Stratification of the Generalized Gauge Orbit Space

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Abstract

The action of Ashtekar's generalized gauge group $\overline{\mathcal{G}}$ on the space $\overline{\mathcal{A}}$ of generalized connections is investigated for compact structure groups $\mathbf{G}$.

First a stratum is defined to be the set of all connections of one and the same gauge orbit type, i.e. the conjugacy class of the centralizer of the holonomy group. Then a slice theorem is proven on $\overline{\mathcal{A}}$. This yields the openness of the strata. Afterwards, a denseness theorem is proven for the strata. Hence, $\overline{\mathcal{A}}$ is topologically regularly stratified by $\overline{\mathcal{G}}$. These results coincide with those of Kondracki and Rogulski for Sobolev connections. As a by-product, we prove that the set of all gauge orbit types equals the set of all (conjugacy classes of) Howe subgroups of $\mathbf{G}$. Finally, we show that the set of all gauge orbits with maximal type has the full induced Haar measure 1.

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1 Introduction

For quite a long time the geometric structure of gauge theories has been investigated. A classical (pure) gauge theory consists of three basic objects: First the set $\mathcal{A}$ of smooth connections ("gauge fields") in a principle fiber bundle, then the set $\mathcal{G}$ of all smooth gauge transforms, i.e. automorphisms of this bundle, and finally the action of $\mathcal{G}$ on $\mathcal{A}$. Physically, two gauge fields that are related by a gauge transform describe one and the same situation. Thus, the space of all gauge orbits, i.e. elements in $\mathcal{A}/\mathcal{G}$, is the configuration space for the gauge theory. Unfortunately, in contrast to $\overline{\mathcal{A}}$, which is an affine space, the space $\mathcal{A}/\mathcal{G}$ has a very complicated structure: It is non-affin, non-compact and infinite-dimensional and it is not a manifold. This causes enormous problems, in particular, when one wants to quantize a gauge theory. One possible quantization method is the path integral quantization. Here one has to find an appropriate measure on the configuration space of the classical theory, hence a measure on $\mathcal{A}/\mathcal{G}$. As just indicated, this is very hard to find. Thus, one has hoped for a better understanding of the structure of $\mathcal{A}/\mathcal{G}$. However, up to now, results are quite rare.

About 20 years ago, the efforts were focussed on a related problem: The consideration of connections and gauge transforms that are contained in a certain Sobolev class (see, e.g., [16]). Now, $\mathcal{G}$ is a Hilbert-Lie group and acts smoothly on $\mathcal{A}$. About 15 years ago, Kondracki and Rogulski [12] found lots of fundamental properties of this action. Perhaps, the most remarkable theorem they obtained was a slice theorem on $\mathcal{A}$. This means, for every orbit $A \circ \mathcal{G} \subseteq \mathcal{A}$ there is an equivariant retraction from a (so-called tubular) neighborhood of A onto $A \circ \mathcal{G}$. Using this theorem they could clarify the structure of the so-called strata. A stratum contains all connections that have the same, fixed type, i.e. the same (conjugacy class of the) stabilizer under the action of $\mathcal{G}$. Using a denseness theorem for the strata, Kondracki and Rogulski proved that the space $\mathcal{A}$ is regularly stratified by the action of $\mathcal{G}$. In particular, all the strata are smooth submanifolds of $\mathcal{A}$.

Despite these results the mathematically rigorous construction of a measure on $\mathcal{A}/\mathcal{G}$ has not been achieved. This problem was solved – at least preliminary – by Ashtekar et al. [1, 2], but, however, not for $\mathcal{A}/\mathcal{G}$ itself. Their idea was to drop simply all smoothness conditions for the connections and gauge transforms. In detail, they first used the fact that a connection can always be reconstructed uniquely by its parallel transports. On the other hand, these parallel transports can be identified with an assignment of elements of the structure group $\mathbf{G}$ to the paths in the base manifold M such that the concatenation of paths corresponds to the product of these group elements. It is intuitively clear that for smooth connections the parallel transports additionally depend smoothly on the paths [14]. But now this restriction is removed for the generalized connections. They are only homomorphisms from the groupoid $\mathcal{P}$ of paths to the structure group $\mathbf{G}$. Analogously, the set $\overline{\mathcal{G}}$ of generalized gauge transforms collects all functions from M to $\mathbf{G}$. Now the action of $\overline{\mathcal{G}}$ to $\overline{\mathcal{A}}$ is defined purely algebraically. Given $\overline{\mathcal{A}}$ and $\overline{\mathcal{G}}$ the topologies induced by the topology of $\mathbf{G}$, one sees that, for compact $\mathbf{G}$, these spaces are again compact. This guarantees the existence of a natural induced Haar measure on $\overline{\mathcal{A}}$ and $\overline{\mathcal{A}/\mathcal{G}}$, the new configuration space for the path integral quantization.

Both from the mathematical and from the physical point of view it is very interesting how the "classical" regular gauge theories are related to the generalized formulation in the Ashtekar framework. First of all, it has been proven that $\mathcal{A}$ and $\mathcal{G}$ are dense subsets in $\overline{\mathcal{A}}$ and $\overline{\mathcal{G}}$, respectively [17]. Furthermore, $\mathcal{A}$ is contained in a set of induced Haar measure zero [15]. These properties coincide exactly with the experiences known from the Wiener or Feynman

path integral. Then the Wilson loop expectation values have been determined for the two-dimensional pure Yang-Mills theory [5, 11] – in coincidence with the known results in the standard framework. In the present paper we continue the investigations on how the results of Kondracki and Rogulski can be extended to the Ashtekar framework. In a previous paper [9] we have already shown that the gauge orbit type is determined by the centralizer of the holonomy group. This closely related to the observations of Kondracki and Sadowski [13]. In the present paper we are going to prove that there is a slice theorem and a denseness theorem for the space of connections in the Ashtekar framework as well. However, our methods are completely different to those of Kondracki and Rogulski.

The outline of the paper is as follows:
After fixing the notations we prove a very crucial lemma in section 4: Every centralizer in a compact Lie group is finitely generated. This implies that every orbit type (being the centralizer of the holonomy group) is determined by a finite set of holonomies of the corresponding connection.
Using the projection onto these holonomies we can lift the slice theorem from an appropriate finite-dimensional $\mathbf{G}^n$ to the space $\overline{\mathcal{A}}$. This is proven in section 5 and it implies the openness of the strata as shown in the following section.
Afterwards, we prove a denseness theorem for the strata. For this we need a construction for new connections from [10]. As a corollary we obtain that the set of all gauge orbit types equals the set of all conjugacy classes of Howe subgroups of $\mathbf{G}$. A Howe subgroup is a subgroup that is the centralizer of some subset of $\mathbf{G}$. This way we completely determine all possible gauge orbit types. This has been succeeded for the Sobolev connections – to the best of our knowlegde – only for $\mathbf{G} = SU(n)$ and low-dimensional M [18].
In Section 8 we show that the slice and the denseness theorem yield again a topologically regular stratification of $\overline{\mathcal{A}}$ as well as of $\overline{\mathcal{A}}/\overline{\mathcal{G}}$. But, in contrast to the Sobolev case, the strata are not proved to be manifolds.
Finally, we show in Section 9 that the generic stratum (it collects the connections of maximal type) is not only dense in $\overline{\mathcal{A}}$, but has also the total induced Haar measure 1. This shows that the Faddeev-Popov determinant for the projection $\overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}/\overline{\mathcal{G}}$ is equal to 1.

2 Preliminaries

As we indicated in [9] the present paper is the final one in a small series of three papers. In the first one [9] we extended the definitions and propositions for $\overline{\mathcal{A}}$, $\overline{\mathcal{G}}$ and $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ made by Ashtekar et al. from the case of graphs [1, 2, 4, 3, 15] and of webs [6] to arbitrarily smooth paths. Moreover, in that paper we determined the gauge orbit type of a connection. In the second paper [10] we investigated properties of $\overline{\mathcal{A}}$ and proved, in particular, the existence of an Ashtekar-Lewandowski measure in our context. Now, we summarize the most important notations, definitions and facts used in the following. For detailed information we refer the reader to the preceding papers [9, 10].
- Let $\mathbf{G}$ be a *compact* Lie group.
- A path (usually denoted by γ or δ) is a piecewise C^r-map from $[0, 1]$ into a connected C^r-manifold M, $\dim M \geq 2$, $r \in \mathbb{N}^+ \cup \{\infty\} \cup \{\omega\}$ arbitrary, but fixed. Additionally, we fix now the decision whether we restrict the paths to be piecewise immersive or not. Paths can be multiplied as usual by concatenation. A graph is a finite union of paths, such that

different paths intersect each other at most in their end points. Paths in a graph are called simple. A path is called finite iff it is up to the parametrization a finite product of simple paths. Two paths are equivalent iff the first one can be reconstructed from the second one by a sequence of reparametrizations or of insertions or deletions of retracings. We will only consider equivalence classes of finite paths and graphs. The set of (classes of) paths is denoted by $\mathcal{P}$, that of paths from x to y by $\mathcal{P}_{xy}$ and that of loops (paths with a fixed initial and terminal point m) by $\mathcal{HG}$, the so-called hoop group.

- A generalized connection $\overline{A} \in \overline{\mathcal{A}}$ is a homomorphism[1] $h_{\overline{A}} : \mathcal{P} \longrightarrow \mathbf{G}$. (We usually write $h_{\overline{A}}$ synonymously for $\overline{A}$.) A generalized gauge transform $\overline{g} \in \overline{\mathcal{G}}$ is a map $\overline{g} : M \longrightarrow \mathbf{G}$. The value $\overline{g}(x)$ of the gauge transform in the point x is usually denoted by g_x. The action of $\overline{\mathcal{G}}$ on $\overline{\mathcal{A}}$ is given by

$$h_{\overline{A}\circ\overline{g}}(\gamma) := g_{\gamma(0)}^{-1} \, h_{\overline{A}}(\gamma) \, g_{\gamma(1)} \text{ for all } \gamma \in \mathcal{P}. \tag{1}$$

  We have $\overline{\mathcal{A}}/\overline{\mathcal{G}} \cong \mathrm{Hom}(\mathcal{HG}, \mathbf{G})/\mathrm{Ad}$.

- Now, let Γ be a graph with $\mathbf{E}(\Gamma) = \{e_1, \ldots, e_E\}$ being the set of edges and $\mathbf{V}(\Gamma) = \{v_1, \ldots, v_V\}$ the set of vertices. The projections onto the lattice gauge theories are defined by

$$
\begin{array}{ccccccc}
\pi_\Gamma : & \overline{\mathcal{A}} & \longrightarrow & \overline{\mathcal{A}}_\Gamma \equiv \mathbf{G}^E & \text{and} & \pi_\Gamma : & \overline{\mathcal{G}} & \longrightarrow & \overline{\mathcal{G}}_\Gamma \equiv \mathbf{G}^V. \\
& \overline{A} & \longmapsto & \left(h_{\overline{A}}(e_1), \ldots, h_{\overline{A}}(e_E)\right) & & & \overline{g} & \longmapsto & \left(g_{v_1}, \ldots, g_{v_V}\right)
\end{array}
$$

  The topologies on $\overline{\mathcal{A}}$ and $\overline{\mathcal{G}}$ are the topologies generated by these projections. Using these topologies the action $\Theta : \overline{\mathcal{A}} \times \overline{\mathcal{G}} \longrightarrow \overline{\mathcal{A}}$ defined by (1) is continuous. Since $\mathbf{G}$ is compact Lie, $\overline{\mathcal{A}}$ and $\overline{\mathcal{G}}$ are compact Hausdorff spaces and consequently completely regular.

