

Quantization of Poisson algebras associated to Lie algebroids

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ABSTRACT. We prove the existence of a strict deformation quantization for the canonical Poisson structure on the dual of an integrable Lie algebroid. It follows that any Lie groupoid C^* -algebra may be regarded as a result of a quantization procedure. The C^* -algebra of the tangent groupoid of a given Lie groupoid G (with Lie algebroid $A(G)$) is the C^* -algebra of a continuous field of C^* -algebras over \mathbb{R} with fibers $A_0 = C^*(A(G)) \simeq C_0(A^*(G))$ and $A_{\hbar} = C^*(G)$ for $\hbar \neq 0$. The same is true for the corresponding reduced C^* -algebras. Our results have applications to, e.g., transformation group C^* -algebras, K -theory, and index theory.

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1. Introduction

The idea of “quantization” has evolved through a number of stages. At the beginning of this century, one meant the fact that at a microscopic scale certain physical quantities (like energy or angular momentum) assume only discrete values. Such discreteness is easily understood within the Hilbert space formalism of

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quantum mechanics, where self-adjoint operators may or may not have a discrete spectrum, and is no longer seen as the defining property of a quantum theory.

Since about 1925, the idea has referred to the passage from a classical to a “corresponding” quantum theory. There are basically two ways to describe this passage, starting either from the Lagrangian or from the Hamiltonian version of classical mechanics.

As first recognized by Feynman, the Lagrangian approach naturally leads to the path-integral formulation of quantum mechanics, and writing down a path integral may be seen as an act of quantization. All quantum-mechanical observables are constructed from so-called transition amplitudes, which correspond to the integration of certain functions with respect to the path integral. Though intuitively appealing, it is often hard to make this step rigorous. Through the nineties, many ideas of Witten originated from the use of the path integral.

Most mathematically precise work on quantization is based on the Hamiltonian formulation of classical mechanics. It was initially believed that the underlying mathematics consisted of symplectic geometry, but since about 1976 it has been understood that classical mechanics should be based on the concept of a Poisson algebra **[Ki, Li]**. This is a commutative algebra with a Lie bracket that turns each element of the algebra into a derivation with respect to the commutative structure; see Definition 4.1 below. Most Poisson algebras one encounters in physics are of the form $C^\infty(P)$, where P is a so-called Poisson manifold; the commutative algebra structure comes from pointwise multiplication, and the Lie bracket is just the Poisson bracket $\{, \}$. Symplectic manifolds form a special, nondegenerate case.

As first recognized by Heisenberg in 1925, the quantum-mechanical observables of a given physical system should form a noncommutative algebra; the noncommutativity leads to the uncertainty relations that form the physical basis of quantum mechanics. A general, heuristic theory incorporating this idea was given in Dirac’s book **[Di]**. The correct mathematical formalism of quantum mechanics, which still stands today, is due to von Neumann **[vN]**, who created the abstract theory of Hilbert spaces and self-adjoint operators for this purpose. The only (but crucial) modification to von Neumann’s formalism has been to allow other C^* -algebras than $\mathfrak{B}(\mathcal{H})$ (the algebra of all bounded operators on a Hilbert space \mathcal{H}) as algebras of observables. This generalization corresponds to admitting superselection rules; in the context of the quantization of finite-dimensional systems this means that one incorporates Poisson manifolds that are not symplectic in the underlying classical theory. The examples in this paper are of such a form.

In the Hamiltonian formalism, quantizing a Poisson algebra A^0 of classical observables amounts to finding a “corresponding” noncommutative associative algebra A of quantum observables, as well as a quantization map $\mathcal{Q} : A^0 \rightarrow A$, subject to certain conditions. Initially, practically all of quantization theory was based on a single idea of Dirac **[Di]**, which he conceived in 1926 during a Sunday walk near Cambridge. Namely, in quantum mechanics the role of the Poisson bracket of the classical theory should be played by $1/i\hbar$ times the commutator. Here \hbar is a small positive number, which in physics is a constant of nature. Hence if a classical observable f is quantized by a quantum observable $\mathcal{Q}(f)$, one expects that “Dirac’s condition”

$$(i\hbar)^{-1}[\mathcal{Q}(f), \mathcal{Q}(g)] = \mathcal{Q}(\{f, g\})$$

holds at some fixed value of \hbar . In other words, the quantization map \mathcal{Q} should be a Lie algebra homomorphism with respect to the Poisson bracket and the (rescaled) commutator.

Geometric quantization [Wo] was an attempt to make this idea precise. Given a “prequantizable” symplectic manifold S with associated Poisson algebra $C^\infty(S)$, this approach produces a Hilbert space $\mathcal{H}(S)$ (the space of L^2 -sections of the pre-quantization line bundle $L(S)$ over S) and a Lie algebra homomorphism \mathcal{Q}^{pre} from $C^\infty(S)$ to a certain algebra $\mathcal{A}(S)$ of unbounded operators on $\mathcal{H}(S)$ that are densely defined on the domain of smooth sections of $L(S)$.

The case where S is a coadjoint orbit in the dual \mathfrak{g}^* of the Lie algebra \mathfrak{g} of a Lie group G has been studied in particular detail, for the following reason. Each element of $X \in \mathfrak{g}$ defines a function \tilde{X} on \mathfrak{g}^* , and hence on S , by $\tilde{X}(\theta) = \theta(X)$. One has

$$(1.1) \quad \{\tilde{X}, \tilde{Y}\} = \widetilde{[X, Y]},$$

so that \mathcal{Q}^{pre} restricts to a Lie algebra homomorphism from \mathfrak{g} to $\mathcal{A}(S)$. Assuming that this (infinitesimal) representation of \mathfrak{g} is integrable to a unitary representation U of G , one finds that U tends to be reducible. In order to obtain an irreducible sub-representation of U , one must restrict $\mathcal{H}(S)$. In the simplest cases one may simply project onto an irreducible subspace with orthogonal projection P in the commutant of $U(G)$. For general $f \in C^\infty(S)$, one then has to define $\mathcal{Q}(f) = P\mathcal{Q}^{\text{pre}}(f)P$; it is clear that this modification destroys the Lie algebra homomorphism property (except on the linear span of the \tilde{X} , since $[\mathcal{Q}^{\text{pre}}(\tilde{X}), P] = 0$).

A vast number of other methods has been proposed to achieve irreducibility; see [Ki] for a recent overview. The conclusion is that one cannot achieve irreducibility of $U(G)$ while preserving the Lie algebra homomorphism property on all smooth functions. In other examples than coadjoint orbits, one finds that the map \mathcal{Q}^{pre} is unsatisfactory for other (though related) reasons, the conclusion being the same: a satisfactory quantization map is only a Lie algebra homomorphism in the stated sense on some subspace of the Poisson algebra of classical observables.

The way out of this dilemma emerged in the seventies, mainly as a consequence of the work of Berezin [Be], Vey [Ve], and Bayen et al. [BFFLS]. It is, quite simply, to require Dirac’s condition only asymptotically, that is, for $\hbar \rightarrow 0$. This, of course, necessitates the dramatic step of defining the quantization data A and \mathcal{Q} for a family $I \subseteq \mathbb{R}$ of values of \hbar that contains 0 as an accumulation point. Hence, given a Poisson algebra A^0 , one now needs to find a family of algebras $\{A_\hbar\}_{\hbar \in I}$ and maps $\mathcal{Q}_\hbar : A^0 \rightarrow A_\hbar$, with $\mathcal{Q}_0 = \text{id}$, such that $(i\hbar)^{-1}[\mathcal{Q}_\hbar(f), \mathcal{Q}_\hbar(g)] \rightarrow \mathcal{Q}_\hbar(\{f, g\})$ in some sense. A method that accomplishes this is generically referred to as a deformation quantization.

There are (at least) two ways to make rigorous sense of the above convergence. The oldest, introduced in the context of quantization theory in [BFFLS], is to define A_\hbar and \mathcal{Q}_\hbar as formal power series in \hbar . This method, called formal deformation quantization, remains the most popular; see [Ko] for a recent highlight.

The second approach, which is based on choosing the A_\hbar to be C^* -algebras, was introduced by Rieffel [Ri2]-[Ri7]. Here \hbar is simply a real number rather than a formal deformation parameter, and one imposes Dirac’s condition asymptotically in norm; see Definition 4.3 below. It is then mathematically natural and physically meaningful to require that the A_\hbar form a continuous field of C^* -algebras over the index set $I \ni \hbar$. In section 4 we give a precise formulation of this approach to

deformation quantization, introducing two appropriate generalizations of Rieffel’s original definition that lie at the basis of the results in the present paper.

Having discussed the development of modern quantization theory in broad outline, we now turn to the class of classical systems (that is, Poisson manifolds) that are quantized in this paper. Our motivation comes from a number of directions.

The first is the quantization theory of coadjoint orbits discussed above. Although the attempt to quantize individual orbits is only successful in special cases, one could try to quantize all coadjoint orbits of a given Lie group at one stroke by quantizing \mathfrak{g}^* (or $C^\infty(\mathfrak{g}^*)$) as a whole. To do so, one starts from the fact that \mathfrak{g}^* is canonically a Poisson manifold when equipped with the so-called Lie–Poisson structure that may be defined by (1.1). As first shown by Rieffel [Ri4] in the special case that G is exponential and nilpotent, one may construct a quantization of $C^\infty(\mathfrak{g}^*)$ in which $A_\hbar = C^*(G)$ for all $\hbar \neq 0$, and the maps \mathcal{Q}_\hbar are essentially given by the pullback of the exponential map from \mathfrak{g} to G , rescaled by $1/\hbar$. We show that this works without any assumption on the Lie group G . The idea that coadjoint orbits in \mathfrak{g}^* are quantized by unitary irreducible representations of G then re-enters through the back door, as follows. The well-known correspondence between nondegenerate representations of $C^*(G)$ and unitary representations of G , preserving irreducibility, is matched by the fact that the irreducible representations of $C^\infty(\mathfrak{g}^*)$ as a Poisson algebra [La3] are precisely given by the coadjoint orbits of G (or covering spaces thereof). In general, there is no correspondence through quantization between the irreducible representations of $C^\infty(\mathfrak{g}^*)$ and the irreducible representations of $C^*(G)$; the correspondence through (deformation) quantization is between the algebras in question themselves.

Secondly, one may quantize a particle moving on a Riemannian manifold M (so that the pertinent Poisson manifold is the cotangent bundle T^*M , which is symplectic) by taking A_\hbar to be the C^* -algebra $\mathfrak{B}_0(L^2(M))$ of compact operators on $L^2(M)$ for all $\hbar \neq 0$, and defining a quantization map \mathcal{Q}_\hbar in terms of the geodesics on M . This result may be generalized by coupling the particle to an external Yang–Mills field with gauge group H ; one then starts from a principal H -bundle P , so that $M = P/H$. The Poisson manifold describing the classical theory is then given by $P = (T^*P)/H$, which is not symplectic unless H is discrete, and when H is compact the quantum algebra of observables turns out to be $\mathfrak{B}_0(L^2(P))^H$; see [La1, La3] for details and the generalization to noncompact H . Taking $P = H = G$ actually reproduces the previous example, since $(T^*G)/G \simeq \mathfrak{g}^*$ as Poisson manifolds, whereas for compact Lie groups one has $\mathfrak{B}_0(L^2(G))^G \simeq C^*(G)$ (and analogously for the noncompact case). The quantization maps in question are also compatible with this specialization.

Thirdly, in the early eighties Lie groupoids and locally compact groupoids started to play a role in C^* -algebras as a result of the work of Connes and of Renault, whereas in the late eighties Lie groupoids entered symplectic and Poisson geometry, as well as quantization theory, through the idea of a symplectic groupoid (cf. [We2]). Against this background, it was an obvious idea that the above quantizations ought to be interpretable in terms of Lie groupoids. The key to this possibility lies in the fact the passage from a Lie groupoid to its convolution C^* -algebra has a classical analogue. Namely, like a Lie group, a Lie groupoid G has an associated “infinitesimal” object, its Lie algebroid [Pr1] $A(G)$, which is a vector bundle over the base space G_0 of G . The dual bundle $A^*(G)$ of $A(G)$ admits

a canonical Poisson structure [CDW, Cou], so that eventually one may associate a Poisson algebra $C^\infty(A^*(G))$ to G .

All threads now come together [La1], in that in each of the three classes of examples above the classical Poisson algebra is of the form $C^\infty(A^*(G))$ and the quantum C^* -algebra is of the form $C^*(G)$, where G is some Lie groupoid. This is obvious in the first example. In the second, one takes G to be the pair groupoid $G = M \times M$ on M , whose Lie algebroid is the tangent bundle TM ; the Poisson structure on the dual bundle T^*M , which is nothing but the cotangent bundle of M , is just its usual canonical structure (which is symplectic). See also [CCFGRV]. Finally, in the third example G is the so-called gauge groupoid of the principal bundle \mathbb{P} [Ma], with associated Lie algebroid $(TP)/H$.

Based on these examples, it was conjectured in [La2] that one should obtain a deformation quantization of this type for any Lie groupoid G . For formal deformation quantization a version of this conjecture was proved in [NWX], and a proof in C^* -algebraic deformation quantization above appeared in [La3, La4, Ra1].

In this paper we give a precise and complete formulation of all mathematical concepts surrounding this class of C^* -algebraic deformation quantizations, and prove two technically distinct versions of the above conjecture, each of which has its merits. The generalization of Connes's tangent groupoid [Co2] \hat{G} to arbitrary Lie groupoids G [HS, We1] plays an important role. A significant intermediate result is that $C^*(\hat{G})$ is the C^* -algebra of a continuous field of C^* -algebras over \mathbb{R} with fibers $A_0 = C^*(A(G)) \simeq C_0(A^*(G))$ and $A_\hbar = C^*(G)$ for $\hbar \neq 0$. The same is true for the corresponding reduced C^* -algebras. See section 6.

The sections of the paper are listed in the table of contents following the abstract. Our main results, Theorems 4.4 and 4.6, are stated at the end of section 4. The remainder of the paper develops the proofs of these theorems.

2. Lie groupoids and Lie algebroids

This section is a brief review of our objects of study, mainly intended to establish our notation. We assume that the reader is familiar with the basic theory of locally compact groupoids (cf. [Re1]), which we always assume to be second countable. We denote a groupoid as a whole by G , the base is called G_0 , which is seen as a subspace of the arrow space (or total space) G_1 under the inclusion map ι . The source and range projections are called s and r , respectively, and for $x \in G_0$ we put

$$(2.1) \quad G^x = r^{-1}(x) \subset G_1.$$

The space of composable arrows is $G_2 = \{(\gamma, \gamma') \in G_1 \times G_1 \mid r(\gamma') = s(\gamma)\}$.

DEFINITION 2.1. *A Lie groupoid is a locally compact groupoid G for which G_1 and G_0 are manifolds, s and r are surjective submersions, and multiplication and inclusion are smooth maps.*

It follows that ι is an immersion, that the inverse is a diffeomorphism, that G_2 is a closed submanifold of $G_1 \times G_1$, that the fibers $r^{-1}(x)$ and $s^{-1}(x)$ are submanifolds of G_1 for all $x \in G_0$, and that all isotropy groups are Lie groups. The basic reference on Lie groupoids is Mackenzie's book [Ma]; also see [Co2, La3].

A Lie group defines its Lie algebra; one may also study Lie algebras in their own right. The situation for Lie groupoids is similar.

DEFINITION 2.2. A Lie algebroid on a manifold M is a vector bundle E over M , which apart from the bundle projection $\tau : E \rightarrow M$ is equipped with a vector bundle map $\rho : E \rightarrow TM$ (called the anchor), as well as with a Lie bracket $[\cdot, \cdot]_E$ on the space $C^\infty(M, E)$ of smooth sections of E , satisfying

$$(2.2) \quad \rho \circ [X, Y]_E = [\rho \circ X, \rho \circ Y],$$

where the right-hand side is the usual commutator of vector fields on $C^\infty(M, TM)$, and

$$(2.3) \quad [X, fY]_E = f[X, Y]_E + ((\rho \circ X)f)Y$$

for all $X, Y \in C^\infty(M, E)$ and $f \in C^\infty(M)$. We generally omit the suffix E on the Lie bracket.

