On finite order variational sequences

Raffaele Vitolo[1]
Department of Mathematics "E. De Giorgi", University of Lecce
Via per Arnesano, 73100 Lecce, Italy
E–mail: vitolo@ilenic.unile.it
Web: www.dm.unile.it/ mongelli/docenti/

[1]This paper has been partially supported by INdAM 'F. Severi'through a senior research fellowship, GNFM of CNR, MURST, Universities of Florence and Lecce.

Abstract

We discuss intrinsic aspects of Krupka's approach to finite–order variational sequences. We give intrinsic isomorphisms of the quotient subsheaves of the short finite–order variational sequence with sheaves of forms on jet spaces of suitable order, obtaining a new finite–order (short exact) variational sequence which is made by sheaves of polynomial differential operators. Moreover, we present an intrinsic formulation for the Helmholtz condition of local variationality using a technique introduced by Kolář that we have adapted to our context. Finally, we provide the minimal order solution to the inverse problem of the calculus of variations and a solution of the problem of the variationally trivial Lagrangian.

Key words: Fibred manifold, jet space, variational sequence, Euler–Lagrange morphism, Helmholtz morphism.
1991 MSC: 58A12, 58A20, 58E30, 58G05.

Contents

Introduction . . . 1

1 Jet spaces **5**
- 1.1 Jet spaces . . . 5
- 1.2 Contact maps . . . 6
- 1.3 Distinguished sheaves of forms . . . 7
- 1.4 Main splitting . . . 9
- 1.5 Horizontal and vertical differential . . . 13

2 Variational sequence **14**
- 2.1 Contact subsequence . . . 14
- 2.2 Variational bicomplex . . . 16

3 Representation of the variational sequence **20**
- 3.1 Lagrangian . . . 21
- 3.2 Euler–Lagrange morphism . . . 21
- 3.3 Helmholtz morphism . . . 28
- 3.4 Inverse problems . . . 32

Appendix: calculus of variations . . . 34

Introduction

It is known that there exist several geometric formulations of the variational calculus. They are inspired by a geometrical version of the Hamilton's principle of least action, stated on a fibred manifold. See, for example, [Gar74, GoSt73, Kru73, Tul75], and for further developments [Cos94, Cra81, Fer83, FeFr82, GaMu82, Kol83, Kru83, MaMo83b, Sau89]. In these papers the leading idea is that one can introduce the variational calculus in a purely differential context. See the Appendix for an introduction to this formalism.

Variational sequences go a step forward according to this guideline [AnDu80, Kup80, OlSh78, Tak79, Tul77, Tul80, Vin77, Vin78]. The basic idea is to interpret the passages from a Lagrangian to its Euler–Lagrange morphism and from an Euler–Lagrange morphism to its conditions of local variationality (Helmholtz' conditions) as morphisms of an exact sequence, namely the variational sequence. This is the framework where a lot of problems and ambiguities of geometrical formulations of Lagrangian field theories and mechanics can be solved. See [Tra96] for a discussion of these problems.

But in [Kup80, OlSh78, Tak79, Tul77, Tul80, Vin77, Vin78] the variational sequence is built over the space of infinite jets of a fibred manifold. This procedure is suggested by the relatively simple structure of such spaces. Only in [AnDu80] there is a partial construction on finite order jets.

This paper deals with Krupka's setting of variational sequence on finite–order jet spaces [Kru90] (for further developments, see [Kru93, Kru95a, Kru95b, KrMu99]). The finite–order variational sequence is produced when one quotients the de Rham sequence on a finite–order jet space by means of an intrinsically defined subsequence. The choice of this subsequence is inspired by the variational calculus; it is made by forms which do not contribute to action–like integrals.

Several papers investigated problems arising from the above construction [Gri99a, Gri99b, Kas99, Mus95, MuKr99, Ste95]. But all of them are not concerned with the intrinsic aspects of the problems that they face.

In this paper (and in [FrPaVi99, FrPaVi99b, Vit95, Vit96a, Vit98a, Vit98b, Vit99a, Vit99b]) our leading idea is to analyse Krupka's variational sequence by means of intrinsic techniques on jet spaces. Namely, we will use the structure form on jet spaces [MaMo83a] and the geometric version of the first variation formula by [Kol83].

In [Vit95, Vit96a], we analysed the particular case of the first–order variational sequence on a fibred manifold whose base is 1–dimensional. This was done in order to reduce technical difficulties. Here, we analyse the most general situation, i.e. the r–th order variational sequence based on a fibred manifold, without any restriction on the dimension of the base. We give isomorphisms of the quotient sheaves of the variational sequence with subsheaves of the sheaves of forms on a jet space of suitable order. This order is always found as the minimal among all possible candidates; this aspect is not present in the infinite jet formalism.

We give a characterisation of the local conditions of local variationality. More precisely, it is known [Bau82, Kru90] that there exists a locally defined geometric object, namely the Helmholtz morphism, whose vanishing is equivalent to the local conditions of local variationality [And86, GiMa90, LaTu77, Kru90, Ton69]. We show that the Helmholtz morphism is intrinsically characterised by means of the Euler–Lagrange morphism. This issue is also present in [Gri99a], with a slightly different proof. In this way, we obtain that the variationality conditions are global and intrinsic. This fact is also due to the intrinsic nature of the variational sequence. Moreover, we obtain an intrinsic geometrical object which plays a role analogous to the role of the momentum of a Lagrangian.