- The holonomy group $\mathbf{H}_{\overline{A}}$ of a connection $\overline{A}$ is defined by $\mathbf{H}_{\overline{A}} := h_{\overline{A}}(\mathcal{HG}) \subseteq \mathbf{G}$, its centralizer is denoted by $Z(\mathbf{H}_{\overline{A}})$. The stabilizer of a connection $\overline{A} \in \overline{\mathcal{A}}$ under the action of $\overline{\mathcal{G}}$ is denoted by $\mathbf{B}(\overline{A})$. We have $\overline{g} \in \mathbf{B}(\overline{A})$ iff $g_m \in Z(\mathbf{H}_{\overline{A}})$ and for all $x \in M$ there is a path $\gamma \in \mathcal{P}_{mx}$ with $h_{\overline{A}}(\gamma) = g_m^{-1} h_{\overline{A}}(\gamma) g_x$. In [9] we proved that $\mathbf{B}(\overline{A})$ and $Z(\mathbf{H}_{\overline{A}})$ are homeomorphic.

- The type of a gauge orbit $\mathbf{E}_{\overline{A}} := \overline{A} \circ \overline{\mathcal{G}}$ is the centralizer of the holonomy group of $\overline{A}$ modulo conjugation in $\mathbf{G}$. (An equivalent definition uses the stabilizer $\mathbf{B}(\overline{A})$ itself.)

3 Partial Ordering of Types

Definition 3.1 A subgroup U of $\mathbf{G}$ is called **Howe subgroup** iff there is a set $V \subseteq \mathbf{G}$ with $U = Z(V)$.

Analogously to the general theory we define a partial ordering for the gauge orbit types [8].

Definition 3.2 Let $\mathcal{T}$ denote the set of all Howe subgroups of $\mathbf{G}$. Let $t_1, t_2 \in \mathcal{T}$. Then $t_1 \leq t_2$ holds iff there are $\mathbf{G}_1 \in t_1$ and $\mathbf{G}_2 \in t_2$ with $\mathbf{G}_1 \supseteq \mathbf{G}_2$.

Obviously, we have

Lemma 3.1 The maximal element in $\mathcal{T}$ is the class $t_{\max}$ of the center $Z(\mathbf{G})$ of $\mathbf{G}$, the minimal is the class $t_{\min}$ of $\mathbf{G}$ itself.

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[1]Homomorphism means $h_{\overline{A}}(\gamma_1\gamma_2) = h_{\overline{A}}(\gamma_1)h_{\overline{A}}(\gamma_2)$ supposed $\gamma_1\gamma_2$ is defined.

Definition 3.3 Let $t \in \mathcal{T}$. We define the following expressions:

$$\begin{array}{rcl}
\overline{\mathcal{A}}_{\geq t} & := & \{\overline{A} \in \overline{\mathcal{A}} \mid \mathrm{Typ}(\overline{A}) \geq t\} \\
\overline{\mathcal{A}}_{=t} & := & \{\overline{A} \in \overline{\mathcal{A}} \mid \mathrm{Typ}(\overline{A}) = t\} \\
\overline{\mathcal{A}}_{\leq t} & := & \{\overline{A} \in \overline{\mathcal{A}} \mid \mathrm{Typ}(\overline{A}) \leq t\}.
\end{array}$$

All the $\overline{\mathcal{A}}_{=t}$ are called **strata**.[2]

4 Reducing the Problem to Finite-Dimensional G-Spaces

4.1 Finiteness Lemma for Centralizers

We start with the crucial

Lemma 4.1 Let U be a subset of a compact Lie group $\mathbf{G}$. Then there exist an $n \in \mathbb{N}$ and $u_1, \ldots, u_n \in U$, such that $Z(\{u_1, \ldots, u_n\}) = Z(U)$.

Proof
- The case $Z(U) = \mathbf{G} = Z(\emptyset)$ is trivial.
- Let $Z(U) \neq \mathbf{G}$. Then there is a $u_1 \in U$ with $Z(\{u_1\}) \neq \mathbf{G}$. Choose now for $i \geq 1$ successively $u_{i+1} \in U$ with $Z(\{u_1, \ldots, u_i\}) \supset Z(\{u_1, \ldots, u_{i+1}\})$ as long as there is such a u_{i+1}. This procedure stops after a finite number of steps, since each non-increasing sequence of compact subgroups in $\mathbf{G}$ stabilizes [8]. (Centralizers are always closed, thus compact.) Therefore there is an $n \in \mathbb{N}$, such that $Z(\{u_1, \ldots, u_n\}) = Z(\{u_1, \ldots, u_n\} \cup \{u\})$ for all $u \in U$. Thus, we have $Z(\{u_1, \ldots, u_n\}) = \bigcap_{u \in U} Z(\{u_1, \ldots, u_n\} \cup \{u\}) = Z(\{u_1, \ldots, u_n\} \cup U) = Z(U)$.
qed

Corollary 4.2 Let $\overline{A} \in \overline{\mathcal{A}}$.
Then there is a finite set $\boldsymbol{\alpha} \subseteq \mathcal{HG}$, such that $Z(\mathbf{H}_{\overline{A}}) = Z(h_{\overline{A}}(\boldsymbol{\alpha}))$.[3]

Proof Due to $\mathbf{H}_{\overline{A}} \subseteq \mathbf{G}$ and the just proven lemma there are an $n \in \mathbb{N}$ and $g_1, \ldots, g_n \in \mathbf{H}_{\overline{A}}$ with $Z(\{g_1, \ldots, g_n\}) = Z(\mathbf{H}_{\overline{A}})$. On the other hand, since $g_1, \ldots, g_n \in \mathbf{H}_{\overline{A}}$, there are $\alpha_1, \ldots, \alpha_n \in \mathcal{HG}$ with $g_i = h_{\overline{A}}(\alpha_i)$ for all $i = 1, \ldots, n$.
qed

4.2 Reduction Mapping

Definition 4.1 Let $\boldsymbol{\alpha} \subseteq \mathcal{HG}$. Then the map

$$\varphi_{\boldsymbol{\alpha}}: \begin{array}{ccc} \overline{\mathcal{A}} & \longrightarrow & \mathbf{G}^{\#\boldsymbol{\alpha}} \\ \overline{A} & \longmapsto & h_{\overline{A}}(\boldsymbol{\alpha}) \end{array}$$

is called **reduction mapping**.

Lemma 4.3 Let $\boldsymbol{\alpha} \subseteq \mathcal{HG}$ be arbitrary.
Then $\varphi_{\boldsymbol{\alpha}}$ is continuous, and for all $\overline{A} \in \overline{\mathcal{A}}$ and $\overline{g} \in \overline{\mathcal{G}}$ we have $\varphi_{\boldsymbol{\alpha}}(\overline{A} \circ \overline{g}) = \varphi_{\boldsymbol{\alpha}}(\overline{A}) \circ g_m$. Here $\mathbf{G}$ acts on $\mathbf{G}^{\#\boldsymbol{\alpha}}$ by the adjoint map.

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[2]The justification for that notation can be found in section 8.

[3]$h_{\overline{A}}(\boldsymbol{\alpha}) := \{h_{\overline{A}}(\alpha_1), \ldots, h_{\overline{A}}(\alpha_n)\} \subseteq \mathbf{G}$ where $n := \#\boldsymbol{\alpha}$. To avoid cumbersome notations we denote also $\big(h_{\overline{A}}(\alpha_1), \ldots, h_{\overline{A}}(\alpha_n)\big) \in \mathbf{G}^n$ by $h_{\overline{A}}(\boldsymbol{\alpha})$. It should be clear from the context what is meant. Furthermore, $\boldsymbol{\alpha}$ is always finite.

Proof • $\varphi_{\boldsymbol{\alpha}} : \overline{\mathcal{A}} \longrightarrow \mathbf{G}^{\#\boldsymbol{\alpha}}$ is as a map into a product space continuous iff $\pi_i \circ \varphi_{\boldsymbol{\alpha}} \equiv \varphi_{\{\alpha_i\}}$ is continuous for all projections $\pi_i : \mathbf{G}^{\#\boldsymbol{\alpha}} \longrightarrow \mathbf{G}$ onto the ith factor. Thus, it is sufficient to prove the continuity of $\varphi_{\{\alpha\}}$ for all $\alpha \in \mathcal{HG}$.

Now decompose α into a product of finitely many edges e_j, $j = 1, \ldots, J$ (i.e., into paths that can be represented as an edge in a graph). Then the mapping $\overline{\mathcal{A}} \longrightarrow \mathbf{G}^J$ with $\overline{A} \longmapsto \left(\pi_{e_1}(\overline{A}), \ldots, \pi_{e_J}(\overline{A}) \right)$ is continuous per definitionem. Since the multiplication in $\mathbf{G}$ is continuous, $\varphi_{\{\alpha\}}$ is continuous, too.

• The compatibility with the group action follows from $h_{\overline{A} \circ \overline{g}}(\boldsymbol{\alpha}) = g_m^{-1} \, h_{\overline{A}}(\boldsymbol{\alpha}) \, g_m$.

qed

4.3 Adjoint Action of G on $\mathbf{G}^n$

In this short subsection we will summarize the most important facts about the adjoint action of $\mathbf{G}$ on $\mathbf{G}^n$ that can be deduced from the general theory of transformation groups (see, e.g., [7]).

First we determine the stabilizer $\mathbf{G}_{\vec{g}}$ of an element $\vec{g} \in \mathbf{G}^n$. We have

$$\mathbf{G}_{\vec{g}} = \{ g \in \mathbf{G} \mid \vec{g} \circ g = \vec{g} \} = \{ g \in \mathbf{G} \mid g^{-1} g_i g = g_i \quad \forall i \} = Z(\{g_1, \ldots, g_n\}).$$

Consequently, we have for the type of the corresponding orbit

$$\mathrm{Typ}(\vec{g}) = [\mathbf{G}_{\vec{g}}] = [Z(\{g_1, \ldots, g_n\})].$$

The slice theorem reads now as follows:

Proposition 4.4 Let $\vec{g} \in \mathbf{G}^n$. Then there is an $S \subseteq \mathbf{G}^n$ with $\vec{g} \in S$, such that:

• $S \circ \mathbf{G}$ is an open neighboorhood of $\vec{g} \circ \mathbf{G}$ and

• there is an equivariant retraction $f : S \circ \mathbf{G} \longrightarrow \vec{g} \circ \mathbf{G}$ with $f^{-1}(\{\vec{g}\}) = S$.

Both on $\overline{\mathcal{A}}$ and on $\mathbf{G}^n$ the type is a Howe subgroup of $\mathbf{G}$. The transformation behaviour of the types under a reduction mapping is stated in the next

Proposition 4.5 Any reduction mapping is type-minorifying, i.e. for all $\boldsymbol{\alpha} \subseteq \mathcal{HG}$ and all $\overline{A} \in \overline{\mathcal{A}}$ we have

$$\mathrm{Typ}\left(\varphi_{\boldsymbol{\alpha}}(\overline{A}) \right) \leq \mathrm{Typ}(\overline{A}).$$

Proof We have $\mathrm{Typ}\left(\varphi_{\boldsymbol{\alpha}}(\overline{A}) \right) = [Z(\varphi_{\boldsymbol{\alpha}}(\overline{A}))] \equiv [Z(h_{\overline{A}}(\boldsymbol{\alpha}))] \leq [Z(\mathbf{H}_{\overline{A}})] = \mathrm{Typ}(\overline{A}).$ **qed**

5 Slice Theorem for $\overline{\mathcal{A}}$

We state now the main theorem of the present paper.

Theorem 5.1 There is a tubular neighbourhood for any gauge orbit.

Equivalently we have: For all $\overline{A} \in \overline{\mathcal{A}}$ there is an $\overline{S} \subseteq \overline{\mathcal{A}}$ with $\overline{A} \in \overline{S}$, such that:

• $\overline{S} \circ \overline{\mathcal{G}}$ is an open neighbourhood of $\overline{A} \circ \overline{\mathcal{G}}$ and

• there is an equivariant retraction $F : \overline{S} \circ \overline{\mathcal{G}} \longrightarrow \overline{A} \circ \overline{\mathcal{G}}$ with $F^{-1}(\{\overline{A}\}) = \overline{S}$.