It is part of the definition of a bundle map that the anchor is fiber-preserving and linear on each fiber. This concept is due to Pradines [Pr2]. The basic reference on Lie algebroids is [Ma]; also see [La3]. The simplest examples of Lie algebroids are tangent bundles TM , where ρ is the identity, and Lie algebras, for which M is a point.

We now explain how one may associate a Lie algebroid $A(G)$ with a given Lie groupoid G [Pr1, Ma, La3]. A left-invariant vector field ξ^L on G_1 is a vector field satisfying $Tr(\xi^L) = 0$ and $TL_\gamma(\xi^L)(\gamma') = \xi^L(\gamma\gamma')$ for all $(\gamma, \gamma') \in G_2$. Here $L_\gamma : G^{s(\gamma)} \rightarrow G^{r(\gamma)}$ is defined for each $\gamma \in G_1$ by

$$(2.4) \quad L_\gamma(\gamma_1) = \gamma\gamma_1.$$

Note that the second condition is well-defined because of the first one. The space of all smooth left-invariant vector fields on G_1 is denoted by $C^\infty(G_1, TG_1)^L$, which is easily shown to be a Lie algebra under the usual commutator borrowed from $C^\infty(G_1, TG_1)$. Also, a left-invariant vector field is obviously determined by its values on the unit space G_0 . The tangent bundle of G_1 at the unit space has a decomposition

$$(2.5) \quad T_x G_1 = T_x G_0 \oplus (\ker Tr)_x,$$

where $\ker Tr \subset TG_1$ is the vector bundle over G_1 whose fiber $(\ker Tr)_\gamma$ above $\gamma \in G_1$ is the kernel of the derivative $T_\gamma r : T_\gamma G_1 \rightarrow T_{r(\gamma)} G_0$ of the range projection $r : G_1 \rightarrow G_0$. The special case $\gamma = x \in G_0$ occurs in (2.5).

DEFINITION 2.3. The Lie algebroid $A(G)$ of a Lie groupoid G is given by the following:

1. The vector bundle $A(G)$ over G_0 is the normal bundle defined by the embedding $G_0 \hookrightarrow G_1$; accordingly, the bundle projection $\tau : A(G) \rightarrow G_0$ is given by s or r (which coincide on G_0).
2. Identifying the normal bundle with $(\ker Tr)|_{G_0}$ by (2.5), the anchor is given by $\rho = Ts : \ker Tr \rightarrow TG_0$.
3. Identifying a section of the normal bundle with an element of $C^\infty(G_1, TG_1)^L$ through the previous item, the Lie bracket $[\cdot, \cdot]_{A(G)}$ is given by the commutator.

The required equality (2.2) is automatically satisfied (as it holds for all vector fields on G_1). We leave the verification of (2.3) to the reader.

It follows from this definition that a Lie algebra \mathfrak{g} is the Lie algebroid of a Lie group G , and that the tangent bundle TG_0 is the Lie algebroid of the pair groupoid $G_1 = G_0 \times G_0$.

For Lie groups one has an exponential map from the Lie algebra to the group. For manifolds M with connection the exponential map is defined on the tangent bundle TM , which it maps into M . As indicated by Pradines [Pr3], these are special cases of a construction that holds for general Lie groupoids G , provided its Lie algebroid $A(G)$ is endowed with a connection. Following [La3, La4], we here present a slightly different construction that is more suitable for our application to deformation quantization; also see [NWX].

First note that the vector bundles $\ker Tr$ and $s^*A(G)$ (over G_1) are isomorphic via the map TL_γ , applied fiberwise. Hence the connection on $A(G)$, with associated horizontal lift $\ell^{A(G)}$, yields a connection on $\ker Tr$ through pull-back. Going through the definitions, one finds that the associated horizontal lift ℓ of a tangent vector $X = \dot{\gamma} = d\gamma(t)/dt|_{t=0}$ in $T_\gamma G_1$ to $Y \in T_\gamma G_1$ is

$$(2.6) \quad \ell_Y(\dot{\gamma}) = \frac{d}{dt}[L_{\gamma(t)*}\ell_{TL_{\gamma^{-1}(Y)}^{A(G)}}(s(\gamma(t)))]_{t=0},$$

which is an element of $T_Y(\ker Tr)$ (here $\ell^{A(G)}(\dots)$ lifts a curve).

Since the bundle $\ker Tr \rightarrow G_1$ has a connection, one can define parallel transport $X \rightarrow X(t)$ on $\ker Tr$ in precisely the same way as on a tangent bundle with affine connection. That is, the flow $X(t)$ is the solution of

$$(2.7) \quad \dot{X}(t) = \ell_{X(t)}(X(t))$$

with initial condition $X(0) = X$. The projection of $X(t)$ to G_1 is a ‘‘geodesic’’ $\gamma_X(t)$.

DEFINITION 2.4. *The left exponential map $\text{Exp}^L : A(G) \rightarrow G_1$ is defined by*

$$(2.8) \quad \text{Exp}^L(X) = \gamma_X(1) = \tilde{\tau}(X(1)),$$

where $\tilde{\tau} : \ker Tr \rightarrow G$ is the restriction of the bundle projection $TG_1 \rightarrow G_1$ to $\ker Tr$. It is assumed that the geodesic flow $X(t)$ on $\ker Tr$ (defined by the connection on $\ker Tr$ pulled back from the one on $A(G)$) is defined at $t = 1$. If not, $\text{Exp}^L(X)$ is undefined.

There is a symmetrized version of Exp^L that plays a key role in our quantization theory. First note that for all $X \in A(G)$ for which $\text{Exp}^L(X)$ is defined one has

$$(2.9) \quad r(\text{Exp}^L(X)) = \tau(X);$$

recall that τ is the bundle projection of the Lie algebroid. To derive this, note that $r(\gamma_X(0)) = \tau(X)$ by construction, and $dr(\gamma_X(t))/dt = Tr(X(t)) = 0$.

We combine this with the obvious $\tau(\frac{1}{2}X) = \tau(-\frac{1}{2}X)$ to infer that

$$r(\text{Exp}^L(\frac{1}{2}X)) = r(\text{Exp}^L(-\frac{1}{2}X)) = s(\text{Exp}^L(-\frac{1}{2}X)^{-1}).$$

Thus the (groupoid) multiplication in (2.10) below is well-defined.

DEFINITION 2.5. *The Weyl exponential map $\text{Exp}^W : A(G) \rightarrow G_1$ is defined by*

$$(2.10) \quad \text{Exp}^W(X) = \text{Exp}^L(-\frac{1}{2}X)^{-1}\text{Exp}^L(\frac{1}{2}X).$$

The following well-known result is a form of the tubular neighbourhood theorem.

PROPOSITION 2.6. *The maps Exp^L and Exp^W are diffeomorphisms from a neighbourhood V of $G_0 \subset A(G)$ (as the zero section) to a neighbourhood W of G_0 in G_1 , such that $\text{Exp}^L(x) = \text{Exp}^W(x) = x$ for all $x \in G_0$.*

Cf. [La3, La4] for a proof.

3. The C^* -algebra of a Lie groupoid

In this section we show that to every Lie groupoid G one can associate a C^* -algebra $C^*(G)$. This C^* -algebra was introduced by Bigonnet [Bg], but the idea to use half densities for an intrinsic construction of the C^* -algebra of a Lie groupoid first appeared in Connes's work [Co1] for the special case of the holonomy groupoid of a foliation. Connes also proposed a general definition in [Co2]; this only works if fiber isomorphisms of the type discussed below are understood. The construction we give in this paper is due to Renault [Re4]; it is essentially the same as that in [La2, La3]. Also see [CW]. Further details on densities as used below may be found in [Tr, La3].

Let V be an n -dimensional vector space and let α be a fixed real number. A density of weight α on V is a function $\rho : \wedge^n(V) \rightarrow \mathbb{C}$ that satisfies

$$\rho(\lambda T) = |\lambda|^\alpha \rho(T)$$

for every $\lambda \in \mathbb{C}$ and $T \in \wedge^n(V)$. The 1-dimensional vector space of the densities of weight α on V is denoted by $|\Omega|^\alpha(V)$. If a density is positive at a point, then it is positive everywhere on $\wedge^n(V) \setminus \{0\}$. Hence one may unambiguously speak of positive densities. We have the canonical isomorphisms $|\Omega|^{\alpha+\beta}(V) \simeq |\Omega|^\alpha(V) \otimes |\Omega|^\beta(V)$ and $|\Omega|^\alpha(V \oplus W) \simeq |\Omega|^\alpha(V) \otimes |\Omega|^\alpha(W)$, where W is another vector space.

Denote the parallelepiped generated by a family v_1, \dots, v_n of vectors in V by $[v_1, \dots, v_n]$. Given a positive 1-density ρ on V , the translation invariant measure μ on V that satisfies $\mu([v_1, \dots, v_n]) = \rho(v_1 \wedge \dots \wedge v_n)$ for every family $\{v_1, \dots, v_n\}$ of vectors is said to be the measure generated by ρ . If $\{e_1, \dots, e_n\}$ is a basis of V and $f \in L^1(V, \mu)$, then

$$(3.1) \quad \int f \left(\sum_{i=1}^n x_i e_i \right) \rho(e_1 \wedge \dots \wedge e_n) dx = \int f d\mu.$$

We now generalize this construction to fibers in vector bundles. Let E be a vector bundle on M , with fibers E_x over $x \in M$. For each $x \in M$, consider $|\Omega|^\alpha(E_x)$, the 1-dimensional vector space of densities of weight α on the vector space E_x . The vector bundle over M with fibers $|\Omega|^\alpha(E_x)$ is denoted by $|\Omega|^\alpha(E)$. The sections of the vector bundle $|\Omega|^\alpha(E)$ over M are called α -densities. An α -density $\mu \in C^\infty(M, |\Omega|^\alpha(E))$ is called positive if $\mu(x)$ is a positive density for every $x \in M$. The function associated to μ in the local frame $e = (e_1, \dots, e_p)$ of E on $U \subset M$ is the map $\mu_e \in C^\infty(U)$ defined by $\mu_e(x) = \mu(x)(e_1(x) \wedge \dots \wedge e_p(x))$.

The set of α -densities on E is a free 1-dimensional module over the ring of functions defined on M . Moreover, $C^\infty(M, |\Omega|^\alpha(E))$ is a free 1-dimensional module over $C^\infty(M)$, and choosing a positive smooth density we can identify densities with functions.

Now let G be a Lie groupoid with Lie algebroid $A(G)$; recall from Definition 2.3 that $A(G)$ is a vector bundle over G_0 . For every real number α , one may associate the vector bundle $|\Omega|^\alpha(A(G))$ to $A(G)$. In addition, form the vector

bundle $|\Omega|^\alpha(\ker Tr)$ of α -densities associated to the vector bundle $\ker Tr \subset TG_1$. (defined after (2.5)).

The groupoid G acts (from the left) on the bundle $|\Omega|^\alpha(\ker Tr)$. To define this action we need a further bit of notation. Let V, W be vector spaces with $\dim V = \dim W$. For every linear map $f : V \rightarrow W$, define $f^\Omega : |\Omega|^\alpha(F) \rightarrow |\Omega|^\alpha(E)$ on $\rho \in |\Omega|^\alpha(F)$, $v_1, v_2, \dots, v_n \in V$ by $(f^\Omega \rho)(v_1 \wedge \dots \wedge v_n) = \rho(f(v_1) \wedge \dots \wedge f(v_n))$. Obviously, if ρ is a positive density then $f^\Omega \rho$ is positive, too.

To the derivative $T_{\gamma\gamma_1} L_{\gamma^{-1}} : T_{\gamma\gamma_1} G^{r(\gamma)} \rightarrow T_{\gamma_1} G^{s(\gamma)}$ of L_γ (cf. (2.4)) one associates the corresponding map between the spaces of densities $(T_{\gamma\gamma_1} L_{\gamma^{-1}})^\Omega : |\Omega|^\alpha(T_{\gamma_1} G^{s(\gamma)}) \rightarrow |\Omega|^\alpha(T_{\gamma\gamma_1} G^{r(\gamma)})$. For simplicity, we denote $(T_{\gamma\gamma_1} L_{\gamma^{-1}})^\Omega$ by γ , and have an action of G on $|\Omega|^\alpha(\ker Tr)$ given by $\gamma : |\Omega|^\alpha(T_{\gamma_1} G^{s(\gamma)}) \rightarrow |\Omega|^\alpha(T_{\gamma\gamma_1} G^{r(\gamma)})$, where $\gamma_1 \in G^{s(\gamma)}$.

Denote the vector bundle $r^*|\Omega|^{1/2}(A(G)) \otimes s^*|\Omega|^{1/2}(A(G))$ over G by \mathfrak{G} . The fiber of \mathfrak{G} over $\gamma \in G$ is $\mathfrak{G}_\gamma = |\Omega|^{1/2}(T_{r(\gamma)} G^{r(\gamma)}) \otimes |\Omega|^{1/2}(T_{s(\gamma)} G^{s(\gamma)})$.

DEFINITION 3.1. *The convolution algebra of a Lie groupoid G is the space $C_c^\infty(G_1, \mathfrak{G})$ of smooth compactly supported sections of the vector bundle \mathfrak{G} over G_0 , with product $f * g$ of $f, g \in C_c^\infty(G_1, \mathfrak{G})$ defined by*

$$(3.2) \quad (f * g)(\gamma) = \int_{G^{r(\gamma)}} (\text{id} \otimes \gamma_1) f(\gamma_1) (\gamma_1 \otimes \text{id}) g(\gamma_1^{-1} \gamma),$$

where id is the identity map, and involution given by

$$(3.3) \quad f^*(\gamma) = \widetilde{f(\gamma^{-1})}.$$

Here the map

$$\sim : |\Omega|^{1/2}(T_x G^x) \otimes |\Omega|^{1/2}(T_y G^y) \rightarrow |\Omega|^{1/2}(T_y G^y) \otimes |\Omega|^{1/2}(T_x G^x)$$

is defined by $a(x) \otimes b(y) \mapsto \overline{b(y)} \otimes a(x)$.

It is not difficult to show that these expressions are well defined. Let us show, for example, that the integral in (3.2) is well defined. We extend the action of G on the bundle $|\Omega|^{1/2}(\ker Tr)$ to the tensor product in an obvious way. For $f, g \in C_c^\infty(G_1, \mathfrak{G})$ we see that $(\text{id} \otimes \gamma_1) f(\gamma_1)$ is an element of $|\Omega|^{1/2}(T_{r(\gamma_1)} G^{r(\gamma_1)}) \otimes |\Omega|^{1/2}(T_{\gamma_1} G^{r(\gamma_1)})$, and that $(\gamma_1 \otimes \text{id}) g(\gamma_1^{-1} \gamma)$ lies in $|\Omega|^{1/2}(T_{\gamma_1} G^{r(\gamma_1)}) \otimes |\Omega|^{1/2}(T_{s(\gamma)} G^{s(\gamma)})$. The integrand in (3.2) is then an element of $|\Omega|^{1/2}(T_{r(\gamma)} G^{r(\gamma)}) \otimes |\Omega|^{1/2}(T_{\gamma_1} G^{r(\gamma_1)}) \otimes |\Omega|^{1/2}(T_{s(\gamma)} G^{s(\gamma)})$, so that it may be integrated as a 1-density on the manifold $G^{r(\gamma)}$. Hence we obtain an element of $|\Omega|^{1/2}(T_{r(\gamma)} G^{r(\gamma)}) \otimes |\Omega|^{1/2}(T_{s(\gamma)} G^{s(\gamma)})$.

