Gauge Orbit Types for Generalized Connections

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Abstract

Different versions for defining Ashtekar's generalized connections are investigated depending on the chosen smoothness category for the paths and graphs – the label set for the projective limit. Our definition covers the analytic case as well as the case of webs.

Then the orbit types of the generalized connections are determined for compact structure groups. The stabilizer of a connection is homeomorphic to the holonomy centralizer, i.e. the centralizer of its holonomy group, and the homeomorphism class of the gauge orbit is completely determined by the holonomy centralizer. Furthermore, the stabilizers of two connections are conjugate in the gauge group if and only if their holonomy centralizers are conjugate in the structure group. Finally, the gauge orbit type of a connection is defined to be the conjugacy class of its holonomy centralizer equivalently to the standard definition via stabilizers.

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1 Introduction

For a few decades the quantization of Yang-Mills theories has been investigated extensively. One of the most important approaches uses functional integration. Here one quantizes a classical theory by introducing an appropriate measure on its configuration space. In gauge theories this space is given by $\mathcal{A}/\mathcal{G}$. Here, originally, $\mathcal{G}$ denoted the set of all (smooth) gauge transforms acting on the space $\mathcal{A}$ of all (smooth) connections. That is why a lot of the efforts has been focussed on clarifying the structure of $\mathcal{A}/\mathcal{G}$. One typical property of $\mathcal{A}/\mathcal{G}$ is that there do not exist global gauge fixings, i.e. smooth sections in $\mathcal{A} \longrightarrow \mathcal{A}/\mathcal{G}$, – the so-called Gribov problem. Other problems are caused by the very difficult structure of $\mathcal{A}/\mathcal{G}$: $\mathcal{A}/\mathcal{G}$ is non-linear, infinite-dimensional and it is usually not a manifold. Thus, results concerning $\mathcal{A}/\mathcal{G}$ are quite scarce up to now. But, should one restrict oneself to the case of smooth connections? Since in a quantization process smoothness is usually lost anyway, it is quite clear that one has to admit also non-smooth connections. This way, about 20 years ago, several authors started the consideration of Sobolev connections. For basic results we refer, e.g., to [19]. By now, the structure, in particular, of the generic stratum of $\mathcal{A}_{\mathrm{Sob}}/\mathcal{G}_{\mathrm{Sob}}$ is quite well-understood. Nevertheless, measure theory did not become easier. Concerning that point, first convincing successes have been gained through the introduction of generalized connections by Ashtekar and Isham [1]. Here one drops completely the "differential" conditions like smoothness or Sobolev integrability and works with the algebraic structure of the space of connections only. The main idea is as follows. A (smooth) connection is uniquely determined by its parallel transports, i.e., by a (smooth [16]) homomorphism from the groupoid of paths to the structure group $\mathbf{G}$. A generalized connection is now simply such a homomorphism, but without the smoothness condition. Analogously, a generalized gauge transform $\overline{g} \in \overline{\mathcal{G}}$ is now a (usually non-smooth) map from the base manifold M to $\mathbf{G}$, and it acts purely algebraically on the space $\overline{\mathcal{A}}$ of generalized connections. One of the main advantages of $\overline{\mathcal{A}}$ is that it is (for compact $\mathbf{G}$) compact and it possesses a natural kinematical measure, the induced Haar measure [2]. Now, the perhaps most important question is how the standard smooth and the new Ashtekar theory are related to each other – mathematically and physically. The first very nice answer was the statement that $\mathcal{A}$ is dense in $\overline{\mathcal{A}}$ [20]. This result is usually expected when one quantizes a theory. Then it has been proven that for the two-dimensional pure Yang-Mills theory the Wilson loop expectation values are in fact the same in the classical as well as in the Ashtekar framework [5, 11]. Now, we are going to investigate the action of the generalized gauge transforms on the space of generalized connections in comparison with its counterpart in the Sobolev case described in detail in [14].

The present paper is the first in a series of three papers.

In the first part of this paper we will give a quite detailed introduction into the algebraic and topological definitions and properties of $\overline{\mathcal{A}}$, $\overline{\mathcal{G}}$ and $\overline{\mathcal{A}}/\overline{\mathcal{G}}$. Here we closely follow Ashtekar and Lewandowski [4, 3] as well as Marolf and Mourão [18]. The most important difference to their definitions is that we do not restrict the paths to be (piecewise) analytic or smooth. For our purpose it is sufficient to fix a category of smoothness from the beginning. This is C^r, where r can be any positive natural number, ∞ (smooth case) or ω (analytical case). We can also consider the corresponding cases $C^{r,+}$ of paths that are (piecewise) immersions. We will show that in a certain sense the case $(\omega, +)$ corresponds to the loop structures introduced by Ashtekar and Lewandowski [2] and the case $(\infty, +)$ corresponds to the webs introduced by Baez and Sawin [6].

Now, the line of our papers ramifies. One branch, described in the second paper [9] of our short series, investigates properties of the space $\overline{\mathcal{A}}$ itself. There we will give a construction method for new connections. Then, as a main result, we will show that an induced Haar measure $d\mu_0$ can be defined for arbitrary smoothness conditions. For this, we introduce the notion of a hyph that generalizes the notion of a web and a graph. We show that the paths of a hyph are holonomically independent and that the set of all hyphs is directed. These two properties yield the well-definedness of $d\mu_0$.

The other branch is followed in the second part of the present paper. It is devoted to the type of the gauge orbit. In the general theory of transformation groups the type of an orbit (or, more precisely, an element of an orbit) is defined by the conjugacy class of its stabilizer (see, e.g., [8]). Here, we will derive the explicit form of the stabilizer for every generalized connection. As we will see, the stabilizer of a connection is homeomorphic to the centralizer of its holonomy group, hence a finite-dimensional Lie group. Since stabilizers are conjugate in $\overline{\mathcal{G}}$ if and only if these centralizers are conjugated in $\mathbf{G}$, the type of an orbit is uniquely determined by a certain equivalence class of a Howe subgroup of the structure group $\mathbf{G}$. (A Howe subgroup of $\mathbf{G}$ is a subgroup that can be written as the centralizer of some subset $V \subseteq \mathbf{G}$.)

In the final paper [10] of this short series we reunite the two branches. There we will see how the results of Kondracki and Rogulski [14] can be extended from the Sobolev framework to the generalized case (for compact $\mathbf{G}$). We will prove that there is a slice theorem for the action of $\overline{\mathcal{G}}$ on $\overline{\mathcal{A}}$. This means that for every connection $\overline{A} \in \overline{\mathcal{A}}$ there is an open and $\overline{\mathcal{G}}$-invariant neighbourhood that can be retracted equivariantly to the orbit $\overline{A} \circ \overline{\mathcal{G}}$. Moreover, we prove that the space $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ is topologically regularly stratified. But, two results for generalized connections go beyond those for Sobolev ones. First, we can explicitly derive the set of all gauge orbit types. This was not known until now for the Sobolev case. However, recently, Rudolph, Schmidt and Volobuev [21] solved this problem for all $SU(n)$-bundles over two-, three- and four-dimensional manifolds. We show that in the Ashtekar framework (the conjugacy class of) every Howe subgroup of $\mathbf{G}$ occurs as a gauge orbit type. Second, we prove that the generic stratum, i.e. the set of all connections whose holonomy centralizers equal the center of $\mathbf{G}$, has the induced Haar measure 1.

In the following, M is always a connected and at least two-dimensional C^r-manifold with $r \in \mathbb{N}^+ \cup \{\infty\} \cup \{\omega\}$ being arbitrary, but fixed. Furthermore, m is an, as well, arbitrary, but fixed point in M and $\mathbf{G}$ is a Lie group.

2   Paths

In the classical approach a connection can be described by the corresponding parallel transports along paths in the base manifold. But, not every assignment of group elements to the paths yields a connection. On the one hand, this map has to be a homomorphism, i.e., products of paths have to lead to products of the parallel transports, and on the other hand, it has to depend in a certain sense continuously on the paths. Moreover, additional topological obstructions may occur. In the Ashtekar approach, however, the second (and the third) condition is dropped. A connection is now simply a homomorphism from the set of paths to the structure group $\mathbf{G}$.

Up to now, it is not clear, whether there is an "optimal" definition for the structure of the

groupoid $\mathcal{P}$ of paths. The first version was given by Ashtekar and Lewandowski [2]. They used piecewise analytical paths. The advantage of this approach was that any finite set of paths forms a finite graph. Hence for two finite graphs there is always a third graph containing both of them, i.e. the set of all graphs forms a directed set. Using this it is easy to prove independence theorems for loops and to define then a natural measure on $\overline{\mathcal{A}}$ and $\overline{\mathcal{A}/\mathcal{G}}$. But, the restriction to analyticity seems a little bit unsatisfactory. Since one has desired from the very beginning to use $\overline{\mathcal{A}}$ for describing quantum gravity, one comes into troubles with the diffeomorphism invariance of this theory: After applying a diffeomorphism a path need no longer be analytical.

That is why Baez and Sawin [6] introduced so-called webs and tassels built by only smooth paths fulfilling certain conditions. Any graph can be written as a web and for any finite number of webs there is a web containing all of them. So the directedness of the label set for the definition of $\overline{\mathcal{A}}$ remaines valid, and, consequently, one can generalize the construction of the natural induced Haar measure and lots of things more.

In this paper we will introduce another definition for paths. Our definition will have the advantage that it does not depend explicitly on the chosen smoothness category labelled by $r \in \mathbb{N}^+ \cup \{\infty\} \cup \{\omega\}$. Moreover, it does not matter, whether we demand the paths to be piecewise immersions (cases $C^{r,+}$) or not. Therefore, in what follows suppose that we have fixed the parameter r from the very beginning. Furthermore, we decide now whether we additionally demand the paths to be piecewise immersions or not. Nevertheless, we write always simply C^r.

2.1 General Case

In this subsection we consider all smoothness categories on one stroke.

Definition 2.1   A **path** is a piecewise C^r-map $\gamma : [0,1] \longrightarrow M$.[1]
The **initial point** is $\gamma(0)$ and the **terminal point** $\gamma(1)$.
Two paths γ_1 und γ_2 can be multiplied iff the terminal point of γ_1 and the initial point of γ_2 coincide. Then the product is given by
$$\gamma_1\gamma_2(t) := \begin{cases} \gamma_1(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}.$$
A path γ is called **trivial** iff im $\gamma \equiv \gamma([0,1])$ is a single point.

An important idea of the Ashtekar program is the assumption that the total information about the continuum theory is encoded in the set of all subtheories on finite lattices. Thus we need the definition of paths and graphs. The set of all paths is hard to manage. That is why we restrict ourselves to special paths.

Definition 2.2   • A path γ has **no self-intersections** iff from $\gamma(\tau_1) = \gamma(\tau_2)$ follows that
  – $\tau_1 = \tau_2$ or
  – $\tau_1 = 0$ and $\tau_2 = 1$ or
  – $\tau_1 = 1$ and $\tau_2 = 0$.

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[1]If we consider piecewise immersed paths, we have to additionally define all $\gamma : [0,1] \longrightarrow M$ that are piecewise constant, i.e. $\gamma \mid_{[\tau_1,\tau_2]} = \{x\}$ for some $x \in M$, or immersive to be a path.

- A path γ' is called **subpath** of a path γ iff there is an affine non-decreasing map $\phi : [0,1] \longrightarrow [0,1]$ with $\gamma' = \gamma \circ \phi$. Iff additionally $\phi(0) = 0$ (or $\phi(1) = 1$), γ' is called **initial path** (or **terminal path**) of γ.

  We define $\gamma^{t,+}(\tau) := \gamma(t + \tau(1-t))$ for all $t \in [0,1)$ and $\gamma^{t,-}(\tau) := \gamma(\tau t)$ for all $t \in (0,1]$ to be the outgoing and incoming subpath of γ in t, respectively.

  If γ is a path without self-intersections then set $\gamma^{x,\pm} := \gamma^{t,\pm}$ for all $x \in \operatorname{im} \gamma$ where t fulfills $\gamma(t) = x$. (We choose $t = 0$ in the +-case if $x = \gamma(0)$. Analogously for $t = 1$.)

- A (finite) **graph** Γ is a (finite) union of paths e_i without self-intersections and of isolated points v_j. The elements of $\mathbf{V}(\Gamma) := \bigcup_i \{e_i(0), e_i(1)\} \cup \bigcup_j \{v_j\}$ are called **vertices**, that of $\mathbf{E}(\Gamma) := \bigcup_i \{e_i\}$ **edges**. A graph Γ is called **connected** iff $\mathbf{V}(\Gamma) \cup \bigcup_{e \in \mathbf{E}(\Gamma)} \operatorname{im} e$ is connected.

- A **path in a graph** Γ is a path in M, that equals a product of edges in Γ and trivial paths (with values in $\mathbf{V}(\Gamma)$), respectively, whereas the product of two consecutive paths has to exist.

  A path γ in M is called **simple** iff there is a finite graph Γ such that γ is a path in Γ.

- A path γ in M is called **finite** iff γ is up to the parametrization[2] equal to a finite product of simple paths.

- Two finite paths γ_1 and γ_2 are called **equivalent** iff there is a finite sequence of finite paths δ_i with $\delta_0 = \gamma_1$ and $\delta_n = \gamma_2$ such that for all $i = 1, \ldots, n$
  - δ_i and δ_{i-1} coincide up to the parametrization or
  - δ_i arises from δ_{i-1} by inserting[3] a retracing or
  - δ_{i-1} arises from δ_i by inserting a retracing.

- The set of all classes of finite paths is denoted by $\mathcal{P}$, that of paths in Γ by $\mathcal{P}_\Gamma$. Furthermore, we write $\mathcal{P}_{xy}$ for the set of all classes of finite paths from x to y. The set of all classes of finite paths having base point m forms the **hoop group** $\mathcal{HG} \equiv \mathcal{P}_{mm}$.

We have immediately from the definitions

Proposition 2.1 The multiplication on $\mathcal{P}$ induced by the multiplication of paths is well-defined and generates a groupoid structure on $\mathcal{P}$.[4]

The hoop group $\mathcal{HG}$ is a subgroup of $\mathcal{P}$.