5.1 The Idea

Our proof imitates in a certain sense the proof of the standard slice theorem (see, e.g., [7]) which is valid for the action of a *finite-dimensional* compact *Lie* group G on a Hausdorff space X. Let us review the main idea of this proof. Given $x \in X$. Let $H \subseteq G$ be the stabilizer of x, i.e., $[H]$ is an orbit type on the G-space X. Now, this situation is simulated on an $\mathbb{R}^n$, i.e., for an appropriate action of G on $\mathbb{R}^n$ one chooses a point with stabilizer H. So the orbits on X and on $\mathbb{R}^n$ can be identified. For the case of $\mathbb{R}^n$ the proof of a slice theorem is not very complicated. The crucial point of the general proof is the usage of the Tietze-Gleason extension theorem because this yields an equivariant extension $\psi : X \longrightarrow \mathbb{R}^n$, mapping one orbit onto the other. Finally, by means of ψ the slice theorem can be lifted from $\mathbb{R}^n$ to X.

What can we learn for our problem? Obviously, $\overline{\mathcal{G}}$ is not a finite-dimensional Lie group. But, we know that the stabilizer $\mathbf{B}(\overline{A})$ of a connection is homeomorphic to the centralizer $Z(\mathbf{H}_{\overline{A}})$ of the holonomy group that is a subgroup of $\mathbf{G}$. Since every centralizer is finitely generated, $Z(\mathbf{H}_{\overline{A}})$ equals $Z(h_{\overline{A}}(\boldsymbol{\alpha}))$ with an appropriate finite $\boldsymbol{\alpha} \in \mathcal{HG}$. This is nothing but the stabilizer of the adjoint action of $\mathbf{G}$ on $\mathbf{G}^n$. Thus, the reduction mapping $\varphi_{\boldsymbol{\alpha}}$ is the desired equivalent for ψ.

We are now looking for an appropriate $\overline{S} \subseteq \overline{\mathcal{A}}$, such that
$$F : \quad \overline{S} \circ \overline{\mathcal{G}} \quad \longrightarrow \quad \overline{A} \circ \overline{\mathcal{G}}$$
$$\overline{A}' \circ \overline{g} \quad \longmapsto \quad \overline{A} \circ \overline{g}$$
is well-defined and has the desired properties.

In order to make F well-defined, we need $\overline{A}' \circ \overline{g} = \overline{A}' \implies \overline{A} \circ \overline{g} = \overline{A}$ for all $\overline{A}' \in \overline{S}$ and $\overline{g} \in \overline{\mathcal{G}}$, i.e. $\mathbf{B}(\overline{A}') \subseteq \mathbf{B}(\overline{A})$. Applying the projections π_x on the stabilizers (see [9]) we get for $\gamma_x \in \mathcal{P}_{mx}$ (let γ_m be the trivial path)
$$h_{\overline{A}'}(\gamma_m)^{-1} Z(\mathbf{H}_{\overline{A}'}) h_{\overline{A}'}(\gamma_x) = \pi_x(\mathbf{B}(\overline{A}')) \subseteq \pi_x(\mathbf{B}(\overline{A})) = h_{\overline{A}}(\gamma_m)^{-1} Z(\mathbf{H}_{\overline{A}}) h_{\overline{A}}(\gamma_x),$$
thus
$$Z(\mathbf{H}_{\overline{A}'}) \subseteq h_{\overline{A}'}(\gamma_m) h_{\overline{A}}(\gamma_m)^{-1} Z(\mathbf{H}_{\overline{A}}) h_{\overline{A}}(\gamma_x) h_{\overline{A}'}^{-1}(\gamma_x) \tag{2}$$
for all $x \in M$. In particular, we have $Z(\mathbf{H}_{\overline{A}'}) \subseteq Z(\mathbf{H}_{\overline{A}})$ for $x = m$.

Now we choose an $\boldsymbol{\alpha} \subseteq \mathcal{HG}$ with $Z(\mathbf{H}_{\overline{A}}) = Z(h_{\overline{A}}(\boldsymbol{\alpha}))$ and an $S \subseteq \mathbf{G}^{\#\boldsymbol{\alpha}}$ and an equivariant retraction $f : S \circ \mathbf{G} \longrightarrow \varphi_{\boldsymbol{\alpha}}(\overline{A}) \circ \mathbf{G}$. Since equivariant mappings magnify stabilizers (or at least do not reduce them), we have $Z(\vec{g}') \subseteq Z(\varphi_{\boldsymbol{\alpha}}(\overline{A}))$ for all $\vec{g}' \in S$.

Therefore, the condition of (2) would be, e.g., fulfilled if we had for all $\overline{A}' \in \overline{S}$
1. $\varphi_{\boldsymbol{\alpha}}(\overline{A}') \in S$ and
2. $h_{\overline{A}'}(\gamma_x) = h_{\overline{A}}(\gamma_x)$ for all $x \in M$,

because the first condition implies $Z(\mathbf{H}_{\overline{A}'}) \subseteq Z(h_{\overline{A}'}(\boldsymbol{\alpha})) \equiv Z(\varphi_{\boldsymbol{\alpha}}(\overline{A}')) \subseteq Z(\varphi_{\boldsymbol{\alpha}}(\overline{A})) = Z(\mathbf{H}_{\overline{A}})$.

We could now choose $\overline{S}$ such that these two conditions are fulfilled. However, this would imply $F^{-1}(\{A\}) \supset \overline{S}$ in general because for $\overline{g} \in \mathbf{B}(\overline{A})$ together with $\overline{A}'$ the connection $\overline{A}' \circ \overline{g}$ is contained in $F^{-1}(\{A\})$ as well,[4] but $\overline{A}' \circ \overline{g}$ needs no longer fulfill the two conditions above. Now it is quite obvious to define $\overline{S}$ as the set of all connections fulfilling these conditions multiplied with $\mathbf{B}(\overline{A})$. And indeed, the well-definedness remains valid.

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[4]We have $F(\overline{A}') = \overline{A} = \overline{A} \circ \overline{g} = F(\overline{A}' \circ \overline{g})$.

5.2 The Proof

Proof

1. Let $\overline{A} \in \overline{\mathcal{A}}$. Choose for $\overline{A}$ an $\boldsymbol{\alpha} \subseteq \mathcal{HG}$ with $Z(\mathbf{H}_{\overline{A}}) = Z(h_{\overline{A}}(\boldsymbol{\alpha}))$ according to Corollary 4.2 and denote the corresponding reduction mapping $\varphi_{\boldsymbol{\alpha}} : \overline{\mathcal{A}} \longrightarrow \mathbf{G}^{\#\boldsymbol{\alpha}}$ shortly by φ.

2. Due to Proposition 4.4 there is an $S \subseteq \mathbf{G}^{\#\boldsymbol{\alpha}}$ with $\varphi(\overline{A}) \in S$, such that
   - $S \circ \mathbf{G}$ is an open neighbourhood of $\varphi(\overline{A}) \circ \mathbf{G}$ and
   - there exists an equivariant mapping f with
     - $f : S \circ \mathbf{G} \longrightarrow \varphi(\overline{A}) \circ \mathbf{G}$ and
     - $f^{-1}(\{\varphi(\overline{A})\}) = S$.

3. We define the mapping
$$\psi : \quad \overline{\mathcal{A}} \quad \longrightarrow \quad \overline{\mathcal{G}},$$
$$\overline{A}' \quad \longmapsto \quad \left(h_{\overline{A}'}(\gamma_x) \right)_{x \in M}$$
whereas for all $x \in M \setminus \{m\}$ the (arbitrary, but fixed) path γ_x runs from m to x and γ_m is the trivial path.

4. As we motivated above we set
$$\overline{S}_0 \quad := \quad \varphi^{-1}(S) \cap \psi^{-1}(\psi(\overline{A})),$$
$$\overline{S} \quad := \quad \left(\varphi^{-1}(S) \cap \psi^{-1}(\psi(\overline{A})) \right) \circ \mathbf{B}(\overline{A}) \quad \equiv \quad \overline{S}_0 \circ \mathbf{B}(\overline{A})$$
and
$$F : \quad \overline{S} \circ \overline{\mathcal{G}} \quad \longrightarrow \quad \overline{A} \circ \overline{\mathcal{G}}.$$
$$\overline{A}' \circ \overline{g} \quad \longmapsto \quad \overline{A} \circ \overline{g}$$

5. F is well-defined.
   - Let $\overline{A}' \circ \overline{g}' = \overline{A}'' \circ \overline{g}''$ with $\overline{A}', \overline{A}'' \in \overline{S}$ and $\overline{g}', \overline{g}'' \in \overline{\mathcal{G}}$. Then there exist $\overline{z}', \overline{z}'' \in \mathbf{B}(\overline{A})$ with $\overline{A}' = \overline{A}'_0 \circ \overline{z}'$ and $\overline{A}'' = \overline{A}''_0 \circ \overline{z}''$ as well as $\overline{A}'_0, \overline{A}''_0 \in \overline{S}_0$.
   - Due to $\overline{S}_0 \subseteq \psi^{-1}(\psi(\overline{A}))$ we have $\psi(\overline{A}'_0) = \psi(\overline{A}) = \psi(\overline{A}''_0)$, i.e. $h_{\overline{A}'_0}(\gamma_x) = h_{\overline{A}}(\gamma_x) = h_{\overline{A}''_0}(\gamma_x)$ for all x.
   - Furthermore, we have
$$
\begin{aligned}
f(\varphi(\overline{A}' \circ \overline{g}')) &= f(\varphi(\overline{A}'_0 \circ \overline{z}' \circ \overline{g}')) & \\
&= f(\varphi(\overline{A}'_0) \circ z'_m \circ g'_m) & (\varphi \text{ "equivariant"}) \\
&= f(\varphi(\overline{A}'_0)) \circ z'_m \circ g'_m & (f \text{ equivariant}) \\
&= \varphi(\overline{A}) \circ z'_m \circ g'_m & (\varphi(\overline{A}'_0) \in S) \\
&= \varphi(\overline{A} \circ \overline{z}') \circ g'_m & (\varphi \text{ "equivariant"}) \\
&= \varphi(\overline{A}) \circ g'_m & (\overline{z}' \in \mathbf{B}(\overline{A}))
\end{aligned}
$$
   and analogously $f(\varphi(\overline{A}'' \circ \overline{g}'')) = \varphi(\overline{A}) \circ g''_m$.
   Therefore, we have $\varphi(\overline{A}) \circ g'_m = \varphi(\overline{A}) \circ g''_m$, i.e. $g''_m (g'_m)^{-1}$ is an element of the stabilizer of $\varphi(\overline{A})$, thus $g''_m (g'_m)^{-1} \in Z(\varphi(\overline{A})) = Z(\mathbf{H}_{\overline{A}})$.
   - Since $\overline{A}'_0 \circ \overline{z}' \circ \overline{g}' = \overline{A}''_0 \circ \overline{z}'' \circ \overline{g}''$, we have $\overline{A}'_0 = \overline{A}''_0 \circ \left(\overline{z}'' \, \overline{g}'' \, (\overline{g}')^{-1} \, (\overline{z}')^{-1} \right)$, and so for all $x \in M$
$$h_{\overline{A}'_0}(\gamma_x) = \left(\overline{z}'' \, \overline{g}'' \, (\overline{g}')^{-1} \, (\overline{z}')^{-1} \right)_m^{-1} h_{\overline{A}''_0}(\gamma_x) \left(\overline{z}'' \, \overline{g}'' \, (\overline{g}')^{-1} \, (\overline{z}')^{-1} \right)_x.$$
   Moreover, since $\left(\overline{g}'' \, (\overline{g}')^{-1} \right)_m \in Z(\mathbf{H}_{\overline{A}})$, we have $\left(\overline{z}'' \, \overline{g}'' \, (\overline{g}')^{-1} \, (\overline{z}')^{-1} \right)_m \in Z(\mathbf{H}_{\overline{A}})$. From $h_{\overline{A}'_0}(\gamma_x) = h_{\overline{A}}(\gamma_x) = h_{\overline{A}''_0}(\gamma_x)$ for all x now $\overline{z}'' \, \overline{g}'' \, (\overline{g}')^{-1} \, (\overline{z}')^{-1} \in \mathbf{B}(\overline{A})$ follows, and thus $\overline{g}'' \, (\overline{g}')^{-1} \in \mathbf{B}(\overline{A})$.
   - By this we have $\overline{A} \circ \overline{g}' = \overline{A} \circ \overline{g}''$, i.e. F is well-defined.