PROPOSITION 3.2. *The map $\lambda \mapsto \lambda|_{G_0}$ is a bijection between the left invariant sections of $|\Omega|^\alpha(\ker Tr)$ and the sections of $|\Omega|^\alpha(A(G))$. This bijection preserves smoothness as well as positivity.*

PROOF. The restriction is well defined, since $|\Omega|^\alpha(\ker Tr)$ contains $|\Omega|^\alpha(A(G))$. To prove that the map in question is bijective, it is sufficient to check that its inverse is $\rho \mapsto \lambda$, where $\rho \in |\Omega|^\alpha(A(G))$ and $\lambda(\gamma) = \gamma \cdot \rho(s(\gamma))$. \equiv

We now define the groupoid C^* -algebra $C^*(G)$ and its reduced counterpart $C_r^*(G)$. Firstly, to endow the convolution $*$ -algebra $C_c^\infty(G_1, \mathfrak{G})$ with a C^* -norm we consider its $*$ -representations on a Hilbert space. Let \mathcal{R} be the set of all $*$ -representations $\pi(C_c^\infty(G_1, \mathfrak{G}))$ for which $f \mapsto \langle \xi, \pi(f)\eta \rangle$ is a Radon measure on

$C_c^\infty(G_1, \mathfrak{G})$ for each pair $\xi, \eta \in H$. One then defines a C^* -norm on $C_c^\infty(G_1, \mathfrak{G})$ by

$$(3.4) \quad \|f\| = \sup_{\pi \in \mathcal{R}} \|\pi(f)\|;$$

cf. [Re1]. The completion of $C_c^\infty(G_1, \mathfrak{G})$ in this norm is the C^* -algebra $C^*(G)$.

Secondly, for every $x \in G_0$ we define an involutive representation π_x of $C_c^\infty(G_1, \mathfrak{G})$ on the Hilbert space $L^2(G^x)$ of half-densities on the manifold G^x by

$$(3.5) \quad [\pi_x(f)\xi](\gamma) = \int_{G^x} (\text{id} \otimes \gamma_1)f(\gamma_1)(\gamma_1 \otimes \text{id})\xi(\gamma_1^{-1}\gamma),$$

for $\gamma \in G^x$, where $\xi \in L^2(G^x)$. Then $\|f\|_r = \sup_{x \in G_0} \|\pi_x(f)\|$ is a norm (cf. [Re1]),

and the completion of $C_c^\infty(G_1, \mathfrak{G})$ in this norm is the reduced C^* -algebra $C_r^*(G)$.

For an amenable groupoid G we have $C_r^*(G) = C^*(G)$. For the proof of this result and a detailed discussion of amenable groupoids see [AR].

Recall the following concept (Def. I.2.2 in [Re1]).

DEFINITION 3.3. *A left Haar system on a locally compact groupoid G is a family of measures $(\lambda^x)_{x \in G_0}$ such that the support of λ^x is G^x , the system is left-invariant under the map $L_\gamma : G^{s(\gamma)} \rightarrow G^{r(\gamma)}$, and for each $\phi \in C_c(G_1)$, the map*

$$(3.6) \quad x \mapsto \lambda(\phi)(x) = \int \phi d\lambda^x$$

defines a continuous function on G_0 .

If G is a Lie groupoid, we say that the Haar system is smooth if each such function is smooth for $\phi \in C_c^\infty(G_1)$.

As explained in [Re1], one can associate C^* -algebras $C^*(G, \lambda)$ and $C_r^*(G, \lambda)$ to a locally compact groupoid G with Haar system (λ^x) . Recall that convolution and involution are given on the dense subalgebra $C_c(G_1)$ by

$$(3.7) \quad f * g(\gamma) = \int_{G^{s(\gamma)}} f(\gamma\gamma_1)g(\gamma_1^{-1})d\lambda^{s(\gamma)}(\gamma_1);$$

$$(3.8) \quad f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

This turns $C_c(G_1)$ into a $*$ -algebra, which we denote by $C_c(G, \lambda)$. Similarly, if G is smooth one has the $*$ -algebra $C_c^\infty(G, \lambda)$. The C^* -algebras $C^*(G, \lambda)$ and $C_r^*(G, \lambda)$ are completions of $C_c(G, \lambda)$ or $C_c^\infty(G, \lambda)$ in suitable C^* -norms.

Not every locally compact groupoid admits a left Haar system; in the theory of groupoid C^* -algebras one therefore usually postulates its existence. For Lie groupoids the situation is more favourable.

PROPOSITION 3.4. *Any Lie groupoid G admits a smooth left Haar system $(\lambda^x)_{x \in G_0}$. The associated convolution $*$ -algebra $C_c^\infty(G, \lambda)$ is isomorphic to $C_c^\infty(G_1, \mathfrak{G})$, and the associated C^* -algebras $C^*(G, \lambda)$ and $C_r^*(G, \lambda)$ are isomorphic to $C^*(G)$ and $C_r^*(G)$, respectively.*

PROOF. Let λ be a smooth positive section of $|\Omega|^1(A(G))$. By Proposition 3.2, λ can be extended to a smooth, positive, and left invariant section λ_G of $|\Omega|^1(\ker Tr)$. Then $\lambda_G^{1/2}$ is a smooth left invariant section of $|\Omega|^{1/2}(\ker Tr)$, and $\gamma \mapsto \lambda_G^{1/2}(r(\gamma)) \otimes \lambda_G^{1/2}(s(\gamma))$ is a smooth section of \mathfrak{G} . For every section $f \in C_c^\infty(G_1, \mathfrak{G})$ there exists a function $f_\lambda \in C_c^\infty(G_1)$ such that

$$(3.9) \quad f(\gamma) = f_\lambda(\gamma)\lambda_G^{1/2}(r(\gamma)) \otimes \lambda_G^{1/2}(s(\gamma)).$$

An easy calculation shows that for $f, g \in C_c^\infty(G_1, \mathfrak{G})$ we have

$$(f * g)(\gamma) = (f_\lambda * g_\lambda)(\gamma) \lambda_G^{1/2}(r(\gamma)) \otimes \lambda_G^{1/2}(s(\gamma)),$$

where

$$(3.10) \quad (f_\lambda * g_\lambda)(\gamma) = \int_{G^{r(\gamma)}} f_\lambda(\gamma_1) g_\lambda(\gamma_1^{-1} \gamma) \lambda_G(\gamma_1).$$

The right-hand side of the last equality is well defined as the integral of the 1-density $f_\lambda(\gamma_1) g_\lambda(\gamma_1^{-1} \gamma) \lambda_G(\gamma_1)$ over the manifold $G^{r(\gamma)}$. It is also easy to see that

$$(3.11) \quad f^*(\gamma) = f_\lambda^*(\gamma) \lambda_G^{1/2}(r(\gamma)) \otimes \lambda_G^{1/2}(s(\gamma)),$$

where $f_\lambda^*(\gamma) = \overline{f_\lambda(\gamma^{-1})}$.

Let $x \in G_0$. The restriction of λ_G to the submanifold G^x is a 1-density, hence it has an associated measure on G^x . As we identify the 1-densities and their associated measures, we use the same notation λ^x for the restriction of λ_G to G^x and the induced measure. The equality $\gamma \cdot \lambda^{s(\gamma)} = \lambda^{r(\gamma)}$ proves that the λ^x form a left Haar system; it is easy to check that each function (3.6) is smooth.

It is easily checked that, under the correspondence $f \leftrightarrow f_\lambda$ defined in (3.9) and the replacement of λ_G by the left Haar system $(\lambda^x)_{x \in G_0}$, the expressions (3.10) and (3.11) are transformed into (3.7) and (3.8), respectively. We leave the proof of the isomorphisms of the completions of the *-algebras in question to the reader. \equiv

4. Strict deformation quantization

As we have seen, the idea of C^* -algebraic deformation quantization is to relate a given Poisson algebra to a family of C^* -algebras.

DEFINITION 4.1. *A Poisson algebra is a complex commutative associative algebra equipped with a Lie bracket $\{, \}$ for the which the Leibniz rule holds; that is, one has*

$$(4.1) \quad \{f, gh\} = \{f, g\}h + g\{f, h\}.$$

A Poisson manifold is a manifold P equipped with a Lie bracket $\{, \}$ on $C^\infty(P)$ that together with pointwise multiplication turns $C^\infty(P)$ into a Poisson algebra.

This definition is due to Lichnerowicz [Li] and Kirillov [Ki]. The Lie bracket in a Poisson algebra is usually called the Poisson bracket. For the theory of Poisson manifolds we refer to these papers, and to the textbooks [MR, Va]. A Poisson algebra is the classical analogue of a C^* -algebra [La3]. This analogy is clear if one sees a C^* -algebra as a non-associative version of a Poisson algebra, in which the commutative product is given by the anti-commutator $xy + yx$ and the Lie bracket is $\{x, y\} = i(xy - yx)$. The Leibniz rule is then satisfied. The analogy is even better when one defines a Poisson algebra as a real algebra, and restricts the above two operations to the self-adjoint part of a C^* -algebra.

In this paper, the key example of a Poisson algebra is provided by the dual bundle E^* of a Lie algebroid E over M (cf. Definition 2.2). This Poisson structure

was discovered in [CDW, Cou]. The corresponding bracket is most easily defined by listing special cases by which it is uniquely determined; these are

$$(4.2) \quad \{f, g\} = 0;$$

$$(4.3) \quad \{\tilde{X}, f\} = (\rho \circ X)f;$$

$$(4.4) \quad \{\tilde{X}, \tilde{Y}\} = \widetilde{[X, Y]}_{A(G)}.$$

Here $f, g \in C^\infty(M)$ are regarded as functions on E^* in the obvious way, and $\tilde{X} \in C^\infty(E^*)$ is defined by a section $X \in C^\infty(M, E)$ through $\tilde{X}(\theta) = \theta(X(\tau^*(\theta)))$. See [CDW] for an intrinsic definition. In particular, a Lie groupoid G canonically determines a Poisson algebra $C^\infty(A^*(G))$. This is the generic Poisson algebra that we are going to quantize.

Our first definition of C^* -algebraic deformation quantization uses the concept of a continuous field of C^* -algebras. We here state a reformulation of Dixmier's familiar definition [Di] due to Kirchberg–S. Wassermann [KW], which is tailor made for our applications.

DEFINITION 4.2. *A continuous field of C^* -algebras $(C, \{A_t, \varphi_t\}_{t \in T})$ over a locally compact Hausdorff space T consists of a C^* -algebra C , a collection of C^* -algebras $\{A_t\}_{t \in T}$, and a set $\{\varphi_t : C \rightarrow A_t\}_{t \in T}$ of surjective $*$ -homomorphisms, such that for all $c \in C$*

1. *the function $t \rightarrow \|\varphi_t(c)\|$ is in $C_0(T)$;*
2. *one has $\|c\| = \sup_{t \in T} \|\varphi_t(c)\|$;*
3. *there is an element $fc \in C$ for any $f \in C_0(T)$ for which $\varphi_t(fc) = f(t)\varphi_t(c)$ for all $t \in T$.*

The continuous cross-sections of the field in the sense of [Di] consist of those elements $\{a_t\}_{t \in T}$ of $\prod_{t \in T} A_t$ for which there is a (necessarily unique) $a \in C$ such that $a_t = \varphi_t(a)$ for all $t \in T$.

Our first definition of C^* -algebraic deformation quantization is now as follows [La3, La4].

DEFINITION 4.3. *A strict deformation quantization of a Poisson manifold P consists of*

1. *a dense Poisson algebra $A^0 \subset C_0(P)$ under the given Poisson bracket $\{, \}$ on P ;*
2. *A subset $I \subseteq \mathbb{R}$ containing 0 as an accumulation point;*
3. *a continuous field of C^* -algebras $(C, \{A_{\hbar}, \varphi_{\hbar}\}_{\hbar \in I})$, with $A_0 = C_0(P)$;*
4. *a linear map $\mathcal{Q} : A^0 \rightarrow C$ that satisfies (with $\mathcal{Q}_{\hbar}(f) \equiv \varphi_{\hbar}(\mathcal{Q}(f))$)*

$$(4.5) \quad \mathcal{Q}_0(f) = f;$$

$$(4.6) \quad \mathcal{Q}_{\hbar}(f^*) = \mathcal{Q}_{\hbar}(f)^*,$$

for all $f \in A^0$ and $\hbar \in I$, and for all $f, g \in A^0$ satisfies Dirac's condition

$$(4.7) \quad \lim_{\hbar \rightarrow 0} \|(i\hbar)^{-1}[\mathcal{Q}_{\hbar}(f), \mathcal{Q}_{\hbar}(g)] - \mathcal{Q}_{\hbar}(\{f, g\})\| = 0.$$

In view of the comment after Definition 4.2, for fixed $f \in A^0$ each family $\{\mathcal{Q}_{\hbar}(f)\}_{\hbar \in I}$ is a continuous cross-section of the continuous field in question. In view of (4.5) this implies, in particular, that

$$(4.8) \quad \lim_{\hbar \rightarrow 0} \|\mathcal{Q}_{\hbar}(f)\mathcal{Q}_{\hbar}(g) - \mathcal{Q}_{\hbar}(fg)\| = 0.$$

This shows that a strict deformation quantization yields asymptotic morphisms in the sense of E -theory [Co2]. See [La3] for an extensive discussion of quantization theory from the above perspective.

Every good definition is the hypothesis of a theorem. Indeed, our first main result [La3, La4] is as follows.

THEOREM 4.4. *Let G be a Lie groupoid, with associated C^* -algebras $C^*(G)$ and $C_r^*(G)$, and Poisson manifold $A^*(G)$. There exists a strict deformation quantization of $A^*(G)$ for which $I = \mathbb{R}$, $A_0 = C_0(A^*(G))$, and $A_{\hbar} = C^*(G)$ for $\hbar \neq 0$. In particular, there exists a continuous field of C^* -algebras over \mathbb{R} with these fibers. The same claim holds with $C^*(G)$ replaced by $C_r^*(G)$.*

A different definition of C^* -algebraic deformation quantization, which generalizes Rieffel's original definition [Ri2], was introduced in [Ra1]. This definition is closer to the notion of formal deformation quantization introduced in [BFFLS] than Definition 4.3.

DEFINITION 4.5. *Let P be a Poisson manifold. A semi-strict deformation quantization of P consists of*

1. a dense Poisson algebra $A^0 \subset C_0(P)$ under the given Poisson bracket $\{, \}$ on P ;
2. A subset $I \subseteq \mathbb{R}$ containing 0 as an accumulation point;
3. For each $h \in I$, an associative product \times_h , an involution *h , and a C^* -seminorm $\|\cdot\|_h$ on A^0 , such that
 - (a) For $h = 0$ we recover the usual product, involution and norm of $C_0(P)$;
 - (b) For each $f \in A^0$, the map $h \mapsto \|f\|_h$ is continuous;
 - (c) For all $f, g \in A^0$ one has

$$(4.9) \quad \lim_{\hbar \rightarrow 0} \|(i\hbar)^{-1}(f \times_{\hbar} g - g \times_{\hbar} f) - \{f, g\}\|_{\hbar} = 0.$$

A strict deformation quantization cannot necessarily be turned into a semi-strict one, since $\mathcal{Q}(A^0)\mathcal{Q}(A^0)$ is not necessarily contained in $\mathcal{Q}(A^0)$, so that one may not be able to transfer the products on the A_{\hbar} to an \hbar -dependent product on A^0 . Conversely, a semi-strict deformation quantization is not necessarily strict, for the seminorms can fail to be norms. However, let $f \in A^0$ be a non-null function. Then $\|f\|_0 \neq 0$ and by the condition 3.2 in Definition 4.5 one has $\|f\|_h \neq 0$ for h in a neighborhood of 0.

For each $h \in I$, $N_h = \{f \in A^0 \mid \|f\|_h = 0\}$ is a two-sided closed ideal in $(A^0, \times_h, {}^*h, \|\cdot\|_h)$, and A^0/N_h is a pre- C^* -algebra, with completion A_{\hbar}^- . Defining \mathcal{Q}_{\hbar} as the canonical projection of A^0 onto A^0/N_h , we almost obtain a strict deformation quantization of P ; the only difficulty is that the A_{\hbar}^- may not form a continuous field of C^* -algebras.