Finally, we obtain a finite–order (short and exact) variational sequence, whose sheaves are constituted by polynomial differential operators. This allows us to give a solution of the problem of the minimal order Lagrangian. Indeed, given a locally variational Euler–Lagrange morphism ϵ of order s, the theory of infinite order variational sequences yields the existence of a (local) Lagrangian of order s inducing ϵ. But the finite order variational sequence provides the minimal order Lagrangian inducing ϵ. The solution of this long–standing problem of the calculus of variations was announced (but not given) by Anderson [And86, And92, AnTh92]. The finite order variational sequence yields a proof of this condition which is of 'structural' nature, rather than of 'computational' nature. We also identify each minimal order variationally trivial Lagrangian by a very simple intrinsic technique. Our result agrees with local results from [Gri99b, KrMu99].

We notice that a short version of this report has already been published in [Vit98a]. The results of this paper has been improved and completed ever since. Indeed, it has been shown [Vit98b, Vit99a] that Krupka's approach to variational sequences can be equivalently reformulated in the context of $\mathcal{C}$–spectral sequences [Vin77, Vin78, Vin84], both in the finite and infinite order case. Also, $\mathcal{C}$–spectral sequences allow to extend the finite order formalism to jets of submanifolds and differential equations, and Green–Vinogradov formula [Vin84] allows us to represent each quotient space of the variational sequence in an intrinsic way [Vit99b]. Finally, symmetries has been fitted into Krupka's framework [FrPaVi99, FrPaVi99b], recovering old results and stating some new results.

We hope that our work could serve as a tool to both mathematical and theoretical physicists for a deeper understanding of Lagrangian formalism.

Preliminaries

In this paper, manifolds and maps between manifolds are C^∞. All morphisms of fibred manifolds (and hence bundles) will be morphisms over the identity of the base manifold, unless otherwise specified.

Let V be a vector space such that $\dim V = n$. Suppose that $V = W_1 \oplus W_2$, with $p_1 : V \to W_1$ and $p_2 : V \to W_2$ the related projections. Then, we have the splitting

(1) $$\overset{m}{\wedge} V = \bigoplus_{k+h=m} \overset{k}{\wedge} W_1 \wedge \overset{h}{\wedge} W_2 \ ,$$

where $\overset{k}{\wedge} W_1 \wedge \overset{h}{\wedge} W_2$ is the subspace of $\overset{m}{\wedge} V$ generated by the wedge products of elements of $\overset{k}{\wedge} W_1$ and $\overset{h}{\wedge} W_2$.

There exists a natural inclusion $\overset{k}{\odot} L(V,V) \subset L(\overset{k}{\wedge} V, \overset{k}{\wedge} V)$. Then, the projections $p_{k,h}$ related to the above splitting turn out to be the maps

$$p_{k,h} = \binom{k}{p} \overset{k}{\odot} p_1 \odot \overset{h}{\odot} p_2 : \overset{m}{\wedge} V \to \overset{k}{\wedge} W_1 \wedge \overset{h}{\wedge} W_2 \ .$$

Let $V' \subset V$ be a vector subspace, and set $W'_1 := p_1(V')$, $W'_2 := p_2(V')$. Then we have

(2) $$V' \subset W'_1 \oplus W'_2 \, ,$$

but the inclusion, in general, is not an equality.

As for sheaves, we will use the definitions and the main results given in [Wel80]. In particular, we will be concerned only with sheaves of $\mathbb{R}$–vector spaces. Thus, by 'sheaf morphism' we will mean morphism of sheaves of $\mathbb{R}$–vector spaces.

Let $\mathcal{P}$ be a presheaf over a topological space X. We will denote by $\overline{\mathcal{P}}$ the sheaf generated by $\mathcal{P}$ in the sense of [Wel80]. This means that $\overline{\mathcal{P}}$ is a completion of $\mathcal{P}$ with respect to the gluing axiom. We will denote by $\mathcal{P}_U$ the set of sections of $\mathcal{P}$ defined on the open subset $U \subset X$. The sum between two local sections $\alpha \in \mathcal{P}$ and $\beta \in \mathcal{P}$ will be defined on the intersection of their domain of definition. If $\mathcal{A}$, $\mathcal{B}$ are two subpresheaves of a presheaf $\mathcal{P}$, then the wedge product $\mathcal{A} \wedge \mathcal{B}$ is defined to be the subpresheaf of sections of $\overset{2}{\wedge}\mathcal{P}$ generated by wedge products of sections of $\mathcal{A}$ and $\mathcal{B}$.

Let $\mathcal{S}$ be a sheaf. We recall that $\mathcal{S}$ is said to be *soft* if each section defined on a closed subset $C \subset X$ can be extended to a section defined on any open subset U such that $C \subset U$. Moreover, $\mathcal{S}$ is said to be *fine* if it admits a partition of unity. A fine sheaf is also a soft sheaf. We recall also that a sequence of sheaves over X is said to be exact if it is locally exact (see [Wel80] for a more precise definition). Finally, we recall that the sheaf of sections of a vector bundle is a fine sheaf, hence a soft sheaf.

Acknowledgements. I would like to thank I. Kolář, D. Krupka, M. Modugno, and J. Štefánek for helpful suggestions.

The commutative diagrams are produced by Paul Taylor's `diagrams` macro package, available in CTAN in `TeX/macros/generic/diagrams/taylor`.