8

6. F is equivariant.
   - Let $\overline{A}'' = \overline{A}' \circ \overline{g}' \in \overline{S} \circ \overline{G}$. Then
$$
\begin{aligned}
F(\overline{A}'' \circ \overline{g}) &= F(\overline{A}' \circ (\overline{g}' \circ \overline{g})) \\
&= \overline{A} \circ (\overline{g}' \circ \overline{g}) \\
&= (\overline{A} \circ \overline{g}') \circ \overline{g} \\
&= F(\overline{A}' \circ \overline{g}') \circ \overline{g} \\
&= F(\overline{A}'') \circ \overline{g}.
\end{aligned}
$$

7. F is retracting.
   - Let $\overline{A}' = \overline{A} \circ \overline{g} \in \overline{A} \circ \overline{G}$. Then $F(\overline{A}') = F(\overline{A} \circ \overline{g}) = \overline{A} \circ \overline{g} = \overline{A}'$.

8. $\overline{S} \circ \overline{G}$ is an open neighbourhood of $\overline{A} \circ \overline{G}$.
   - Obviously, $\overline{A} \circ \overline{G} \subseteq \overline{S} \circ \overline{G}$.
   - We have $\overline{S} \circ \overline{G} = \varphi^{-1}(S \circ \mathbf{G})$.

     "$\subseteq$" Let $\overline{A}'' = \overline{A}' \circ \overline{g} \in \overline{S}_0 \circ \overline{G} = \overline{S} \circ \overline{G}$.

     Then we have $\varphi(\overline{A}'') = \varphi(\overline{A}' \circ \overline{g}) = \varphi(\overline{A}') \circ g_m \in S \circ \mathbf{G}$ because $\varphi(\overline{S}_0) \subseteq S$. Thus, $\overline{A}'' \in \varphi^{-1}(S \circ \mathbf{G})$.

     "$\supseteq$" − Let $\overline{A}'' \in \varphi^{-1}(S \circ \mathbf{G})$, i.e. $\varphi(\overline{A}'') = \vec{g}'' \circ g$ with appropriate $\vec{g}'' \in S$ and $g \in \mathbf{G}$.

     − Choose some $\overline{g}$ with $g_m = g$.
     Then $\varphi(\overline{A}'' \circ \overline{g}^{-1}) = \varphi(\overline{A}'') \circ g_m^{-1} = \vec{g}'' \in S$.
     Now set $\overline{A}''' := \overline{A}'' \circ \overline{g}^{-1}$.

     − Using $g'_x := \left(h_{\overline{A}'''}(\gamma_x)\right)^{-1} h_{\overline{A}}(\gamma_x)$ and $\overline{A}' := \overline{A}''' \circ \overline{g}'$ we get

     a) $\varphi(\overline{A}') = \varphi(\overline{A}''') \in S$ because of $g'_m = e_{\mathbf{G}}$ and

     b) $h_{\overline{A}'}(\gamma_x) = h_{\overline{A}'''}(\gamma_x)\, g'_x = h_{\overline{A}}(\gamma_x)$ for all $x \in M$.

     Thus, we have $\overline{A}' \in \overline{S}_0 \subseteq \overline{S}$ and $\overline{A}'' = \overline{A}''' \circ \overline{g} = \overline{A}' \circ ((\overline{g}')^{-1} \circ \overline{g}) \in \overline{S} \circ \overline{G}$.
   - Consequently, $\overline{S} \circ \overline{G} = \varphi^{-1}(S \circ \mathbf{G})$ is as a preimage of an open set again open because of the continuity of φ.

9. F is continuous.
   - We consider the following diagram

$$
\begin{array}{ccccc}
\overline{S} \circ \overline{G} & \xrightarrow{\ F\ } & \overline{A} \circ \overline{G} & & \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} & & \\
S \circ \mathbf{G} & \xrightarrow{\ f\ } & \varphi(\overline{A}) \circ \mathbf{G} & \xrightarrow[\cong]{\ \tau_{\mathbf{G}}\ } & Z(\mathbf{H}_{\overline{A}}) \backslash \mathbf{G}
\end{array}
$$

(3)

$$
\begin{array}{ccc}
\overline{A}' \circ \overline{g} & \xmapsto{\ F\ } & \overline{A} \circ \overline{g} \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\
\varphi(\overline{A}') \circ g_m & \xmapsto{\ f\ } \varphi(\overline{A}) \circ g_m & \xmapsto{\ \tau_{\mathbf{G}}\ } [g_m]_{Z(\mathbf{H}_{\overline{A}})}
\end{array}
$$

It is commutative due to $\varphi(\overline{S} \circ \overline{G}) \subseteq S \circ \mathbf{G}$, $\varphi(\overline{A} \circ \overline{G}) \subseteq \varphi(\overline{A}) \circ \mathbf{G}$ and the definition of F. $\tau_{\mathbf{G}}$ is the canonical homeomorphism between the orbit of $\varphi(\overline{A})$ and the quotient of the acting group $\mathbf{G}$ by the stabilizer of $\varphi(\overline{A})$.

Since φ, f and $\tau_{\mathbf{G}}$ are continuous, the map
$$F' := \tau_{\mathbf{G}} \circ \varphi \circ F : \quad \overline{S} \circ \overline{\mathcal{G}} \quad \longrightarrow \quad Z(\mathbf{H}_{\overline{A}}) \backslash \mathbf{G}$$
$$\overline{A}' \circ \overline{g} \quad \longmapsto \quad [g_m]_{Z(\mathbf{H}_{\overline{A}})}$$
is continuous.

- Now, we consider the map
$$F'' : \quad (\overline{S} \circ \overline{\mathcal{G}}) \times \mathbf{G} \quad \longrightarrow \quad \overline{\mathcal{G}}.$$
$$(\overline{A}' \circ \overline{g}', g_m) \quad \longmapsto \quad \left(h_{\gamma_x}(\overline{A})^{-1} \, g_m \, h_{\gamma_x}(\overline{A}' \circ \overline{g}')\right)_{x \in M}$$

F'' is continuous because
$$\pi_x \circ F'' : \quad (\overline{S} \circ \overline{\mathcal{G}}) \times \mathbf{G} \quad \longrightarrow \quad \mathbf{G} \times \mathbf{G} \quad \xrightarrow{\text{mult.}} \quad \mathbf{G}$$
$$(\overline{A}'', g_m) \quad \longmapsto \quad (h_{\gamma_x}(\overline{A}''), g_m) \quad \longmapsto \quad h_{\gamma_x}(\overline{A})^{-1} \, g_m \, h_{\gamma_x}(\overline{A}'')$$
is obviously continuous for all $x \in M$.

- F'' induces a map F''' via the following commutative diagram

$$
\begin{array}{ccc}
(\overline{S} \circ \overline{\mathcal{G}}) \times \mathbf{G} & \xrightarrow{\quad F'' \quad} & \overline{\mathcal{G}} \\
\scriptstyle{\text{id} \times \pi_{Z(\mathbf{H}_{\overline{A}})}} \Big\downarrow & & \Big\downarrow \scriptstyle{\pi_{\mathbf{B}(\overline{A})}} \\
(\overline{S} \circ \overline{\mathcal{G}}) \times Z(\mathbf{H}_{\overline{A}}) \backslash \mathbf{G} & \xrightarrow{\quad F''' \quad} & \mathbf{B}(\overline{A}) \backslash \overline{\mathcal{G}}
\end{array}
\quad,
$$

i.e., $F'''(\overline{A}'', [g_m]_{Z(\mathbf{H}_{\overline{A}})}) = \left[\left(h_{\gamma_x}(\overline{A})^{-1} \, g_m \, h_{\gamma_x}(\overline{A}'')\right)_{x \in M}\right]_{\mathbf{B}(\overline{A})}$.

  - F''' is well-defined.
    Let $g_{2,m} = z g_{1,m}$ with $z \in Z(\mathbf{H}_{\overline{A}})$. Then
$$
\begin{aligned}
F'''(\overline{A}'', [g_{2,m}]_{Z(\mathbf{H}_{\overline{A}})}) &= \left[\left(h_{\gamma_x}(\overline{A})^{-1} \, g_{2,m} \, h_{\gamma_x}(\overline{A}'')\right)_{x \in M}\right]_{\mathbf{B}(\overline{A})} \\
&= \left[\left(h_{\gamma_x}(\overline{A})^{-1} \, z \, g_{1,m} \, h_{\gamma_x}(\overline{A}'')\right)_{x \in M}\right]_{\mathbf{B}(\overline{A})} \\
&= \left[\left(z_x \, h_{\gamma_x}(\overline{A})^{-1} \, g_{1,m} \, h_{\gamma_x}(\overline{A}'')\right)_{x \in M}\right]_{\mathbf{B}(\overline{A})} \\
&= F'''(\overline{A}'', [g_{1,m}]_{Z(\mathbf{H}_{\overline{A}})}),
\end{aligned}
$$
    because $(z_x)_{x \in M} := (h_{\gamma_x}(\overline{A})^{-1} \, z \, h_{\gamma_x}(\overline{A}))_{x \in M} \in \mathbf{B}(\overline{A})$ for $z \in Z(\mathbf{H}_{\overline{A}})$.
  - F''' is continuous, because $\text{id} \times \pi_{Z(\mathbf{H}_{\overline{A}})}$ is open and surjective and $\pi_{\mathbf{B}(\overline{A})}$ and F'' are continuous.

- For $\overline{A}' \in \overline{S}$ there is an $\overline{A}'_0 \in \overline{S}_0$ and a $\overline{g}' \in \mathbf{B}(\overline{A})$ with $\overline{A}' = \overline{A}'_0 \circ \overline{g}'$. Thus, we have $h_{\gamma_x}(\overline{A}'_0) = h_{\gamma_x}(\overline{A})$ and
$$
\begin{aligned}
F'''(\overline{A}' \circ \overline{g}, [g_m]) &= \left[\left(h_{\gamma_x}(\overline{A})^{-1} \, g_m \, h_{\gamma_x}(\overline{A}'_0 \circ \overline{g}' \circ \overline{g})\right)_{x \in M}\right]_{\mathbf{B}(\overline{A})} \\
&= \left[\left(h_{\gamma_x}(\overline{A})^{-1} \, g_m \, g_m^{-1}(g'_m)^{-1} h_{\gamma_x}(\overline{A}) g'_x g_x\right)_{x \in M}\right]_{\mathbf{B}(\overline{A})} \\
&= \left[\left(h_{\gamma_x}(\overline{A})^{-1} h_{\gamma_x}(\overline{A} \circ g') \, g_x\right)_{x \in M}\right]_{\mathbf{B}(\overline{A})} \\
&= \left[(g_x)_{x \in M}\right]_{\mathbf{B}(\overline{A})} \\
&= [\overline{g}]_{\mathbf{B}(\overline{A})}
\end{aligned}
$$
  where we used $\overline{g}' \in \mathbf{B}(\overline{A})$.