The counterpart of Theorem 4.4 for strict deformation quantization is as follows [Ra1].

THEOREM 4.6. *Let $A(G)$ be the Lie algebroid of a Lie groupoid, with associated Poisson manifold $A^*(G)$. There exists a semi-strict deformation quantization of $A^*(G)$ over $I = \mathbb{R}$.*

As explained above, this theorem does not follow from Theorem 4.4, though the proofs have many elements in common. We now briefly outline the organization of the proofs. Sections 5 and 6 address the continuous field of C^* -algebras claimed to

exist in Theorem 4.4, and in similar vein contain the heart of the proof of condition (b) of Definition 4.5 in Theorem 4.6. Section 7 develops the theory of the Fourier transform on vector bundles. In section 8 this theory is used to define the map \mathcal{Q} of Theorem 4.4 as well as the product \times_{\hbar} , the involution $*_{\hbar}$, and the seminorm $\|\cdot\|_{\hbar}$ of Theorem 4.6. The proofs of both theorems are then complete up to Dirac's condition (4.7) and (4.9). Section 9 develops local techniques that give detailed information on the Poisson structure on $A^*(G)$. Finally, (4.7) and (4.9) are proved in section 10.

5. Continuous fields of groupoids

Rieffel has developed useful techniques for proving continuity of fields of C^* -algebras occurring in examples of strict deformation quantization in the context of groups [Ri1]. The main result of this section, Theorem 5.5, generalizes these results to the context of groupoids. This theorem, which is an application of results of Blanchard [Bl], gives information on the field of C^* -algebras associated to a continuous field of locally compact groupoids. Although the examples studied in this paper concern Lie groupoids, we prove Theorem 5.5 for the general case of locally compact groupoids.

We first mention a known result. Recall that, when $E \subset G_0$, G_E is the subgroupoid of G consisting of all $\gamma \in G_1$ for which $r(\gamma) \in E$ and $s(\gamma) \in E$.

PROPOSITION 5.1. *Let G be a locally compact groupoid with left Haar system $(\lambda^x)_{x \in G_0}$, and let U be an open invariant subset of G_0 . Write $F = G_0 \setminus U$.*

1. *The following sequence is exact:*

$$(5.1) \quad 0 \longrightarrow C^*(G_U) \xrightarrow{e} C^*(G) \xrightarrow{i^*} C^*(G_F) \longrightarrow 0.$$

Here e is firstly defined as map from $C_c(G_U)$ to $C_c(G_1)$ by extending a function on G_U to one on G_1 by making it zero on the complement of G_U , and secondly extended to a map from $C^(G_U)$ to $C^*(G)$ by continuity. Similarly, i^* is firstly defined from $C_c(G_1)$ to $C_c(G_F)$ as the pullback of the inclusion $G_F \hookrightarrow G_1$, and then extended by continuity.*

2. *If the groupoid G_F is amenable, then the following sequence is exact:*

$$(5.2) \quad 0 \longrightarrow C_r^*(G_U) \xrightarrow{e} C_r^*(G) \xrightarrow{i^*} C_r^*(G_F) \longrightarrow 0.$$

This result was given by Torpe [To] in the case of C^* -algebras associated to foliations, and by Hilsun and Skandalis [HS] in the general case; also see [Re1], p.102. A complete proof, based on Renault's theorem of disintegration of representations [Re2], may be found in [Ra1]. A counterexample that shows that the second sequence of reduced C^* -algebras is not necessarily exact for non-amenable G_F is given in [Re3].

DEFINITION 5.2. *A field of groupoids is a triple (G, T, p) , with G a groupoid, T a set, and $p : G \rightarrow T$ a surjective map such that $p = p_0 \circ r = p_0 \circ s$, where $p_0 = p|_{G_0}$. If G is locally compact, T Hausdorff, and p continuous and open, we say that (G, T, p) is a continuous field of locally compact groupoids. If G is a Lie groupoid, T a manifold, and p a submersion, (G, T, p) is called a smooth field of Lie groupoids.*

If (G, T, p) is a field of groupoids and $Y \subset T$, then $A = p_0^{-1}(Y)$ is an invariant subset of G_0 , and $G_A = p^{-1}(Y)$ is a subgroupoid of G . In the case of a continuous field of locally compact groupoids,

$$(5.3) \quad G(t) = p^{-1}(\{t\})$$

is a closed locally compact subgroupoid of G for every $t \in T$. In the case of a smooth field of Lie groupoids, $G(t)$ is a Lie groupoid for every $t \in T$.

In this section, (G, T, p) is a continuous field of locally compact groupoids over a locally compact Hausdorff space T , for which there exists a left Haar system $(\lambda^x)_{x \in G_0}$ on G .

Let $C_0(T)$ be the C^* -algebra of continuous functions on T that vanish at infinity. We define a structure of $C_0(T)$ left module on the space of continuous compactly supported functions $C_c(G_1)$ by $(fa)(\gamma) = f(p(\gamma))a(\gamma)$, where $f \in C_0(T)$ and $a \in C_c(G_1)$. For every $f \in C_0(T)$ and $a, b \in C_c(G_1)$, the following equalities hold:

$$(5.4) \quad \begin{aligned} f(a * b) &= (fa) * b = a * (fb); \\ (fa)^* &= f^* a^*. \end{aligned}$$

We also have $C_0(T)C_c(G_1) = C_c(G_1)$. Indeed, for every $a \in C_c(G_1)$, there is a function $f \in C_c(T)$ such that $f = 1$ on $p(\text{supp } a)$, and then $a = fa \in C_0(T)C_c(G_1)$.

An easy rewriting of Lemma 1.13 from [Re1] proves

LEMMA 5.3. *For every representation $\pi : C_c(G_1) \rightarrow \mathcal{L}(\mathcal{H})$ that is continuous for the inductive limit topology there is a unique continuous representation $\phi : C_0(T) \rightarrow \mathcal{L}(\mathcal{H})$ such that $\pi(fa) = \phi(f)\pi(a)$.*

Using the previous lemma for $f \in C_0(T)$, $a \in C_c(G_1)$, and π continuous representation of $C_c(G_1)$, we have $\|\pi(fa)\| \leq \|\phi(f)\| \|\pi(a)\| \leq \|f\| \|a\|$, hence $\|fa\| \leq \|f\| \|a\|$. For every $f \in C_0(T)$, the map $C_c(G_1) \ni a \mapsto fa \in C_c(G_1)$ has a continuous extension on $C^*(G)$. This provides a $C_0(T)$ Banach module structure on $C^*(G)$.

Corollary 1.9 in [B1] shows that $C_0(T)C^*(G)$ is closed in $C^*(G)$. But one has $C_0(T)C^*(G) \supset C_0(T)C_c(G_1) = C_c(G_1)$, and we obtain that $C^*(G)$ is a nondegenerate module. For every $a, b \in C^*(G)$ and every $f \in C_0(T)$, the conditions $f(a * b) = (fa) * b = a * (fb)$ and $(fa)^* = f^* a^*$ are easily verified, hence $C^*(G)$ is a $C_0(T)$ *-algebra (cf. [B1]). Similar arguments prove that $C_r^*(G)$ is a $C_0(T)$ *-algebra.

LEMMA 5.4. *Let T, Y be topological spaces, $p : Y \rightarrow T$ an open onto map, and $\varphi : Y \rightarrow \mathbb{R}$ an upper semicontinuous function such that for every $t \in T$ one has $\sup\{\varphi(y) | p(y) = t\} < \infty$. Then the function $\psi : T \rightarrow \mathbb{R}$ given by $\psi(t) = \sup_{p(y)=t} \varphi(y)$ is lower semi-continuous.*

PROOF. Let t_i be a net converging in T to t , and pick $a \in \mathbb{R}$. We show that $\psi(t_i) \leq a$ implies $\psi(t) \leq a$. Let $y \in Y$ be such that $p(y) = t$. A lemma on open maps (cf. [FD], p.126) proves the existence of a net $y_i \in Y$, for which $p(y_i) = t_i$ and $y_i \rightarrow y$. For every i , $\varphi(y_i) \leq \psi(t_i) \leq a$. But φ is lower semicontinuous, so $\varphi(y) \leq a$ and $\psi(t) \leq a$. \equiv

We now come to the main result of this section.

THEOREM 5.5. *Let (G, T, p) be a continuous field of locally compact groupoids, take $a \in C_c(G_1)$, and, for each $t \in T$, write $a_t = a|_{G(t)}$. Then*

1. The map $t \mapsto \|a_t\|_{C^*(G(t))}$ is upper semicontinuous.
2. The map $t \mapsto \|a_t\|_{C_r^*(G(t))}$ is lower semicontinuous.

PROOF. 1. For $t \in T$, $C_t(T) = \{f \in C_0(T) \mid f(t) = 0\}$ is a closed two sided ideal, hence $C_t(T)C^*(G)$ is a closed two sided ideal in $C^*(G)$. Corollary 1.9 in [Bl] applied to the $C_t(T)$ Banach module $C^*(G)$ shows that $C_t(T)C^*(G)$ is closed in $C^*(G)$. Let $\pi_t : C^*(G) \rightarrow C^*(G)/C_t(T)C^*(G)$ be the quotient map. By Lemma 1.10 from [Bl] one has $\|\pi_t(a)\| = \inf_{f \in C_0(T)} \|(1-f+f(t))a\|$; hence the map $t \mapsto \|\pi_t(a)\|$ is upper semicontinuous as the infimum of a family of continuous functions.

Denote the closed invariant subset $p_0^{-1}(\{t\})$ of G_0 by F , and let $U = G_0 \setminus F$. Obviously, $G(t) = G_F$. We claim that

$$(5.5) \quad i(C_c(G_U)) = C_t(T)C_c(G),$$

where $i : C^*(G_U) \rightarrow C^*(G)$ is as in Proposition 5.1. Let $a \in i(C_c(G_U))$. In T there is an open neighborhood V of the compact set $p(\text{supp } a)$ such that $t \notin V$. We take $f \in C_c(T)$ with $f = 1$ on $p(\text{supp } a)$ and $\text{supp } f \subset V$, and then $a = fa \in C_t(T)C_c(G_1)$. The other inclusion is obvious.

Taking the norm closure in (5.5), we obtain $i(C^*(G_U)) = C_t(T)C^*(G)$. With Proposition 5.1 this shows that $C^*(G(t)) = C^*(G)/C_t(T)C^*(G)$, so $\|\pi_t(a)\| = \|a_t\|$, which ends the proof.

2. The left regular representation $\pi_x^L(f) : L^2(G_1, \lambda^x) \rightarrow L^2(G_1, \lambda^x)$ of $C_c(G_1)$ associated to $x \in G_0$ is given by

$$(5.6) \quad \pi_x^L(f)\xi(\gamma) = \int_{G^x} f(\gamma\gamma_1)\xi(\gamma_1^{-1})d\lambda^x(\gamma_1).$$

Set $\|\xi\|_\infty = \sup_{x \in G_0} \|\xi\|_{L^2(G_1, \lambda^x)}$; then

$$\|\pi_x^L(f)\| = \sup \{ \langle \pi_x^L(f)\xi, \eta \rangle \mid \xi, \eta \in C_c(G_1), \|\xi\|_\infty < \infty, \|\eta\|_\infty < \infty \}.$$

The map $x \mapsto \langle \pi_x^L(f)\xi, \eta \rangle$ is continuous for each $f \in C_c(G_1)$ and $\xi, \eta \in C_c(G_1)$, hence $x \mapsto \|\pi_x^L(f)\|$ is lower semicontinuous. But $\|a_t\| = \sup_{t \in p^{-1}(t)} \|\pi_x^L(a_t)\|$, and

Lemma 5.4 ends the proof. ≡

COROLLARY 5.6. *If (G, T, p) is a continuous field of locally compact groupoids and $a \in C_c(G_1)$, then*

1. The maps $t \mapsto \|a_t\|$ and $t \mapsto \|a_t\|_r$ are continuous at every $t \in T$ for which the groupoid $G(t)$ is amenable.
2. If G is amenable, then the maps $t \mapsto \|a_t\|$ and $t \mapsto \|a_t\|_r$ are continuous.

PROOF. 1. Straightforward.

2. By Proposition 5.2.3 from [AR], G is amenable if and only if $G(t)$ is amenable for every $t \in T$. ≡

6. The tangent groupoid

The tangent groupoid of a manifold was introduced by Connes [Co2]. This idea was generalized to arbitrary Lie groupoids by Hilsun and Skandalis [HS] under the name normal groupoid, and by Weinstein [We1] under the name blowup. Connes's construction corresponds to the pair groupoid of a manifold. We here use the name "tangent groupoid" also for the general case.

DEFINITION 6.1. *Let G be a Lie groupoid with Lie algebroid $A(G)$. The tangent groupoid \hat{G} is a Lie groupoid with base $\hat{G}_0 = [0, 1] \times G_0$, defined by the following structures.*

- *As a set, $\hat{G}_1 = A(G) \cup \{(0, 1] \times G_1\}$. We write elements of \hat{G} as pairs (\hbar, u) , where $u \in A(G)$ for $\hbar = 0$ and $u \in G$ for $\hbar \neq 0$. Thus $A(G)$ is identified with $\{0\} \times A(G)$.*
- *As a groupoid, $\hat{G} = \{0 \times A(G)\} \cup \{(0, 1] \times G\}$. Here $A(G)$ is regarded as a Lie groupoid over G_0 with $s = r = \tau$ (the bundle projection of $A(G)$), and addition in the fibers as the groupoid multiplication. The groupoid operations in $(0, 1] \times G$ are those in G . For example, for $\hbar \neq 0$ one has*

$$(6.1) \quad \hat{r}(\hbar, u) = (\hbar, r(u)).$$

- *The smooth structure on \hat{G}_1 , making it a manifold with boundary, is as follows. To start, the open subset $\mathcal{O}_1 = (0, 1] \times G_1 \subset \hat{G}_1$ inherits the product manifold structure. Let $G_0 \subset V \subset A(G)$ and $\iota(G_0) \subset W \subset G_1$, as in Theorem 2.6. Let \mathcal{O} be the open subset of $[0, 1] \times A(G)$ (equipped with the product manifold structure; this is a manifold with boundary, since $[0, 1]$ is), defined as $\mathcal{O} = \{(\hbar, X) \mid \hbar X \in V\}$. Note that $\{0\} \times A(G) \subset \mathcal{O}$. The map $\psi : \mathcal{O} \rightarrow \hat{G}_1$ is defined by*

$$(6.2) \quad \begin{aligned} \psi(0, X) &= (0, X); \\ \psi(\hbar, X) &= (\hbar, \text{Exp}^W(\hbar X)). \end{aligned}$$

Since $\text{Exp}^W : V \rightarrow W$ is a diffeomorphism (cf. Proposition 2.6) we see that ψ is a bijection from \mathcal{O} to $\mathcal{O}_2 = \{0 \times A(G)\} \cup \{(0, 1] \times W\}$. This defines the smooth structure on \mathcal{O}_2 in terms of the smooth structure on \mathcal{O} . Since \mathcal{O}_1 and \mathcal{O}_2 cover \hat{G}_1 , this specifies a smooth structure on \hat{G}_1 .

Using Exp^L rather than Exp^W in this definition, one obtains the smooth structure defined in [HS], which is equivalent to the one constructed above. A proof that the change of coordinates from \mathcal{O}_1 to \mathcal{O}_2 on their intersection is smooth may be found in [HS].

We omit the proof that \hat{G} is a Lie groupoid; see [HS], and, for full details, [Ra1]. It follows that \hat{G} is a smooth field of Lie groupoids over \mathbb{R} .