Chapter 1

Jet spaces

In this chapter we recall some facts on jet spaces. We start with the definition of jet space, then we introduce the contact maps. We study the natural sheaves of forms on jet spaces which arise from the fibring and the contact maps. Finally, we introduce the horizontal and vertical differential of forms on jet spaces.

1.1 Jet spaces

Our framework is a fibred manifold

$$\pi : \boldsymbol{Y} \to \boldsymbol{X} \,,$$

with $\dim \boldsymbol{X} = n$ and $\dim \boldsymbol{Y} = n + m$.

We deal with the tangent bundle $T\boldsymbol{Y} \to \boldsymbol{Y}$, the tangent prolongation $T\pi : T\boldsymbol{Y} \to T\boldsymbol{X}$ and the vertical bundle $V\boldsymbol{Y} \to \boldsymbol{Y}$.

Moreover, for $0 \leq r$, we are concerned with the r–jet space $J_r\boldsymbol{Y}$; in particular, we set $J_0\boldsymbol{Y} \equiv \boldsymbol{Y}$. We recall the natural fibrings

$$\pi^r_s : J_r\boldsymbol{Y} \to J_s\boldsymbol{Y} \,, \qquad \pi^r : J_r\boldsymbol{Y} \to \boldsymbol{X} \,,$$

and the affine bundle

$$\pi^r_{r-1} : J_r\boldsymbol{Y} \to J_{r-1}\boldsymbol{Y}$$

associated with the vector bundle

$$\odot^r T^*\boldsymbol{X} \underset{J_{r-1}\boldsymbol{Y}}{\otimes} V\boldsymbol{Y} \to J_{r-1}\boldsymbol{Y} \,,$$

for $0 \leq s \leq r$. A detailed account of the theory of jets can be found in [MaMo83a, Kup80, Sau89].

Charts on $\boldsymbol{Y}$ adapted to the fibring are denoted by (x^λ, y^i). Greek indices $\lambda, \mu, \ldots$ run from 1 to n and label base coordinates, Latin indices $i, j, \ldots$ run from 1 to m and label fibre coordinates, unless otherwise specified. We denote by $(\partial_\lambda, \partial_i)$ and (d^λ, d^i), respectively, the local bases of vector fields and 1–forms on $\boldsymbol{Y}$ induced by an adapted chart.

We denote multi–indices of dimension n by underlined latin letters such as $\underline{p} = (p_1, \dots, p_n)$, with $0 \leq p_1, \dots, p_n$; by identifying the index λ with a multi–index according to

$$\lambda \simeq (p_1, \dots, p_\lambda, \dots, p_n) \equiv (0, \dots, 1, \dots, 0)\,,$$

we can write

$$\underline{p} + \lambda = (p_1, \dots, p_\lambda + 1, \dots, p_n)\,.$$

We also set $|\underline{p}| := p_1 + \dots + p_n$ and $\underline{p}! := p_1! \dots p_n!$.

The charts induced on $J_r\boldsymbol{Y}$ are denoted by $(x^0, y^i_{\underline{p}})$, with $0 \leq |\underline{p}| \leq r$; in particular, if $|\underline{p}| = 0$, then we set $y^i_{\underline{0}} \equiv y^i$. The local vector fields and forms of $J_r\boldsymbol{Y}$ induced by the fibre coordinates are denoted by $(\partial^{\underline{p}}_i)$ and $(d^i_{\underline{p}})$, $0 \leq |\underline{p}| \leq r, 1 \leq i \leq m$, respectively.

1.2 Contact maps

A fundamental role is played in the theory of variational sequences by the "contact maps" on jet spaces (see [MaMo83a]). Namely, for $1 \leq r$, we consider the natural injective fibred morphism over $J_r\boldsymbol{Y} \to J_{r-1}\boldsymbol{Y}$

$$\text{д}_r : J_r\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X} \to TJ_{r-1}\boldsymbol{Y}\,,$$

and the complementary surjective fibred morphism

$$\vartheta_r : J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} TJ_{r-1}\boldsymbol{Y} \to VJ_{r-1}\boldsymbol{Y}\,,$$

whose coordinate expression are

$$\begin{aligned}
\text{д}_r &= d^\lambda \otimes \text{д}_{r\lambda} = d^\lambda \otimes (\partial_\lambda + y^j_{\underline{p}+\lambda} \partial^{\underline{p}}_j)\,, && 0 \leq |\underline{p}| \leq r-1,\\
\vartheta_r &= \vartheta^j_{\underline{p}} \otimes \partial^{\underline{p}}_j = (d^j_{\underline{p}} - y^j_{\underline{p}+\lambda} d^\lambda) \otimes \partial^{\underline{p}}_j\,, && 0 \leq |\underline{p}| \leq r-1\,.
\end{aligned}$$

We stress that

$$\text{д}_r \rfloor \vartheta_r = \vartheta_r \rfloor \text{д}_r = 0 \tag{1.1}$$

$$(\vartheta_r)^2 = \vartheta_r \qquad\qquad (\text{д}_r)^2 = \text{д}_r \tag{1.2}$$

The transpose of the map ϑ_r is the injective fibred morphism over $J_r\boldsymbol{Y} \to J_{r-1}\boldsymbol{Y}$

$$\vartheta^*_r : J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} V^*J_{r-1}\boldsymbol{Y} \to T^*J_r\boldsymbol{Y}\,.$$