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[2]Two paths γ_1 and γ_2 are equal up to the parametrization iff there is a bijective $\Pi : [0,1] \longrightarrow [0,1]$ with $\Pi(0) = 0$ and $\gamma_2 = \gamma_1 \circ \Pi$ such that Π and Π^{-1} are C^r.

[3]This means, there is a $\tau \in [0,1]$ and a finite path δ such that

$$\delta_i(t) = \begin{cases} \delta_{i-1}(\frac{1}{2}t) & \text{for} & 0 \leq t \leq \frac{1}{2}\tau \\ \delta(4(t - \frac{1}{2}\tau)) & \text{for } \frac{1}{2}\tau & \leq t \leq \frac{1}{2}\tau + \frac{1}{4} \\ \delta(4(\frac{1}{2}\tau + \frac{1}{2} - t)) & \text{for } \frac{1}{2}\tau + \frac{1}{4} \leq t \leq \frac{1}{2}\tau + \frac{1}{2} \\ \delta_{i-1}(\frac{1}{2}t - \frac{1}{2}) & \text{for } \frac{1}{2}\tau + \frac{1}{2} \leq t \leq & 1 \end{cases}.$$

In the following, we denote by a retracing of a path γ a subpath of the form $\delta\delta^{-1}$ with a certain finite δ.

[4]This means, roughly speaking, $\mathcal{P}$ possesses all properties of a group: associativity, existence of unit elements and of the inverse. But, the product need not be defined for all paths.

Remark 1. One can define an analogous equivalence relation on the set of paths in a fixed graph: Two paths would be "Γ-equivalent", iff they arise from each other by reparametrizations or by inserting or cancelling of retracings contained in Γ. Obviously, two paths in Γ are equivalent, if they are Γ-equivalent. On the other hand, one can also prove that two paths contained in Γ are already Γ-equivalent if they are equivalent.

Consequently, we can identify $\mathcal{P}_\Gamma$ and the set of all Γ-equivalence classes of paths in Γ. In other words: $\mathcal{P}_\Gamma$ is the groupoid that is generated freely by the set of all edges of Γ.

2. In what follows we usually say instead of "finite connected graph" simply "graph" and instead of "finite path" only "path". Moreover, by a path we always mean – if not explicitly the converse is said – an equivalence class of paths.

3. Finally, we identify two graphs if the (corresponding) edges are equivalent. Since edges are per def. free of retracings, this simply means that the edges are equal up to the parametrization.

4. Note that the paths $\gamma_1(\tau) := \tau$ and $\gamma_2(\tau) := \tau^2$ in $\mathbb{R}(\subseteq \mathbb{R}^n)$ are not equivalent. This comes from the fact that $\Pi : \tau \longrightarrow \tau^2$ is C^r, but $\Pi^{-1} : \tau \longrightarrow \sqrt{\tau}$ is not. (As well, it is not possible to transform γ_1 into γ_2 successively inserting or deleting retracings as in Definition 2.2.) Furthermore, one sees that $\gamma_1 \circ \gamma_2^{-1}$ is an example for a path with retracings that is not equivalent to a path without.

5. If we restricted ourselves to piecewise analytical paths, i.e. the smoothness category $(\omega, +)$ from the very beginning, every path would be finite. [2]

The main assumption quoted above suggests the usage of finite graphs as an index set for the subtheories. But, these theories are not "independent". Roughly speaking, a subtheory defined on a smaller lattice arises by projecting the theory defined on the bigger lattice.

Definition 2.3 Let Γ_1 and Γ_2 be two graphs. Γ_1 is **smaller or equal** Γ_2 ($\Gamma_1 \leq \Gamma_2$) iff each edge of Γ_1 is (up to the parametrization) a product of edges of Γ_2 and the vertex sets fulfill $\mathbf{V}(\Gamma_1) \subseteq \mathbf{V}(\Gamma_2)$.

Obviously, $\leq$ is a partial ordering.

2.2   Immersive Case

In the case of piecewise immersed paths we can define another equivalence relation for finite paths. Here we use the fact that any piecewise immersed path can be parametrized proportionally to the arc length:

Definition 2.4 We shortly call a path a **pal-path** iff it is parametrized *p*roportionally to the *a*rc *l*ength.

Two finite paths γ_1 and γ_2 are called **equivalent** iff there is a finite sequence of finite paths δ_i with $\delta_0 = \gamma_1$ and $\delta_n = \gamma_2$ such that for all $i = 1, \ldots, n$
- δ_i and δ_{i-1} coincide when parametrized proportionally to the arc length[5] or

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[5]This definition seems to require a certain Riemannian structure on M. But, on the one hand, each manifold can be given a Riemannian structure. On the other hand, the definition of equivalence does not

- δ_i arises from δ_{i-1} by inserting a retracing or
- δ_{i-1} arises from δ_i by inserting a retracing.

Lemma 2.2
1. Two finite paths γ_1 and γ_2 are equivalent if they can be obtained from each other by a reparametrization.
2. Each nontrivial finite path is equivalent to a pal-path without retracings.

Proof
1. Clear.
2. We prove this inductively on the number n of simple paths γ_i that the finite path γ is decomposed into. We will even prove that γ is equivalent to a pal-path γ' that can be decomposed (up to the parametrization) into $n' \leq n$ simple paths and that has no retracings.
   For $n = 1$ we have nothing to prove. Thus, let $n \geq 2$. First free $\gamma_0 := \prod_{i=1}^{n-1} \gamma_i$ off the retracings using the induction hypothesis. We get a pal-path $\gamma_0' \sim \prod_{i=1}^{n'-1} \gamma_i'$ with the desired properties and $n' \leq n$. Denote by γ' the pal-path corresponding to $\gamma_0' \gamma_n$. Obviously, $\gamma' \sim \gamma$. Suppose, γ' is not free of retracings. Let $\delta \delta^{-1}$ be a retracing. Then a part of the retracing $\delta \delta^{-1}$ has to be in γ_n. Since γ_n is simple (and w.l.o.g. non-trivial), the terminal point of δ cannot be in int γ_n. Since by assumption γ_0' is free of retracings, the terminal point has to be the initial point of γ_n, and thus δ^{-1} is (if necessary, after an appropriate [affine] reparametrization) an initial path of γ_n. Assume now δ to be maximal, i.e., any δ "containing" terminal path δ' of γ_0' that yields a retracing in γ' is equal to δ.[6] Now, cancel out the retracing: If δ is not a (genuine) subpath of $\gamma_{n'-1}'$ (i.e., "exceeds" or equals it), define γ_n' to be the "remaining" part of γ_n "outside" $(\gamma_{n'-1}')^{-1}$; then $\gamma'' := \left(\prod_{i=1}^{n'-2} \gamma_i' \right) \circ \gamma_n' \sim \gamma'$ consists of at most $n' - 2 + 1 < n$ finite paths. The induction hypothesis gives the assertion. Suppose now that δ is a (genuine) subpath of $\gamma_{n'-1}'$. Then define the pal-path γ'' by $\left(\prod_{i=1}^{n'-2} \gamma_i' \right) \gamma_{n'-1}'' \circ \gamma_n'$, where γ_n' denotes the "remaining" part of γ_n outside of δ^{-1} and $\gamma_{n'-1}''$ that of $\gamma_{n'-1}'$ outside of δ. By the maximality of δ, γ'' contains no retracings. $\gamma'' \sim \gamma' \sim \gamma$ yields the assertion. **qed**

Most of the constructions in the following as well as most of those in the subsequent papers [9, 10] do actually not depend on the choice of the equivalence relation for the paths. But, the second one can only be used for piecewise immersed paths. Therefore, in what follows, we will use the general equivalence relation given in the last subsection.

3 Gauge Theory on the Lattice

In this section we will transfer the lattice gauge theory given by Ashtekar and Lewandowski [4, 3] to our case. The algebraic definitions for the connections, gauge transforms and the

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depend on the chosen Riemannian metric: if two paths coincide w.r.t. to the arc length to the first metric then they obviously coincide w.r.t. to the arc length of the other metric. Thus, this definition is indeed completely determined by the manifold structure of M.

[6]Such a δ exists: Assume that every pal-subpath δ_τ^{-1} of γ_n corresponding to the parameter interval $[0, \tau]$ with $\tau < T$ yields a retracing. (Such a T exists, because there exists some retracing.) By the continuity of every path and the fact that the paths arising here and so all their subpaths are pal, also δ_T^{-1} has to yield a retracing. Consequently, there is a maximal T.

action of the latter ones follow these authors closely. In the last two subsections we will state some assertions mainly on the basic properties of the action of the gauge transforms and the projections onto smaller graphs.

3.1 Algebraic Definition

We use the standard definition: Globally connections are parallel transports, i.e. $\mathbf{G}$-valued homomorphisms of paths in M, and gauge transforms are $\mathbf{G}$-valued functions over M. The lattice versions now come from restricting the domain of definition to edges and vertices in a graph.

Definition 3.1 Let Γ be a graph. We define
$\overline{\mathcal{A}}_\Gamma := \mathrm{Hom}(\mathcal{P}_\Gamma, \mathbf{G})$ ... set of all connections on Γ and
$\overline{\mathcal{G}}_\Gamma := \mathrm{Maps}(\mathbf{V}(\Gamma), \mathbf{G})$ ... set of all gauge transforms on Γ.
Here, $\mathrm{Hom}(\mathcal{P}_\Gamma, \mathbf{G})$ denotes the set of all homomorphisms from the groupoid $\mathcal{P}_\Gamma$ freely generated by the edges of Γ into the structure group and $\mathrm{Maps}(\mathbf{V}(\Gamma), \mathbf{G})$ the set of all maps from the set of all vertices of Γ into the structure group.

In the classical case the action of a gauge transform on a connection can be described by the corresponding action on the parallel transports:
$$h_A(\gamma) \longmapsto g_{\gamma(0)}^{-1} h_A(\gamma) g_{\gamma(1)}.$$
By simply restricting onto the lattice we receive the action of $\overline{\mathcal{G}}_\Gamma$ on $\overline{\mathcal{A}}_\Gamma$ by
$$\Theta_\Gamma : \quad \overline{\mathcal{A}}_\Gamma \times \overline{\mathcal{G}}_\Gamma \quad \longrightarrow \quad \overline{\mathcal{A}}_\Gamma$$
$$(h_\Gamma, g_\Gamma) \quad \longmapsto \quad h_\Gamma \circ g_\Gamma$$
with $h_\Gamma \circ g_\Gamma(\gamma) := g_\Gamma(\gamma(0))^{-1} \, h_\Gamma(\gamma) \, g_\Gamma(\gamma(1))$ for all paths γ in Γ.

Definition 3.2 For each graph Γ we define
$\overline{\mathcal{A}/\mathcal{G}}_\Gamma := \overline{\mathcal{A}}_\Gamma / \overline{\mathcal{G}}_\Gamma$ ... set of all equivalence classes of connections in Γ.

3.2 Topological Definition

It is obvious that the groupoid $\mathcal{P}_\Gamma$ is always freely generated by the edges e_i of Γ. Hence, the set $\overline{\mathcal{A}}_\Gamma = \mathrm{Hom}(\mathcal{P}_\Gamma, \mathbf{G})$ can be identified via $h \longmapsto \left(h(e_1), \ldots, h(e_{\#\mathbf{E}(\Gamma)}) \right)$ with $\mathbf{G}^{\#\mathbf{E}(\Gamma)}$ and can so be given a natural topology. Analogously, we use that naturally $\overline{\mathcal{G}}_\Gamma = \mathrm{Maps}(\mathbf{V}(\Gamma), \mathbf{G})$ can be identified via $g \longmapsto (g(x))_{x \in \mathbf{V}(\mathbf{G})}$ with $\mathbf{G}^{\#\mathbf{V}(\Gamma)}$. So $\overline{\mathcal{G}}_\Gamma$ is by means of the pointwise multiplication a topological group. We have immediately

Proposition 3.1 For all graphs Γ the action $\Theta_\Gamma : \overline{\mathcal{A}}_\Gamma \times \overline{\mathcal{G}}_\Gamma \longrightarrow \overline{\mathcal{A}}_\Gamma$ is continuous.

Proof Θ_Γ as a map from $\mathbf{G}^{\#\mathbf{E}(\Gamma)} \times \mathbf{G}^{\#\mathbf{V}(\Gamma)}$ to $\mathbf{G}^{\#\mathbf{E}(\Gamma)}$ is a concatenation of multiplications, hence continuous.   **qed**

Corollary 3.2 $\overline{\mathcal{A}/\mathcal{G}}_\Gamma = \overline{\mathcal{A}}_\Gamma / \overline{\mathcal{G}}_\Gamma$ is a Hausdorff space. $\overline{\mathcal{A}/\mathcal{G}}_\Gamma$ is compact for compact $\mathbf{G}$.

It is well-known that connections are dual to paths and equivalence classes of connections are dual to closed paths. This is again confirmed by

Proposition 3.3 $\overline{\mathcal{A}/\mathcal{G}}_\Gamma$ is isomorphic to $\operatorname{Hom}(\mathcal{HG}_{x,\Gamma},\mathbf{G})/\mathrm{Ad}$, hence isomorphic to $\mathbf{G}^{\dim \pi_1(\Gamma)}/\mathrm{Ad}$, for each graph Γ and for each vertex x in Γ.

Here $\mathcal{HG}_{x,\Gamma}$ is the set of all (classes of) path(s) in Γ starting and ending in x, and $\pi_1(\Gamma)$ is the fundamental group of Γ.