PROPOSITION 6.2. *Let G be a Lie groupoid with tangent groupoid \hat{G} . Writing $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, the following sequences are exact:*

$$(6.3) \quad 0 \rightarrow C_0(\mathbb{R}^*) \otimes C^*(G) \rightarrow C^*(\hat{G}) \rightarrow C_0(A^*(G)) \rightarrow 0;$$

$$(6.4) \quad 0 \rightarrow C_0(\mathbb{R}^*) \otimes C_r^*(G) \rightarrow C_r^*(\hat{G}) \rightarrow C_0(A^*(G)) \rightarrow 0.$$

PROOF. Consider the open invariant subset $U = G_0 \times \mathbb{R}^*$ of \hat{G}_0 , with complement $F = G_0 \setminus U$. Then $\hat{G}_U = G \times \mathbb{R}^*$ and $\hat{G}_F = A(G) \times \{0\}$. By Proposition 5.1 we have the exact sequence

$$0 \rightarrow C^*(G \times \mathbb{R}^*) \rightarrow C^*(\hat{G}) \rightarrow C^*(A(G)) \rightarrow 0.$$

Gelfand's theorem implies $C^*(A(G)) = C_0(A^*(G))$ (also cf. section 7), and since $C^*(G \times \mathbb{R}^*)$ is the C^* -algebra of the trivial field of the C^* -algebras $C^*(G)$ on \mathbb{R}^* , we have $C^*(G \times \mathbb{R}^*) = C_0(\mathbb{R}^*) \otimes C^*(G)$.

The second exact sequence is a consequence of Proposition 5.1 as well, since $A(G)$ is commutative as a groupoid (cf. Definition 6.1), and therefore amenable [AR]. \equiv

COROLLARY 6.3. (cf. [Co2]) *Let M be a manifold. Then the tangent groupoid $\widehat{M \times M} = (M \times M \times \mathbb{R}^*) \cup (TM \times \{0\})$ is amenable, and the sequence*

$$(6.5) \quad 0 \rightarrow C_0(\mathbb{R}^*) \otimes \mathcal{K}(L^2(M)) \rightarrow C^*(\widehat{M \times M}) \rightarrow C_0(T^*M) \rightarrow 0$$

is exact.

PROOF. By Proposition 5.2.3 in [AR], a groupoid bundle is amenable if and only if all its fibers are amenable. Since the Lie algebroid $A(G)$ is amenable as a groupoid, the amenability of \hat{G} is reduced to the amenability of the groupoid G . In this case, $C^*(\hat{G}) = C_r^*(\hat{G})$, cf. [Re1, AR]. Now use Proposition 6.2 for the amenable groupoid $G = M \times M$ to finish the proof. \equiv

Recall Definition 4.2, whose notation we adopt.

THEOREM 6.4. *Let G be a Lie groupoid with tangent groupoid \hat{G} . For $\hbar \in \mathbb{R}$, define*

$$(6.6) \quad \begin{aligned} G(0) &= A(G); \\ G(\hbar) &= G \quad \forall \hbar \neq 0. \end{aligned}$$

Define $\hat{\varphi}_\hbar : C_c^\infty(\hat{G}) \rightarrow C_c^\infty(G(\hbar))$ as the pullback of the inclusion $G(\hbar) \hookrightarrow \hat{G}$ (cf. Definition 6.1); in other words, $\hat{\varphi}_\hbar(\hat{f})$ is the restriction of $\hat{f} \in C_c^\infty(\hat{G})$ to $G(\hbar)$.

1. Each $\hat{\varphi}_\hbar$ may be extended by continuity to a surjective *-homomorphism $\varphi_\hbar : C^*(\hat{G}) \rightarrow C^*(G(\hbar))$, and also to a surjective *-homomorphism $\varphi_{(r)\hbar} : C_r^*(\hat{G}) \rightarrow C_r^*(G(\hbar))$.
2. The C^* -algebras $C = C^*(\hat{G})$ and $A_\hbar = C^*(G(\hbar))$, and the maps φ_\hbar form a continuous field of C^* -algebras over $I = \mathbb{R}$.
3. The C^* -algebras $C = C_r^*(\hat{G})$ and $A_\hbar = C_r^*(G(\hbar))$, and the maps $\varphi_{(r)\hbar}$ form a continuous field of C^* -algebras over $I = \mathbb{R}$.

- PROOF.**
1. Using the definition of the norms in question, one checks that each map $\hat{\varphi}_\hbar$ is contractive. The *-homomorphism property is obvious on $C_c^\infty(\hat{G})$ from the definition of \hat{G} and $C^*(G)$, and extends by continuity.
 2. Condition 1 in Definition 4.2 follows at $\hbar \neq 0$ since the field of C^* -algebras is trivial away from 0. At $\hbar = 0$ continuity follows from Corollary 5.6, since $A(G)$, being a commutative groupoid, is amenable [AR].
 3. The reduced case is proved in the same way. \equiv

For a different proof of this theorem, note that each family $\hat{\varphi}_\hbar(\hat{f})$, $\hat{f} \in C_c^\infty(\hat{G})$, is stable under convolution and involution, and dense in $C_{(r)}^*(G)$. One then uses Prop. 10.3.2 in [Di] and the equivalence between Kirchberg–S. Wassermann’s Definition 4.2 and Dixmier’s definition Def. 10.3.1 in [Di] of a continuous field of C^* -algebras. Condition 10.1.2 (ii) in [Di] is then proved in the same way as above.

A different proof of Theorem 6.4, which makes no use of the results in section 5 is possible as well [La3, La4]. This proof is based on the following lemma, due Lee; cf. Theorem 4 in [Le].

LEMMA 6.5. *Let C be a C^* -algebra, and let $\psi : \text{Prim}(C) \rightarrow T$ be a continuous and open map from the primitive spectrum $\text{Prim}(C)$ (equipped with the Jacobson topology **[Di]**) to a locally compact Hausdorff space T . Define $I_t = \cap \psi^{-1}(t)$; i.e., $c \in I_t$ iff $\pi_I(c) = 0$ for all $I \in \psi^{-1}(t)$ (here $\pi_I(C)$ is the irreducible representation whose kernel is I). Note that I_t is a closed two-sided ideal in C .*

Taking $A_t = C/I_t$ and $\varphi_t : C \rightarrow A_t$ to be the canonical projection, $(C, \{A_t, \varphi_t\}_{t \in T})$ is a continuous field of C^ -algebras.*

For a proof see **[Le]** or **[ENN1]**. We apply this lemma with $C = C^*(\hat{G})$ and $T = I = \mathbb{R}$. In order to verify the assumption in the lemma, we first note that $I_0 \simeq C_0((0, 1]) \otimes C^*(G)$, as follows from a glance at the topology of \hat{G} . Hence $\text{Prim}(I_0) = (0, 1] \times \text{Prim}(C^*(G))$, with the product topology. Furthermore, one has $C^*(\hat{G})/I_0 \simeq C^*(A(G)) \simeq C_0(A^*(G))$. Hence $\text{Prim}(C^*(\hat{G})/I_0) \simeq A^*(G)$. Using this in Prop. 3.2.1 in **[Di]**, with $A = C^*(\hat{G})$ and I the ideal I_0 generated by those $f \in C_c^\infty(\hat{G})$ that vanish at $\hbar = 0$, yields the decomposition

$$(6.7) \quad \text{Prim}(C^*(\hat{G})) \simeq A^*(G) \cup \{(0, 1] \times \text{Prim}(C^*(G))\},$$

in which $A^*(G)$ is closed. This does not provide the full topology on $\text{Prim}(C^*(\hat{G}))$, but it is sufficient to know that $A^*(G)$ is not open. If it were, $(0, 1] \times \text{Prim}(C^*(G))$ would be closed in $\text{Prim}(C^*(\hat{G}))$, and this possibility can be excluded by looking at the topology of \hat{G} and the definition of the Jacobson topology.

Using (6.7), we can define a map $\psi : \text{Prim}(C^*(\hat{G})) \rightarrow [0, 1]$ by $\psi(I) = 0$ for all $I \in A^*(G)$ and $\psi(\hbar, I) = \hbar$ for $\hbar \neq 0$ and $I \in \text{Prim}(C^*(G))$. It is clear from the preceding considerations that ψ is continuous and open. Using this in Lemma 6.5, one sees that I_\hbar is the ideal in $C^*(\hat{G})$ generated by those $\hat{f} \in C_c^\infty(\hat{G})$ vanish at \hbar . Hence $A_0 \simeq C_0(A^*(G))$, as above, and $A_\hbar \simeq C^*(G)$ for $\hbar \neq 0$. Theorem 6.4 then follows from Lemma 6.5.

We now give a left Haar system for the tangent groupoid \hat{G} . Firstly, pick a positive section $\mu \in C^\infty(G_0, |\Omega|^1(A(G)))$, and let λ be the associated smooth, positive, left invariant section of $|\Omega|^1(\ker Tr)$, as in Proposition 3.2; hence $\mu = \lambda|_{G_0}$. Now define a section $\hat{\lambda} \in C^\infty(\hat{G}_1, |\Omega|^1(\ker T\hat{r}))$ by

$$(6.8) \quad \begin{aligned} \hat{\lambda}(x, X, 0) &= \lambda(x); \\ \hat{\lambda}(\gamma, \hbar) &= |\hbar|^{-p} \lambda(\gamma). \end{aligned}$$

Here p is the dimension of the typical fiber of $A(G)$; the factor $|\hbar|^{-p}$ is necessary in order to have a smooth system also at $\hbar = 0$, as is easily verified using the manifold structure on \hat{G} . This section is smooth, positive and left invariant, so that it defines a left Haar system $(\hat{\lambda}^{(\hbar, x)})_{(\hbar, x) \in \hat{G}_0}$ for \hat{G} by Proposition 3.4. The $*$ -algebraic structure on $C_c^\infty(\hat{G}, \hat{\lambda})$ defined by (3.7) and (3.8) with (6.8) becomes

$$(6.9) \quad \hat{f} * \hat{g}(0, \xi_x) = \int_{\tau^{-1}(x)} \hat{f}(0, \xi - \eta_x) \hat{g}(0, \eta_x) d\mu^x(\eta_x);$$

$$(6.10) \quad \hat{f} * \hat{g}(\hbar, \gamma) = |\hbar|^{-p} \int_{G^{s(\gamma)}} \hat{f}(\hbar, \gamma\gamma_1) \hat{g}(\hbar, \gamma_1^{-1}) d\lambda^{s(\gamma)}(\gamma_1);$$

$$(6.11) \quad \hat{f}^*(0, \xi) = \overline{\hat{f}(0, -\xi)};$$

$$(6.12) \quad \hat{f}^*(\hbar, \gamma) = \overline{\hat{f}(\hbar, \gamma^{-1})}.$$

Here $\xi_x \in \tau^{-1}(x)$, and μ^x is the measure on $\tau^{-1}(x) \subset A(G)$ generated by the density $\mu(x)$ (cf. (3.1)). Finally, (λ^x) is the left Haar system on G defined by λ according to Proposition 3.4. These formulae should be compared with (3.7) and (3.8).

7. The Fourier transform on vector bundles

In this section we extend the notion of a Fourier transform from \mathbb{R}^n to vector bundles, and show that it remains an isomorphism of a suitably defined Schwartz space of rapidly decreasing functions.

Let M be a manifold of dimension n , and let E be a vector bundle over M with fiber dimension p . Now E is a commutative Lie groupoid if the source and range projections are both equal to the bundle projection, and groupoid multiplication is addition in each fiber. The C^* -algebra $C^*(E)$ is then commutative, too. Using the proposition on page 582 of [FD], it can be shown that the maximal ideal space of $C^*(E)$ is E^* . The Gelfand transform is an isomorphism between the C^* -algebras $C^*(E)$ and $C_0(E^*)$. Its explicit form is a fiberwise Fourier transform $\mathcal{F} : C^*(E) \rightarrow C_0(E^*)$. On the convolution algebra $C_c^\infty(E, \mathfrak{E})$ the Fourier transform is given, for $\theta_x \in E_x^* \subset E^*$, by

$$(\mathcal{F}\rho)(\theta_x) = \int_{E_x} e^{-i\langle \theta_x, \xi_x \rangle} \rho(\xi_x).$$

It is easier to write formulas for functions rather than densities. To do so, fix a positive 1-density $\mu \in C^\infty(M, |\Omega|^1(E))$; cf. the proof of Proposition 3.4. For every $x \in M$, the density $\mu(x) \in |\Omega|^1(E_x)$ defines a translation invariant measure μ^x on E_x , and the family $(\mu^x)_{x \in M}$ is a smooth Haar system for E . The Fourier transform is denoted by $\mathcal{F}_\mu : C^*(E, \mu) \rightarrow C_0(E^*)$ in order to emphasize its dependence on μ . For $f \in L^1(E, d\mu^x)$, $\theta_x \in E_x^*$, we then have

$$(7.1) \quad (\mathcal{F}_\mu f)(\theta_x) = \int f(\xi_x) e^{-i\langle \theta_x, \xi_x \rangle} d\mu^x(\xi_x).$$

We now generalize the definition given by Rieffel in the case of the trivial bundle $M \times \mathbb{R}^p$ (cf. [Ri3]) to define a Schwartz space on E . But first we fix some notations.

The variables in $\mathbb{R}^n \times \mathbb{R}^p$ are denoted (u, v) , and for $\beta \in \mathbb{N}^p$ we put $\partial_v^\beta F = \frac{\partial^{\beta_1 + \dots + \beta_p} F}{\partial v_1^{\beta_1} \dots \partial v_p^{\beta_p}}$. The Fourier transform \mathcal{F}_u on $\mathbb{R}^n \times \mathbb{R}^p$ with u constant is given by

$$(\mathcal{F}_u F)(u, w) = \int F(u, v) e^{-i\langle v, w \rangle} dv.$$

Let (e_1, \dots, e_p) be a local frame of E , with dual frame (e_1^*, \dots, e_p^*) of E^* , let (q, λ) be local coordinates on E , and finally let (q, ϵ) be local coordinates of E^* . The expression of a function f on E in local coordinates is denoted by the corresponding capital letter F . Then (3.1) for $f \in L^1(E, \mu^x)$ becomes

$$(7.2) \quad \int f \left(\sum_{i=1}^n v_i e_i(x) \right) \mu_\epsilon(x) dv = \int f d\mu^x.$$

DEFINITION 7.1. *We say that a function $f : E \rightarrow \mathbb{C}$ is M -compactly supported when $\pi(\text{supp } f)$ is relatively compact in M . The set of continuous M -compactly supported functions is called $C_{c,M}(E)$.*

A function $f \in C_{c,M}(E)$ is said to be rapidly decreasing if $\lambda^\alpha f(q, \lambda)$ is bounded for every $\alpha \in \mathbb{N}^p$, where $\lambda^\alpha = \lambda_1^{\alpha_1} \dots \lambda_p^{\alpha_p}$, with $\lambda_1, \dots, \lambda_p$ the coordinates of λ in the frame (e_1, \dots, e_p) .

This definition is independent of the frame. Indeed, let (e_1, \dots, e_p) be a frame of E on an open subset U of M , and suppose that $\lambda^\alpha f(q, \lambda)$ is bounded for every $\alpha \in \mathbb{N}^p$. If (e'_1, \dots, e'_p) is another frame of E on U , let $g : U \rightarrow GL(p, \mathbb{R})$ be the function given by $e' = eg$. If we write the same point of E in the two frames, its coordinates are related by $\lambda'_i = \sum_j \lambda_j g_{ji}$. To show that $\lambda'^\alpha f(q, \lambda')$ is bounded, remark that expanding $\lambda_1^{\alpha_1} \dots \lambda_p^{\alpha_p}$ we obtain a polynomial in $\lambda_1, \dots, \lambda_p$ with continuous functions on M that vanishes outside the compact closure of $\pi(\text{supp} f)$ as coefficients.