We have the remarkable vector subbundle

$$\operatorname{im}\vartheta^*_r \subset J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*J_{r-1}\boldsymbol{Y} \subset T^*J_r\boldsymbol{Y} \ , \tag{1.3}$$

and, for $0 \leq t \leq s \leq r$, the fibred inclusions

$$J_r\boldsymbol{Y} \underset{J_t\boldsymbol{Y}}{\times} \operatorname{im}\vartheta^*_t \subset J_r\boldsymbol{Y} \underset{J_s\boldsymbol{Y}}{\times} \operatorname{im}\vartheta^*_s \subset \operatorname{im}\vartheta^*_r \ . \tag{1.4}$$

The above vector subbundle $\operatorname{im}\vartheta^*_r$ yields the splitting [MaMo83a]

$$J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*J_{r-1}\boldsymbol{Y} = \left(J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*\boldsymbol{X}\right) \oplus \operatorname{im}\vartheta^*_r \ . \tag{1.5}$$

1.3 Distinguished sheaves of forms

We are concerned with some distinguished sheaves of forms on jet spaces.

Remark 1.3.1. The manifold $\boldsymbol{Y}$ is a differentiable retract of $J_r\boldsymbol{Y}$, hence the de Rham cohomologies of $\boldsymbol{Y}$ and $J_r\boldsymbol{Y}$ are isomorphic. Therefore, we reduce sheaves on $J_r\boldsymbol{Y}$ to sheaves on $\boldsymbol{Y}$ by considering for each sheaf $\mathcal{S}$ on $J_r\boldsymbol{Y}$ the sheaf induced by $\mathcal{S}$ by restricting to the tube topology on $J_r\boldsymbol{Y}$, i.e. , the topology generated by open sets of the kind $(\pi^r_0)^{-1}(\boldsymbol{U})$, with $\boldsymbol{U} \subset \boldsymbol{Y}$ open in $\boldsymbol{Y}$. So, from now on, the sheaves of forms on $J_r\boldsymbol{Y}$ and the related subsheaves will be considered as sheaves over the topological space $\boldsymbol{Y}$ of the above kind. $\square$

Let $0 \leq k$.

1. First of all, for $0 \leq r$, we consider the standard sheaf $\overset{k}{\Lambda}_r$ of *k–forms* on $J_r\boldsymbol{Y}$

$$\alpha : J_r\boldsymbol{Y} \to \overset{k}{\wedge} T^*J_r\boldsymbol{Y} \ .$$

2. Then, for $0 \leq s \leq r$, we consider the sheaves $\overset{k}{\mathcal{H}}_{(r,s)}$ and $\overset{k}{\mathcal{H}}_r$ of *horizontal forms*, i.e. of local fibred morphisms over $J_r\boldsymbol{Y} \to J_s\boldsymbol{Y}$ and $J_r\boldsymbol{Y} \to \boldsymbol{X}$ of the type

$$\alpha : J_r\boldsymbol{Y} \to \overset{k}{\wedge} T^*J_s\boldsymbol{Y} \qquad \text{and} \qquad \beta : J_r\boldsymbol{Y} \to \overset{k}{\wedge} T^*\boldsymbol{X} \ ,$$

respectively. In coordinates, if $0 < k \leq n$, then

$$\begin{aligned}\alpha &= \alpha^{\underline{p}_1 \dots \underline{p}_h}_{i_1 \dots i_h}{}_{\lambda_{h+1}\dots\lambda_k}\, d^{i_1}_{\underline{p}_1} \wedge \dots \wedge d^{i_h}_{\underline{p}_h} \wedge d^{\lambda_{h+1}} \wedge \dots \wedge d^{\lambda_k} \\ \beta &= \beta_{\lambda_1 \dots \lambda_k}\, d^{\lambda_1} \wedge \dots \wedge d^{\lambda_k} \ ;\end{aligned}$$

if $k > n$, then

$$\alpha = \alpha^{\underline{p}_1 \dots \underline{p}_{k-n+l}}_{i_1 \dots i_{k-n+l}}{}_{\lambda_{l+1}\dots\lambda_n}\, d^{i_1}_{\underline{p}_1} \wedge \dots \wedge d^{i_{k-n+l}}_{\underline{p}_{k-n+l}} \wedge d^{\lambda_{l+1}} \wedge \dots \wedge d^{\lambda_n} \ ,$$

Here, the coordinate functions are sections of $\overset{0}{\Lambda}_r$, and the indices' range is $0 \leq |\underline{p}_j| \leq s$, $0 \leq h \leq k$ and $0 \leq l \leq n$. We remark that, in the coordinate expression of α, the indices λ_j are suppressed if $h = k$ or $l = n$, and the indices $\substack{\underline{p}_j \\ i_j}$ are suppressed if $h = 0$.

Clearly $\overset{k}{\mathcal{H}}_{(r,r)} = \overset{k}{\Lambda}_r$ and $\overset{k}{\mathcal{H}}_r = 0$ for $k > n$.