Proof Define
$$J:\ \overline{\mathcal{A}/\mathcal{G}}_\Gamma \longrightarrow \operatorname{Hom}(\mathcal{HG}_{x,\Gamma},\mathbf{G})/\mathrm{Ad}.$$
$$[h] \longmapsto [h\ |_{\mathcal{HG}_{x,\Gamma}}]_{\mathrm{Ad}}$$

- J is well-defined.
  If $h' = h'' \circ g$, then $h'(\alpha) = g(x)^{-1}h''(\alpha)g(x)$ for all $\alpha \in \mathcal{HG}_{x,\Gamma}$, i.e. $h'\ |_{\mathcal{HG}_{x,\Gamma}} = h''\ |_{\mathcal{HG}_{x,\Gamma}} \circ_{\mathrm{Ad}} g(x)$.
- J is injective.
  Let $J(h') = J(h'')$, i.e., let there exist a $g \in \mathbf{G}$ such that $h'(\alpha) = g^{-1}h''(\alpha)g$ for all $\alpha \in \mathcal{HG}_{x,\Gamma}$. Choose for all vertices $y \neq x$ a path γ_y from x to y, set $\gamma_x := 1$ and set $g(y) := h''(\gamma_y)^{-1}\, g\, h'(\gamma_y)$ for all y. Now, $h' = h'' \circ (g(y))_{y\in\mathbf{V}(\Gamma)}$ is clear.
- J is surjective.
  Let $[h] \in \operatorname{Hom}(\mathcal{HG}_{x,\Gamma},\mathbf{G})/\mathrm{Ad}$ be given. Choose an $h \in [h]$ and as above for all vertices y a path γ_y and some $g_y \in \mathbf{G}$. For each $\gamma \in \mathcal{P}_\Gamma$ set $h_0(\gamma) := g_{\gamma(0)}^{-1}\, h(\gamma_{\gamma(0)}\gamma\gamma_{\gamma(1)}^{-1})\, g_{\gamma(1)}$. We have $J(h_0) = [h]$.

Since $\mathcal{HG}_{x,\Gamma}$ is isomorphic to $\pi_1(\Gamma)$, hence a free group with $\dim \pi_1(\Gamma)$ generators [11, 2], we have $\overline{\mathcal{A}/\mathcal{G}}_\Gamma \cong \mathbf{G}^{\dim \pi_1(\Gamma)}/\mathrm{Ad}$. **qed**

3.3 Relations between the Lattice Theories

If one constructs a global theory from its subtheories one has to guarantee that these subtheories are "consistent". This means, e.g., that the projection of a connection onto a smaller graph has to be already defined by its projection onto a bigger graph. So we need projections onto the subtheories induced by the partial ordering on the set of graphs.

Definition 3.3 Let $\Gamma_1 \leq \Gamma_2$.
We define
$$\pi_{\Gamma_1}^{\Gamma_2}:\ \overline{\mathcal{A}}_{\Gamma_2} \longrightarrow \overline{\mathcal{A}}_{\Gamma_1},$$
$$h \longmapsto h\ |_{\mathcal{P}_{\Gamma_1}}$$
$$\pi_{\Gamma_1}^{\Gamma_2}:\ \overline{\mathcal{G}}_{\Gamma_2} \longrightarrow \overline{\mathcal{G}}_{\Gamma_1}$$
$$g \longmapsto g\ |_{\mathbf{V}(\Gamma_1)}$$
and
$$\pi_{\Gamma_1}^{\Gamma_2}:\ \overline{\mathcal{A}/\mathcal{G}}_{\Gamma_2} \longrightarrow \overline{\mathcal{A}/\mathcal{G}}_{\Gamma_1}.$$
$$[h] \longmapsto [h\ |_{\mathcal{P}_{\Gamma_1}}]$$

We denote all the three maps by one and the same symbol because it should be clear in the following what map is meant.

Obviously, from $h' = h'' \circ g$ on Γ_2 follows $h'\ |_{\mathcal{P}_{\Gamma_1}} = h''\ |_{\mathcal{P}_{\Gamma_1}} \circ g\ |_{\mathbf{V}(\Gamma_1)}$ on Γ_1, i.e. $\pi_{\Gamma_1}^{\Gamma_2}$ is well-defined. Furthermore, we have

Proposition 3.4 Let $\Gamma_1 \leq \Gamma_2 \leq \Gamma_3$. Then $\pi_{\Gamma_1}^{\Gamma_2}\pi_{\Gamma_2}^{\Gamma_3} = \pi_{\Gamma_1}^{\Gamma_3}$.

Finally, we write down the projections by operations on the structure group in order to see topological properties.

Let again $\Gamma_1 \leq \Gamma_2$. First we decompose each edge e_i of Γ_1 into edges f_j of Γ_2: $e_i = \prod_{k_i=1}^{K_i} f_{j(i,k_i)}^{\epsilon(i,k_i)}$.
With this we get for the map between the connections $(n := \#\mathbf{E}(\Gamma_1))$

$$\pi_{\Gamma_1}^{\Gamma_2}: \qquad \mathbf{G}^{\#\mathbf{E}(\Gamma_2)} \qquad \longrightarrow \qquad \mathbf{G}^{\#\mathbf{E}(\Gamma_1)}.$$
$$\left(g_1, \ldots, g_{\#\mathbf{E}(\Gamma_2)}\right) \quad \longmapsto \quad \left(\prod_{k_1=1}^{K_1} g_{j(1,k_1)}^{\epsilon(1,k_1)}, \ldots, \prod_{k_n=1}^{K_n} g_{j(n,k_n)}^{\epsilon(n,k_n)}\right)$$

On the level of gauge transforms the description is very easy: $\pi_{\Gamma_1}^{\Gamma_2}$ projects $(g_v)_{v \in \mathbf{V}(\Gamma_2)}$ onto those elements belonging to vertices in Γ_1. For classes of connections an analogous formula as for connections holds: First choose two free generating systems $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of $\mathcal{HG}_{x_1,\Gamma_1}$ and $\mathcal{HG}_{x_2,\Gamma_2}$, respectively, and then a path γ from x_2 to x_1 in the bigger graph Γ_2. Thus we get decompositions $\alpha_i = \gamma^{-1}\left(\prod_{k_i=1}^{K_i} \beta_{j(i,k_i)}^{\epsilon(i,k_i)}\right)\gamma$. Hence, $(n_i := \dim \pi_1(\Gamma_i))$

$$\pi_{\Gamma_1}^{\Gamma_2}: \qquad \mathbf{G}^{n_2}/\mathrm{Ad} \qquad \longrightarrow \qquad \mathbf{G}^{n_1}/\mathrm{Ad}.$$
$$\left[g_1, \ldots, g_{n_2}\right]_{\mathrm{Ad}} \quad \longmapsto \quad \left[\prod_{k_1=1}^{K_1} g_{j(1,k_1)}^{\epsilon(1,k_1)}, \ldots, \prod_{k_{n_1}=1}^{K_{n_1}} g_{j(n_1,k_{n_1})}^{\epsilon(n_1,k_{n_1})}\right]_{\mathrm{Ad}}$$

Proposition 3.5 $\pi_{\Gamma_1}^{\Gamma_2}$ is continuous, open and surjective.

Proof The surjectivity is clear for all three cases.

The continuity is trivial for the first two cases and follows in the third because the projections $\mathbf{G}^n \longrightarrow \mathbf{G}^n/\mathrm{Ad}$ are open, continuous and surjective (see [8]) and the map from $\mathbf{G}^{n_2}$ to $\mathbf{G}^{n_1}$ corresponding to $\pi_{\Gamma_1}^{\Gamma_2}$ is obviously continuous.

The openness follows immediately in the case of gauge transforms because projections onto factors of a direct product are open anyway. In the case of connections one additionally needs the openness of the multiplication in $\mathbf{G}$: Each edge in Γ_1 is a product of edges in Γ_2, i.e., after possibly renumbering we have $e_i = f_{i,1} \cdots f_{i,K_i}$. Thus, $\pi_{\Gamma_1}^{\Gamma_2}(g_{1,1}, \ldots, g_{n,K_n}, \ldots) = (g_{1,1} \cdots g_{1,K_1}, \ldots, g_{n,1} \cdots g_{n,K_n})$. Let now W be open in $\overline{\mathcal{A}}_{\Gamma_2} = \mathbf{G}^{\#\mathbf{E}(\Gamma_2)}$. Then W is a union of sets of the form $W_{1,1} \times \cdots \times W_{n,K_n} \times \cdots$, i.e., $\pi_{\Gamma_1}^{\Gamma_2}(W)$ is a union of sets of the form $(W_{1,1} \cdots W_{1,K_1}) \times \cdots \times (W_{n,1} \cdots W_{n,K_n})$. But these are open, i.e., $\pi_{\Gamma_1}^{\Gamma_2}$ is open. The openness of $\pi_{\Gamma_1}^{\Gamma_2}: \overline{\mathcal{A}/\mathcal{G}}_{\Gamma_2} \longrightarrow \overline{\mathcal{A}/\mathcal{G}}_{\Gamma_1}$ follows now because the map $\pi_{\Gamma_1}^{\Gamma_2}: \overline{\mathcal{A}}_{\Gamma_2} \longrightarrow \overline{\mathcal{A}}_{\Gamma_1}$ is open and the projections $\overline{\mathcal{A}}_\Gamma \longrightarrow \overline{\mathcal{A}/\mathcal{G}}_\Gamma$ are continuous, open and surjective. **qed**

4 Continuum Gauge Theory

For completeness in the first subsection we will briefly quote the definitions of $\overline{\mathcal{A}}$, $\overline{\mathcal{G}}$ and $\overline{\mathcal{A}/\mathcal{G}}$ from [4] and in the second we summarize the most important facts about these spaces. In the last two subsections we will first investigate the topological properties of the action of $\overline{\mathcal{G}}$ on $\overline{\mathcal{A}}$ and of the projections onto the lattice gauge theories and then prove that the connections etc. are algebraically described exactly in the same form both for our definition of paths and for that of Ashtekar and Lewandowski [2].

4.1 Definition of $\overline{\mathcal{A}}$, $\overline{\mathcal{G}}$ and $\overline{\mathcal{A}/\mathcal{G}}$

By means of the continuity of the projections $\pi_{\Gamma_1}^{\Gamma_2}$ the spaces $(\overline{\mathcal{A}}_\Gamma)_\Gamma$, $(\overline{\mathcal{G}}_\Gamma)_\Gamma$ and $(\overline{\mathcal{A}/\mathcal{G}}_\Gamma)_\Gamma$ are projective systems of topological spaces. This leads to the crucial [4]

Definition 4.1 Generalized Gauge Theories

- $\overline{\mathcal{A}} := \varprojlim_{\Gamma} \overline{\mathcal{A}}_{\Gamma}$ is the space of **generalized connections**.
  The elements of $\overline{\mathcal{A}}$ are usually denoted by $\overline{A}$ or $h_{\overline{A}}$.
- $\overline{\mathcal{G}} := \varprojlim_{\Gamma} \overline{\mathcal{G}}_{\Gamma}$ is the space of **generalized gauge transforms**.
  The elements of $\overline{\mathcal{G}}$ are usually denoted by $\overline{g}$.
- $\overline{\mathcal{A}/\mathcal{G}} := \varprojlim_{\Gamma} \overline{\mathcal{A}/\mathcal{G}}_{\Gamma}$ is the space of **generalized equivalence classes of connections**.

Explicitly this means

$$\overline{\mathcal{A}} = \{(h_{\Gamma})_{\Gamma} \in \underset{\Gamma}{\times} \overline{\mathcal{A}}_{\Gamma} \mid \pi_{\Gamma_1}^{\Gamma_2} h_{\Gamma_2} = h_{\Gamma_1} \text{ for all } \Gamma_1 \le \Gamma_2\},$$

$$\overline{\mathcal{G}} = \{(g_{\Gamma})_{\Gamma} \in \underset{\Gamma}{\times} \overline{\mathcal{G}}_{\Gamma} \mid \pi_{\Gamma_1}^{\Gamma_2} g_{\Gamma_2} = g_{\Gamma_1} \text{ for all } \Gamma_1 \le \Gamma_2\} \tag{1}$$

as well as

$$\overline{\mathcal{A}/\mathcal{G}} = \{([h_{\Gamma}])_{\Gamma} \in \underset{\Gamma}{\times} \overline{\mathcal{A}/\mathcal{G}}_{\Gamma} \mid \pi_{\Gamma_1}^{\Gamma_2} [h_{\Gamma_2}] = [h_{\Gamma_1}] \text{ for all } \Gamma_1 \le \Gamma_2\}.$$

We denote

$$
\begin{aligned}
\pi_{\Gamma}: \quad & \overline{\mathcal{A}} \quad \longrightarrow \quad \overline{\mathcal{A}}_{\Gamma}, \\
& (h_{\Gamma'})_{\Gamma'} \quad \longmapsto \quad h_{\Gamma} \\
\pi_{\Gamma}: \quad & \overline{\mathcal{G}} \quad \longrightarrow \quad \overline{\mathcal{G}}_{\Gamma} \\
& (g_{\Gamma'})_{\Gamma'} \quad \longmapsto \quad g_{\Gamma}
\end{aligned}
$$

and

$$
\begin{aligned}
\pi_{\Gamma}: \quad & \overline{\mathcal{A}/\mathcal{G}} \quad \longrightarrow \quad \overline{\mathcal{A}/\mathcal{G}}_{\Gamma}. \\
& ([h_{\Gamma'}])_{\Gamma'} \quad \longmapsto \quad [h_{\Gamma}]
\end{aligned}
$$

4.2 Topological Characterization of $\overline{\mathcal{A}}$, $\overline{\mathcal{G}}$ and $\overline{\mathcal{A}/\mathcal{G}}$

We have [4, 13]

Theorem 4.1
1. $\overline{\mathcal{A}}, \overline{\mathcal{G}}$ and $\overline{\mathcal{A}/\mathcal{G}}$ are completely regular Hausdorff spaces and, for compact **G**, compact.
2. For every principle fibre bundle over M with structure group **G** the regular connections (gauge transforms, equivalence classes of connections) are also generalized connections (gauge transforms, equivalence classes of generalized connections). This means the maps $\mathcal{A} \longrightarrow \overline{\mathcal{A}}$, $\mathcal{G} \longrightarrow \overline{\mathcal{G}}$ and $\mathcal{A}/\mathcal{G} \longrightarrow \overline{\mathcal{A}/\mathcal{G}}$ are embeddings.
3. Let X be a topological space.
   A map $f : X \longrightarrow \overline{\mathcal{A}}$ is continuous iff $\pi_{\Gamma} \circ f : X \longrightarrow \overline{\mathcal{A}}_{\Gamma} \equiv \mathbf{G}^{\#\mathbf{E}(\Gamma)}$ is continuous for all graphs Γ.
   The analogous assertion holds for maps from X to $\overline{\mathcal{G}}$ and $\overline{\mathcal{A}/\mathcal{G}}$, respectively, as well.
4. π_{Γ} is continuous for all graphs Γ.
5. $\overline{\mathcal{G}}$ is a topological group.

