a_0 P_n + P_{n-1})/Q_n) Q_n^2 = (-1)^{n-1}. \tag{3.4}$$

To see that $(a_0 Q_n + P_n, Q_n)$ is the fundamental solution to (3.1), notice that

$$\sqrt{F_j} = [\,a_0(t_0, t_1, \cdots, t_{\lfloor \frac{n+1}{2} \rfloor}); \overline{a_1(t_1), \cdots, a_n(t_1), 2a_0(t_0, t_1, \cdots, t_{\lfloor \frac{n+1}{2} \rfloor})}].$$

This has period $n+1$ and the nth approximant is $a_0 + P_n/Q_n = (a_0 Q_n + P_n)/Q_n$ and by the theory of the Pell equation $(a_0 Q_n + P_n, Q_n)$ is the fundamental solution to (3.1). □

As regards fundamental units in quadratic fields there is the following theorem on page 119 of [6]:

Theorem 3. *Let D be a square-free, positive rational integer and let $K = \mathbb{Q}(\sqrt{D})$. Denote by ϵ_0 the fundamental unit of K which exceeds unity, by s the period of the continued fraction expansion for $\sqrt{D}$, and by P/Q the $(s-1)$-th approximant of it.*

If $D \not\equiv 1 \mod 4$ or $D \equiv 1 \mod 8$, then

$$\epsilon_0 = P + Q\sqrt{D}.$$

However, if $D \equiv 5 \mod 8$, then

$$\epsilon_0 = P + Q\sqrt{D}.$$

or

$$\epsilon_0^3 = P + Q\sqrt{D}.$$

Finally, the norm of ϵ_0 is positive if the period s is even and negative otherwise.

It is easy, working modulo 4, to determine simple conditions (on t_0) which make $F_j \equiv 2$ or $3 \mod 4$ and thus to say further, for a particular set of choices of $t_1, \cdots, t_{\lfloor \frac{n+1}{2} \rfloor}$ and for all odd or even t_0, that if F_j is square-free, then $a_0 Q_n + P_n + \sqrt{F_j} Q_n$ is the fundamental unit in $\mathbb{Q}[\sqrt{F_j}\ ]$. For example, suppose that n is even and that the original Q_{n-1} determined from the permissible zero-one sequence is also even (so that P_{n-1} and Q_n are both odd and $P_n = Q_{n-1}$ is even). Then the multi-variable form of Q_{n-1} evaluated in *Step 3* will also have all even coefficients. Suppose $\frac{Q_{n-1}}{2} \equiv c_0 + \sum t_{i'} \mod 2$. (Here c_0 may be 0 and the sum $\sum t_{i'}$ may contain some, all or none of the t_i's) It is easy to see that $F_j(t_0, t_1, \cdots, t_{\lfloor \frac{n+1}{2} \rfloor}) \equiv (c_0 + \sum t_{i'} + t_0)^2 + 1 \mod 4$. Even more simply, if the original original Q_{n-1} as in *Step 1* is odd (here also the case n is even is considered) then P_{n-1} as evaluated in *Step 3* is even and it is not difficult to show that in fact $P_{n-1} \equiv 2 \mod 4$ (since for n even $P_n Q_{n-1} - P_{n-1} Q_n = -1$) and that Q_n is odd, which leads to $F_j(t_0, t_1, \cdots, t_{\lfloor \frac{n+1}{2} \rfloor}) \equiv t_0^2 + 1 \mod 4$. Similar relations hold in the case where n is odd.

The polynomials constructed in theorem (2) take values in only one parity class, if all the variables are positive. However, given any two parity classes, there are multi-variable Fermat-Pell polynomials that take values in those two classes.

Theorem 4. *Let n be any fixed positive integer large enough so that the set of positive integers whose square roots have a continued fraction expansion of period $n+1$ can be divided into more than one parity class.*

(i) Given any two parity classes of integers whose square roots have continued fraction expansions of period $n+1$, there are multi-variable Fermat - -Pell polynomials, which can be constructed, that take values in both parity classes;

(ii) These polynomials have a polynomial continued fraction expansion which can be explicitly determined;

(iii) If $F = F(t_0, c, t_1, \cdots, t_{\lfloor \frac{n+1}{2} \rfloor})$ *is any such polynomial then the fundamental polynomial solution*

$$C = C(t_0, c, t_1, \cdots, t_{\lfloor \frac{n+1}{2} \rfloor}), H = H(t_0, c, t_1, \cdots, t_{\lfloor \frac{n+1}{2} \rfloor})$$

to

$$C^2 - FH^2 = (-1)^{n-1} \tag{3.5}$$

can be explicitly determined.