If $0 \leq q \leq r$, then pull–back by π^r_q yields the sheaf inclusions

$$\overset{k}{\mathcal{H}}_q \simeq \pi^{r*}_q \overset{k}{\mathcal{H}}_q \subset \overset{k}{\mathcal{H}}_r \subset \overset{k}{\mathcal{H}}_{(r,t)} \subset \overset{k}{\mathcal{H}}_{(r,s)} \subset \overset{k}{\Lambda}_r \ ,$$
$$\overset{k}{\Lambda}_s \simeq \pi^{r*}_s \overset{k}{\Lambda}_s \subset \overset{k}{\mathcal{H}}_{(r,s)} \subset \overset{k}{\Lambda}_r \ .$$

The above inclusions are proper inclusions if $t < s < r$ and $q < r$. Indeed, not all sections of the pull–back of a bundle (like $J_r\boldsymbol{Y} \underset{J_s\boldsymbol{Y}}{\times} T^*J_s\boldsymbol{Y}$) are the pull–back of some section of the bundle itself. In fact, we deal with two different operations: pull–back of bundles and pull–back of sections (forms).

3. For $0 \leq s < r$, we consider the subsheaf $\overset{k}{\mathcal{C}}_{(r,s)} \subset \overset{k}{\mathcal{H}}_{(r,s)}$ of *contact* forms, i.e. of local fibred morphisms over $J_r\boldsymbol{Y} \to J_s\boldsymbol{Y}$ of the type

$$\alpha : J_r\boldsymbol{Y} \to \overset{k}{\wedge} \operatorname{im} \vartheta^*_{s+1} \subset \overset{k}{\wedge} T^*J_s\boldsymbol{Y} \, .$$

Due to the injectivity of ϑ^*_{s+1}, the subsheaf $\overset{k}{\mathcal{C}}_{(r,s)}$ turns out to be the sheaf of local fibred morphisms $\alpha \in \overset{k}{\mathcal{H}}_{(r,s)}$ which factorise as $\alpha = \overset{k}{\wedge}\vartheta^*_{s+1} \circ \tilde{\alpha}$, through the composition

$$J_r\boldsymbol{Y} \xrightarrow{\tilde{\alpha}} J_{s+1}\boldsymbol{Y} \underset{J_s\boldsymbol{Y}}{\times} \overset{k}{\wedge} V^*J_s\boldsymbol{Y} \xrightarrow{\overset{k}{\wedge}\vartheta_{s+1}} \overset{k}{\wedge} T^*J_s\boldsymbol{Y} \, .$$

Thus, $\alpha \in \overset{k}{\mathcal{C}}_{(r,s)}$ if and only if its coordinate expression is of the type

$$\alpha = \alpha^{\underline{p}_1 \dots \underline{p}_k}_{i_1 \dots i_k} \, \vartheta^{i_1}_{\underline{p}_1} \wedge \ldots \wedge \vartheta^{i_k}_{\underline{p}_k} \qquad 0 \leq |\underline{p}_1|, \ldots, |\underline{p}_k| \leq s \, ,$$

with $\alpha^{\underline{p}_1 \dots \underline{p}_k}_{i_1 \dots i_k} \in \overset{0}{\Lambda}_r$.

If $0 \leq s < r \leq r'$, $s \leq s'$, then we have the inclusions (see (1.3) and (1.4))

$$\overset{k}{\mathcal{C}}_{(r,s)} \subset \overset{k}{\mathcal{C}}_{(r',s')} \ .$$

4. Furthermore, we consider the subsheaf $\overset{k}{\mathcal{H}}{}^P_r \subset \overset{k}{\mathcal{H}}_r$ of local fibred morphisms $\alpha \in \overset{k}{\mathcal{H}}_r$ such that α is a polynomial fibred morphism over $J_{r-1}\boldsymbol{Y} \to \boldsymbol{X}$ of degree k. Thus, in coordinates, $\alpha \in \overset{k}{\mathcal{H}}{}^P_r$ if and only if $\alpha_{\lambda_1,\ldots,\lambda_k} : J_r\boldsymbol{Y} \to I\!R$ is a polynomial map of degree k with respect to the coordinates $y^i_{\underline{p}}$, with $|\underline{p}| = r$.

5. Finally, we consider the subsheaf $\overset{k}{\mathcal{C}}_r \subset \overset{k}{\mathcal{C}}_{(r+1,r)}$ of local fibred morphisms $\alpha \in \overset{k}{\mathcal{C}}_{(r+1,r)}$ such that $\tilde{\alpha}$ projects down on $J_r\boldsymbol{Y}$. Thus, in coordinates, $\alpha \in \overset{k}{\mathcal{C}}_r$ if and only if $\alpha^{\underline{p}_1 \dots \underline{p}_k}_{i_1 \dots i_k} \in \overset{0}{\Lambda}_r$.

1.4 Main splitting

The maps $д_r$ and ϑ_r induce two important derivations of degree 0 (see [Sau89, Cos94]), namely the interior products by $д_r$ and ϑ_r

$$i_h \equiv i_{д_{r+1}} : \overset{k}{\Lambda}_r \to \overset{k}{\Lambda}_{r+1}\,, \qquad i_v \equiv i_{\vartheta_{r+1}} : \overset{k}{\Lambda}_r \to \overset{k}{\Lambda}_{r+1}\,,$$

which make sense taking into account the natural inclusions $J_r\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T^*\boldsymbol{X} \subset T^*J_r\boldsymbol{Y}$ and $VJ_r\boldsymbol{Y} \subset TJ_r\boldsymbol{Y}$.

The fibred splitting (1.5) yields a fundamental sheaf splitting.