We shall postpone the discussion whether the space $\mathcal{A}$ is dense in $\overline{\mathcal{A}}$ or not for several reasons. This, in fact, depends crucially on the chosen smoothness category and equivalence relation for the paths. It should be clear that – provided $\gamma_1(\tau) := \tau$ and $\gamma_2(\tau) := \tau^2$ are seen to be non-equivalent – the denseness is unlikely: No classical smooth connection A can distinguish between these paths. So we will discuss this a bit more in detail in the subsequent paper

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[9]. As well, we will show there that π_Γ is also open and surjective. But all that requires some technical efforts that are absolutely not necessary for the actual goal of this paper – the determination of the gauge orbit types.

Proof 1. The property of being compact, Hausdorff or completely regular is maintained by forming product spaces and by the transition to closed subsets. Thus the assertion follows from the corresponding properties of the structure group **G**.
2. The embedding property follows from Giles' reconstruction theorem [12] and [1].
3. See, e.g., [13].
4. Since $\mathrm{id} : \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}$ etc. is continuous, this follows from the facts just proven.
5. The multiplication on $\overline{\mathcal{G}}$ is defined by $(g_\Gamma)_\Gamma \circ (g'_\Gamma)_\Gamma = (g_\Gamma \circ g'_\Gamma)_\Gamma$. With this $\overline{\mathcal{G}}$ is a group with unit $(e_\Gamma)_\Gamma$ and inverse $\left((g_\Gamma)_\Gamma\right)^{-1} = (g_\Gamma^{-1})_\Gamma$. The multiplication $\mathrm{m} : \overline{\mathcal{G}} \times \overline{\mathcal{G}} \longrightarrow \overline{\mathcal{G}}$ is continuous due to the continuity criterion above: $\pi_\Gamma \circ \mathrm{m} = \mathrm{m}_\Gamma \circ (\pi_\Gamma \times \pi_\Gamma)$ is continuous for all Γ, because the multiplication m_Γ on $\overline{\mathcal{G}}_\Gamma$ is continuous. **qed**

4.3 Action of Gauge Transforms on Connections

Because of the consistency of the actions of $\overline{\mathcal{G}}_\Gamma$ on $\overline{\mathcal{A}}_\Gamma$ one can also define an action of $\overline{\mathcal{G}}$ on $\overline{\mathcal{A}}$. One simply sets [4]

$$\Theta : \quad \overline{\mathcal{A}} \times \overline{\mathcal{G}} \quad \longrightarrow \quad \overline{\mathcal{A}}.$$
$$\left((h_\Gamma)_\Gamma, (g_\Gamma)_\Gamma\right) \quad \longmapsto \quad (h_\Gamma \circ g_\Gamma)_\Gamma$$

Theorem 4.2 1. The action Θ of $\overline{\mathcal{G}}$ on $\overline{\mathcal{A}}$ is continuous.
2. The maps
$$\overline{A} : \quad \overline{\mathcal{G}} \quad \longrightarrow \quad \overline{\mathcal{A}} \qquad \text{and} \qquad \overline{g} : \quad \overline{\mathcal{A}} \quad \longrightarrow \quad \overline{\mathcal{A}}$$
$$\overline{g} \quad \longmapsto \quad \overline{A} \circ \overline{g} \qquad\qquad\qquad\qquad \overline{A} \quad \longmapsto \quad \overline{A} \circ \overline{g}$$
are continuous.
3. The canonical projection $\pi_{\overline{\mathcal{A}}/\overline{\mathcal{G}}} : \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}/\overline{\mathcal{G}}$ is continuous and open and for compact **G** also closed and proper.
4. The map $\qquad \pi_\Gamma : \quad \overline{\mathcal{A}}/\overline{\mathcal{G}} \quad \longrightarrow \quad \overline{\mathcal{A}}_\Gamma/\overline{\mathcal{G}}_\Gamma \qquad$ is well-defined and
$$[(h_{\Gamma'})_{\Gamma'}] \quad \longmapsto \quad [h_\Gamma]$$
continuous.

Proof 1. $\pi_\Gamma \circ \Theta = \Theta_\Gamma \circ (\pi_\Gamma \times \pi_\Gamma) : \overline{\mathcal{A}} \times \overline{\mathcal{G}} \longrightarrow \overline{\mathcal{A}}_\Gamma$ as a concatenation of continuous maps on the right-hand side is continuous for any graph Γ. By the continuity criterion for maps to $\overline{\mathcal{A}}$ in Theorem 4.1, Θ is continuous.
2. Follows from the continuity of Θ.
3. Follows because Θ is a continuous action of a (compact) topological group $\overline{\mathcal{G}}$ on the Hausdorff space $\overline{\mathcal{A}}$. [8]
4. π_Γ is well-defined. Namely, let $\overline{A}' = \overline{A} \circ \overline{g}$, i.e. $(h'_{\Gamma'})_{\Gamma'} = (h_{\Gamma'} \circ g_{\Gamma'})_{\Gamma'}$, thus $h'_{\Gamma'} = h_{\Gamma'} \circ g_{\Gamma'}$ for all graphs Γ'. Then $[h'_\Gamma] = [h_\Gamma]$.
The continuity of $\pi_\Gamma : \overline{\mathcal{A}}/\overline{\mathcal{G}} \longrightarrow \overline{\mathcal{A}}_\Gamma/\overline{\mathcal{G}}_\Gamma$ follows from the continuity of $\pi_\Gamma : \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}_\Gamma$ and $\pi_{\overline{\mathcal{A}}_\Gamma/\overline{\mathcal{G}}_\Gamma}$ as well as from the continuity criterion for the quotient topology because the diagram

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$$\begin{array}{ccc} \overline{\mathcal{A}} & \xrightarrow{\pi_{\overline{\mathcal{A}/\mathcal{G}}}} & \overline{\mathcal{A}/\mathcal{G}} \\ \pi_\Gamma \downarrow & & \downarrow \pi_\Gamma \\ \overline{\mathcal{A}}_\Gamma & \xrightarrow{\pi_{\overline{\mathcal{A}}_\Gamma/\overline{\mathcal{G}}_\Gamma}} & \overline{\mathcal{A}}_\Gamma/\overline{\mathcal{G}}_\Gamma \end{array}$$

is commutative. **qed**

We note that for a compact structure group $\mathbf{G}$ and for analytic paths $\overline{\mathcal{A}/\mathcal{G}}$ and $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ are even homeomorphic (cf. [4, 3]).

4.4 Algebraic Characterization of $\overline{\mathcal{A}}$, $\overline{\mathcal{G}}$ and $\overline{\mathcal{A}/\mathcal{G}}$

In this subsection we will show that our choice of the definition of paths leads to the same results as the definitions in [2] do.

Theorem 4.3 1. We have $\overline{\mathcal{A}} \cong \mathrm{Hom}(\mathcal{P}, \mathbf{G})$.[7]
Here, $\mathrm{Hom}(\mathcal{P}, \mathbf{G})$ is the set of all maps $h : \mathcal{P} \longrightarrow \mathbf{G}$, that fulfill $h(\gamma_1\gamma_2) = h(\gamma_1)h(\gamma_2)$ for all multipliable paths $\gamma_1, \gamma_2 \in \mathcal{P}$.

2. We have $\overline{\mathcal{G}} \cong \bigtimes_{x \in M} \mathbf{G} \equiv \mathrm{Maps}(M, \mathbf{G})$.
The isomorphism is even a homeomorphism of topological groups.

3. The action of gauge transforms on the connections is given by
$$h_{\overline{A} \circ \overline{g}}(\gamma) := g_{\gamma(0)}^{-1}\, h_{\overline{A}}(\gamma)\, g_{\gamma(1)} \text{ for all } \gamma \in \mathcal{P}. \tag{2}$$
$h_{\overline{A}} : \mathcal{P} \longrightarrow \mathbf{G}$ is the homomorphism corresponding to $\overline{A} \in \overline{\mathcal{A}}$ and g_x the component of the gauge transform $\overline{g} \in \overline{\mathcal{G}}$ in x.

4. We have $\overline{\mathcal{A}/\mathcal{G}} \cong \mathrm{Hom}(\mathcal{HG}, \mathbf{G})/\mathrm{Ad}$.
Here, $\mathrm{Hom}(\mathcal{HG}, \mathbf{G})$ is the set of all homomorphisms $h : \mathcal{HG} \longrightarrow \mathbf{G}$.

Proof 1. Define
$$\begin{array}{rccc} I : & \mathrm{Hom}(\mathcal{P}, \mathbf{G}) & \longrightarrow & \overline{\mathcal{A}}. \\ & h & \longmapsto & (h\,|_{\mathcal{P}_\Gamma})_\Gamma \end{array}$$

- I is injective.
From $h_1 \neq h_2$ follows the existence of a $\gamma \in \mathcal{P}$ with $h_1(\gamma) \neq h_2(\gamma)$. Since γ equals $\prod \gamma_i$ with appropriate simple γ_i, we have $\prod h_1(\gamma_i) \neq \prod h_2(\gamma_i)$, hence $h_1(\gamma_i) \neq h_2(\gamma_i)$ for some γ_i. Choose a finite graph Γ such that γ_i is a path in Γ. Here we have $h_1\,|_{\mathcal{P}_\Gamma}(\gamma_i) = h_1(\gamma_i) \neq h_2(\gamma_i) = h_2\,|_{\mathcal{P}_\Gamma}(\gamma_i)$, i.e. $I(h_1) \neq I(h_2)$.

- I is surjective.
Let $(h_\Gamma)_\Gamma$ be given. We consider first not classes of paths, but the paths itself. Construct for any simple $\gamma \in \mathcal{P}$ a graph Γ containing γ. Define $h(\gamma) := h_\Gamma(\gamma)$. For general $\gamma \in \mathcal{P}$ define $h(\gamma) := \prod h(\gamma_i)$ according to some decomposition of γ into simple paths γ_i.
This construction is well-defined: First one easily realizes that it is independent of the decomposition of γ into finite paths (thus also of the parametrization).[8] Hence obviously, h is a homomorphism. Thus, also $h(\gamma'\delta\delta^{-1}\gamma'') =$

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[7]This justifies the notation $h_{\overline{A}}$ for a connection $\overline{A}$.
[8]Namely, let $\prod \gamma_i'$ and $\prod \gamma_j''$ be two decompositions of γ. The terminal points of γ_i' and γ_j'' correspond to certain values of the parameters τ_i' and τ_j', respectively, of the path γ. Order these values to a sequence (τ_k)

$h(\gamma'\gamma'')$ etc., i.e., h is constant on equivalence classes of paths. Consequently, $h : \mathcal{P} \longrightarrow \mathbf{G}$ is a well-defined homomorphism with $I(h) = (h_\Gamma)_\Gamma$.

2. Set
$$I : \quad \mathrm{Maps}(M, \mathbf{G}) \quad \longrightarrow \quad \overline{\mathcal{G}}.$$
$$(g_x)_{x \in M} \quad \longmapsto \quad \left((g_x)_{x \in \mathbf{V}(\Gamma)}\right)_\Gamma$$

Obviously, I is bijective and a group homomorphism.

The topology on $\mathrm{Maps}(M, \mathbf{G}) = \bigtimes_{x \in M} \mathbf{G}$ is generated by the preimages $\pi_y^{-1}(U)$ of open $U \subseteq \mathbf{G}$, by which $\pi_y : (g_x)_{x \in M} \longmapsto g_y$ is continuous. Hence, $\pi_\Gamma \circ I = \pi_{v_1} \times \cdots \times \pi_{v_{\#\mathbf{V}(\Gamma)}}$ is continuous for all Γ, i.e., I is continuous. Due to the continuity criterion for maps into product spaces, also I^{-1} is continuous because for all y the map $\pi_y \circ I^{-1} = \pi_\Gamma$ (Γ consists only of the vertex y) is continuous.

3. This follows immediately from the preceding steps.

4. Use the map
$$J : \quad \overline{\mathcal{A}/\mathcal{G}} \quad \longrightarrow \quad \mathrm{Hom}(\mathcal{HG}, \mathbf{G})/\mathrm{Ad} \qquad \text{and repeat the steps of}$$
$$[h] \quad \longmapsto \quad [h \mid_{\mathcal{HG}}]_{\mathrm{Ad}}$$
the proof of Proposition 3.3. **qed**

In the following we will usually write a gauge transform in the form $\overline{g} = (g_x)_{x \in M}$. Furthermore we have again by the continuity criterion for maps into product spaces

Corollary 4.4 Let X be a topological space.
A map $f : X \longrightarrow \overline{\mathcal{G}}$ is continuous iff $\pi_x \circ f : X \longrightarrow \mathbf{G}$ is continuous for all $x \in M$.
π_x is continuous for all $x \in X$.

Remark If we work in the $(\omega, +)$-category for the paths, i.e., we only consider piecewise analytical graphs, all the definitions and results coincide completely with those of Ashtekar and Lewandowski in [2, 4, 3].

5 Graphs vs. Webs

In this section we will compare the consequences of our definition of paths to that of webs [6, 7, 17]. Within this section we only consider the smooth category[9] $(\infty, +)$ for paths. Note that, within this section, a path is simply a piecewise immersive and C^∞-map from $[0, 1]$ to M, i.e. it is not an equivalence class. But it is still finite as before.

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and construct a decomposition of γ into simple paths δ_k such that δ_k corresponds to the segment $\gamma \mid_{[\tau_k, \tau_{k+1}]}$. Now, on the one hand, γ equals up to the parametrization $\prod \delta_k$, but, on the other hand, each γ'_i and γ''_j equals up to the parametrization a product $\delta_\kappa \circ \delta_{\kappa+1} \circ \cdots \circ \delta_\lambda$ with certain κ, λ.