- Now, F is the concatenation of the following continuous maps:
$$F : \quad \overline{S} \circ \overline{\mathcal{G}} \quad \xrightarrow{\text{id} \times F'} \quad (\overline{S} \circ \overline{\mathcal{G}}) \times Z(\mathbf{H}_{\overline{A}}) \backslash \mathbf{G} \quad \xrightarrow{F'''} \quad \mathbf{B}(\overline{A}) \backslash \overline{\mathcal{G}} \quad \xrightarrow{\tau_{\overline{\mathcal{G}}}} \quad \overline{A} \circ \overline{\mathcal{G}},$$
$$\overline{A}' \circ \overline{g} \quad \longmapsto \quad (\overline{A}' \circ \overline{g}, [g_m]_{Z(\mathbf{H}_{\overline{A}})}) \quad \longmapsto \quad [\overline{g}]_{\mathbf{B}(\overline{A})} \quad \longmapsto \quad \overline{A} \circ \overline{g}$$
  where $\tau_{\overline{\mathcal{G}}}$ is the canonical homeomorphism between the orbit $\overline{A} \circ \overline{\mathcal{G}}$ and the

acting group $\overline{\mathcal{G}}$ modulo the stabilizer $\mathbf{B}(\overline{A})$ of $\overline{A}$.

Hence, F is continuous.

10. We have $F^{-1}(\{\overline{A}\}) = \overline{S}$.

- "⊆" Let $\overline{A}' \in F^{-1}(\{\overline{A}\})$, i.e. $F(\overline{A}') = \overline{A}$.
  - By the commutativity of (3) we have $f(\varphi(\overline{A}')) = \varphi(F(\overline{A}')) = \varphi(\overline{A})$, hence $\overline{A}' \in \varphi^{-1}(f^{-1}(\varphi(\overline{A}))) = \varphi^{-1}(S)$.
  - Define $g_x := h_{\overline{A}'}(\gamma_x)^{-1} h_{\overline{A}}(\gamma_x)$ and $\overline{A}'' := \overline{A}' \circ \overline{g}$. Then we have $\varphi(\overline{A}'') = \varphi(\overline{A}') \in S$, i.e. $\overline{A}'' \in \varphi^{-1}(S)$, and $h_{\overline{A}''}(\gamma_x) = h_{\overline{A}}(\gamma_x)$ for all x, i.e. $\overline{A}'' \in \psi^{-1}(\psi(\overline{A}))$. By this, $\overline{A}'' \in \overline{S}_0$.
  - Consequently, $F(\overline{A}'') = \overline{A} = F(\overline{A}')$ and therefore also $\overline{A} \circ \overline{g} = F(\overline{A}') \circ \overline{g} = F(\overline{A}' \circ \overline{g}) = F(\overline{A}'') = \overline{A}$, i.e. $\overline{g} \in \mathbf{B}(\overline{A})$.

    Thus, $\overline{A}' = \overline{A}'' \circ \overline{g}^{-1} \in \overline{S}_0 \circ \mathbf{B}(\overline{A}) = \overline{S}$.
- "⊇" Let $\overline{A}' \in \overline{S}$. Then $F(\overline{A}') = F(\overline{A}' \circ 1) = \overline{A} \circ 1 = \overline{A}$, i.e. $\overline{A}' \in F^{-1}(\{\overline{A}\})$.

qed

6 Openness of the Strata

Proposition 6.1 $\overline{\mathcal{A}}_{\geq t}$ is open for all $t \in \mathcal{T}$.

Corollary 6.2 $\overline{\mathcal{A}}_{=t}$ is open in $\overline{\mathcal{A}}_{\leq t}$ for all $t \in \mathcal{T}$.

Proof Since $\overline{\mathcal{A}}_{=t} = \overline{\mathcal{A}}_{\geq t} \cap \overline{\mathcal{A}}_{\leq t}$, $\overline{\mathcal{A}}_{=t}$ is open w.r.t. to the relative topology on $\overline{\mathcal{A}}_{\leq t}$. **qed**

Corollary 6.3 $\overline{\mathcal{A}}_{\leq t}$ is compact for all $t \in \mathcal{T}$.

Proof $\overline{\mathcal{A}} \setminus \overline{\mathcal{A}}_{\leq t} = \bigcup_{t' \in \mathcal{T}, t' \nleq t} \overline{\mathcal{A}}_{=t'} = \bigcup_{t' \in \mathcal{T}, t' \nleq t} \overline{\mathcal{A}}_{\geq t'}$ is open because $\overline{\mathcal{A}}_{\geq t'}$ is open for all $t' \in \mathcal{T}$.

Thus, $\overline{\mathcal{A}}_{\leq t}$ is closed and therefore compact. **qed**

The proposition on the openness of the strata can be proven in two ways: first as a simple corollary of the slice theorem on $\overline{\mathcal{A}}$, but second directly using the reduction mapping. Thus, altogether the second variant needs less effort.

Proof Proposition 6.1

We have to show that any $\overline{A} \in \overline{\mathcal{A}}_{\geq t}$ has a neighbourhood that again is contained in $\overline{\mathcal{A}}_{\geq t}$. So, let $\overline{A} \in \overline{\mathcal{A}}_{\geq t}$.

- Variant 1

  Due to the slice theorem there is an open neighbourhood U of $\overline{A} \circ \overline{\mathcal{G}}$, and so of $\overline{A}$, too, and an equivariant retraction $F : U \longrightarrow \overline{A} \circ \overline{\mathcal{G}}$. Since every equivariant mapping reduces types, we have $\mathrm{Typ}(\overline{A}') \geq \mathrm{Typ}(\overline{A}) = t$ for all $\overline{A}' \in U$, thus $U \subseteq \overline{\mathcal{A}}_{\geq t}$.

- Variant 2

  Choose again for $\overline{A}$ an $\boldsymbol{\alpha} \subseteq \mathcal{HG}$ with
  $$\mathrm{Typ}(\overline{A}) = [Z(\mathbf{H}_{\overline{A}})] = [Z(h_{\overline{A}}(\boldsymbol{\alpha}))] \equiv [Z(\varphi_{\boldsymbol{\alpha}}(\overline{A}))] = \mathrm{Typ}(\varphi_{\boldsymbol{\alpha}}(\overline{A})).$$
  Due to the slice theorem for general transformation groups there is an open, invariant neighbourhood U' of $\varphi_{\boldsymbol{\alpha}}(\overline{A})$ in $\mathbf{G}^{\#\boldsymbol{\alpha}}$ and an equivariant retraction $f : U' \longrightarrow \varphi_{\boldsymbol{\alpha}}(\overline{A}) \circ \mathbf{G}$. Since $\varphi_{\boldsymbol{\alpha}}(\overline{A})$ and f are type-reducing, we have
  $$\mathrm{Typ}(\overline{A}') \geq \mathrm{Typ}(\varphi_{\boldsymbol{\alpha}}(\overline{A}')) \geq \mathrm{Typ}\big(f(\varphi_{\boldsymbol{\alpha}}(\overline{A}'))\big) = \mathrm{Typ}(\varphi_{\boldsymbol{\alpha}}(\overline{A})) = \mathrm{Typ}(\overline{A})$$
  for all $\overline{A}' \in U := \varphi_{\boldsymbol{\alpha}}^{-1}(U')$, i.e. $U \subseteq \overline{\mathcal{A}}_{\geq t}$. Obviously, U contains $\overline{A}$ and is open as a preimage of an open set. **qed**

7 Denseness of the Strata

The next theorem we want to prove is that the set $\overline{\mathcal{A}}_{=t}$ is not only open, but also dense in $\overline{\mathcal{A}}_{\leq t}$. This assertion does – in contrast to the slice theorem and the openness of the strata – not follow from the general theory of transformation groups. We have to show this directly on the level of $\overline{\mathcal{A}}$.

As we will see in a moment, the next proposition will be very helpful.

Proposition 7.1   Let $\overline{A} \in \overline{\mathcal{A}}$ and Γ_i be finitely many graphs.
Then there is for any $t \geq \mathrm{Typ}(\overline{A})$ an $\overline{A}' \in \overline{\mathcal{A}}$ with $\mathrm{Typ}(\overline{A}') = t$ and $\pi_{\Gamma_i}(\overline{A}) = \pi_{\Gamma_i}(\overline{A}')$ for all i.

Namely, we have

Corollary 7.2 $\overline{\mathcal{A}}_{=t}$ is dense in $\overline{\mathcal{A}}_{\leq t}$ for all $t \in \mathcal{T}$.

Proof   Let $\overline{A} \in \overline{\mathcal{A}}_{\leq t} \subseteq \overline{\mathcal{A}}$. We have to show that any neighbourhood U of $\overline{A}$ contains an $\overline{A}'$ having type t. It is sufficient to prove this assertion for all graphs Γ_i and all $U = \bigcap_i \pi_{\Gamma_i}^{-1}(W_i)$ with open $W_i \subseteq \mathbf{G}^{\# \mathbf{E}(\Gamma_i)}$ and $\pi_{\Gamma_i}(\overline{A}) \in W_i$ for all $i \in I$ with finite I, because any general open U contains such a set.
Now let Γ_i and U be chosen as just described. Due to Proposition 7.1 above there exists an $\overline{A}' \in \overline{\mathcal{A}}$ with $\mathrm{Typ}(\overline{A}') = t \geq \mathrm{Typ}(\overline{A})$ and $\pi_{\Gamma_i}(\overline{A}) = \pi_{\Gamma_i}(\overline{A}')$ for all i, i.e. with $\overline{A}' \in \overline{\mathcal{A}}_{=t}$ and $\overline{A}' \in \pi_{\Gamma_i}^{-1}\left(\pi_{\Gamma_i}(\{\overline{A}\})\right) \subseteq \pi_{\Gamma_i}^{-1}(W_i)$ for all i, thus, $\overline{A}' \in \bigcap_i \pi_{\Gamma_i}^{-1}(W_i) = U$.
                                                                                                qed

Along with the proposition about the openness of the strata we get

Corollary 7.3   For all $t \in \mathcal{T}$ the closure of $\overline{\mathcal{A}}_{=t}$ w.r.t. $\overline{\mathcal{A}}$ is equal to $\overline{\mathcal{A}}_{\leq t}$.

Proof   Denote the closure of F w.r.t. E by $\mathrm{Cl}_E(F)$.
Due to the denseness of $\overline{\mathcal{A}}_{=t}$ in $\overline{\mathcal{A}}_{\leq t}$ we have $\mathrm{Cl}_{\overline{\mathcal{A}}_{\leq t}}(\overline{\mathcal{A}}_{=t}) = \overline{\mathcal{A}}_{\leq t}$. Since the closure is compatible with the relative topology, we have $\overline{\mathcal{A}}_{\leq t} = \mathrm{Cl}_{\overline{\mathcal{A}}_{\leq t}}(\overline{\mathcal{A}}_{=t}) = \overline{\mathcal{A}}_{\leq t} \cap \mathrm{Cl}_{\overline{\mathcal{A}}}(\overline{\mathcal{A}}_{=t})$, i.e. $\overline{\mathcal{A}}_{\leq t} \subseteq \mathrm{Cl}_{\overline{\mathcal{A}}}(\overline{\mathcal{A}}_{=t})$. But, due to Corollary 6.3, $\overline{\mathcal{A}}_{\leq t} \supseteq \overline{\mathcal{A}}_{=t}$ itself is closed in $\overline{\mathcal{A}}$. Hence, $\overline{\mathcal{A}}_{\leq t} \supseteq \mathrm{Cl}_{\overline{\mathcal{A}}}(\overline{\mathcal{A}}_{=t})$.
                                                                                                qed

7.1   How to Prove Proposition 7.1?

Which ideas will the proof of Proposition 7.1 be based on? As in the last two sections we get help from the finiteness lemma for centralizers. Namely, let $\boldsymbol{\alpha} \subseteq \mathcal{HG}$ be chosen such that $\mathrm{Typ}(\overline{A}) = [Z(\mathbf{H}_{\overline{A}})] = [Z(\varphi_{\boldsymbol{\alpha}}(\overline{A}))]$. $t \geq \mathrm{Typ}(\overline{A})$ is finitely generated as well. Thus, we have to construct a connection whose type is determined by $\varphi_{\boldsymbol{\alpha}}(\overline{A})$ and the generators of t. For this we use the induction on the number of generators of t. In conclusion, we have to construct inductively from $\overline{A}$ new connections $\overline{A}_i$, such that $\overline{A}_{i-1}$ coincides with $\overline{A}_i$ at least along the paths that pass $\boldsymbol{\alpha}$ or that lie in the graphs Γ_i. But, at the same time, there has to exist a path e, such that $h_{\overline{A}_i}(e)$ equals the ith generator of t.
Now, it should be obvious that we get help from the construction method for new connections introduced in [10]. Before we do this we recall an important notation used there.