Lemma 1.4.1. *We have the splitting*

$$\overset{1}{\mathcal{H}}_{(r+1,r)} = \overset{1}{\mathcal{H}}_{r+1} \oplus \overset{1}{\mathcal{C}}_{(r+1,r)}\ ,$$

where the projection on the first factor and on the second factor are given, respectively, by

$$H : \overset{1}{\mathcal{H}}_{(r+1,r)} \to \overset{1}{\mathcal{H}}_{r+1} : \alpha \mapsto i_h\alpha\ ,$$
$$V : \overset{1}{\mathcal{H}}_{(r+1,r)} \to \overset{1}{\mathcal{C}}_{(r+1,r)} : \alpha \mapsto i_v\alpha\ .$$

If $\alpha \in \overset{1}{\mathcal{H}}_{(r+1,r)}$ has the coordinate expression $\alpha = \alpha_\lambda d^\lambda + \alpha^{\underline{p}}_i d^i_{\underline{p}}$ $(0 \le \underline{p} \le r)$, then

$$H(\alpha) = (\alpha_\lambda + y^i_{\underline{p}}\alpha^{\underline{p}}_i)\ d^\lambda\ , \qquad V(\alpha) = \alpha^{\underline{p}}_i \vartheta^i_{\underline{p}}\ .$$

Proposition 1.4.1. *The above splitting of $\overset{1}{\mathcal{H}}_{(r+1,r)}$ induces the splitting*

$$\overset{k}{\mathcal{H}}_{(r+1,r)} = \bigoplus_{l=0}^{k} \overset{k-l}{\mathcal{C}}_{(r+1,r)} \wedge \overset{l}{\mathcal{H}}_{r+1}$$

(see Preliminaries).

We recall that, in the above splitting, direct summands with $l > n$ vanish.

We set H to be the projection of the above splitting on the factor with the highest degree of the horizontal factor.

Proposition 1.4.2. *If $k \leq n$, then we have*

$$H : \overset{k}{\mathcal{H}}_{(r+1,r)} \to \overset{k}{\mathcal{H}}_{r+1} : \alpha \mapsto \frac{1}{k!} \square^k д_{r+1}(\alpha) \,;$$

if $k > n$, then we have

$$H : \overset{k}{\mathcal{H}}_{(r+1,r)} \to \overset{k-n}{\mathcal{C}}_{(r+1,r)} \wedge \overset{n}{\mathcal{H}}_{r+1} : \alpha \mapsto \frac{1}{(k-n)!\, n!} \left(\square^{k-n} \vartheta_{r+1} \square^n д_{r+1} \right) (\alpha) \,.$$

Proof. See Preliminaries. $\boxed{QED}$

We set also

$$V := Id - H$$

to be the projection complementary to H.

Remark 1.4.1. If $k \leq n$, then we have the coordinate expression

$$H(\alpha) = y^{i_1}_{\underline{p}_1 + \lambda_1} \dots y^{i_h}_{\underline{p}_h + \lambda_h} \alpha^{\underline{p}_1 \dots \underline{p}_h}_{i_1 \dots i_h \, \lambda_{h+1} \dots \lambda_k} d^{\lambda_1} \wedge \dots \wedge d^{\lambda_k} \,,$$

with $0 \leq h \leq k$. If $k > n$, then we have

$$\begin{aligned} H(\alpha) = \sum y^{j_1}_{\underline{q}_1 + \lambda_1} \dots y^{j_l}_{\underline{q}_l + \lambda_l} \alpha^{\underline{p}_1 \widehat{\dots} \underline{p}_{k-n+l} \underline{q}_1 \dots \underline{q}_l}_{i_1 \widehat{\dots} i_{k-n+l} \, j_1 \dots j_l \, \lambda_{l+1} \dots \lambda_n} \\ \vartheta^{i_1}_{\underline{p}_1} \wedge \widehat{\dots} \wedge \vartheta^{i_{k-n+l}}_{\underline{p}_{k-n+l}} \wedge d^{\lambda_1} \wedge \dots \wedge d^{\lambda_n} \,, \end{aligned}$$

where $0 \leq l \leq n$ and the sum is over the subsets

$$\{ {}^{j_1}_{\underline{q}_1} \dots {}^{j_l}_{\underline{q}_l} \} \subset \{ {}^{i_1}_{\underline{p}_1} \dots {}^{i_{k-n+l}}_{\underline{p}_{k-n+l}} \} \,,$$

and $\widehat{\dots}$ stands for suppressed indexes (and corresponding contact forms) belonging to one of the above subsets. $\square$

Now, we apply the conclusion of inclusion (2) of Preliminaries to the subsheaf $\overset{k}{\Lambda}_r \subset \overset{k}{\mathcal{H}}_{(r+1,r)}$. To this aim, we want to find the image of $\overset{k}{\Lambda}_r$ under the projections of the above splitting.

We denote the restrictions of H, V to $\overset{k}{\Lambda}_r$ by h, v.

Next theorem is devoted to a characterisation of the image of $\overset{k}{\Lambda}_r$ under H.

Theorem 1.4.1. *Let $0 < k \leq n$, and denote*

$$\overset{k}{\mathcal{H}}{}^h_{r+1} := h(\overset{k}{\Lambda}_r) \,.$$

Then, we have the inclusion $\overset{k}{\mathcal{H}}{}^h_{r+1} \subset \overset{k}{\mathcal{H}}{}^P_{r+1}$.