Now let $\widetilde{\Gamma}'_i$ be that graph w.r.t. that γ'_i is simple. Construct hereof the graph Γ'_i by inserting the terminal points of all the γ''_j as vertices. Finally, let Γ_k be the graph spanned by δ_k. Thus, $\Gamma_k, \widetilde{\Gamma}'_i \leq \Gamma'_i$, and we have

$$\begin{aligned} h(\gamma'_i) &= h_{\widetilde{\Gamma}'_i}(\gamma'_i) \\ &= h_{\Gamma'_i}(\gamma'_i) \\ &= h_{\Gamma'_i}(\delta_\kappa \circ \delta_{\kappa+1} \circ \cdots \circ \delta_\lambda) \\ &= h_{\Gamma'_i}(\delta_\kappa)\, h_{\Gamma'_i}(\delta_{\kappa+1}) \cdots h_{\Gamma'_i}(\delta_\lambda) \\ &= h_{\Gamma_\kappa}(\delta_\kappa)\, h_{\Gamma_{\kappa+1}}(\delta_{\kappa+1}) \cdots h_{\Gamma_\lambda}(\delta_\lambda). \end{aligned}$$

Using the analogous relation for γ''_j we have $\prod h(\gamma'_i) = \prod h_{\Gamma_k}(\delta_k) = \prod h(\gamma''_j)$. Thus, $h(\gamma)$ does not depend on the decomposition.

[9]Remember, the $+$ means that all the paths are piecewise immersions.

Let us briefly quote the basic properties of webs. A web consists of a finite number of so-called tassels. A tassel T with base point $p \in M$ is a finite, ordered set of curves c_i (piecewise immersive smooth maps[10] from $[0,1]$ to M) that fulfills certain properties:

1. $c_i(0) = p$ for all i (common initial point).
2. c_i is an embedding (in particular, has no self-intersections).
3. There is a positive constant $k_i \in \mathbb{R}$ for each i such that $c_i(t) = c_j(s)$ implies $k_i t = k_j s$ (consistent parametrization).
4. Define $\text{Type}(x) := \{i \in I \mid x \in \text{im } c_i\}$ for all $x \in X$. Then, for all $J \subset I$ the set $\text{Type}^{-1}(\{J\})$ is empty or has p as an accumulation point.

Thus, in our notation, each c_i is a simple path. A web is now a finite collection of tassels such that no path of one tassel contains the base point of another tassel. The following theorem on curves proven by Baez and Sawin [6] will be crucial:

Theorem 5.1 Given a finite set C of curves. Then there is a web w, such that every curve $c \in C$ is equivalent to a finite product of paths $\gamma \in w$ and their inverses.

This, namely, leads immediately to the following

Proposition 5.2 Every curve is equivalent to a finite path.

Thus, our restriction to finite paths is actually no restriction.

Proof Let there be given an arbitrary curve $\gamma : [a,b] \longrightarrow M$. By the preceding theorem γ depends on some web W, i.e., there is a family of curves c_i being simple paths such that γ equals (modulo equivalence, i.e. up to reparametrizations, cf. [6]) a finite product of the curves c_i and their inverses. By Definition 2.2, γ is finite.   **qed**

This means, roughly speaking, the sets of paths the connections are based on are the same for the webs and our case $(\infty, +)$. But this yields the equality of our definition of $\overline{\mathcal{A}}$ and that of Baez and Sawin.

Theorem 5.3 Suppose $\mathbf{G}$ to be compact and semi-simple. Then $\overline{\mathcal{A}}_{\text{Web}}$ and $\overline{\mathcal{A}}_{(\infty,+)}$, i.e. the spaces of generalized connections defined by webs [6] and by Definition 4.1, respectively, are homeomorphic.

Proof Using the proposition above we see analogously to the proof of Theorem 4.3 that
$$I_{\text{Web}} : \begin{array}{ccc} \text{Hom}(\mathcal{P}, \mathbf{G}) & \longrightarrow & \overline{\mathcal{A}}_{\text{Web}} \\ h & \longmapsto & (h \mid_w)_w \end{array}$$
is a bijection. Thus, $\mathcal{I} := I_{\text{Web}} \circ I^{-1} : \overline{\mathcal{A}}_{(\infty,+)} \longrightarrow \overline{\mathcal{A}}_{\text{Web}}$ is a bijection, too. We are left with the proof that $\mathcal{I}$ is a homeomorphism. For this it is sufficient to prove that each element of a subbase of the one topology has an open image in the other topology. Possible subbases for $\overline{\mathcal{A}}_{(\infty,+)}$ and $\overline{\mathcal{A}}_{\text{Web}}$ are the families of all sets of the type $\pi_\Gamma^{-1}(W_\Gamma)$ and $\pi_w^{-1}(W_w)$, respectively. Hereby, w is a web and $W_w \subseteq \mathbf{G}^k$, k being the number of paths in w, open[11]; Γ is a graph and $W_\Gamma \subseteq \mathbf{G}^{\#\mathbf{E}(\Gamma)}$ an element of a certain subbase,

---

[10]Thus the notion of a curve coincides with our notion of a general, usually non-finite path.

[11]This is the point where we need the semi-simplicity and compactness of $\mathbf{G}$, because only for these assumptions it is proven [17] up to now that the projection $\pi_w \mid_\mathcal{A}: \overline{\mathcal{A}}_{\text{Web}} \supseteq \mathcal{A} \longrightarrow \mathbf{G}^k$ is surjective, i.e. $\mathcal{A}_w = \mathbf{G}^k$. Otherwise, it would be possible that $\pi_w(\mathcal{A})$ is a non-open Lie subgroup of $\mathbf{G}^k$. So the sets $\pi_w^{-1}(W_w)$ do no longer create a subbase.

e.g., a set of type $W_\Gamma = W_1 \times \cdots \times W_{\#\mathbf{E}(\Gamma)}$ with open $W_i \in \mathbf{G}$. Thus, we can take as a subbase for $\overline{\mathcal{A}}_{(\infty,+)}$ simply all sets $\pi_c^{-1}(W)$ where c is a simple path, i.e. a graph, and $W \subseteq \mathbf{G}$ is open. Since every web is a collection of a finite number of simple paths, we get completely analogously that the family of all $\pi_c^{-1}(W)$ is a subbase for $\overline{\mathcal{A}}_{\text{Web}}$. The only difference here is that c has to be simple with different initial and terminal point. We are therefore left with the proof that $\mathcal{I}(\pi_c^{-1}(W))$ is open in $\overline{\mathcal{A}}_{\text{Web}}$ for all simple, closed paths c and all open W, which is, however, quite easy. Decompose c into two paths c_1 and c_2 (with different initial and terminal points) which span the graph Γ. Then $\mathcal{I}(\pi_c^{-1}(W)) = \mathcal{I}(\pi_\Gamma^{-1}((\pi_c^\Gamma)^{-1}(W)))$. By the continuity of π_c^Γ the set $(\pi_c^\Gamma)^{-1}(W)$ is open in $\mathbf{G}^2$, i.e. a union of sets of the type $W_1 \times W_2$, but $\mathcal{I}(\pi_\Gamma^{-1}(W_1 \times W_2))$ is open as discussed above. **qed**

Remark We note that the homeomorphy of $\overline{\mathcal{A}}_{\text{Web}}$ and $\overline{\mathcal{A}}_{(\infty,+)}$ remains valid also for arbitrary Lie groups $\mathbf{G}$. But, for this proof we need the surjectivity of π_w for all webs as mentioned in Footnote 11. This, on the other hand, will be discussed in a subsequent paper [9].

6   Determination of the Gauge Orbit Types

Now we come to the main part of this paper. In contrast to the general theory above let now $\mathbf{G}$ be a *compact* Lie group throughout this section. The goal of this section is the classification of the generalized connections by the type of their $\overline{\mathcal{G}}$-orbits. In contrast to the theory of classical connections in principal fiber bundles, topological subtleties do not play an important rôle – a generalized connection is only an (algebraic) homomorphism from the groupoid $\mathcal{P}$ of paths into the structure group $\mathbf{G}$, and the generalized gauge transforms are simply mappings from M to $\mathbf{G}$. Thus, also the theory of generalized gauge orbits is governed completely by the algebraic structure of the action of $\overline{\mathcal{G}}$ on $\overline{\mathcal{A}}$:

$$\boxed{h_{\overline{A} \circ \overline{g}}(\gamma) = g_x^{-1}\, h_{\overline{A}}(\gamma)\, g_y} \qquad \text{for all } \overline{A} \in \overline{\mathcal{A}},\, \overline{g} \in \overline{\mathcal{G}},\, \gamma \in \mathcal{P}_{xy}. \tag{3}$$

For each element $\overline{g}$ of the stabilizer $\mathbf{B}(\overline{A})$ of a connection $\overline{A}$ the following must be fulfilled:

$$h_{\overline{A}}(\gamma) = h_{\overline{A} \circ \overline{g}}(\gamma) = g_x^{-1}\, h_{\overline{A}}(\gamma)\, g_y \qquad \text{for all } \gamma \in \mathcal{P}_{xy}, \tag{4}$$

hence, in particular,
- $h_{\overline{A}}(\alpha) = g_m^{-1}\, h_{\overline{A}}(\alpha)\, g_m$ for all $\alpha \in \mathcal{HG} \equiv \mathcal{P}_{mm}$ and
- $h_{\overline{A}}(\gamma_x) = g_m^{-1}\, h_{\overline{A}}(\gamma_x)\, g_x$ for all $x \in M$, whereas γ_x is for any x some fixed path from m to x.

Thanks any path $\gamma \in \mathcal{P}_{xy}$ can be written as $\gamma_x^{-1}\, (\gamma_x \gamma \gamma_y^{-1})\, \gamma_y$, i.e. as a product of paths in $\mathcal{HG}$ and $\{\gamma_x\}$, both conditions are even equivalent to (4). From the first condition follows that g_m has to commute with all holonomies $h_{\overline{A}}(\alpha)$, i.e. g_m is contained in the centralizer $Z(\mathbf{H}_{\overline{A}})$ of the holonomy group of $\overline{A}$. Writing the second condition as

$$g_x = h_{\overline{A}}(\gamma_x)^{-1}\, g_m\, h_{\overline{A}}(\gamma_x) \qquad \text{for all } x \in M, \tag{5}$$

we see that an element $\overline{g}$ of the stabilizer of $\overline{A}$ is already completely determined by its value in the point m, i.e. by an element of the holonomy centralizer $Z(\mathbf{H}_{\overline{A}})$. From this the isomorphy of $\mathbf{B}(\overline{A})$ and $Z(\mathbf{H}_{\overline{A}})$ follows immediately.

Due to general theorems of the theory of transformation groups the gauge orbit $\overline{A} \circ \overline{\mathcal{G}}$ is homeomorphic to the factor space $\mathbf{B}(\overline{A}) \backslash \overline{\mathcal{G}}$. Since $\mathbf{B}(\overline{A})$ and $Z(\mathbf{H}_{\overline{A}}) \cong Z(\mathbf{H}_{\overline{A}}) \times \{e_{\overline{\mathcal{G}}_0}\}$ are

homeomorphic,[12] we get for the moment *heuristically*

$$\mathbf{B}(\overline{A})\backslash\overline{\mathcal{G}} \cong \left(Z(\mathbf{H}_{\overline{A}}) \times \{e_{\overline{\mathcal{G}}_0}\}\right)\backslash\left(\mathbf{G}\times\overline{\mathcal{G}}_0\right) \cong \left(Z(\mathbf{H}_{\overline{A}})\backslash\mathbf{G}\right)\times\overline{\mathcal{G}}_0.$$

We will prove that the left and the right space are indeed homeomorphic, i.e. the homeomorphism type of a gauge orbit is already determined by that of $Z(\mathbf{H}_{\overline{A}})\backslash\mathbf{G}$. Consequently, two connections have homeomorphic gauge orbits, in particular, if the holonomy centralizers are conjugate.

Finally, we can prove that the stabilizers of two connections are conjugate w.r.t. $\overline{\mathcal{G}}$ iff the corresponding holonomy centralizers are conjugate w.r.t. $\mathbf{G}$. This allows us to define the type of a connection not only (as known from the general theory of transformation groups) by the $\overline{\mathcal{G}}$-conjugacy class of its stabilizer $\mathbf{B}(\overline{A})$, but equivalently by the $\mathbf{G}$-conjugacy class of its holonomy centralizer $Z(\mathbf{H}_{\overline{A}})$.

After all, we again mention that in the following $\mathbf{G}$ is a compact Lie group. The purely algebraic results, of course, are valid also without this assumption.

6.1   Stabilizer of a Connection

Definition 6.1   Let be $\overline{A} \in \overline{\mathcal{A}}$. Then $\mathbf{E}_{\overline{A}} := \overline{A}\circ\overline{\mathcal{G}} \equiv \{\overline{A}' \in \overline{\mathcal{A}} \mid \exists \overline{g} \in \overline{\mathcal{G}} : \overline{A}' = \overline{A}\circ\overline{g}\}$ is called **gauge orbit** of $\overline{A}$.

Obviously, two gauge orbits are equal or disjoint.
We need some notations.

Definition 6.2   Let $\overline{A} \in \overline{\mathcal{A}}$ be given.
1. The **holonomy group $\mathbf{H}_{\overline{A}}$** of $\overline{A}$ is equal to $h_{\overline{A}}(\mathcal{HG}) \subseteq \mathbf{G}$.
2. The centralizer $Z(\mathbf{H}_{\overline{A}})$ of the holonomy group, also called **holonomy centralizer** of $\overline{A}$, is the set of all elements in $\mathbf{G}$ that commute with all elements in $\mathbf{H}_{\overline{A}}$.
3. The **base centralizer $\mathbf{B}(\overline{A})$** of $\overline{A}$ is the set of all elements $\overline{g} = (g_x)_{x\in M}$ in $\overline{\mathcal{G}}$ such that $h_{\overline{A}}(\gamma) = g_m^{-1}\,h_{\overline{A}}(\gamma)\,g_x$ for all $x \in M$ and all paths γ from m to x.