Proof. As in *Step 2* in theorem (2) a polynomial sequence $\{a_1, \cdots, a_n\}$ is created. Suppose $L_1 = \{b_1 \cdots, b_n\}$ and $L_2 = \{c_1, \cdots, c_n\}$ are the permissible sequences associated with the two parity classes. Let $i_1, \cdots, i_k$ be those positions $\leq \lfloor \frac{n+1}{2} \rfloor$ at which the sequences agree. For each of these i_r's set $a_{i_r}(t_{i_r}) = a_{n+1-i_r}(t_{i_r}) = 2t_{i_r} + 1$, if c_{i_r} is odd and set $a_{i_r}(t_{i_r}) = a_{n+1-i_r}(t_{i_r}) = 2t_{i_r} + 2$, if c_{i_r} is even. Subdivide the remaining positions (those positions $\leq \lfloor \frac{n+1}{2} \rfloor$ at which L_1 and L_2 differ) into two subsets: those at which L_1 has a 0 and L_2 has a 1 and those at which L_1 has a 1 and L_2 has a 0.

Suppose i_j is a position of the first kind. Let $a_{i_j}(c, t_{i_j}) = a_{n+1-i_j}(c, t_{i_j}) = c + 2 + 2t_{i_j}$. Repeat this for all the positions i_j in this first set. Likewise, Suppose i_j is a position of the second kind. In this case let $a_{i_j}(c, t_{i_j}) = a_{n+1-i_j}(c, t_{\lfloor \frac{n+1}{2} \rfloor}) = c + 1 + 2t_{i_j}$.This is also repeated for all the positions i_j in this second set. *Step 3* and *Step 4* are then carried out as above. The rest of the proof is identical to theorem (2). Denote the polynomial produced by

$$F := F(t_0, c, t_1, \cdots, t_{\lfloor \frac{n+1}{2} \rfloor}). \tag{3.6}$$

As in theorem (2), if c and all the t_i's are non-negative, $1 \leq i \leq \lfloor \frac{n+1}{2} \rfloor$ and $t_0 > (-1)^{n+1} Q_{n-1} P_{n-1} / (2Q_n)$ then

$$\sqrt{F} = [\,a_0; \overline{a_1, ..., a_n, 2a_0}],$$

where the a_i's, $1 \leq i \leq n$ are as defined just above and a_0 is as defined in equation (3.3).

Under these conditions also the parity class of $F(t_0, c, t_1, \cdots, t_{\lfloor \frac{n+1}{2} \rfloor})$ will depend only on the parity of c. As in theorem (2) the fundamental polynomial solution to

$$C^2 - F(t_0, c, t_1, \cdots, t_{\lfloor \frac{n+1}{2} \rfloor}) H^2 = (-1)^{n-1}$$

is given by $C = a_0 Q_n + P_n, \;\; H = Q_n$. □

4. A Worked Example

As an example, consider those positive integers whose square-roots have continued fraction expansion with period of length 9. Thus the symmetric part of the period has length 8 and it is necessary to check the $2^4 = 16$

zero-one sequences to determine which are permissible. (This checking is done in essentially the same way as in Example 1 above.) There are 11 valid sequences:

$$\begin{array}{c} 0,0,0,0,0,0,0,0\\ 0,0,0,1,1,0,0,0\\ 0,0,1,1,1,1,0,0\\ 0,1,0,0,0,0,1,0\\ 0,1,0,1,1,0,1,0\\ 0,1,1,1,1,1,1,0\\ 1,0,0,1,1,0,0,1\\ 1,0,1,0,0,1,0,1\\ 1,0,1,1,1,1,0,1\\ 1,1,0,0,0,0,1,1\\ 1,1,1,0,0,1,1,1 \end{array}$$

The ninth of these is considered in more detail (Each of the others can be dealt with in a similar way). For clarity the letters a, b, c and d are used instead of t_1, t_2, t_3 and t_4. Evaluating the continued fraction