DEFINITION 7.2. We say that a function $f \in C^\infty(E)$ is of Schwartz type on E if f is M -compactly supported and $\partial_\lambda^\beta f = \frac{\partial^{\beta_1 + \dots + \beta_p} f}{\partial \lambda_1^{\beta_1} \dots \partial \lambda_p^{\beta_p}}$ is rapidly decreasing for every $\beta = (\beta_1, \dots, \beta_p) \in \mathbb{N}^p$. The set of Schwartz functions on E is denoted by $S(E)$.

PROPOSITION 7.3. $S(E)$ is a dense $*$ -subalgebra of $C^*(E, \mu)$ and $S(E^*)$ is a dense $*$ -subalgebra of $C_0(E^*)$.

PROOF. We show that $S(E)$ is closed under convolution. For f, g in $S(E)$ it is obvious that $f * g$ is M -compactly supported. Using a partition of unity argument, it easily follows that $f * g \in S(E)$. It is trivial that $S(E)$ is closed under involution. Now $S(E)$ contains the smooth compactly supported functions on E , and by a result from [FD], page 140, $C_c^\infty(E)$ is dense in $C^*(E, \mu)$; it follows that $S(E)$ is dense in $C^*(E, \mu)$.

For the second part of the proposition, $S(E^*)$ is obviously a $*$ -subalgebra of $C_0(E^*)$ containing the dense subalgebra $C_c^\infty(E^*)$ of $C_0(E^*)$. \equiv

LEMMA 7.4. If $f \in S(E)$, then $\mathcal{F}_\mu f \in S(E^*)$

PROOF. The restriction of f to E_x is integrable because it is rapidly decreasing, so (7.1) makes sense in this case. It is then easy to show that $\mathcal{F}_\mu f$ is smooth and M -compactly supported.

Choosing a suitable partition of unity, we may assume that the projection onto M of the support of f is a subset of the domain of a chart $\alpha : U \rightarrow \mathbb{R}^n$ of M . We can also suppose that there exists a frame (e_1, \dots, e_p) of E on U .

The expression of f in the corresponding local coordinates of E is given by $F : \alpha(U) \times \mathbb{R}^p \rightarrow \mathbb{C}$, $F(u, v) = f(\sum v_i e_i(\alpha^{-1}(u)))$. It is straightforward to see that the local expression of $\mathcal{F}_\mu f$, denoted by \hat{F} , satisfies $\hat{F}(u, w) = \mu_e(\alpha^{-1}(u))(\mathcal{F}_u F)(u, w)$. The relation $w^\alpha \partial_w^\beta \hat{F}(u, w) = \mu_e(\alpha^{-1}(u))(-i)^{|\alpha|+|\beta|} \mathcal{F}_u(\partial_v^\alpha (v^\beta F))(u, w)$ reduces the problem to the fact that $\mathcal{F}_u G$ is bounded for every G rapidly decreasing on the trivial bundle $\mathbb{R}^n \times \mathbb{R}^p$ of base \mathbb{R}^n .

But $|\mathcal{F}_u G(u, w)| \leq \int |F(u, v)| dv$, and the proof is finished by the remark that on the right side we have a continuous compactly supported function in u . \equiv

For a vector space V and a nowhere vanishing element ρ of $|\Omega|^\alpha(V)$, let ρ^* be the element of $|\Omega|^\alpha(V^*)$ that satisfies $\rho^*(v_1^* \wedge \dots \wedge v_n^*) = \frac{1}{\rho(v_1 \wedge \dots \wedge v_n)}$ for every

pair of dual bases $\{v_1, \dots, v_n\}$ on V and $\{v_1^*, \dots, v_n^*\}$ on V^* . Then $\mu_*(x) = (\mu(x))^*$ defines a 1-density $\mu_* \in C^\infty(M, |\Omega|^1(E^*))$, and hence a left Haar system $(\mu_*^x)_{x \in M}$ on E^* (seen as a Lie groupoid in the same way as E).

LEMMA 7.5. *The restriction of the Fourier transform \mathcal{F}_μ to $S(E)$ is a bijection onto $S(E^*)$, with inverse*

$$(7.3) \quad (\mathcal{F}_\mu^{-1}g)(\xi_x) = (2\pi)^{-p} \int g(\theta_x) e^{i\langle \theta_x, \xi_x \rangle} d\mu_*^x(\theta_x).$$

PROOF. Compute in local coordinates. ≡

Using these two lemmas we have

PROPOSITION 7.6. *\mathcal{F}_μ is an algebra isomorphism between $S(E)$ with the convolution $*$ -algebra structure inherited from $C^*(E, \mu)$ and $S(E^*)$ with the $*$ -algebra structure borrowed from $C_0(E^*)$.*

In the following lemma, which is used in the proof of Proposition 9.1, we list a series of properties of the Fourier transform on $S(E)$.

LEMMA 7.7. *If $f, g \in S(E)$, $\phi \in S(E^*)$ and $a \in C^\infty(M)$, then*

1. $[(a \circ \pi)f]^\wedge = (a \circ \pi)\hat{f}$;
2. $\frac{\partial \hat{f}}{\partial q_j}(\theta_x) = \frac{\partial f}{\partial q_j}(\theta_x) + \frac{\partial \ln \mu_e}{\partial q_j}(x)\hat{f}(\theta_x)$;
3. $\frac{\partial \hat{f}}{\partial \epsilon_i}(\theta_x) = -i(\xi_i \hat{f})(\theta_x)$;
4. $\frac{\partial \phi}{\partial \lambda_i}(\xi_x) = i(w_i \phi)(\xi_x)$;
5. $i\theta_k \hat{f}(\theta_x) = \left(\frac{\partial f}{\partial \lambda_k} \right)^\wedge(\theta_x)$.

Here $(\cdot)^\wedge$ is the Fourier transform \mathcal{F}_μ of (\cdot) and $(\cdot)^\vee$ is the inverse Fourier transform \mathcal{F}_μ^{-1} of (\cdot) .

The Schwartz functions do not form the only class of interest to us.

DEFINITION 7.8. *The Paley–Wiener functions on E^* , denoted by $C_{PW}^\infty(E^*)$, consist all functions in $S(E^*)$ whose (inverse) Fourier transform is in $C_c^\infty(E)$.*

We use this definition for $E = A(G)$; the Poisson algebra A^0 in Theorems 4.4 and 4.6 is $C_{PW}^\infty(A^*(G))$. Writing the Poisson bracket and the pointwise product in terms of the Fourier transform, one quickly establishes that A^0 is indeed a Poisson algebra (cf. section 9).

8. Weyl quantization

Having constructed the continuous field of C^* -algebras called for in Theorem 4.4 in section 6, it remains to define a Poisson algebra $A^0 \subset C_0(A^*(G))$ and a map $\mathcal{Q} : A^0 \rightarrow C^*(\hat{G})$, or equivalently, a family of involutive maps $\mathcal{Q}_\hbar : A^0 \rightarrow C^*(G)$ satisfying (4.7). These maps will, in addition, provide the data of Theorem 4.6. We do so by a generalization of Weyl quantization on $T^*\mathbb{R}^n$; cf. [La3].

We pick a positive 1-density $\mu \in C^\infty(G_0, |\Omega|^1(A(G)))$, with associated left Haar system $(\lambda^x)_{x \in G_0}$ on G (see section 3), left Haar system $(\hat{\lambda}^{(\hbar, x)})_{(\hbar, x) \in \hat{G}_0}$ on \hat{G} (see section 6), and left Haar system $(\mu^x)_{x \in G_0}$ on $A(G)$ (see section 7). We shall

work with the concrete C^* -algebras $C^*(G, \lambda)$ and $C^*(\hat{G}, \hat{\lambda})$ rather than with their intrinsically defined versions $C^*(G)$ and $C^*(\hat{G})$; see section 3. Moreover, since the argument below is the same for the reduced C^* -algebras, we will not consider that case explicitly.

Now choose some function $\kappa \in C^\infty(A(G), \mathbb{R})$ with support in V (cf. Proposition 2.6), equaling unity in some smaller tubular neighbourhood of G_0 , as well as satisfying $\kappa(-\xi) = \kappa(\xi)$ for all $\xi \in A(G)$.

DEFINITION 8.1. *Let G be a Lie groupoid with Lie algebroid $A(G)$. We put*

$$(8.1) \quad A^0 = C_{PW}^\infty(A^*(G)) \subset A_0 = C_0(A^*(G)).$$

For $\hbar \neq 0$, the Weyl quantization of $f \in A^0$ is the element $\mathcal{Q}_\hbar^W(f) \in C_c^\infty(G_1)$, regarded as a dense subalgebra of $C^*(G, \lambda)$, defined by

$$(8.2) \quad \begin{aligned} \mathcal{Q}_\hbar^W(f)(\text{Exp}^W(\xi)) &= |\hbar|^{-p} \kappa(\xi) \mathcal{F}_\mu^{-1} f(\xi/\hbar) \quad \forall \xi \in V; \\ \mathcal{Q}_\hbar^W(f)(\gamma) &= 0 \quad \forall \gamma \notin W. \end{aligned}$$

Here the Weyl exponential $\text{Exp}^W : A(G) \rightarrow G$ is defined in (2.10).

This definition is possible by virtue of Proposition 2.6. (An analogous definition using Exp^L would not satisfy (4.6); such a definition would generalize the Kohn–Nirenberg calculus of pseudodifferential operators rather than the Weyl calculus.) By our choice of A^0 , the operator $\mathcal{Q}_\hbar^W(f)$ is independent of κ for small enough \hbar (depending on f).

PROPOSITION 8.2. *For each $f \in A^0$, the operator $\mathcal{Q}_\hbar^W(f)$ of Definition 8.1 satisfies $\mathcal{Q}_\hbar^W(f)^* = \mathcal{Q}_\hbar^W(f^*)$, and the family $\{\mathcal{Q}_\hbar^W(f)\}_{\hbar \in \mathbb{R}}$, with $\mathcal{Q}_0^W(f) = f$, is a continuous cross-section of both continuous fields of C^* -algebras in Theorem 6.4.*

PROOF. It is immediate from (3.8) and (2.10) that for real-valued $f \in A^0$ the operator $\mathcal{Q}_\hbar^W(f)$ is self-adjoint in $C^*(G, \lambda)$; this implies the first claim.

Take $f \in C_{PW}^\infty(A^*(G))$. The function $\mathcal{Q}^W(f)$ on \hat{G}_1 that is defined by

$$(8.3) \quad \begin{aligned} \mathcal{Q}^W(f)(0, \xi) &= \mathcal{F}_\mu^{-1} f(\xi); \\ \mathcal{Q}^W(f)(\hbar, \text{Exp}^W(\xi)) &= \kappa(\xi) \mathcal{F}_\mu^{-1} f(\xi/\hbar) \quad \forall \xi \in V; \\ \mathcal{Q}^W(f)(\gamma) &= 0 \quad \forall \gamma \notin W \end{aligned}$$

and is in $C_c^\infty(\hat{G}_1)$; cf. Definition 6.1. In other words, $\mathcal{Q}^W(f)$ is an element of $C^*(\hat{G}, \hat{\lambda})$.

For $\hat{f} \in C_c^\infty(\hat{G})$, define the restriction maps

$$(8.4) \quad \begin{aligned} \hat{\varphi}_0(\hat{f}) : \theta &\mapsto \mathcal{F}_\mu \hat{f}(0, \theta); \\ \hat{\varphi}_\hbar(\hat{f}) : \gamma &\mapsto \hat{f}(\hbar, \gamma) \quad (\hbar \neq 0). \end{aligned}$$

Then $\hat{\varphi}_0$ is a surjective $*$ -homomorphisms from $C_c^\infty(\hat{G}_1, \hat{\lambda})$ to $C_{PW}^\infty(A^*(G), \mu)$, and each $\hat{\varphi}_\hbar$ is a surjective $*$ -homomorphisms from $C_c^\infty(\hat{G}_1, \hat{\lambda})$ to $C_c^\infty(G, |\hbar|^{-p}\lambda)$. These maps are contractive, and extend by continuity to surjective $*$ -homomorphisms from $C^*(\hat{G}, \hat{\lambda})$ to $C_0(A^*(G))$ and $C^*(G, |\hbar|^{-p}\lambda)$, respectively. However, in Theorem 4.4 we have $A_0 = C_0(A^*(G))$ and $A_\hbar = C^*(G, \lambda)$ for $\hbar \neq 0$. Hence the maps φ_\hbar of Definition 4.2 should be taken as

$$(8.5) \quad \begin{aligned} \varphi_0(\hat{f}) : \theta &\mapsto \mathcal{F}_\mu \hat{f}(0, \theta); \\ \varphi_\hbar(\hat{f}) : \gamma &\mapsto |\hbar|^{-p} \hat{f}(\hbar, \gamma) \quad (\hbar \neq 0). \end{aligned}$$

These maps extend to surjective $*$ -homomorphisms from $C^*(\hat{G}, \hat{\lambda})$ to $C_0(A^*(G))$ and $C^*(G, \lambda)$, respectively. It follows that $A_{\hbar} \simeq C^*(\hat{G}, \hat{\lambda}) / \ker \varphi_{\hbar}$ for all $\hbar \in \mathbb{R}$, and that $\mathcal{Q}_{\hbar}^W(f) = \varphi_{\hbar}(\mathcal{Q}^W(f))$. The proposition follows. \equiv

We have now proved Theorem 4.4 up to Dirac's condition, and turn to Theorem 4.6. We in addition need to choose the tubular neighbourhood W of $G_0 \subset G_1$ so that $\gamma\gamma' \subset W$ whenever $(\gamma, \gamma') \in G_2 \cap (W \times W)$.

We define a semi-strict deformation quantization of $A^*(G)$ in the following way. For given $f \in A^0 = C_{\text{PW}}^{\infty}(A^*(G))$, we choose a cutoff function χ_f on \mathbb{R} that is 1 in a neighbourhood of 0 and has support well inside the set of values of \hbar for which $\mathcal{Q}^W(f)$ as defined in (8.3) is independent of κ . It is clear from the choice of A^0 that this can be done. For each $\hbar \in \mathbb{R}^*$, we then have:

- an involution $f^{*\hbar} = \overline{f}$ (independent of \hbar);
- a semi-norm $\|f\|_{\hbar} = \chi_f(\hbar) \|\mathcal{Q}_{\hbar}^W(f)\|_{C^*(G, \lambda)}$;
- a product \times_{\hbar} defined by the condition

$$\mathcal{Q}_{\hbar}^W(f \times_{\hbar} g) = \chi_f \chi_g(\hbar) \mathcal{Q}_{\hbar}^W(f) * \mathcal{Q}_{\hbar}^W(g),$$

where $*$ is the product in $C^*(G, \lambda)$.

The existence of the product follows from a detailed but straightforward analysis of the support properties of $\mathcal{Q}_{\hbar}^W(f) * \mathcal{Q}_{\hbar}^W(g)$, leading to the conclusion that, seen as a function on G_1 for fixed \hbar , under the stated assumptions its support is contained inside W ; see [Ra1]. Hence it can be pulled back using Exp^W and can subsequently be Fourier-transformed so as to yield the desired function $f \times_{\hbar} g$.

The fact that $*_{\hbar}$ is indeed an involution is a consequence of the self-adjointness property (4.6) of \mathcal{Q}_{\hbar}^W . The same comment applies to the C^* -property of the semi-norm. In general, these are not norms, since for a given $f \neq 0$ there may well exist values of \hbar such that $\chi(\hbar) \mathcal{Q}_{\hbar}^W(f) = 0$, seen as a function on $G(\hbar) \subset \hat{G}$.

The conditions defining a semi-strict deformation quantization, except (4.9), are now trivially satisfied as a consequence of Theorem 4.4 as proved so far. A direct proof of condition (b) in Definition 4.5 is also immediate, based on Theorem 6.4.

9. The local structure of the Poisson bracket

In this section, taken from [Ra1], we express the structure of Lie algebroids E , as well as the Poisson bracket on $C^{\infty}(E^*)$, in local coordinates. This is of interest in itself, but in the context of quantization it is a key tool for proving Dirac's condition (4.7) or (4.9).