Note that for regular connections the holonomy group defined above is exactly the holonomy group known from the classical theory. We get immediately from the definitions

Lemma 6.1   Let be $\overline{A} \in \overline{\mathcal{A}}$ and $\overline{g} \in \overline{\mathcal{G}}$.
1. The holonomy group $\mathbf{H}_{\overline{A}}$ is a subgroup of $\mathbf{G}$.
2. $Z(\mathbf{H}_{\overline{A}})$ is a closed subgroup of $\mathbf{G}$.
3. We have $\mathbf{H}_{\overline{A}\circ\overline{g}} = g_m^{-1}\,\mathbf{H}_{\overline{A}}\,g_m$ and $Z(\mathbf{H}_{\overline{A}\circ\overline{g}}) = g_m^{-1}\,Z(\mathbf{H}_{\overline{A}})\,g_m$.
4. We have $\overline{g} \in \mathbf{B}(\overline{A})$ iff
   a) $g_m \in Z(\mathbf{H}_{\overline{A}})$ and
   b) for all $x \in M$ there is a path γ from m to x with $h_{\overline{A}}(\gamma) = g_m^{-1}\,h_{\overline{A}}(\gamma)\,g_x$.

Proof   1.   This is an obvious consequence of homomorphy property of $h_{\overline{A}} : \mathcal{HG} \longrightarrow \mathbf{G}$.
2.   Trivial.

---

[12]The subgroup $\overline{\mathcal{G}}_0 \subseteq \overline{\mathcal{G}}$ is defined by $\pi_m^{-1}(e_{\mathbf{G}})$. This means, it contains all gauge transforms that are trivial in m. Obviously, we have $\overline{\mathcal{G}} \cong \mathbf{G} \times \overline{\mathcal{G}}_0$.

3. This follows immediately from $h_{\overline{A}\circ\overline{g}}(\alpha) = g_m^{-1}h_{\overline{A}}(\alpha)g_m$ for all $\alpha \in \mathcal{HG}$.

4. $\Longrightarrow$ We have to prove only $g_m \in Z(\mathbf{H}_{\overline{A}})$, but this is clear because we have $h_{\overline{A}}(\alpha) = g_m^{-1}h_{\overline{A}}(\alpha)g_m$ for all $\alpha \in \mathcal{HG}$ by assumption.

$\Longleftarrow$ Let $x \in M$ be fixed and δ be an arbitrary path from m to x. Choose a γ such that $h_{\overline{A}}(\gamma) = g_m^{-1}\,h_{\overline{A}}(\gamma)\,g_x$. Then $\alpha := \delta\gamma^{-1} \in \mathcal{HG}$ and

$$
\begin{aligned}
g_m^{-1}\,h_{\overline{A}}(\delta)\,g_x &= g_m^{-1}\,h_{\overline{A}}(\alpha\gamma)\,g_x \\
&= g_m^{-1}\,h_{\overline{A}}(\alpha)\,h_{\overline{A}}(\gamma)\,g_x \\
&= h_{\overline{A}}(\alpha)\,g_m^{-1}\,h_{\overline{A}}(\gamma)\,g_x \qquad \text{(since } g_m \in Z(\mathbf{H}_{\overline{A}})) \\
&= h_{\overline{A}}(\alpha)\,h_{\overline{A}}(\gamma) \qquad\qquad \text{(by the choice of } \gamma) \\
&= h_{\overline{A}}(\delta).
\end{aligned}
$$

qed

Now we can determine the stabilizer of a connection.

Proposition 6.2 For all $\overline{A} \in \overline{\mathcal{A}}$ and all $\overline{g} \in \overline{\mathcal{G}}$ we have
$$\overline{A} \circ \overline{g} = \overline{A} \Longleftrightarrow \overline{g} \in \mathbf{B}(\overline{A}).$$

Proof Per def. we have
$$\overline{A}\circ\overline{g} = \overline{A} \Longleftrightarrow \forall x,y \in M, \gamma \in \mathcal{P}_{xy} : h_{\overline{A}}(\gamma) = h_{\overline{A}\circ\overline{g}}(\gamma) = g_x^{-1}\,h_{\overline{A}}(\gamma)\,g_y. \qquad (6)$$

$\Longrightarrow$ Let $\overline{A}\circ\overline{g} = \overline{A}$. Due to (6) $g_m^{-1}\,h_{\overline{A}}(\alpha)\,g_m = h_{\overline{A}}(\alpha)$ holds for all $\alpha \in \mathcal{P}_{mm} \equiv \mathcal{HG}$, i.e. $g_m \in Z(\mathbf{H}_{\overline{A}})$. Again by (6) we have $h_{\overline{A}}(\gamma_x) = g_m^{-1}\,h_{\overline{A}}(\gamma_x)\,g_x$ for all $x \in M$ and all $\gamma \in \mathcal{P}_{mx}$. Thus, $\overline{g} \in \mathbf{B}(\overline{A})$.

$\Longleftarrow$ Let $\overline{g} \in \mathbf{B}(\overline{A})$ and $x,y \in M$ be given. Choose some $\gamma_x \in \mathcal{P}_{mx}, \gamma_y \in \mathcal{P}_{my}$. Then for all $\gamma \in \mathcal{P}_{xy}$ the following holds:

$$
\begin{aligned}
g_x^{-1}\,h_{\overline{A}}(\gamma)g_y &= g_x^{-1}\,h_{\overline{A}}(\gamma_x^{-1}\gamma_x\gamma\gamma_y^{-1}\gamma_y)\,g_y \\
&= g_x^{-1}\,h_{\overline{A}}(\gamma_x^{-1})\,g_m\,g_m^{-1}\,h_{\overline{A}}(\gamma_x\gamma\gamma_y^{-1})\,g_m\,g_m^{-1}\,h_{\overline{A}}(\gamma_y)\,g_y \\
&= (g_m^{-1}\,h_{\overline{A}}(\gamma_x)\,g_x)^{-1}\,h_{\overline{A}}(\gamma_x\gamma\gamma_y^{-1})\,(g_m^{-1}\,h_{\overline{A}}(\gamma_y)\,g_y) \\
&\qquad\qquad\qquad \text{(since } \gamma_x\gamma\gamma_y^{-1} \in \mathcal{HG} \text{ and } g_m \in Z(\mathbf{H}_{\overline{A}})) \\
&= h_{\overline{A}}(\gamma_x)^{-1}\,h_{\overline{A}}(\gamma_x\gamma\gamma_y^{-1})\,h_{\overline{A}}(\gamma_y) \\
&= h_{\overline{A}}(\gamma).
\end{aligned}
$$

By (6) we have $\overline{A}\circ\overline{g} = \overline{A}$. **qed**

Since for compact transformation groups every stabilizer is closed (see, e.g., [8]), we have using the proposition above

Corollary 6.3 $\mathbf{B}(\overline{A})$ is a closed, hence compact subgroup of $\overline{\mathcal{G}}$.

Furthermore, by the lemma above we get $\overline{A}\circ\overline{g}_1 = \overline{A}\circ\overline{g}_2 \Longleftrightarrow \overline{A}\circ\overline{g}_1\circ\overline{g}_2^{-1} = \overline{A} \Longleftrightarrow \overline{g}_1\circ\overline{g}_2^{-1} \in \mathbf{B}(\overline{A})$, i.e. we can identify $\mathbf{E}_{\overline{A}}$ and $\mathbf{B}(\overline{A})\backslash\overline{\mathcal{G}}$ by

$$
\begin{aligned}
\tau: \ \mathbf{B}(\overline{A})\backslash\overline{\mathcal{G}} &\longrightarrow \mathbf{E}_{\overline{A}}. \\
[\overline{g}] &\longmapsto \overline{A}\circ\overline{g}
\end{aligned}
$$

Again by the general theory of compact transformation groups we get [8]

Proposition 6.4 $\tau: \mathbf{B}(\overline{A})\backslash\overline{\mathcal{G}} \longrightarrow \mathbf{E}_{\overline{A}}$ is an equivariant isomorphism between compact Hausdorff spaces.

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6.2 Isomorphy of $\mathbf{B}(\overline{A})$ and $Z(\mathbf{H}_{\overline{A}})$

In the next subsection we shall determine the homeomorphism class of a gauge orbit $\mathbf{E}_{\overline{A}}$. For that purpose, we should use the base centralizer. But, this object seems – at least for the first moment – to be quite inaccessible from the algebraic point of view. However, looking carefully at its definition (Def. 6.2) one sees that for given $\overline{A}$ due to $h_{\overline{A}}(\gamma) = g_m^{-1}h_{\overline{A}}(\gamma)g_x$ the value of g_x is already determined by $g_m \in Z(\mathbf{H}_{\overline{A}})$. Therefore, the base centralizer is completely determined by the holonomy centralizer.

Proposition 6.5 For any $\overline{A} \in \overline{\mathcal{A}}$ the map
$$\phi : \begin{array}{ccc} \mathbf{B}(\overline{A}) & \longrightarrow & Z(\mathbf{H}_{\overline{A}}) \\ \overline{g} & \longmapsto & g_m \end{array}$$
is an isomorphism of Lie groups.
(The topologies on $\mathbf{B}(\overline{A})$ and $Z(\mathbf{H}_{\overline{A}})$ are the relative ones induced by $\overline{\mathcal{G}}$ and $\mathbf{G}$, respectively.)

Proof
- Obviously, ϕ is a homomorphism.
- Surjectivity
  Let $g \in Z(\mathbf{H}_{\overline{A}})$. Choose for each $x \in M$ a path γ_x from m to x (w.l.o.g. γ_m is the trivial path) and define
  $$g_x := h_{\overline{A}}(\gamma_x)^{-1}\, g\, h_{\overline{A}}(\gamma_x). \tag{7}$$
  Obviously, $\overline{g} = (g_x) \in \overline{\mathcal{G}}$ and $\phi(\overline{g}) = g$. By Lemma 6.1, 4. we have $\overline{g} \in \mathbf{B}(\overline{A})$ because
  1. $g_m = h_{\overline{A}}(\gamma_m)^{-1}\, g\, h_{\overline{A}}(\gamma_m) = g \in Z(\mathbf{H}_{\overline{A}})$ by the triviality of $\gamma_m \in \mathcal{HG}$ and
  2. $h_{\overline{A}}(\gamma_x) = g_m^{-1}\, h_{\overline{A}}(\gamma_x)\, g_x$ for the γ_x chosen above.
- Injectivity
  Clear, because g_x is uniquely determined by $\overline{A}$ and so g_m is due to $h_{\overline{A}}(\gamma_x) = g_m^{-1}\, h_{\overline{A}}(\gamma_x)\, g_x$.
- Continuity of ϕ
  ϕ is the restriction of $\pi_m : \overline{\mathcal{G}} \longrightarrow \overline{\mathcal{G}}_m \equiv \mathbf{G}$ to $\mathbf{B}(\overline{A})$. The continuity of ϕ is now a consequence of the continuity of π_m.
- Continuity of ϕ^{-1}
  $\phi : \mathbf{B}(\overline{A}) \longrightarrow Z(\mathbf{H}_{\overline{A}})$ is a continuous and bijective map of a compact space onto a Hausdorff space. Therefore, ϕ^{-1} is continuous. **qed**

Finally, we note that obviously the isomorphism ϕ does not depend on the special choice of the paths γ_x.

6.3 Determination of the Homeomorphism Class

As we have seen in the last subsection $\mathbf{B}(\overline{A})$ and $Z(\mathbf{H}_{\overline{A}}) \times \{e_{\overline{\mathcal{G}}_0}\}$ are homeomorphic subgroups of $\overline{\mathcal{G}}$. One could conjecture that consequently
$$\mathbf{B}(\overline{A}) \backslash \overline{\mathcal{G}} \text{ and } \left(Z(\mathbf{H}_{\overline{A}}) \times \{e_{\overline{\mathcal{G}}_0}\} \right) \backslash \left(\mathbf{G} \times \overline{\mathcal{G}}_0 \right) \cong \left(Z(\mathbf{H}_{\overline{A}}) \backslash \mathbf{G} \right) \times \overline{\mathcal{G}}_0$$
are homeomorphic. But, this is not clear at all. For instance, $2\mathbb{Z}$ and $3\mathbb{Z}$ are isomorphic, but $\mathbb{Z}/2\mathbb{Z} = \{0,1\}$ and $\mathbb{Z}/3\mathbb{Z} = \{0,1,2\}$ are not. Nevertheless, in our case the claimed relation holds:

Proposition 6.6 For any $\overline{A} \in \overline{\mathcal{A}}$ there is a homeomorphism
$$\Psi_0 : \overline{\mathcal{G}}_0 \times {}_{Z(\mathbf{H}_{\overline{A}})}\big\backslash\, \mathbf{G} \longrightarrow {}_{\mathbf{B}(\overline{A})}\big\backslash\, \overline{\mathcal{G}}.$$

Hence, the homeomorphism type of $\mathbf{E}_{\overline{A}}$ is not only determined by ${}_{\mathbf{B}(\overline{A})}\backslash\, \overline{\mathcal{G}}$, but already by ${}_{Z(\mathbf{H}_{\overline{A}})}\backslash\, \mathbf{G}$.