(4.1)
$$0+\frac{1}{2a+1+}\,\frac{1}{2b+2+}\,\frac{1}{2c+1+}\,\frac{1}{2d+1+}\,\frac{1}{2d+1+}\,\frac{1}{2c+1+}\,\frac{1}{2b+2+}\,\frac{1}{2a+1}$$

it is found that

$$\begin{aligned} P_8 = Q_7 = &-1-2\,d+\\ &2\,(3+4\,a+2\,b+4\,a\,b)(4+3\,b+4\,c+4\,b\,c+6\,d+4\,b\,d+8\,c\,d+8\,b\,c\,d)\\ &+4\,(3+2\,b+4\,(1+b)\,c)\,(2+b+3\,c+2\,b\,c+a\,(3+2\,b+4\,(1+b)\,c))\times\\ &\qquad (1+2\,d+2\,d^2),\\ P_7 = &\,4\,(1+b)\,(4+3\,b+4\,c+4\,b\,c+6\,d+4\,b\,d+8\,c\,d+8\,b\,c\,d)+\\ &\qquad 2\,(3+2\,b+4\,(1+b)\,c)^2\,(1+2\,d+2\,d^2)\text{ and}\\ Q_8 = &\,8\,(2+b+3\,c+2\,b\,c+a\,(3+2\,b+4\,(1+b)\,c))^2\,(1+2\,d+2\,d^2)+\\ &(3+4\,a+2\,b+4\,a\,b)\,(3+4\,c+4\,d+8\,c\,d+(2+4\,a)\,(4+3\,b+\\ &\qquad 4\,c+4\,b\,c+6\,d+4\,b\,d+8\,c\,d+8\,b\,c\,d)). \end{aligned}$$

Since n is 8 (even) and Q_8 is odd (so t_0 cannot take half-integer values), in this case $F_9(t_0, a, b, c, d)$ is defined by

$$F_9(t_0,a,b,c,d) = (Q_7P_7/2+t_0Q_8)^2+2t_0P_8+P_7^2 \tag{4.2}$$

and

$$\sqrt{F_9(t_0,a,b,c,d)} = [Q_7P_7/2 + t_0Q_8; \overline{2a+1, 2b+2, 2c+1, 2d+1, 2d+1,}$$
$$\overline{2c+1, 2b+2, 2a+1, 2(Q_7P_7/2+t_0Q_8)}],$$

this expansion being valid for all $a, b, c, d \geq 0$ and all $t_0 > -Q_7P_7/(2Q_8)$ and in particular for all $t_0 \geq 0$. In these ranges

$$C = (Q_7P_7/2 + t_0Q_8)Q_8 + P_8, \; H = Q_8$$

gives the fundamental polynomial solution to

$$C^2 - F_9H^2 = -1.$$

$$F_9(t_0,a,b,c,d) = (Q_7P_7/2 + t_0Q_8)^2 + 2t_0P_8 + P_7^2 \equiv (1 + t_0^2) \mod 4$$

so that if $(Q_7P_7/2 + t_0Q_8)^2 + 2t_0P_8 + P_7^2$ is a square-free number for some particular $a, b, c, d \geq 0$ and some odd $t_0 > -Q_7P_7/(2Q_8)$, then

$$(Q_7P_7/2 + t_0Q_8)Q_8 + P_8 + \sqrt{(Q_7P_7/2 + t_0Q_8)^2 + 2t_0P_8 + P_7^2}\,Q_8$$

is the fundamental unit in $\mathbb{Q}\left(\sqrt{(Q_7P_7/2 + t_0Q_8)^2 + 2t_0P_8 + P_7^2}\right)$.

5. Mystification, Fermat-Pell polynomials of a single variable and more on odd-even

Clearly it is possible to "mystify" this process by replacing each t_i by some polynomial $g_i(t_i)$ taking only positive values or by replacing $2t_i$ (recalling that the continued fraction expansion contains only terms like $2t_i + 1$ or $2t_i + 2$) by some polynomial $g_i(t_i)$ taking only even non-negative values or by setting $t_i = t_i(X_1, X_2, \cdots, X_k)$, $1 \leq i \leq \lfloor \frac{n+1}{2} \rfloor$, a polynomial in the X_j's taking only positive values, where the X_j's can be independent variables and k can be as large as desired and so on.