Recall Definition 2.2 of a Lie algebroid. Let $\{e_1, e_2, \dots, e_p\}$ be a local frame on $U \subset M$ for E , and let $(q_1, \dots, q_n, \lambda_1, \dots, \lambda_p)$ be local coordinates of E , with the q_i 's local coordinates for M and the λ_j 's local coordinates for the fibers associated to the frame $\{e_1, e_2, \dots, e_p\}$. Then, locally, the fact that E is a Lie algebroid implies the existence of structure functions c_{ijk} and a_{ij} in $C^{\infty}(U)$ such that

$$(9.1) \quad [e_i, e_j] = \sum_k c_{ijk} e_k;$$

$$(9.2) \quad \rho(e_i) = \sum_j a_{ij} \frac{\partial}{\partial q_j}.$$

The Poisson bracket (4.2) - (4.4) of the functions $\phi, \psi \in C^\infty(E^*)$ is then locally given by

$$\{\phi, \psi\} = \sum_{i,j} \rho(e_i)(q_j) \left(\frac{\partial \phi}{\partial \epsilon_i} \frac{\partial \psi}{\partial q_j} - \frac{\partial \phi}{\partial q_j} \frac{\partial \psi}{\partial \epsilon_i} \right) + \sum_{i,j} \widetilde{[e_i, e_j]} \frac{\partial \phi}{\partial \epsilon_i} \frac{\partial \psi}{\partial \epsilon_j};$$

recall the notation explained below (4.4). Taking the value at the point $(x, \alpha) \in E_x^*$ and using the structure functions of the Lie algebroid, we obtain

$$\begin{aligned} \{\phi, \psi\}(x, \alpha) &= \sum_{i,j} a_{ij}(x) \left(\frac{\partial \phi}{\partial \epsilon_i}(x, \alpha) \frac{\partial \psi}{\partial q_j}(x, \alpha) - \frac{\partial \phi}{\partial q_j}(x, \alpha) \frac{\partial \psi}{\partial \epsilon_i}(x, \alpha) \right) \\ (9.3) \quad &+ \sum_{i,j,k} c_{ijk}(x) \epsilon_k(\alpha) \frac{\partial \phi}{\partial \epsilon_i}(x, \alpha) \frac{\partial \psi}{\partial \epsilon_j}(x, \alpha). \end{aligned}$$

Fix a positive 1-density $\mu \in C^\infty(M, |\Omega|^1(E))$. We define a Poisson bracket on the dense subalgebra $S(E)$ of $C^*(E, \mu)$, by transporting the Poisson bracket of $C^\infty(E^*)$ using the Fourier transform \mathcal{F}_μ . That is,

$$(9.4) \quad \{f, g\}_\mu = \mathcal{F}_\mu^{-1}(\{\mathcal{F}_\mu f, \mathcal{F}_\mu g\}).$$

PROPOSITION 9.1. *Eq. (9.4) defines a Poisson bracket on the convolution algebra $S(E)$. The Poisson bracket of $f, g \in S(E)$ is explicitly given by*

$$\begin{aligned} \{f, g\}_\mu(\xi_x) &= -i \sum_{i,j} a_{ij}(x) \left(\xi_i f * \frac{\partial g}{\partial q_j} - \xi_i g * \frac{\partial f}{\partial q_j} \right)(\xi_x) \\ &- i \sum_{i,j} a_{ij}(x) \frac{\partial \ln \mu_e}{\partial q_j}(x) (\xi_i f * g - \xi_i g * f)(\xi_x) \\ (9.5) \quad &+ i \sum_{i,j,k} c_{ijk}(x) \frac{\partial}{\partial \lambda_k} (\xi_i f * \xi_j g)(\xi_x). \end{aligned}$$

PROOF. Eq. (9.3) proves that $S(E^*)$ is stable under the Poisson bracket $\{\cdot, \cdot\}$, hence (9.4) is well defined. Since the Fourier transform is an algebra isomorphism between $S(E)$ and $S(E^*)$, it can easily be shown that we have a Poisson algebra structure on $S(E)$. Eq. (9.5) is a consequence of Lemma 7.7 and (9.3). \equiv

We now specialize to the case of relevance to us, where $E = A(G)$ is the Lie algebroid of a Lie groupoid G . We analyze the local structure of G in the neighborhood of a fixed point $x_0 \in G_0 \subset G$ in a convenient parametrization; also cf. [NWX].

The map r is a submersion at $x_0 \in G_0 \subset G_1$, hence there exists an open neighborhood U of 0 in \mathbb{R}^n , an open neighborhood V of 0 in \mathbb{R}^m , and parametrizations $\psi : U \times V \rightarrow G_1$ and $\varphi : U \rightarrow G_0$ such that

$$(9.6) \quad \begin{aligned} 1. & \psi(0, 0) = x_0; \\ 2. & r(\psi(u, v)) = \varphi(u); \\ 3. & \psi(U \times \{0\}) = \psi(U \times V) \cap G_0. \end{aligned}$$

It follows from the first two conditions that $\varphi(u) = \psi(u, 0)$. To the parametrization ψ of the Lie groupoid G one can associate a parametrization $\theta : U \times \mathbb{R}^m \rightarrow A(G)$ of the neighborhood $A(G)_{\varphi(U)}$ of the fiber $A(G)_{x_0}$ in $A(G)$, given by $\theta(u, v) = (\varphi(u), \frac{\partial \psi}{\partial v}(u, 0)v)$.

For every $x_0 \in G_0$ there exists a neighborhood $\psi(U \times V)$ of x_0 in G that is diffeomorphic to the neighborhood $\theta(U \times V)$ of $(x_0, 0)$ in $A(G)$ by $\alpha = \psi \circ \theta^{-1}$. Moreover, $\alpha(A(G)_x) \subset G^x$ for each $x \in \varphi(U)$. This result can be formulated in a stronger form, based on the existence of an exponential map for a Lie groupoid; see section 2. Namely, taking α equal to the restriction of Exp^L (cf. Definition 2.4) to V , and α_x as the restriction of α to $A(G)_x \cap V$, one achieves that $\alpha(A(G)_x \cap V) = G^x \cap W$ and that $\alpha'_x(0)$ is the identity of $A(G)_x$.

The submersion $\sigma = \varphi^{-1} \circ s \circ \psi : U \times V \rightarrow U$ is the local expression of the source map s in the parametrization ψ . One has $\sigma(u, 0) = u$, since $\varphi(\sigma(u, 0)) = s(\psi(u, 0)) = \psi(u, 0) = \varphi(u)$. Similar expression may be given for the multiplication and the inversion.

The following theorem gives a version of the Baker-Campbell-Hausdorff formula for Lie groupoids. This will enter the proof of Dirac's condition.

THEOREM 9.2. *Let G be a Lie groupoid. Then*

1. $(\psi(u, v), \psi(u_1, w)) \in G_2$ if and only if $u_1 = \sigma(u, v)$. In that case, their product is of the form $\psi(u, v)\psi(\sigma(u, v), w) = \psi(u, p(u, v, w))$, where $p : U \times V \times V \rightarrow V$ is a smooth map which has the expansion

$$(9.7) \quad p(u, v, w) = v + w + B(u, v, w) + O_3(u, v, w),$$

with B bilinear in (v, w) , and $O_3(u, v, w)$ of the order of $\|(v, w)\|^3$.

2. Let $(u, v) \in U \times V$ be such that $\psi(u, v)^{-1} \in \psi(U \times V)$. Then $\psi(u, v)^{-1} = \psi(\sigma(u, v), w)$, where w satisfies $p(u, v, w) = 0$. Moreover, $w = -v + B(u, v, v) + O_3(u, v)$, with $O_3(u, v)$ of the order of $\|v\|^3$.

PROOF. We merely sketch the proof. For details cf. [Ra4].

1. Set $g = \psi(u, v)$ and $h = \psi(u_1, w)$. With $s(g) = \varphi(\sigma(u, v))$ and $r(h) = \varphi(u_1)$, we have $(g, h) \in G_2$ if and only if $u_1 = \sigma(u, v)$. Also, $r(gh) = \varphi(u)$ implies the existence of a unique $p(u, v, w) \in V$ such that $\psi(u, v)\psi(\sigma(u, v), w) = \psi(u, p(u, v, w))$. Hence we obtain a map $p : U \times V \times V \rightarrow V$ that satisfies $p(u, 0, w) = w$ and $p(u, v, 0) = v$. Using these equations in a Taylor expansion of p yields (9.7).
2. Similar to 1. ≡

We can now give explicit formulae for the structure functions of the Lie algebroid $A(G)$. First, remark that the family $\{e_1, e_2, \dots, e_m\}$ defined by $e_i(\varphi(u)) = \theta(u, f_i)$, $i = \overline{1, m}$, where $\{f_1, f_2, \dots, f_m\}$ is the canonical basis of \mathbb{R}^m , is a frame of $A(G)$ on $\varphi(U)$. Also, recall that the anchor of $A(G)$ is given by $\rho = Ts$ (cf. Definition 2.3.2), that the local coordinate functions of G_0 are $q_j = pr_j \circ \varphi^{-1}$, and that B_1, \dots, B_m are the coordinates of $B : U \times V \times V \rightarrow V$ in the base $\{f_1, f_2, \dots, f_m\}$ of \mathbb{R}^m . Finally, recall that $a_{ij} = \rho(e_i)(q_j)$, and that the c_{ijk} are given by $[e_i, e_j] = \sum c_{ijk} e_k$.

PROPOSITION 9.3. *For each $u \in U$, the structure functions of the Lie algebroid $A(G)$ are given by $a_{ij}(\varphi(u)) = \frac{\partial \sigma_j}{\partial v_i}(u, 0)$ and $c_{ijk}(\varphi(u)) = B_k(u, f_i, f_j) - B_k(u, f_j, f_i)$.*

PROOF. Direct calculations. See [Ra1], [Ra4] for details. ≡

We now return to the Poisson bracket on $A(G)$. Recall Proposition 9.1, and put $E = A(G)$. The next proposition gives a local expression of the Poisson bracket on $S(A(G))$.

Let $\lambda \in C^\infty(G_0, |\Omega|^1(A(G)))$ be a positive 1-density and Λ the local expression in the parametrization θ of the function associated to λ in the local frame $\{e_1, e_2, \dots, e_m\}$.

PROPOSITION 9.4. *For $f, g \in S(A(G))$, let $h = \{f, g\}_\lambda$ be the Poisson bracket of f and g given by (9.5). Denote the local expressions of f, g, h in the parametrization θ by F, G, H , respectively. Then*

$$\begin{aligned} H(u, v) &= i \int F(u, w)G(u, v - w) \sum_j [B_j(u, w, f_j) - B_j(u, f_j, w)] \Lambda(u)dw \\ &+ i \int \sum_k [B_k(u, w, v) - B_k(u, v, w)] F(u, w) \frac{\partial G}{\partial v_k}(u, v - w) \Lambda(u)dw \\ &- i \sum_{i,j} \frac{\partial \sigma_j}{\partial v_i}(u, 0) \frac{\partial \Lambda}{\partial u_j}(u) \int w_i [F(u, w)G(u, v - w) - G(u, w)F(u, v - w)] dw \\ &- i \sum_{i,j} \frac{\partial \sigma_j}{\partial v_i}(u, 0) \int w_i \left[F(u, w) \frac{\partial G}{\partial u_j}(u, v - w) - G(u, w) \frac{\partial F}{\partial u_j}(u, v - w) \right] \Lambda(u)dw. \end{aligned}$$

PROOF. Replacing the structure functions in (9.5) by their expressions calculated in Proposition 9.3, we can directly identify the first two lines of (9.5) with the last two in the formula of $H(u, v)$. Denote the expression in the last line of (9.5) by l . We then have

$$\begin{aligned} l(\theta(u, v)) &= - \sum_{i,j,k} c_{ijk}(\varphi(u)) \frac{\partial}{\partial v_k} \left(\int w_i F(u, w)(v_j - w_j)G(u, v - w)\Lambda(u)dw \right) \\ &= \sum_{i,j} \int (B_j(u, f_j, w_i f_i) - B_j(u, w_i f_i, f_j))F(u, w)G(u, v - w)\Lambda(u)dw \\ &+ \sum_{i,j,k} \int [B_k(u, (v_j - w_j)f_j, w_i f_i) - B_k(u, w_i f_i, (v_j - w_j)f_j)] \\ &\quad \cdot F(u, w) \frac{\partial G}{\partial v_k}(u, v - w)\Lambda(u)dw \\ &= \sum_j \int (B_j(u, f_j, w) - B_j(u, w, f_j))F(u, w)G(u, v - w)\Lambda(u)dw \\ &+ \sum_k \int (B_k(u, v - w, w) - B_k(u, w, v - w))F(u, w) \frac{\partial G}{\partial v_k}(u, v - w)\Lambda(u)dw. \end{aligned}$$

By bilinearity one has $B_k(u, w, v - w) - B_k(u, v - w, w) = B_k(u, w, v) - B_k(u, v, w)$; this finishes the proof. \equiv

10. Proof of Dirac's condition

We now use the results of the preceding section to prove (4.7) and (4.9) in the case at hand. We start with two lemmas.

Let ψ be a parametrization of G as in section 9, and denote the function associated to λ in the local frame of TG generated by ψ by λ_0 . Put $\Lambda = \lambda_0 \circ \psi$.

LEMMA 10.1. *One has $\lambda_0(\gamma) = \frac{\lambda_0(s(\gamma))}{|\det J_{L_\gamma}(s(\gamma))|}$ for every γ in the image of the parametrization ψ .*

PROOF. The left invariance of λ means $\lambda(\gamma) = \gamma \cdot \lambda(s(\gamma))$. It only remains to write the action of G on $|\Omega|^1(\ker Tr)$ in the local parametrization ψ . We leave this to the reader. \equiv

LEMMA 10.2. *Let $\mu_{u,v}(t) = \lambda_0(\psi(u, tv))$. Then*

$$\mu'_{u,v}(0) = \sum_{i,j} \frac{\partial \Lambda}{\partial u_i}(u, 0) \frac{\partial \sigma_i}{\partial v_j}(u, 0) v_j - [B_1(u, v, f_1) + \dots + B_m(u, v, f_m)] \Lambda(u, 0).$$

PROOF. By Lemma 10.1, $\mu_{u,v}(t) = \frac{\lambda_0(\psi(\sigma(u, tv), 0))}{|\det J_{L_{\psi(u, tv)}}(\psi(\sigma(u, tv), 0))|}$. Write

$$\begin{aligned} a_{u,v}(t) &= \lambda_0(\psi(\sigma(u, tv), 0)); \\ b_{u,v}(t) &= |\det J_{L_{\psi(u, tv)}}(\psi(\sigma(u, tv), 0))|. \end{aligned}$$

We have $a_{u,v} = \Lambda(\sigma(u, tv), 0)$ and $a'_{u,v}(0) = \sum_{i,j} \frac{\partial \Lambda}{\partial u_i}(u, 0) \frac{\partial \sigma_i}{\partial v_j}(u, 0) v_j$.