Before we will prove this proposition, we shall motivate our choice of the homeomorphism. First we again choose for each $x \in M$ a path γ_x from m to x where w.l.o.g. γ_m is the trivial path. By equation (7) we get a homomorphism
$$\phi' : \quad \mathbf{G} \quad \longrightarrow \quad \overline{\mathcal{G}}$$
$$g \quad \longmapsto \quad \left(h_{\overline{A}}(\gamma_x)^{-1}\, g\, h_{\overline{A}}(\gamma_x)\right)_{x \in M}$$
with $\phi'(Z(\mathbf{H}_{\overline{A}})) = \mathbf{B}(\overline{A})$ and therefore a map from ${}_{Z(\mathbf{H}_{\overline{A}})}\backslash\, \mathbf{G}$ to ${}_{\mathbf{B}(\overline{A})}\backslash\, \overline{\mathcal{G}}$. Furthermore, we have $\phi'(\mathbf{G})\, \overline{\mathcal{G}}_0 = \overline{\mathcal{G}} \cong \phi'(\mathbf{G}) \times \overline{\mathcal{G}}_0$ with $\overline{g} \longmapsto \left(\phi'(g_m), \phi'(g_m)^{-1}\overline{g}\right)$. Although there is no group structure on ${}_{\mathbf{B}(\overline{A})}\backslash\, \overline{\mathcal{G}}$ – in general, $\mathbf{B}(\overline{A})$ is only a subgroup and not a normal subgroup of $\overline{\mathcal{G}}$ –, there is at least a canonical right action of $\overline{\mathcal{G}}$ and $\overline{\mathcal{G}}_0$, respectively, by $[\overline{g}] \circ \overline{g}' := [\overline{g}\, \overline{g}']$. Thus, $(\overline{g}, [g]) \longmapsto [\phi'(g)] \circ \overline{g}$ is a good candidate to become our desired homeomorphism.

Proof First we choose some path γ_x from m to x for each $x \in M$ where w.l.o.g. γ_m is trivial. Now we define
$$\Psi_0 : \quad \overline{\mathcal{G}}_0 \times {}_{Z(\mathbf{H}_{\overline{A}})}\big\backslash\, \mathbf{G} \quad \longrightarrow \quad {}_{\mathbf{B}(\overline{A})}\big\backslash\, \overline{\mathcal{G}}$$
$$\left((g_x)_{x\in M}, [g]\right) \quad \longmapsto \quad \left[\phi'(g)\,(g_x)_{x\in M}\right]$$
with $g_m = e_{\mathbf{G}}$.

1. Ψ_0 is well-defined.

   Let $g_1 \sim g_2$, i.e. $g_1 = zg_2$ for some $z \in Z(\mathbf{H}_{\overline{A}})$. Define $\overline{g} := (g_x)_{x\in M} \in \overline{\mathcal{G}}_0$. Then we have

$$
\begin{aligned}
\Psi_0\left((g_x)_{x\in M}, [g_1]\right) &= \left[\phi'(g_1)\, \overline{g}\right] \\
&= \left[\phi'(zg_2)\, \overline{g}\right] \\
&= \left[\phi'(z)\, \phi'(g_2)\, \overline{g}\right] \qquad \text{(Homomorphy property of } \phi') \\
&= \left[\phi'(g_2)\, \overline{g}\right] \qquad (\phi'(Z(\mathbf{H}_{\overline{A}})) = \mathbf{B}(\overline{A}) \text{ by Proposition 6.5)} \\
&= \Psi_0\left((g_x)_{x\in M}, [g_2]\right).
\end{aligned}
$$

2. Ψ_0 is injective.

   Let $\Psi_0\left((g_{1,x})_{x\in M}, [g_1]\right) = \Psi_0\left((g_{2,x})_{x\in M}, [g_2]\right)$. Then there exists a $\overline{z} \in \mathbf{B}(\overline{A})$ with $\phi'(g_1)_x\, g_{1,x} = z_x\, \phi'(g_2)_x\, g_{2,x}$, i.e.
$$h_{\overline{A}}(\gamma_x)^{-1}\, g_1\, h_{\overline{A}}(\gamma_x)\, g_{1,x} = z_x\, h_{\overline{A}}(\gamma_x)^{-1}\, g_2\, h_{\overline{A}}(\gamma_x)\, g_{2,x}$$
   for all $x \in M$. Thus,
   - for $x = m$: $g_1 = z_m g_2$, i.e. $[g_1] = [g_2]$, and
   - for $x \neq m$:

$$
\begin{aligned}
g_{1,x} &= h_{\overline{A}}(\gamma_x)^{-1}\, g_1^{-1}\, h_{\overline{A}}(\gamma_x)\, z_x\, h_{\overline{A}}(\gamma_x)^{-1}\, g_2\, h_{\overline{A}}(\gamma_x)\, g_{2,x} \\
&= h_{\overline{A}}(\gamma_x)^{-1}\, g_1^{-1}\, z_m\, g_2\, h_{\overline{A}}(\gamma_x)\, g_{2,x} \\
&= h_{\overline{A}}(\gamma_x)^{-1}\, h_{\overline{A}}(\gamma_x)\, g_{2,x} \\
&= g_{2,x},
\end{aligned}
$$

   i.e. Ψ_0 is injective.

3. Ψ_0 is surjective.

Let $[\tilde{g}] \in \mathbf{B}(\overline{A}) \backslash \overline{\mathcal{G}}$ be given. Define $g_x := (\phi'(\tilde{g}_m)^{-1}\tilde{g})_x$ for all $x \in M$. Then we have $\Psi_0\big((g_x)_{x \in M}, [\tilde{g}_m]\big) = [\tilde{g}]$.

4. Ψ_0^{-1} is continuous.

It is sufficient to prove that the projections $\mathrm{pr}_i \circ \Psi_0^{-1}$ of Ψ_0^{-1} to the factors $\overline{\mathcal{G}}_0$ ($i = 1$) and $Z(\mathbf{H}_{\overline{A}}) \backslash \mathbf{G}$ ($i = 2$) are continuous.

a) $\mathrm{pr}_1 \circ \Psi_0^{-1}$ is continuous.

For all $x \in M \setminus \{m\}$ the map

$$\overline{\mathcal{G}} \xrightarrow{\ \pi_{mx}\ } \mathbf{G} \times \mathbf{G} \xrightarrow{\ \text{mult.}\ } \mathbf{G}$$
$$\tilde{g} \longmapsto (g_m, g_x) \longmapsto (h_{\overline{A}}(\gamma_x)^{-1} g_m^{-1} h_{\overline{A}}(\gamma_x) g_x)$$

is a composition of continuous maps and consequently continuous itself. Since $\pi_{\mathbf{B}(\overline{A})} : \overline{\mathcal{G}} \longrightarrow \mathbf{B}(\overline{A}) \backslash \overline{\mathcal{G}}$ is open and surjective, we get the continuity of $\pi_x \circ \mathrm{pr}_1 \circ \Psi_0^{-1}$ for all $x \in M \setminus \{m\}$ by $\pi_x \circ (\mathrm{pr}_1 \circ \Psi_0^{-1}) \circ \pi_{\mathbf{B}(\overline{A})} = \text{mult.} \circ \pi_{mx}$. For $x = m$ the statement is trivial. Thus, $\mathrm{pr}_1 \circ \Psi_0^{-1}$ is continuous.

b) $\mathrm{pr}_2 \circ \Psi_0^{-1}$ is continuous.

We use $\pi_{Z(\mathbf{H}_{\overline{A}})} \circ \pi_m = (\mathrm{pr}_2 \circ \Psi_0^{-1}) \circ \pi_{\mathbf{B}(\overline{A})} : \overline{\mathcal{G}} \longrightarrow Z(\mathbf{H}_{\overline{A}}) \backslash \mathbf{G}$. The statement now follows because $\pi_{\mathbf{B}(\overline{A})}$ is an open and surjective map and $\pi_{Z(\mathbf{H}_{\overline{A}})}$ and π_m are continuous.

5. Ψ_0 is a homeomorphism because Ψ_0^{-1} is continuous and bijective. **qed**

Thus we get the following important result: The homeomorphism class of a gauge orbit of a connection is completely determined by its holonomy centralizer. Finally, we should emphasize that, in general, the homeomorphism Ψ_0 is not an equivariant map w.r.t. the canonical action of $\overline{\mathcal{G}}$ on $\overline{\mathcal{G}}_0 \times Z(\mathbf{H}_{\overline{A}}) \backslash \mathbf{G}$.

6.4 Criteria for the Homeomorphy of Gauge Orbits

It is well known that orbits of general transformation groups are classified by the conjugacy classes of their stabilizers. This would effect in our case that the gauge orbits are characterized by the conjugacy class of their corresponding base centralizer w.r.t. $\overline{\mathcal{G}}$. As we have already seen, the base centralizer of a connection $\overline{A}$ is isomorphic to the holonomy centralizer of $\overline{A}$ and the homeomorphism type of the gauge orbit is completely determined by that of $Z(\mathbf{H}_{\overline{A}}) \backslash \mathbf{G}$. Now we are going to show that base centralizers are conjugate w.r.t. $\overline{\mathcal{G}}$ if and only if the corresponding holonomy centralizers are conjugate w.r.t. $\mathbf{G}$. This will allow us to define the type of a gauge orbit $\mathbf{E}_{\overline{A}}$ to be the conjugacy class of $Z(\mathbf{H}_{\overline{A}})$ w.r.t. $\mathbf{G}$. The investigation of the set of all these classes is much easier than in the case of classes in $\overline{\mathcal{G}}$.

We want to prove the following

Proposition 6.7 Let $\overline{A}_1, \overline{A}_2 \in \overline{\mathcal{A}}$ be two generalized connections. Then the following statements are equivalent:

1. $Z(\mathbf{H}_{\overline{A}_1})$ and $Z(\mathbf{H}_{\overline{A}_2})$ are conjugate in $\mathbf{G}$.
2. $\mathbf{B}(\overline{A}_1)$ and $\mathbf{B}(\overline{A}_2)$ are conjugate in $\overline{\mathcal{G}}$.

It would be quite easy to prove this directly using Proposition 6.5. Nevertheless, we do not want to do this. Instead, we shall first derive some concrete criteria for the homeomorphy of two gauge orbits. Finally, the just claimed proposition will be a nice by-product.

Proposition 6.8 Let $\overline{A}_1, \overline{A}_2 \in \overline{\mathcal{A}}$ be two generalized connections. Furthermore, let there exist an isomorphism $\Psi : \overline{\mathcal{G}} \longrightarrow \overline{\mathcal{G}}$ of topological groups with $\Psi\big(\mathbf{B}(\overline{A}_1)\big) = \mathbf{B}(\overline{A}_2)$.

Then the map

$$\Phi : \quad \begin{array}{ccc} \mathbf{E}_{\overline{A}_1} & \longrightarrow & \mathbf{E}_{\overline{A}_2} \\ \overline{A}_1 \circ \overline{g} & \longmapsto & \overline{A}_2 \circ \Psi(\overline{g}) \end{array}$$

is a homeomorphism compatible with the action of $\overline{\mathcal{G}}$.

Proof
- Φ is well-defined.
  Let $\overline{A}_1 \circ \overline{g} = \overline{A}_1 \circ \overline{g}'$. Then we have $\overline{A}_1 \circ (\overline{g} \circ \overline{g}'^{-1}) = \overline{A}_1$, i.e. $\overline{g} \circ \overline{g}'^{-1} \in \mathbf{B}(\overline{A}_1)$ by Proposition 6.2. By assumption we have $\Psi(\overline{g} \circ \overline{g}'^{-1}) = \Psi(\overline{g}) \circ \Psi(\overline{g}')^{-1} \in \mathbf{B}(\overline{A}_2)$, i.e. $\overline{A}_2 \circ \Psi(\overline{g}) = \overline{A}_2 \circ \Psi(\overline{g}')$.
- Since Ψ is a group isomorphism, Φ is again an isomorphism that is compatible with the action of $\overline{\mathcal{G}}$.
- For the proof of the homeomorphy property of Φ we consider the following commutative diagram:

$$
\begin{array}{ccc}
\mathbf{E}_{\overline{A}_1} & \xrightarrow{\;\Phi\;} & \mathbf{E}_{\overline{A}_2} \\
{\scriptstyle \tau_1}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \tau_2} \\
\mathbf{B}(\overline{A}_1)\backslash\overline{\mathcal{G}} & \xrightarrow{\;\Omega\;} & \mathbf{B}(\overline{A}_2)\backslash\overline{\mathcal{G}}
\end{array}
\qquad
\begin{array}{ccc}
\overline{A}_1 \circ \overline{g} & \xmapsto{\;\Phi\;} & \overline{A}_2 \circ \overline{g} \\
{\scriptstyle \tau_1}\uparrow & & \uparrow{\scriptstyle \tau_2} \\
[\overline{g}]_{\mathbf{B}(\overline{A}_1)} & \xmapsto{\;\Omega\;} & [\Psi(\overline{g})]_{\mathbf{B}(\overline{A}_2)}
\end{array}
\;.
$$

Since τ_1 and τ_2 are homeomorphisms, it is sufficient to prove the homeomorphy property for Ω.

- Ω is well-defined and bijective due to $\Omega = \tau_2 \circ \Phi \circ \tau_1^{-1}$.
- Ω is continuous.
  The map $\pi_{\mathbf{B}(\overline{A})} : \overline{\mathcal{G}} \longrightarrow \mathbf{B}(\overline{A})\backslash\overline{\mathcal{G}}$ is an orbit space projection for all $\overline{A} \in \overline{\mathcal{A}}$ and consequently surjective, continuous and open. Using $\Omega \circ \pi_{\mathbf{B}(\overline{A}_1)} = \pi_{\mathbf{B}(\overline{A}_2)} \circ \Psi$ we see for any open $U \subseteq \mathbf{B}(\overline{A}_2)\backslash\overline{\mathcal{G}}$ that $\Omega^{-1}(U) = \pi_{\mathbf{B}(\overline{A}_1)}(\Psi^{-1}(\pi_{\mathbf{B}(\overline{A}_2)}^{-1}(U))) \subseteq \mathbf{B}(\overline{A}_1)\backslash\overline{\mathcal{G}}$ is again open.
- Ω^{-1} is continuous by the same reason as above.
  Thus, Ω is a homeomorphism. **qed**

To simplify the speech in the following we state

Definition 6.3 Let $\mathbf{G}$ be a Lie group (topological group) and let U_1 and U_2 be closed subgroups of $\mathbf{G}$.

U_1 and U_2 are called **extendibly isomorphic (w.r.t. $\mathbf{G}$)** iff there is an isomorphism $\psi : \mathbf{G} \longrightarrow \mathbf{G}$ of Lie groups (topological groups) with $\psi(U_1) = U_2$.[13]

In Proposition 6.8 we compared gauge orbits w.r.t. their base centralizers. Now we will compare them using their holonomy centralizers. In order to manage this we need an extendibility lemma.