Definition 7.1   Let $\gamma_1, \gamma_2 \in \mathcal{P}$.

We say that γ_1 and γ_2 have the same initial segment (shortly: $\gamma_1 \uparrow\uparrow \gamma_2$) iff there exist $0 < \delta_1, \delta_2 \le 1$ such that $\gamma_1 \mid_{[0,\delta_1]}$ and $\gamma_2 \mid_{[0,\delta_2]}$ coincide up to the parametrization.

We say analogously that the final segment of γ_1 coincides with the initial segment of γ_2 (shortly: $\gamma_1 \downarrow\uparrow \gamma_2$) iff there exist $0 < \delta_1, \delta_2 \le 1$ such that $\gamma_1^{-1} \mid_{[0,\delta_1]}$ and $\gamma_2 \mid_{[0,\delta_2]}$ coincide up to the parametrization.

Iff the corresponding relations are not fulfilled, we write $\gamma_1 \not\uparrow\!\uparrow \gamma_2$ and $\gamma_1 \not\downarrow\!\uparrow \gamma_2$, respectively.

Finally, we recall the decomposition lemma.

Lemma 7.4   Let $x \in M$ be a point. Any $\gamma \in \mathcal{P}$ can be written (up to parametrization) as a product $\prod \gamma_i$ with $\gamma_i \in \mathcal{P}$, such that

- int $\gamma_i \cap \{x\} = \emptyset$ or
- int $\gamma_i = \{x\}$.

7.2   Successive Magnifying of the Types

In order to prove Proposition 7.1 we need the following lemma for magnifying the types. Hereby, we will use explicitly the construction of a new connection $\overline{A}'$ from $\overline{A}$ as given in [10].

Lemma 7.5   Let Γ_i be finitely many graphs, $\overline{A} \in \overline{\mathcal{A}}$ and $\alpha \subseteq \mathcal{HG}$ be a finite set of paths with $Z(\mathbf{H}_{\overline{A}}) = Z(h_{\overline{A}}(\alpha))$. Furthermore, let $g \in \mathbf{G}$ be arbitrary.

Then there is an $\overline{A}' \in \overline{\mathcal{A}}$, such that:

- $h_{\overline{A}'}(\alpha) = h_{\overline{A}}(\alpha)$,
- $\pi_{\Gamma_i}(\overline{A}') = \pi_{\Gamma_i}(\overline{A})$ for all i,
- $h_{\overline{A}'}(e) = g$ for an $e \in \mathcal{HG}$ and
- $Z(\mathbf{H}_{\overline{A}'}) = Z(\{g\} \cup h_{\overline{A}}(\alpha))$.

Proof   1.   Let $m' \in M$ be some point that is neither contained in the images of Γ_i nor in that of α, and join m with m' by some path γ. Now let e' be some closed path in M with base point m' and without self-intersections, such that

$$\text{im } e' \cap \left(\text{int } \gamma \cup \text{im }(\alpha) \cup \bigcup \text{im }(\Gamma_i) \right) = \emptyset. \tag{4}$$

Obviously, there exists such an e' because M is supposed to be at least two-dimensional. Set $e := \gamma\, e'\, \gamma^{-1} \in \mathcal{HG}$ and $g' := h_{\overline{A}}(\gamma)^{-1} g h_{\overline{A}}(\gamma)$.

Finally, define a connection $\overline{A}'$ for $\overline{A}$, e' and g' as follows:

2.   Construction of $\overline{A}'$

- Let $\delta \in \mathcal{P}$ be for the moment a "genuine" path (i.e., not an equivalence class) that does not contain the initial point $e'(0) \equiv m'$ of e' as an inner point. Explicitly we have int $\delta \cap \{e'(0)\} = \emptyset$. Define

$$h_{\overline{A}'}(\delta) := \begin{cases} g'\, h_{\overline{A}}(e')^{-1}\, h_{\overline{A}}(\delta)\, h_{\overline{A}}(e')\, g'^{-1}, & \text{for } \delta \uparrow\uparrow e' \text{ and } \delta \downarrow\uparrow e' \\ g'\, h_{\overline{A}}(e')^{-1}\, h_{\overline{A}}(\delta), & \text{for } \delta \uparrow\uparrow e' \text{ and } \delta \not\downarrow\!\uparrow e' \\ \quad\quad h_{\overline{A}}(\delta)\, h_{\overline{A}}(e')\, g'^{-1}, & \text{for } \delta \not\uparrow\!\uparrow e' \text{ and } \delta \downarrow\uparrow e' \\ \quad\quad h_{\overline{A}}(\delta), & \text{else} \end{cases}.$$

- For every trivial path δ set $h_{\overline{A}'}(\delta) = e_{\mathbf{G}}$.

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- Now, let $\delta \in \mathcal{P}$ be an arbitrary path. Decompose δ into a finite product $\prod \delta_i$ due to Lemma 7.4 such that no δ_i contains the point $e'(0)$ in the interior supposed δ_i is not trivial. Here, set $h_{\overline{A}'}(\delta) := \prod h_{\overline{A}'}(\delta_i)$.

  We know from [10] that $\overline{A}'$ is indeed a connection.

3. The assertion $\pi_{\Gamma_i}(\overline{A}') = \pi_{\Gamma_i}(\overline{A})$ for all i is an immediate consequence of the construction because im $(\Gamma_i) \cap$ int $e' = \emptyset$. As well, we get $h_{\overline{A}'}(\boldsymbol{\alpha}) = h_{\overline{A}}(\boldsymbol{\alpha})$.

4. Moreover, from (4), the fact that e' has no self-intersections and the definition of $\overline{A}'$ we get $h_{\overline{A}'}(\gamma) = h_{\overline{A}}(\gamma)$ and so
$$h_{\overline{A}'}(e) = h_{\overline{A}'}(\gamma)\, h_{\overline{A}'}(e')\, h_{\overline{A}'}(\gamma^{-1}) = h_{\overline{A}}(\gamma)\, g'\, h_{\overline{A}}(\gamma)^{-1} = g.$$

5. We have $Z(\mathbf{H}_{\overline{A}'}) = Z(\{g\} \cup \mathbf{H}_{\overline{A}})$.

   "$\subseteq$" Let $f \in Z(\mathbf{H}_{\overline{A}'})$, i.e. $f\, h_{\overline{A}'}(\alpha) = h_{\overline{A}'}(\alpha)\, f$ for all $\alpha \in \mathcal{HG}$.
   - From $h_{\overline{A}'}(e) = g$ follows $fg = gf$, i.e. $f \in Z(\{g\})$.
   - From im $e' \cap$ im $(\boldsymbol{\alpha}) = \emptyset$ follows $h_{\overline{A}}(\alpha_i) = h_{\overline{A}'}(\alpha_i)$, i.e. $f \in Z(h_{\overline{A}}(\alpha_i))$ for all i.

   Thus, $f \in Z(\{g\}) \cap Z(h_{\overline{A}}(\boldsymbol{\alpha})) = Z(\{g\} \cup \mathbf{H}_{\overline{A}})$.

   "$\supseteq$" Let $f \in Z(\{g\} \cup \mathbf{H}_{\overline{A}})$.
   - Let α' be a path from m' to m', such that int $\alpha' \cap \{m'\} = \emptyset$ or int $\alpha' = \{m'\}$. Set $\alpha := \gamma\, \alpha'\, \gamma^{-1}$. Then by construction we have
$$\begin{aligned} h_{\overline{A}'}(\alpha) &= h_{\overline{A}'}(\gamma)\, h_{\overline{A}'}(\alpha')\, h_{\overline{A}'}(\gamma)^{-1} \\ &= h_{\overline{A}}(\gamma)\, h_{\overline{A}'}(\alpha')\, h_{\overline{A}}(\gamma)^{-1}. \end{aligned}$$

   There are four cases:
   - $\alpha' \not\mathrel{\rightrightarrows} e'$ and $\alpha' \not\mathrel{\rightrightarrows} e'$:
$$\begin{aligned} h_{\overline{A}'}(\alpha) &= h_{\overline{A}}(\gamma)\, h_{\overline{A}}(\alpha')\, h_{\overline{A}}(\gamma)^{-1} = h_{\overline{A}}(\gamma\, \alpha'\, \gamma^{-1}) \\ &= h_{\overline{A}}(\alpha). \end{aligned}$$

   - $\alpha' \mathrel{\uparrow\uparrow} e'$ and $\alpha' \not\mathrel{\rightrightarrows} e'$:
$$\begin{aligned} h_{\overline{A}'}(\alpha) &= h_{\overline{A}}(\gamma)\, g'\, h_{\overline{A}}(e')^{-1}\, h_{\overline{A}}(\alpha')\, h_{\overline{A}}(\gamma)^{-1} \\ &= g\, h_{\overline{A}}(\gamma)\, h_{\overline{A}}(e')^{-1}\, h_{\overline{A}}(\alpha')\, h_{\overline{A}}(\gamma)^{-1} \\ &= g\, h_{\overline{A}}(\gamma e'^{-1} \alpha' \gamma^{-1}). \end{aligned}$$

   - $\alpha' \not\mathrel{\rightrightarrows} e'$ and $\alpha' \mathrel{\downarrow\uparrow} e'$:
$$\begin{aligned} h_{\overline{A}'}(\alpha) &= h_{\overline{A}}(\gamma)\, h_{\overline{A}}(\alpha')\, h_{\overline{A}}(e')\, (g')^{-1} h_{\overline{A}}(\gamma)^{-1} \\ &= h_{\overline{A}}(\gamma)\, h_{\overline{A}}(\alpha')\, h_{\overline{A}}(e') h_{\overline{A}}(\gamma)^{-1}\, g^{-1} \\ &= h_{\overline{A}}(\gamma \alpha' e' \gamma^{-1})\, g^{-1}. \end{aligned}$$

   - $\alpha' \mathrel{\uparrow\uparrow} e'$ and $\alpha' \mathrel{\downarrow\uparrow} e'$:
$$\begin{aligned} h_{\overline{A}'}(\alpha) &= h_{\overline{A}}(\gamma)\, g'\, h_{\overline{A}}(e')^{-1}\, h_{\overline{A}}(\alpha')\, h_{\overline{A}}(e')\, (g')^{-1}\, h_{\overline{A}}(\gamma)^{-1} \\ &= g\, h_{\overline{A}}(\gamma)\, h_{\overline{A}}(e')^{-1}\, h_{\overline{A}}(\alpha')\, h_{\overline{A}}(e')\, h_{\overline{A}}(\gamma)^{-1}\, g^{-1} \\ &= g\, h_{\overline{A}}(\gamma e'^{-1} \alpha' e' \gamma^{-1})\, g^{-1}. \end{aligned}$$

   Thus, in each case we get $f \in Z(\{h_{\overline{A}'}(\alpha)\})$.
   - Now, let $\alpha \in \mathcal{HG}$ be arbitrary and $\alpha' := \gamma^{-1} \alpha \gamma$.
   By the Decomposition Lemma 7.4 there is a decomposition $\alpha' = \prod \alpha'_i$ with int $\alpha'_i \cap \{m'\} = \emptyset$ or int $\alpha'_i = \{m'\}$ for all i. Thus, $\alpha = \gamma\left(\prod \alpha'_i\right)\gamma^{-1} = \prod\left(\gamma \alpha'_i \gamma^{-1}\right)$. Using the result just proven we get $f \in Z\left(\left\{h_{\overline{A}'}\left(\prod\left(\gamma \alpha'_i \gamma^{-1}\right)\right)\right\}\right) = Z(\{h_{\overline{A}'}(\alpha)\})$.

Thus, $f \in Z(\mathbf{H}_{\overline{A}'})$.