Moreover, the sheaf $\overset{k}{\mathcal{H}}{}^{h}_{r+1}$ *admits the following characterisation: a section* $\alpha \in \overset{k}{\mathcal{H}}{}^{P}_{r+1}$ *is a section of the subsheaf* $\overset{k}{\mathcal{H}}{}^{h}_{r+1}$ *if and only if there exists a section* $\beta \in \overset{k}{\Lambda}_r$ *such that*

$$(j_r s)^* \beta = (j_{r+1} s)^* \alpha$$

for each section $s : \boldsymbol{X} \to \boldsymbol{Y}$.

Proof. If $s : \boldsymbol{X} \to \boldsymbol{Y}$ is a section, then the following identities

$$(j_r s)^* \beta = (j_{r+1} s)^* h(\beta) \,, \qquad (j_{r+1} s)^* v(\beta) = 0 \,,$$

yield

$$\alpha = h(\beta) \quad \Leftrightarrow \quad (j_r s)^* \beta = (j_{r+1} s)^* \alpha$$

for all $\alpha \in \overset{k}{\mathcal{H}}{}^{P}_{r+1}$ and $\beta \in \overset{k}{\Lambda}_r$. QED

Remark 1.4.2. It comes from the above Theorem that not any section of $\overset{k}{\mathcal{H}}{}^{P}_{r+1}$ is a section of $\overset{k}{\mathcal{H}}{}^{h}_{r+1}$; indeed, a section of $\overset{k}{\mathcal{H}}{}^{P}_{r+1}$ in general contains 'too many monomials' with respect to a section of $\overset{k}{\mathcal{H}}{}^{h}_{r+1}$. This can be seen by means of the following example. Consider a one–form $\beta \in \overset{1}{\Lambda}_0$. Then we have the coordinate expressions

$$\beta = \beta_\lambda d^\lambda + \beta_i d^i \,, \qquad h(\beta) = (\beta_\lambda + y^i_\lambda \beta_i) d^\lambda \,.$$

If $\alpha \in \overset{1}{\mathcal{H}}{}^{P}_{1}$, then we have the coordinate expression

$$\alpha = (\alpha_\lambda + y^j_\mu \alpha^{\mu}_{j\,\lambda}) d^\lambda \,.$$

It is evident that, in general, there does not exists $\beta \in \overset{1}{\Lambda}_r$ such that $h(\beta) = \alpha$. □

Corollary 1.4.1. *Let* $\dim \boldsymbol{X} = 1$. *Then we have*

$$\overset{1}{\mathcal{H}}{}^{h}_{r+1} = \overset{1}{\mathcal{H}}{}^{P}_{r+1} \,.$$

Proof. From the above coordinate expressions. See also [Kru95a, Vit95]. QED

Lemma 1.4.2. *The sheaf morphisms* H, V *restrict on the sheaf* $\overset{k}{\Lambda}_r$ *to the surjective sheaf morphisms*

$$h : \overset{1}{\Lambda}_r \to \overset{1}{\mathcal{H}}{}^{h}_{r+1} \,, \qquad v : \overset{1}{\Lambda}_r \to \overset{1}{\mathcal{C}}_r \,.$$

PROOF. The restriction of H has already been studied. As for the restriction of V, it is easy to see by means of a partition of the unity that it is surjective on $\overset{1}{\mathcal{C}}_r$. QED

Theorem 1.4.2. *The splitting of proposition 1.4.1 yields the inclusion*

$$\overset{k}{\Lambda}_r \subset \bigoplus_{l=0}^{k} \overset{k-l}{\mathcal{C}}_r \wedge \overset{l}{\mathcal{H}}{}^h_{r+1} \,,$$

and the splitting projections restrict to surjective maps.

PROOF. In fact, for any $i \leq k$ the restriction of the projection

$$\overset{k}{\mathcal{H}}_{(r+1,r)} \to \overset{k-l}{\mathcal{C}}_{(r+1,r)} \wedge \overset{l}{\mathcal{H}}_{r+1}$$

of the splitting of proposition 1.4.1 to the sheaf $\overset{k}{\Lambda}_r$ takes the form

$$\overset{k}{\Lambda}_r \to \overset{k-l}{\mathcal{C}}_r \wedge \overset{l}{\mathcal{H}}{}^h_{r+1} \subset \overset{k-l}{\mathcal{C}}_{(r+1,r)} \wedge \overset{l}{\mathcal{H}}_{r+1} \,.$$

The above inclusion can be tested in coordinates. For the sake of simplicity, let us consider a global section $\alpha \in \overset{k-l}{\mathcal{C}}_r \wedge \overset{l}{\mathcal{H}}{}^h_{r+1}$ where $0 \leq l \leq n$. We have the coordinate expression

$$\alpha = y^{j_1}_{\underline{q}_1+\lambda_1} \dots y^{j_h}_{\underline{q}_h+\lambda_h} \alpha^{\underline{p}_1 \dots \underline{p}_{k-l} \underline{q}_1 \dots \underline{q}_h}_{i_1 \dots i_{k-l}\, j_1 \dots j_h \lambda_{h+1} \dots \lambda_l} \\ \vartheta^{i_1}_{\underline{p}_1} \wedge \dots \wedge \vartheta^{i_{k-l}}_{\underline{p}_{k-l}} \wedge d^{\lambda_1} \wedge \dots \wedge d^{\lambda_l} \,,$$