The local expression of the multiplication map $L_{\psi(u, tv)} : G^{\varphi(\sigma(u, tv))} \rightarrow G^{\varphi(u)}$ is given by $p(u, tv, \cdot)$ and this shows, after some calculation,

$$b_{u,v}(t) = \begin{vmatrix} 1 + tB_1(u, v, f_1) & tB_1(u, v, f_2) & \dots & tB_1(u, v, f_m) \\ tB_2(u, v, f_1) & 1 + tB_2(u, v, f_2) & \dots & tB_2(u, v, f_m) \\ \dots & \dots & \dots & \dots \\ tB_m(u, v, f_1) & tB_m(u, v, f_2) & \dots & 1 + tB_m(u, v, f_m) \end{vmatrix}.$$

Taking the derivative we obtain $b'_{u,v}(0) = B_1(u, v, f_1) + B_2(u, v, f_2) + \dots + B_m(u, v, f_m)$, and to finish the proof remark that $a_{u,v}(0) = \Lambda(u, 0)$, $b_{u,v}(0) = 1$. \equiv

Now let $(\gamma, \hbar) \in \hat{G}$, with $\hbar \neq 0$. The smooth structure in a neighborhood of (γ, \hbar) is given by a parametrization of the form $\hat{\psi} = \psi \times id : U \times \mathbb{R} \rightarrow \hat{G}$, where $\psi : U \rightarrow G$ is a parametrization of G in a neighborhood of γ and U is an open subset of \mathbb{R}^{n+m} . Write $\alpha = \hat{\psi}^{-1} : \hat{\psi}(U \times \mathbb{R}) \rightarrow U \times \mathbb{R}$, and define an element of $T_{(\gamma, \hbar)} \hat{G}$ by

$$\frac{\partial}{\partial \hbar} \hat{f}|_{(\gamma, \hbar)} = \frac{\partial \hat{f}}{\partial \alpha_{n+m+1}}(\gamma, \hbar).$$

Here $\hat{f} \in C^\infty(\hat{G})$. This definition is independent of the parametrization ψ of G . Thus we obtain a vector field $\frac{\partial}{\partial \hbar} \in C^\infty(\hat{G}, T\hat{G})$.

The following result is the key lemma towards the proof of Dirac's condition.

LEMMA 10.3. *Let $\hat{f}, \hat{g} \in C_c^\infty(\hat{G}, \hat{\lambda})$ (seen as a convolution algebra), and write $\hat{f}_0 = \hat{f}|_{\hat{G}(0)}$, $\hat{g}_0 = \hat{g}|_{\hat{G}(0)}$ for their the restrictions to the Lie algebroid $A(G)$. Then*

$$\frac{\partial}{\partial \hbar} [\hat{f}, \hat{g}]|_{\hbar=0} = i\{\hat{f}_0, \hat{g}_0\}_\mu.$$

PROOF. The proof is only sketched; for details cf. [Ra1, Ra3]. Using a partition of unity, one may assume that \hat{f} and \hat{g} have their support contained in the image of the parametrization $\hat{\psi}$. Let \hat{k} be the convolution of \hat{f} and \hat{g} , and let K be the expression of \hat{k} in the parametrization $\hat{\psi}$. After some calculation and the use

of (9.7) we obtain

$$\begin{aligned} K(u, v, \hbar) &= \int F(u, w, \hbar) G(\sigma(u, \hbar w), v - w + \hbar B(u, w, w)) \\ &\quad - \hbar B(\sigma(u, \hbar w), w, v) + O(\hbar^2, \hbar) \mu_{u, w}(\hbar) dw. \end{aligned}$$

This leads to an expression for the commutator $\hat{f} * \hat{g} - \hat{g} * \hat{f}$. We then differentiate K with respect to \hbar at $\hbar = 0$, use Lemma 10.2 and some changes of variables, and eventually recover the local expression of the Poisson bracket as given in Proposition 9.4. \equiv

We are now in a position to prove Dirac's condition (4.9) and (4.7). We only deal with the former; the latter is proved in exactly the same way. The essence of the proof is that Dirac's condition follows from the continuity condition (b) in Definition 4.5; in the context of Definition 4.3 this condition is satisfied as a consequence of the definition of a continuous field of C^* -algebras.

Define $\hat{V} \subset A(\hat{G})$ as V in Proposition 2.6, but now for the tangent groupoid \hat{G} rather than G . For $f, g \in C_c^\infty(A(G))$, let $\varphi : \hat{V} \rightarrow \mathbb{C}$ be defined by

$$\begin{aligned} \varphi(x, X, 0) &= 0; \\ \varphi(x, X, \hbar) &= \frac{(f \times_{\hbar} g)(x, X) - (g \times_{\hbar} f)(x, X)}{\hbar} - i \{f, g\}(x, X) \forall \hbar \neq 0. \end{aligned}$$

Lemma 10.3 shows that $\varphi \in C^\infty(\hat{V})$, and arguments like those in the discussion of the product $f \times_{\hbar} g$ in section 8 show that $\text{supp } \varphi$ is compact in \hat{V} . Write $\hat{\varphi} = \varphi \circ \hat{\alpha}^{-1} \in C_c^\infty(\hat{\alpha}(\hat{V}))$. One may regard φ as an element of $C_c^\infty(A(G) \times \mathbb{R})$ and $\hat{\varphi}$ as an element of $C_c^\infty(\hat{G})$ by extending these functions by 0 outside \hat{V} and $\hat{\alpha}(\hat{V})$, respectively. Remark that for every \hbar one has function $\varphi(\cdot, \hbar) \in A^0$. Applying Theorem 6.4 and recalling the notation (6.6), where we look at the $G(\hbar)$ as subgroupoids of \hat{G} , one then has

$$\lim_{\hbar \rightarrow 0} \|\varphi(\cdot, \hbar)\|_{\hbar} = \lim_{\hbar \rightarrow 0} \left\| \hat{\varphi}|_{G(\hbar)} \right\|_{C^*(G(\hbar), \lambda)} = \left\| \hat{\varphi}|_{G(0)} \right\|_{C^*(G(0), \mu)} = 0.$$

This finishes the proof of Dirac's condition. Having giving the other parts of the proof in section 8, this finishes the proof of Theorems 4.4 and 4.6.

11. Examples and comments

Since a vast number of interesting C^* -algebras are defined by some Lie groupoid, one obtains a large reservoir of examples of our theorems, of which in this section we merely scratch the surface.

EXAMPLE 11.1. *Quantization of the Lie-Poisson structure on a dual Lie algebra*

This example was already introduced in the Introduction. A Lie group is a Lie groupoid with $G_0 = e$. A left-invariant Haar measure on G provides a left Haar system; the ensuing convolution algebra $C^*(G)$ is the usual group C^* -algebra. The Poisson structure on the dual Lie algebra $A^*(G)$ is the well-known Lie-Poisson structure [MR, Va]. No connection is needed to define the exponential map, and one has

$$(11.1) \quad \text{Exp}^L(X) = \text{Exp}^W(X) = \text{Exp}(X),$$

where $X \in \mathfrak{g}$ and $\text{Exp} : \mathfrak{g} \rightarrow G$ is the usual exponential map. We obtain a semi-strict or strict deformation quantization of $A^*(G)$ for any Lie group G , but the situation is particularly favourable when G is exponential (in that Exp is a diffeomorphism). One may then omit the cutoff functions κ and χ in section 8, and $\mathcal{Q}_\hbar^W(f)$ in (8.2) is simply given by

$$(11.2) \quad \mathcal{Q}_\hbar^W(f) : \text{Exp}(X) \rightarrow (2\pi\hbar)^{-n} \int_{\mathfrak{g}^*} e^{i\langle \theta, X \rangle / \hbar} f(\theta) d^n \theta.$$

This is precisely Rieffel's prescription [Ri4]; his assumption that G be nilpotent is now seen to be unnecessary in order to obtain a strict deformation quantization.

Moreover, in the exponential case our semi-strict quantization is easily shown to be strict, since the semi-norms $\|\cdot\|_\hbar$ are now norms. More generally, if a Lie groupoid is diffeomorphic to its Lie algebroid then the semi-strict deformation quantization we have given is always strict; see [Ra1].

EXAMPLE 11.2. *Transformation group C^* -algebras*

Let a Lie group H act smoothly on a manifold M . The transformation groupoid (or action groupoid) $G = H \times M$ is defined by the operations $s(x, m) = x^{-1}m$ and $r(x, m) = m$, so that $((x, m), (y, m')) \in G_2$ when $m' = x^{-1}m$. In that case, $(x, m) \cdot (y, x^{-1}m) = (xy, m)$. The inclusion is $m \mapsto (e, m)$, and the inverse is $(x, m)^{-1} = (x^{-1}, x^{-1}m)$.

Each left-invariant Haar measure dx on H leads to a left Haar system on G . The corresponding groupoid C^* -algebra is the usual transformation group C^* -algebra $C^*(G) = C^*(H, M)$, cf. [Re1].

The Lie algebroid $\mathcal{H} \times M$ (where \mathcal{H} is the Lie algebra of H) is a trivial bundle over M , with anchor $\rho(X, m) = -\xi_X(m)$ (the fundamental vector field on M defined by $X \in \mathcal{H}$). Identifying sections of $\mathcal{H} \times M$ with \mathcal{H} -valued functions $X(\cdot)$ on M , the Lie bracket on $C^\infty(M, \mathcal{H} \times M)$ is

$$(11.3) \quad [X, Y]_{\mathcal{H} \times M}(m) = [X(m), Y(m)]_{\mathcal{H}} + \xi_Y X(m) - \xi_X Y(m).$$

The associated Poisson bracket coincides with the semi-direct product bracket defined in [KM].

The trivial connection on $\mathcal{H} \times M \rightarrow M$ yields the exponential maps

$$(11.4) \quad \text{Exp}^L(X, m) = (\text{Exp}(X), m);$$

$$(11.5) \quad \text{Exp}^W(X, m) = (\text{Exp}(X), \text{Exp}(\frac{1}{2}X)m).$$

The cutoff function κ in (8.2) is independent of m , and coincides with the function appearing in Example 11.1. For small enough \hbar , a function $f \in C_{\text{PW}}^\infty(\mathcal{H}^* \times M)$ is then quantized by

$$(11.6) \quad \mathcal{Q}_\hbar^W(f) : (\text{Exp}(X), m) \rightarrow (2\pi\hbar)^{-n} \int_{\mathcal{H}^*} e^{i\langle \theta, X \rangle / \hbar} f(\theta, \text{Exp}(-\frac{1}{2}X)m) d^n \theta.$$

When $H = \mathbb{R}^n$ and M has a H -invariant measure, the map $f \rightarrow \mathcal{Q}_\hbar^W(f)$ is equivalent to the deformation quantization considered by Rieffel [Ri2], who already proved that it is strict.

Finally, when H is exponential we are in the situation discussed at the end of the previous example, so that our semi-strict deformation quantization is strict.

EXAMPLE 11.3. *Weyl quantization on $T^*\mathbb{R}^n$*

As already remarked in the Introduction, the tangent bundle TM of a manifold M is the Lie algebroid of the pair groupoid $G = M \times M$. Hence our formalism produces a (semi) strict deformation quantization of the cotangent bundle T^*M of any manifold with linear connection.

In order to understand Weyl quantization in the light of our formalism, we must change some signs: we add minus signs in front of the right-hand sides of (4.3) and (4.4), and change $f(\xi/\hbar)$ in (8.2) to $f(-\xi/\hbar)$ (cf. [La3]) for the rationale behind these signs). For $M = \mathbb{R}^n$ with flat connection (8.2) then simply becomes

$$(11.7) \quad \mathcal{Q}_\hbar^W(f)\Psi(x) = (2\pi\hbar)^{-n} \int_{T^*\mathbb{R}^n} e^{ip(x-y)/\hbar} f(p, \frac{1}{2}(x+y))\Psi(y)d^n p d^n y,$$

where $\Psi \in L^2(\mathbb{R}^n)$; we here use the fact that $C^*(\mathbb{R}^n \times \mathbb{R}^n) = \mathfrak{B}_0(L^2(\mathbb{R}^n))$. This is precisely Weyl’s original prescription, now written in a form that emphasizes its geometric origin. For $\frac{1}{2}(x+y)$ is the midpoint of the geodesic from x to y , and $x-y$ is a tangent vector to the geodesic at this midpoint. Looked at in this way, Weyl quantization may easily be generalized to manifolds with connection [La1, La3], and forms a special case of what we have called Weyl quantization for general Lie groupoids in this paper. This answers Rieffel’s Question 20 in [Ri5].

The fact that this quantization is strict, and in particular satisfies (4.7), had already been proved by Rieffel [Ri6]. The associated continuous field of C^* -algebras has fibers $A_0 = C_0(T^*\mathbb{R}^n)$ and $A_\hbar = \mathfrak{B}_0(L^2(\mathbb{R}^n))$ for $\hbar \neq 0$. The C^* -algebra C in Definition 4.2 is $C^*(H_n)$, the group algebra of the simply connected Heisenberg group on \mathbb{R}^n ; also see [ENN1]. This is indeed the C^* -algebra of the tangent groupoid of \mathbb{R}^n .

When M be Riemannian and exponential (in that the exponential map is a diffeomorphism between the tangent bundle TM and the pair groupoid $M \times M$), our semi-strict deformation quantization is strict [Ra1]. As is well known, for $M = \mathbb{R}^n$ with flat metric the product \times_\hbar is then equal to the Moyal product [Mo, BFFLS]. For a detailed analysis of this situation in the setting of the present paper see [CCFGRV].

We close with two remarks.

REMARK 11.4. There are clear connections between index theory (in the sense of Atiyah–Singer), C^* -algebraic K-theory, and quantization; see, e.g., [Fe, Hi, ENN2, Ro]. Moreover, Connes [Co2] discovered a beautiful proof of the index theorem that is based on the tangent groupoid of a manifold.

Let M be a compact Riemannian manifold, with associated cosphere bundle S^*M and C^* -algebra of bounded classical pseudodifferential operators $\Psi^0(M)$. The interpretation of index theory in terms of the K-theory of C^* -algebras comes from the short exact sequence of C^* -algebras

$$(11.8) \quad 0 \rightarrow \mathfrak{B}_0(L^2(M)) \rightarrow \Psi^0(M) \xrightarrow{\sigma} C(S^*M) \rightarrow 0;$$

here σ is the symbol map. Any short exact sequence $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$ leads to a connecting map $\partial : K^1(B) \rightarrow K_0(J)$, so that for (11.8) one obtains

$$\partial : K_1(C(S^*M)) = K^1(S^*(M)) \rightarrow K_0(\mathfrak{B}_0) = \mathbb{Z}.$$

The analytic index of an elliptic pseudodifferential operator $P \in \Psi^0(M)$ is given by $\partial([\sigma(P)]) \in \mathbb{Z}$, where, for $f \in C(X)$, $[f]$ is the class in $K^1(X)$ defined by f . One may prove the Atiyah–Singer index theorem from the existence of a strict deformation quantization $\mathcal{Q}_\hbar : C_0(T^*M) \rightarrow \mathfrak{B}_0(L^2(M))$; cf. [Hi].

Seen in the light of Lie groupoids and quantization, it is obvious that the above data may be generalized [MP] by replacing the pair groupoid on M by an arbitrary Lie groupoid G . Then $\mathfrak{B}_0(L^2(M))$ is replaced by $C^*(G)$, the cosphere bundle S^*M is replaced by a cosphere bundle $S^*A(G)$ in $A^*(G)$, and the appropriate generalization $\Psi^0(G)$ (with some abuse of notation) of $\Psi^0(M)$ has recently been defined as well [NWX, MP]. Thus (11.8) is generalized to

$$(11.9) \quad 0 \rightarrow C^*(G) \rightarrow \Psi^0(M) \xrightarrow{\sigma} C_0(S^*A(G)) \rightarrow 0;$$

see [MP].

One may therefore expect that our strict deformation quantization of $A^*(G)$ eventually leads to an index theorem, where the index now takes values in $K_0(C^*(G))$.

REMARK 11.5. There is no analogue of Lie's Third Theorem for Lie algebroids. Examples given in [AM] show the existence of Lie algebroids that are not associated to any Lie groupoid. For an arbitrary Lie algebroid $A(G)$, Pradines proved (cf. [Pr4]) that one can merely construct a local Lie groupoid G which has $A(G)$ as its Lie algebroid. In this paper, we always assume that we have a given Lie groupoid G with Lie algebroid $A(G)$. In order to use the ideas from this paper to quantize all Lie algebroids, we would need to extend two constructions, the C^* -algebra and the tangent groupoid which can be associated to every Lie groupoid, to the context of local Lie groupoids. The construction of the tangent groupoid has been extended to local Lie groupoids in the context of pseudo-differential operators by Nistor, Weinstein and Xu (cf. [NWX]). The missing ingredient in order to extend our results to arbitrary Lie algebroids is therefore the construction of the C^* -algebra of a local Lie groupoid.

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