Finally of course one can obtain single-variable Fermat-Pell polynomials by replacing the original variables $t_0, t_i, 1 \leq i \leq \lfloor \frac{n+1}{2} \rfloor$ by polynomials in a single variable. If it is desired that the period of the continued fraction expansion of the new single-variable Fermat-Pell polynomial should stay the same as that of the originating multi-variable polynomial then the domain of the single variable should be restricted so that the polynomials replacing each of the t_i's take only positive values as in the multi-variable case and the polynomial replacing t_0 must be such that the a_0 term stays positive for all allowed values of the new single variable.

For example, letting $a = s, b = 0, c = s, d = 0$ and $t_0 = s$ in the polynomial (4.2) above produces the single-variable Fermat-Pell polynomial

$$g(s) = 639557 + 6858268\,s + 33078145\,s^2 +$$
$$94534688\,s^3 + 177380352\,s^4 + 228442240\,s^5 + 204593408\,s^6 +$$
$$125870080\,s^7 + 50925568\,s^8 + 12238848\,s^9 + 1327104\,s^{10}$$

which has the continued fraction expansion (valid for all $s \geq 0$)

$$\sqrt{g(s)} = [799 + 4289\,s + 9184\,s^2 + 9856\,s^3 + 5312\,s^4 + 1152\,s^5; \overline{2s+1, 2, 2s+1, 1, 1, 2s+1, 2, 2s+1, 2(799 + 4289\,s + 9184\,s^2 + 9856\,s^3 + 5312\,s^4 + 1152\,s^5)}].$$

$g(s) \equiv (1+s^2) \mod 4$ so when s is odd and positive and $g(s)$ is square-free

$$51982 + 534625\,s + 2429840\,s^2 + 6408000\,s^3 + 10812928\,s^4 + 12115200\,s^5 + 9019392\,s^6 + 4304896\,s^7 + 1196032\,s^8 + 147456\,s^9 + \sqrt{g(s)}(65 + 320\,s + 576\,s^2 + 448\,s^3 + 128\,s^4)$$

is the fundamental unit in $\mathbb{Q}[\sqrt{g(s)}]$. For example, letting $s = 1$ gives that $47020351 + 1537\sqrt{935888258}$ is the fundamental unit in $\mathbb{Q}[\sqrt{935888258}]$.

Starting with the continued fraction

$$0 + \frac{1}{2a+1+}\,\frac{1}{2b+2+}\,\frac{1}{c+2e+}\,\frac{1}{2d+1+}\,\frac{1}{2d+1+}\,\frac{1}{c+2e+}\,\frac{1}{2b+2+}\,\frac{1}{2a+1}$$

and following the same steps as above with the continued fraction (4.1) a multi-variable Fermat-Pell polynomial is developed which takes values in the parity classes associated with permissible sequences 7 and 9. Letting $a = b = d = e = t = 0$ one gets the single-variable Fermat-Pell polynomial

$$g(c) = 4325 + 28140\,c + 83652\,c^2 + 147440\,c^3 + 168000\,c^4 + 126528\,c^5 + 61504\,c^6 + 17664\,c^7 + 2304\,c^8$$

with continued fraction expansion

$$\sqrt{g(c)} = [65 + 214c + 288c^2 + 184c^3 + 48c^4; \overline{1, 2, c, 1, 1, c, 2, 1, 2(65 + 214c + 288c^2 + 184c^3 + 48c^4)}],$$

valid for $c \geq 1$.

6. Concluding Remarks

Every Fermat-Pell polynomial in one variable, s say, that eventually has a continued fraction expansion of fixed period length can be found from (3.2), if it takes values in only one parity class for all sufficiently large s, and from (3.6), if it takes values in two parity class for all sufficiently large s. (Recall remark (i) after theorem (1))

Of course none of this does anything to answer Schinzel's question of whether every Fermat-Pell polynomial in one variable has a continued fraction expansion. Neither does it provide a criterion (such as Schinzel's in the degree-two case) for deciding if a polynomial of arbitrarily high even degree is a Fermat-Pell polynomial. Perhaps it raises another question - Does every multi-variable Fermat-Pell polynomial have a continued fraction expansion?

Does every multi-variable Fermat-Pell polynomial have a continued fraction expansion, assuming every Fermat-Pell polynomial in one variable does?

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