Due to the definition of $\boldsymbol{\alpha}$ we have $Z(\mathbf{H}_{\overline{A}'}) = Z(\{g\} \cup h_{\overline{A}}(\boldsymbol{\alpha}))$.   **qed**

7.3 Construction of Arbitrary Types

Finally, we can now prove the desired proposition.

Proof Proposition 7.1

- Let $t \in \mathcal{T}$ and $t \geq \text{Typ}(\overline{A})$. Then there exist a Howe subgroup $V' \subseteq \mathbf{G}$ with $t = [V']$ and a $g \in \mathbf{G}$, such that $Z(\mathbf{H}_{\overline{A}}) \supseteq g^{-1}V'g =: V$. Since V is a Howe subgroup, we have $Z(Z(V)) = V$ and so by Lemma 4.1 there exist certain $u_0, \dots, u_k \in Z(V) \subseteq \mathbf{G}$, such that $V = Z(Z(V)) = Z(\{u_0, \dots, u_k\})$.

- Now let $Z(\mathbf{H}_{\overline{A}}) = Z(h_{\overline{A}}(\boldsymbol{\alpha}))$ with an appropriate $\boldsymbol{\alpha} \subseteq \mathcal{HG}$ as in Corollary 4.2. Because of $V \subseteq Z(\mathbf{H}_{\overline{A}})$ we have $V = V \cap Z(\mathbf{H}_{\overline{A}}) = Z(\{u_0, \dots, u_k\}) \cap Z(h_{\overline{A}}(\boldsymbol{\alpha})) = Z(\{u_0, \dots, u_k\} \cup h_{\overline{A}}(\boldsymbol{\alpha}))$.

- We now use inductively Lemma 7.5. Let $\overline{A}_0 := \overline{A}$ and $\boldsymbol{\alpha}_0 := \boldsymbol{\alpha}$. Construct for all $j = 0, \dots, k$ a connection $\overline{A}_{j+1}$ and an $e_j \in \mathcal{HG}$ from $\overline{A}_j$ and $\boldsymbol{\alpha}_j$ by that lemma, such that $\pi_{\Gamma_i}(\overline{A}_{j+1}) = \pi_{\Gamma_i}(\overline{A}_j)$ for all i, $h_{\overline{A}_{j+1}}(\boldsymbol{\alpha}_j) = h_{\overline{A}_j}(\boldsymbol{\alpha}_j)$, $h_{\overline{A}_{j+1}}(e_j) = u_j$ and $Z(\mathbf{H}_{\overline{A}_{j+1}}) = Z(\{u_j\} \cup h_{\overline{A}_j}(\boldsymbol{\alpha}_j))$.

  Setting $\boldsymbol{\alpha}_{j+1} := \boldsymbol{\alpha}_j \cup \{e_j\}$ we get $Z(\mathbf{H}_{\overline{A}_{j+1}}) = Z(\{u_j\} \cup h_{\overline{A}_j}(\boldsymbol{\alpha}_j)) = Z(h_{\overline{A}_{j+1}}(\boldsymbol{\alpha}_{j+1}))$.

  Finally, we define $\overline{A}' := \overline{A}_{k+1}$.

  Now, we get $\pi_{\Gamma_i}(\overline{A}') = \pi_{\Gamma_i}(\overline{A})$ for all i, $h_{\overline{A}'}(\boldsymbol{\alpha}) = h_{\overline{A}}(\boldsymbol{\alpha})$ and $h_{\overline{A}'}(e_j) = u_j$. Thus,

$$
\begin{aligned}
Z(\mathbf{H}_{\overline{A}'}) &= Z(h_{\overline{A}'}(\boldsymbol{\alpha}_{k+1})) \\
&= Z(h_{\overline{A}'}(\{e_0, \dots, e_k\} \cup h_{\overline{A}'}(\boldsymbol{\alpha}))) \\
&= Z(\{u_0, \dots, u_k\} \cup h_{\overline{A}}(\boldsymbol{\alpha})) \\
&= V,
\end{aligned}
$$

i.e., $\text{Typ}(\overline{A}') = [V] = t$.   **qed**

The proposition just proven has a further immediate consequence.

Corollary 7.6 $\overline{\mathcal{A}}_{=t}$ is non-empty for all $t \in \mathcal{T}$.

Proof Let $\overline{A}$ be the trivial connection, i.e. $h_{\overline{A}}(\alpha) = e_{\mathbf{G}}$ for all $\alpha \in \mathcal{P}$. The type of $\overline{A}$ is $[\mathbf{G}]$, thus minimal, i.e. we have $t \geq \text{Typ}(\overline{A})$ for all $t \in \mathcal{T}$. By means of Proposition 7.1 there is an $\overline{A}' \in \overline{\mathcal{A}}$ with $\text{Typ}(\overline{A}') = t$.   **qed**

This corollary solves the problem which gauge orbit types exist for generalized connections.

Theorem 7.7 The set of all gauge orbit types on $\overline{\mathcal{A}}$ is the set of all conjugacy classes of Howe subgroups of $\mathbf{G}$.

Furthermore we have

Corollary 7.8 Let Γ be some graph. Then $\pi_\Gamma(\overline{\mathcal{A}}_{=t_{\max}}) = \pi_\Gamma(\overline{\mathcal{A}})$. In other words: π_Γ is surjective even on the generic connections.

Proof π_Γ is surjective on $\overline{\mathcal{A}}$ as proven in [10]. By Proposition 7.1 there is now an $\overline{A}'$ with $\text{Typ}(\overline{A}') = t_{\max}$ and $\pi_\Gamma(\overline{A}') = \pi_\Gamma(\overline{A})$.   **qed**

8 Stratification of $\overline{\mathcal{A}}$

First we recall the general definition of a stratification [12].

Definition 8.1 A countable family $\mathcal{S}$ of non-empty subsets of a topological space X is called **stratification** of X iff $\mathcal{S}$ is a covering for X and for all $U, V \in \mathcal{S}$ we have
- $U \cap V \neq \emptyset \Longrightarrow U = V$,
- $\overline{U} \cap V \neq \emptyset \Longrightarrow \overline{U} \supseteq V$ and
- $\overline{U} \cap V \neq \emptyset \Longrightarrow \overline{V} \cap (U \cup V) = V$.

The elements of such a stratification $\mathcal{S}$ are called **strata**.

A stratification is called **topologically regular** iff for all $U, V \in \mathcal{S}$
$$U \neq V \text{ and } \overline{U} \cap V \neq \emptyset \Longrightarrow \overline{V} \cap U = \emptyset.$$

Theorem 8.1 $\mathcal{S} := \{\overline{\mathcal{A}}_{=t} \mid t \in \mathcal{T}\}$ is a topologically regular stratification of $\overline{\mathcal{A}}$.

Analogously, $\{(\overline{\mathcal{A}/\mathcal{G}})_{=t} \mid t \in \mathcal{T}\}$ is a topologically regular stratification of $\overline{\mathcal{A}/\mathcal{G}}$.

Proof
- Obviously, $\mathcal{S}$ is a covering of $\overline{\mathcal{A}}$.
- For a compact Lie group the set of all types, i.e. all conjugacy classes of Howe subgroups of $\mathbf{G}$, is at most countable (cf. [12]).
- Moreover, from $\overline{\mathcal{A}}_{=t_1} \cap \overline{\mathcal{A}}_{=t_2} \neq \emptyset$ immediately follows $\overline{\mathcal{A}}_{=t_1} = \overline{\mathcal{A}}_{=t_2}$.
- Due to Corollary 7.3 we have[5] $\mathrm{Cl}(\overline{\mathcal{A}}_{=t_1}) = \overline{\mathcal{A}}_{\leq t_1}$, i.e. from $\mathrm{Cl}(\overline{\mathcal{A}}_{=t_1}) \cap \overline{\mathcal{A}}_{=t_2} \neq \emptyset$ follows $t_2 \leq t_1$ and thus $\mathrm{Cl}(\overline{\mathcal{A}}_{=t_1}) \supseteq \overline{\mathcal{A}}_{=t_2}$.
- Analogously we get $\mathrm{Cl}(\overline{\mathcal{A}}_{=t_2}) \cap (\overline{\mathcal{A}}_{=t_1} \cup \overline{\mathcal{A}}_{=t_2}) = \overline{\mathcal{A}}_{\leq t_2} \cap (\overline{\mathcal{A}}_{=t_1} \cup \overline{\mathcal{A}}_{=t_2}) = \overline{\mathcal{A}}_{=t_2}$.
- As well, from $\mathrm{Cl}(\overline{\mathcal{A}}_{=t_1}) \cap \overline{\mathcal{A}}_{=t_2} \neq \emptyset$ and $\overline{\mathcal{A}}_{=t_1} \neq \overline{\mathcal{A}}_{=t_2}$ follows $t_1 > t_2$, i.e. $\mathrm{Cl}(\overline{\mathcal{A}}_{=t_2}) \cap \overline{\mathcal{A}}_{=t_1} = \emptyset$.

Consequently, $\mathcal{S}$ is a topologically regular stratification of $\overline{\mathcal{A}}$. **qed**

For a regular stratification it would be required that each stratum carries the structure of a manifold that is compatible with the topology of the total space. In contrast to the case of the classical gauge orbit space [12], this is not fulfilled for generalized connections.

9 Non-complete Connections

We shall round off that paper with the proof that the set of the so-called non-complete connections is contained in a set of measure zero. This section actually stands a little bit separated from the context because it is the only section that is not only algebraic and topological, but also measure theoretical.

Definition 9.1 Let $\overline{A} \in \overline{\mathcal{A}}$ be a connection.
1. $\overline{A}$ is called **complete** $\Longleftrightarrow \mathbf{H}_{\overline{A}} = \mathbf{G}$.
2. $\overline{A}$ is called **almost complete** $\Longleftrightarrow \overline{\mathbf{H}_{\overline{A}}} = \mathbf{G}$.
3. $\overline{A}$ is called **non-complete** $\Longleftrightarrow \overline{\mathbf{H}_{\overline{A}}} \neq \mathbf{G}$.

Obviously, we have

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[5] $\mathrm{Cl}(U)$ denotes again the closure of U, here w.r.t. $\overline{\mathcal{A}}$.

Lemma 9.1 If $\overline{A} \in \overline{\mathcal{A}}$ is complete (almost complete, non-complete), so $\overline{A} \circ \overline{g}$ is complete (almost complete, non-complete) for all $\overline{g} \in \overline{\mathcal{G}}$.

Thus, the total information about the completeness of a connection is already contained in its gauge orbit. Now, to the main assertion of this section.

Proposition 9.2 Let $N := \{\overline{A} \in \overline{\mathcal{A}} \mid \overline{A} \text{ non-complete}\}$. Then N is contained in a set of μ_0-measure zero whereas μ_0 is the induced Haar measure on $\overline{\mathcal{A}}$. [2, 6, 10]

Since N is gauge invariant, we have

Corollary 9.3 Let $[N] := \{[\overline{A}] \in \overline{\mathcal{A}}/\overline{\mathcal{G}} \mid \overline{A} \text{ non-complete}\}$. Then $[N]$ is contained in a set of μ_0-measure zero.

For the proof of the proposition we still need the following

Lemma 9.4 Let $U \subseteq \mathbf{G}$ be measurable with $\mu_{\text{Haar}}(U) > 0$ and $N_U := \{\overline{A} \in \overline{\mathcal{A}} \mid \mathbf{H}_{\overline{A}} \subseteq \mathbf{G} \setminus U\}$.
Then N_U is contained in a set of μ_0-measure zero.