Let the holonomy centralizers of two connections be extendibly isomorphic, i.e. let there exist a $\psi : \mathbf{G} \longrightarrow \mathbf{G}$ with $\psi(Z(\mathbf{H}_{\overline{A}_1})) = Z(\mathbf{H}_{\overline{A}_2})$. By $\Psi := \phi_2^{-1} \circ \psi \circ \phi_1$ the base centralizers are isomorphic. Extending Ψ to $\overline{\mathcal{G}}$ we get

---

[13]If misunderstanding seems to be unlikely, we simply drop "w.r.t. $\mathbf{G}$" and write "extendibly isomorphic".

Lemma 6.9 Let $\overline{A}_1, \overline{A}_2 \in \overline{\mathcal{A}}$ be two generalized connections. Then the following statement holds:

If $Z(\mathbf{H}_{\overline{A}_1})$ and $Z(\mathbf{H}_{\overline{A}_2})$ are extendibly isomorphic, then $\mathbf{B}(\overline{A}_1)$ and $\mathbf{B}(\overline{A}_2)$ are also extendibly isomorphic.

We have explicitly: Let $\psi : \mathbf{G} \longrightarrow \mathbf{G}$ be an isomorphism of Lie groups with $\psi\big(Z(\mathbf{H}_{\overline{A}_1})\big) = Z(\mathbf{H}_{\overline{A}_2})$. Furthermore, let γ_x be an arbitrary, but fixed path in M for each $x \in M$. Then we have:

- The map $\Psi : \overline{\mathcal{G}} \longrightarrow \overline{\mathcal{G}}$ defined by

$$\Psi(\overline{g})_x := h_{\overline{A}_2}(\gamma_x)^{-1} \; \psi\Big(h_{\overline{A}_1}(\gamma_x)g_x h_{\overline{A}_1}(\gamma_x)^{-1}\Big) \; h_{\overline{A}_2}(\gamma_x) \tag{8}$$

  is an isomorphism of topological groups.

- $\Psi\mid_{\mathbf{B}(\overline{A}_1)}$ is an isomorphism of Lie groups between $\mathbf{B}(\overline{A}_1)$ and $\mathbf{B}(\overline{A}_2)$. Furthermore, $\Psi\mid_{\mathbf{B}(\overline{A}_1)}$ is independent of the choice of the paths γ_x.

Proof Let $Z(\mathbf{H}_{\overline{A}_1})$ and $Z(\mathbf{H}_{\overline{A}_2})$ be extendibly isomorphic with the corresponding isomorphism ψ.

- Obviously, we have $\Psi(\overline{g}) \in \overline{\mathcal{G}}$ and Ψ is a homomorphism of groups. Moreover, Ψ is bijective with the inverse

$$\Psi^{-1}(\overline{g})_x = h_{\overline{A}_1}(\gamma_x)^{-1} \; \psi^{-1}\Big(h_{\overline{A}_2}(\gamma_x)g_x h_{\overline{A}_2}(\gamma_x)^{-1}\Big) \; h_{\overline{A}_1}(\gamma_x). \tag{9}$$

  To prove the continuity of Ψ it is sufficient to prove the continuity of $\pi_x \circ \Psi$ for all x. Hence, let $U \subseteq \mathbf{G}$ be open. Then we have

$$
\begin{aligned}
&(\pi_x \circ \Psi)^{-1}(U) \\
=\; &\{\overline{g} \in \overline{\mathcal{G}} \mid (\pi_x \circ \Psi)(\overline{g}) = \Psi(\overline{g})_x = \\
&\qquad h_{\overline{A}_2}(\gamma_x)^{-1} \; \psi\Big(h_{\overline{A}_1}(\gamma_x)g_x h_{\overline{A}_1}(\gamma_x)^{-1}\Big) \; h_{\overline{A}_2}(\gamma_x) \in U\} \\
=\; &\pi_x^{-1}\Big(h_{\overline{A}_1}(\gamma_x)^{-1} \; \psi^{-1}(h_{\overline{A}_2}(\gamma_x)) \; \psi^{-1}(U) \; \psi^{-1}(h_{\overline{A}_2}(\gamma_x)^{-1}) \; h_{\overline{A}_1}(\gamma_x)\Big).
\end{aligned}
$$

  Since ψ is a homeomorphism and π_x is continuous, $(\pi_x \circ \Psi)^{-1}(U)$ is open. The continuity of Ψ is now a consequence of Corollary 4.4, that of Ψ^{-1} is clear.

- Let ϕ_i be the isomorphism for $\overline{A}_i$ ($i = 1, 2$) corresponding to Proposition 6.5. Then we have

$$\Psi\mid_{\mathbf{B}(\overline{A}_1)} = \phi_2^{-1} \circ \psi \circ \phi_1 : \mathbf{B}(\overline{A}_1) \longrightarrow \mathbf{B}(\overline{A}_2).$$

  Since ϕ_1, ϕ_2 and ψ are Lie isomorphisms and, moreover, independent of the choice of the γ_x, $\Psi\mid_{\mathbf{B}(\overline{A}_1)}$ is again an isomorphism of Lie groups that is independent of the choice of the γ_x.

Thus, $\mathbf{B}(\overline{A}_1)$ and $\mathbf{B}(\overline{A}_2)$ are extendibly isomorphic. **qed**

The next lemma is obvious.

Lemma 6.10 Let $\overline{A}_1, \overline{A}_2 \in \overline{\mathcal{A}}$ be two generalized connections. Then $Z(\mathbf{H}_{\overline{A}_1})$ and $Z(\mathbf{H}_{\overline{A}_2})$ are extendibly isomorphic provided they are conjugate w.r.t. $\mathbf{G}$.

Now we can prove Proposition 6.7.

Proof Proposition 6.7

- Let $Z(\mathbf{H}_{\overline{A}_1})$ and $Z(\mathbf{H}_{\overline{A}_2})$ be conjugate and thus also extendibly isomorphic. The map $\Psi : \overline{\mathcal{G}} \longrightarrow \overline{\mathcal{G}}$ from Lemma 6.9 fulfills now

$$\Psi(\overline{g'}) = \Big((h_{\overline{A}_1}(\gamma_x)^{-1} \, g \, h_{\overline{A}_2}(\gamma_x))^{-1} \, g'_x \, (h_{\overline{A}_1}(\gamma_x)^{-1} \, g \, h_{\overline{A}_2}(\gamma_x))\Big)_{x \in M}$$

where $g \in \mathbf{G}$ was chosen such that $Z(\mathbf{H}_{\overline{A}_2}) = (\mathrm{Ad}\, g) Z(\mathbf{H}_{\overline{A}_1})$. We define $\overline{g} :=$ $\left(h_{\overline{A}_1}(\gamma_x)^{-1}\, g\, h_{\overline{A}_2}(\gamma_x) \right)_{x \in M}$. Hence, the map $\Psi : \overline{\mathcal{G}} \longrightarrow \overline{\mathcal{G}}$ from Lemma 6.9 is simply $\mathrm{Ad}\, \overline{g}$. Moreover, $\mathrm{Ad}\, \overline{g}$ maps $\mathbf{B}(\overline{A}_1)$ isomorphically onto $\mathbf{B}(\overline{A}_2)$. Thus, $\mathbf{B}(\overline{A}_2) = (\mathrm{Ad}\, \overline{g}) \mathbf{B}(\overline{A}_1)$.

- Let $\mathbf{B}(\overline{A}_1)$ and $\mathbf{B}(\overline{A}_2)$ be conjugate, i.e. let there exist a $\overline{g} \in \overline{\mathcal{G}}$ with $\mathbf{B}(\overline{A}_2) = \overline{g}^{-1} \mathbf{B}(\overline{A}_1) \overline{g}$. Then we obviously have $Z(\mathbf{H}_{\overline{A}_2}) = g_m^{-1} Z(\mathbf{H}_{\overline{A}_1}) g_m$. **qed**

Let us summarize:

Theorem 6.11 Let $\overline{A}_1, \overline{A}_2 \in \overline{\mathcal{A}}$ be two generalized onnections. Then the following implication chain holds:

$$\begin{aligned} & \mathbf{B}(\overline{A}_1) \text{ and } \mathbf{B}(\overline{A}_2) \text{ are conjugate in } \overline{\mathcal{G}}. \\ \Longleftrightarrow \quad & Z(\mathbf{H}_{\overline{A}_1}) \text{ and } Z(\mathbf{H}_{\overline{A}_2}) \text{ are conjugate in } \mathbf{G}. \\ \Longrightarrow \quad & Z(\mathbf{H}_{\overline{A}_1}) \text{ und } Z(\mathbf{H}_{\overline{A}_2}) \text{ are extendibly isomorphic.} \\ \Longrightarrow \quad & \mathbf{B}(\overline{A}_1) \text{ und } \mathbf{B}(\overline{A}_2) \text{ are extendibly isomorphic.} \\ \Longrightarrow \quad & \text{The gauge orbits } \mathbf{E}_{\overline{A}_1} \text{ and } \mathbf{E}_{\overline{A}_2} \text{ are homeomorphic.} \end{aligned}$$

This theorem has an interesting and perhaps a little bit surprising consequence: Even *after* projecting $\overline{\mathcal{A}}$ down to $\overline{\mathcal{A}}/\overline{\mathcal{G}} \equiv \mathrm{Hom}(\mathcal{HG}, \mathbf{G})/\mathrm{Ad}$ the complete knowledge about the homeomorphism class of the corresponding gauge orbit is conserved. Naively one would suggest that after projecting the total gauge orbit onto one single point this information should be lost. But, the homeomorphism class is already determined by giving the holonomy centralizer, that, the other way round, can be, up to a global conjugation, reconstructed from $[\overline{A}]$.

Proposition 6.12 For each $[\overline{A}] \in \overline{\mathcal{A}}/\overline{\mathcal{G}}$ the homeomorphism class of the gauge orbit corresponding to $[\overline{A}]$ can be reconstructed from $[\overline{A}]$.

7 Discussion or How to Define the Gauge Orbit Type

If we ignored the usual definition of the type of an orbit in a general G-space, then Theorem 6.11 would open us several possibilities to define the type of a gauge orbit. If the type should characterize as "uniquely" as possible the homeomorphism class of the gauge orbit, then it would be advisable to define the base centralizer modulo extendible isomorphy to be the type. But, even this choice would not guarantee that two gauge orbits with different type are in fact non-homeomorphic. Moreover, the base centralizers as subgroups of $\overline{\mathcal{G}}$ are not so easily controllable as centralizers in $\mathbf{G}$ are. Thus, we will take the holonomy centralizer for the definition. It remains only the question, whether we should take the centralizer modulo conjugation or modulo extendible isomorphy. We have to collect conjugate centralizers in one type anyway in order to make points of one orbit be of the same type. (Note, that the holonomy centralizers of two gauge equivalent connections are generally not equal but only conjugate.)

If we now include the general definition of an orbit type into our considerations again, it will be clear that we shall use the centralizer modulo conjugation. But, since two connections have one and the same (usual) orbit type iff their base centralizers are conjugate, i.e. iff their holonomy centralizers are conjugate, we define the *gauge* orbit type by

Definition 7.1 The **type** of a gauge orbit $\mathbf{E}_{\overline{A}}$ is the holonomy centralizer of $\overline{A}$ modulo conjugation.

We emphasize that this definition of the type of the gauge orbit $\mathbf{E}_{\overline{A}}$ is – as mentioned above – independent of the choice of the connection $\overline{A}' \in \mathbf{E}_{\overline{A}}$. In fact, if $\overline{A}'$ is gauge equivalent to $\overline{A}$, by Lemma 6.1 there is a $g \in \mathbf{G}$ with $Z(\mathbf{H}_{\overline{A}'}) = g^{-1} Z(\mathbf{H}_{\overline{A}}) g$. Hence, the holonomy centralizers of $\overline{A}$ and $\overline{A}'$ are conjugate. Thus, we can assign to each $[\overline{A}] \in \overline{\mathcal{A}}/\overline{\mathcal{G}}$ a unique gauge orbit type. Using Theorem 6.11 we get immediately

Corollary 7.1 Two gauge orbits with the same type are homeomorphic.

Finally, we want to give a further justification for our definition of the gauge orbit type. Let us consider regular connections. In the literature there are two different definitions for the type of a "classical" gauge orbit: On the one hand [14], one chooses the total stabilizer of $A \in \mathcal{A}$ in $\mathcal{G}$. On the other hand [15], one sees first that the pointed gauge group $\mathcal{G}_0$ (the set of all gauge transforms that are the identity on a fixed fibre) is a normal and closed subgroup in $\mathcal{G}$. Obviously, $G := \mathcal{G}/\mathcal{G}_0$ can be identified with the structure group $\mathbf{G}$. Moreover, the action of $\mathcal{G}_0$ on $\mathcal{A}$ is free, proper and smooth. This way one gets an action of G, the "essential part" of the gauge transforms, on the space $\mathcal{A}/\mathcal{G}_0$. Now, the gauge orbit types are the conjugacy classes of stabilizers being closed subgroups of $G \cong \mathbf{G}$. This definition corresponds to our choice of the centralizer of the holonomy group. Due to the statements proven above these two descriptions are equivalent if we consider generalized connections, but in general *not* if we work in the classical framework. There it is under certain circumstances possible [15] that two connections though have conjugate holonomy centralizers, but this conjugation cannot be lifted to a conjugation of the base centralizers. The deeper reason behind this is that the gauge transform $\overline{g} = \left(h_{A_1}(\gamma_x)^{-1} \, g \, h_{A_2}(\gamma_x) \right)_{x \in M}$ (cf. proof of Proposition 6.7) generally is not a classical gauge transform, i.e. it is not smooth. Nevertheless, in case of the definition using the holonomy group we have

Corollary 7.2 The gauge orbit type is conserved by the embedding $\mathcal{A} \hookrightarrow \overline{\mathcal{A}}$.

But, note that this does not mean at all that the classical and the generalized gauge orbit of a classical connection itself are equal or at least homeomorphic.

8 Acknowledgements

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