Proof
- Let $k \in \mathbb{N}$ and Γ_k be some connected graph with one vertex m and k edges $\alpha_1, \ldots, \alpha_k \in \mathcal{HG}$.[6] Furthermore, let $\quad \pi_k : \begin{array}{ccc} \overline{\mathcal{A}} & \longrightarrow & \mathbf{G}^k. \\ \overline{A} & \longmapsto & (h_{\overline{A}}(\alpha_1), \ldots, h_{\overline{A}}(\alpha_k)) \end{array}$

- Denote now by $N_{k,U} := \pi_k^{-1}((\mathbf{G} \setminus U)^k)$ the set of all connections whose holonomies on Γ_k are not contained in U. Per constructionem we have $N_U \subseteq N_{k,U}$.
- Since the characteristic function $\chi_{N_{k,U}}$ for $N_{k,U}$ is obviously a cylindrical function, we get

$$\mu_0(N_{k,U}) \quad = \quad \int_{\overline{\mathcal{A}}} \chi_{N_{k,U}} \, d\mu_0 \quad = \quad \int_{\overline{\mathcal{A}}} \pi_k^*(\chi_{(\mathbf{G} \setminus U)^k}) \, d\mu_0$$

$$= \quad \int_{\mathbf{G}^k} \chi_{(\mathbf{G} \setminus U)^k} \, d\mu_{\text{Haar}}^k \quad = \quad [\mu_{\text{Haar}}(\mathbf{G} \setminus U)]^k.$$

- From $N_U \subseteq N_{k,U}$ for all k follows $N_U \subseteq \bigcap_k N_{k,U}$. But, $\mu_0(\bigcap_k N_{k,U}) \leq \mu_0(N_{k,U}) = \mu_{\text{Haar}}(\mathbf{G} \setminus U)^k$ for all k, i.e. $\mu_0(\bigcap_k N_{k,U}) = 0$, because $\mu_{\text{Haar}}(\mathbf{G} \setminus U) = 1 - \mu_{\text{Haar}}(U) < 1$. **qed**

Proof Proposition 9.2
- Let $(\epsilon_k)_{k \in \mathbb{N}}$ be some null sequence. Furthermore, let $\{U_{k,i}\}_i$ be for each k a finite covering of $\mathbf{G}$ by open sets $U_{k,i}$ whose respective diameters are smaller than ϵ_k. Now define $N' := \bigcup_k \left(\bigcup_i N_{U_{k,i}} \right)$.
- Since $U_{k,i}$ is open and $\mathbf{G}$ is compact, $U_{k,i}$ is measureable with $\mu_{\text{Haar}}(U_{k,i}) > 0$. Due to Lemma 9.4 we have $N_{U_{k,i}} \subseteq N^*_{U_{k,i}}$ with $\mu_0(N^*_{U_{k,i}}) = 0$ for all k, i; thus $N' \subseteq N^* := \bigcup_k \left(\bigcup_i N^*_{U_{k,i}} \right)$ with $\mu_0(N^*) = 0$.
- We are left to show $N \subseteq N'$.
  Let $\overline{A} \in N$. Then there is an open $U \subseteq \mathbf{G}$ with $\mathbf{H}_{\overline{A}} \subseteq \mathbf{G} \setminus U$.
  Now let $m \in U$. Then $\epsilon := \text{dist}(m, \partial U) > 0$. Choose k such that $\epsilon_k < \epsilon$. Then choose a $U_{k,i}$ with $m \in U_{k,i}$. We get for all $x \in U_{k,i}$: $d(x, m) \leq \text{diam } U_{k,i} < \epsilon_k < \epsilon$, i.e. $x \in U$. Consequently, $U_{k,i} \subseteq U$ and thus $\mathbf{H}_{\overline{A}} \subseteq \mathbf{G} \setminus U_{k,i}$, i.e. $\overline{A} \in N'$. **qed**

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[6]Such a graph does indeed exist for $\dim M \geq 2$. For instance, take k circles K_i with centers in $(\frac{1}{i}, 0, \ldots)$ and radii $\frac{1}{i}$. By means of an appropriate chart mapping around m these circles define a graph with the desired properties.

Corollary 9.5 The set of all generic connections (i.e. connections of maximal type) has μ_0-measure 1.

Proof Every almost complete connection $\overline{A}$ has type $[Z(\mathbf{H}_{\overline{A}})] = [Z(\mathbf{G})] = t_{\max}$. (Observe that the centralizer of a set $U \subseteq \mathbf{G}$ equals that of the closure $\overline{U}$.) Since $\overline{\mathcal{A}}_{=t_{\max}}$ is open due to Proposition 6.1, thus measurable, Proposition 9.2 yields the assertion.

qed

The last assertion is very important: It justifies the definition of the natural induced Haar measure on $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ (cf. [2, 10]). Actually, there were (at least) two different possibilities for this. Namely, let X be some general topological space equipped with a measure μ and let G be some topological group acting on X. The problem now is to find a natural measure μ_G on the orbit space X/G. On the one hand, one could simply define $\mu_G(U) := \mu(\pi^{-1}(U))$ for all measurable $U \subseteq X/G$. ($\pi : X \longrightarrow X/G$ is the canonical projection.) But, on the other hand, one also could stratify the orbit space. For instance, in the easiest case we could have $X = X/G \times G$. In general, one gets (roughly speaking) $X = \bigcup\bigl(V/G \times_{G_V} \backslash G\bigr)$ whereas $\bigcup V$ is an appropriate disjoint decomposition of X and G_V characterizes the type of the orbits on V. Now one naively defines $\mu_G(U) := \sum_V \frac{\mu(\pi^{-1}(U) \cap V)}{\mu_{G,V}(G/G_V)} := \sum_V \mu\bigl(\pi^{-1}(U) \cap V\bigr)\mu_V(G_V)$, where μ_V measures the "size" of the stabilizer G_V in G. This second variant is nothing but the transformation of the measures using the Faddeev-Popov determinant (i.e. the Jacobi determinant) $\frac{d\mu}{d\mu_G}$. In contrast to the first method, here the orbit space and not the total space is regarded to be primary. For a uniform distribution of the measure over all points of the total space the image measure on the orbit space needs no longer be uniformly distributed; the orbits are weighted by size. But, for the second method the uniformity is maintained. In other words, the gauge freedom does not play any rôle when the Faddeev-Popov method is used.

Nevertheless, we see in our concrete case of $\pi_{\overline{\mathcal{A}}/\overline{\mathcal{G}}} : \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}/\overline{\mathcal{G}}$ that both methods are equivalent because the Faddeev-Popov determinant is equal to 1 (at least outside a set of μ_0-measure zero). This follows immediately from the slice theorem and the corollary above that the generic connections have total measure 1.

10   Summary and Discussion

In the present paper and its predecessor [9] we gained a lot of information about the structure of the generalized gauge orbit space within the Ashtekar framework. The most important tool was the theory of compact transformation groups on topological spaces. This enabled us to investigate the action of the group of generalized gauge transforms on the space of generalized connections. Our considerations were guided by the results of Kondracki and Rogulski [12] about the structure of the classical gauge orbit space for Sobolev connections. The methods used there are however fundamentally different from ours. Within the Ashtekar approach most of the proofs are purely algebraic or topological; in the classical case the methods are especially based on the theory of fiber bundles, i.e. analysis and differential geometry.

In a preceding paper [9] we proved that the $\overline{\mathcal{G}}$-stabilizer $\mathbf{B}(\overline{A})$ of a connection $\overline{A}$ is isomorphic to the $\mathbf{G}$-centralizer $Z(\mathbf{H}_{\overline{A}})$ of the holonomy group of $\overline{A}$. Furthermore, two connections have conjugate $\overline{\mathcal{G}}$-stabilizers if and only if their holonomy centralizers are conjugate. Thus, the type of a generalized connection can be defined equivalently both by the $\overline{\mathcal{G}}$-conjugacy class of

$\mathbf{B}(\overline{A})$ (as known from the general theory of transformation groups) and by the $\mathbf{G}$-conjugacy class of $Z(\mathbf{H}_{\overline{A}})$. This is a significant difference to the classical case.

The reduction of our problem from structures in $\overline{\mathcal{G}}$ to those in $\mathbf{G}$ was the crucial idea in the present paper. Since stabilizers in compact groups are even generated by a finite number of elements, we could model the gauge orbit type $[Z(\mathbf{H}_{\overline{A}})]$ on a finite-dimensional space. Using an appropriate mapping we lifted the corresponding slice theorem to a slice theorem on $\overline{A}$. This is the main result of our paper. Collecting connections of one and the same type we got the so-called strata whose openness was an immediate consequence of the slice theorem. In the next step we showed that the natural ordering on the set of the types encodes the topological properties of the strata. More precisely, we proved that the closure of a stratum contains (besides the stratum itself) exactly the union of all strata having a smaller type. This implied that this decomposition of $\overline{A}$ is a topologically regular stratification.

All these results hold in the classical case as well. This is very remarkable because our proofs used partially completely different ideas. However, two results of this paper go beyond the classical theorems. First, we were able to determine the full set of all gauge orbit types occurring in $\overline{A}$. This set is known for Sobolev connections – to the best of our knowlegde – only for certain bundles. Recently, Rudolph, Schmidt and Volobuev solved this problem completely for $SU(n)$-bundels P over two-, three- and four-dimensional manifolds [18]. The main problem in the Sobolev case is the non-triviality of the bundle P. This can exclude orbit types that occur in the trivial bundle $M \times SU(n)$. But, this problem is irrelevant for the Ashtekar framework: *Every* regular connection in *every* $\mathbf{G}$-bundle over M is contained in $\overline{A}$ [2]. This means, in a certain sense, we only have to deal with trivial bundles. Second, in the Ashtekar framework there is a well-defined natural measure on $\overline{A}$. Using this we could show that the generic stratum has the total measure one; this is not true in the classical case. The proposition above implies now that the Faddeev-Popov determinant for the transformation from $\overline{A}$ to $\overline{A}/\overline{\mathcal{G}}$ is equal to 1. This, on the other hand, justifies the definition of the induced Haar measure on $\overline{A}/\overline{\mathcal{G}}$ by projecting the corresponding measure for $\overline{A}$ which has been discussed in detail in section 9.

Hence, we were able to "transfer" the classical theory of strata in a certain sense (almost) completely to the Ashtekar program. We emphasize that all assertions are valid for *each* compact structure group – *both* in the analytical *and* in the C^r-smooth case.

What could be next steps in this area? An important – and in this paper completely ignored – item is the physical interpretation of the gained knowledge. So we will conclude our paper with a few ideas that could link mathematics and physics:

- *Topology*

  What is the topological structure of the strata? Are they connected or is $\overline{A}$ connected itself (at least for connected $\mathbf{G}$)? Is $\overline{A}_{=t}$ globally trivial over $(\overline{A}/\overline{\mathcal{G}})_{=t}$, at least for the generic stratum with $t = t_{\max}$? What sections do exist in these bundles, i.e. what gauge fixings do exist in $\overline{A}$?

  These problems are closely related to the so-called Gribov problem, the non-existence of global gauge fixings for classical connections in principal fiber bundles with compact, non-commutative structure group (see, e.g., [19]). From this lots of difficulties result for the quantization of such a Yang-Mills theory that are not circumvented up to now.

- *Algebraic topology*

  Is there a meaningful, i.e. especially non-trivial cohomology theory on $\overline{\mathcal{A}}$?[7] Is it possible to construct this way characteristic classes or even topological invariants?

- *Measure theory*

  How are arbitrary measures distributed over single strata? In other words: What properties do measures have that are defined by the choice of a measure on each single stratum? This is extremely interesting, in particular, from the physical point of view because the choice of a μ_0-absolutely continuous measure μ on $\overline{\mathcal{A}}$ corresponds to the choice of an action functional S on $\overline{\mathcal{A}}$ by $\int_{\overline{\mathcal{A}}} f \, d\mu = \int_{\overline{\mathcal{A}}} f \, e^{-S} \, d\mu_0$. According to Lebesgue's decomposition theorem all measures whose support is not fully contained in the generic stratum have singular parts.

Finally, we have to stress that the present paper only investigates the case of pure gauge theories. Of course, this is physically not satisfying. Therefore the next goal should be the inclusion of matter fields. A first step has already been done by Thiemann [20] whereas the aspects considered in the present paper did not play any rôle in Thiemann's paper.

Acknowledgements

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