where $0 \leq |\underline{p}_i|, |\underline{q}_i| \leq r$ and $0 \leq h \leq n$. If $\{\psi_i\}$ is a partition of the unity on $\overset{0}{\Lambda}_r$ subordinate to a coordinate atlas, let

$$\tilde{\alpha}_i := \psi_i \, \tilde{\alpha}^{\underline{s}_1 \dots \underline{s}_r}_{t_1 \dots t_r \, \lambda_{r+1} \dots \lambda_k} d^{t_1}_{\underline{s}_1} \wedge \dots \wedge d^{t_r}_{\underline{p}_r} \wedge d^{\lambda_{r+1}} \wedge \dots \wedge d^{\lambda_k} \,,$$

where the set of pairs of indices $\{^{t_1}_{\underline{s}_1} \dots {}^{t_r}_{\underline{s}_r}\}$ is a permutation of the set of pairs of indices $\{^{i_1}_{\underline{p}_1} \dots {}^{i_{k-l}}_{\underline{p}_{k-l}} {}^{j_1}_{\underline{q}_1} \dots {}^{j_l}_{\underline{q}_l}\}$. Then

1. $\sum_i \tilde{\alpha}_i$ is a global section of $\overset{k}{\Lambda}_r$;
2. the projection of $\sum_i \tilde{\alpha}_i$ on $\overset{k-l}{\mathcal{C}}_r \wedge \overset{l}{\mathcal{H}}{}^h_{r+1}$ is α.

The proof is analogous for $k > n$. QED

We remark that, in general, the above inclusion is a proper inclusion: in general, a sum of elements of the direct summands is not an element of $\overset{k}{\Lambda}_r$.

Corollary 1.4.2. *The sheaf morphism H restricts on the sheaf $\overset{k}{\Lambda}_r$ to the surjective sheaf morphisms*

$$h : \overset{k}{\Lambda}_r \to \overset{k}{\mathcal{H}}{}^h_{r+1} \qquad k \leq n \,,$$
$$h : \overset{k}{\Lambda}_r \to \overset{k-n}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1} \qquad k > n \,.$$

1.5 Horizontal and vertical differential

The derivations i_h, i_v, and the exterior differential d yield two derivations of degree one (see [Sau89, Cos94]). Namely, we define the *horizontal* and *vertical differential* to be the sheaf morphisms

$$
\begin{aligned}
d_h &:= i_h \circ d - d \circ i_h : \overset{k}{\Lambda}_r \to \overset{k}{\Lambda}_{r+1} \,, \\
d_v &:= i_v \circ d - d \circ i_v : \overset{k}{\Lambda}_r \to \overset{k}{\Lambda}_{r+1} \,,
\end{aligned}
$$

It can be proved (see [Sau89]) that d_h and d_v fulfill the properties

$$
\begin{gathered}
d_h^2 = d_v^2 = 0 \;, \qquad d_h \circ d_v + d_v \circ d_h = 0 \;, \\
d_h + d_v = (\pi^{r+1}_r)^* \circ d \;, \\
(j_{r+1}s)^* \circ d_v = 0 \;, \qquad d \circ (j_r s)^* = (j_{r+1}s)^* \circ d_h \;.
\end{gathered}
$$

The action of d_h and d_v on functions $f : J_r\boldsymbol{Y} \to I\!R$ and one–forms on $J_r\boldsymbol{Y}$ uniquely characterises d_h and d_v. We have the coordinate expressions

$$
\begin{gathered}
d_h f = (\text{д}_{r+1})_\lambda . f d^\lambda = (\partial_\lambda f + y^i_{\underline{p}+\lambda} \partial^{\underline{p}}_i f) d^\lambda \,, \\
d_h d^\lambda = 0 \,, \qquad d_h d^i_{\underline{p}} = -d^i_{\underline{p}+\lambda} \wedge d^\lambda \,, \qquad d_h \vartheta^i_{\underline{p}} = -\vartheta^i_{\underline{p}+\lambda} \wedge d^\lambda \,, \\
d_v f = \partial^{\underline{p}}_i f \vartheta^i_{\underline{p}} \;, \\
d_v d^\lambda = 0 \,, \qquad d_v d^i_{\underline{p}} = d^i_{\underline{p}+\lambda} \wedge d^\lambda \,, \qquad d_v \vartheta^i_{\underline{p}} = 0 \,.
\end{gathered}
$$

We note that

$$
-d^i_{\underline{p}+\lambda} \wedge d^\lambda = -\vartheta^i_{\underline{p}+\lambda} \wedge d^\lambda + y^i_{\underline{p}+\lambda+\mu} d^\mu \wedge d^\lambda = -\vartheta^i_{\underline{p}+\lambda} \wedge d^\lambda \,.
$$

Finally, next Proposition analyses the relationship of d_h and d_v with the splitting of Proposition 1.4.1.

Proposition 1.5.1. *We have*

$$
\begin{gathered}
d_h \left(\overset{k}{\mathcal{H}}_r \right) \subset \overset{k+1}{\mathcal{H}}_{r+1} \,, \qquad d_v \left(\overset{k}{\mathcal{H}}_r \right) \subset \overset{1}{\mathcal{C}}_r \wedge \overset{k}{\mathcal{H}}_r \,, \\
d_h \left(\overset{k}{\mathcal{C}}_{(r,r-1)} \wedge \overset{h}{\mathcal{H}}_r \right) \subset \overset{k}{\mathcal{C}}_{(r+1,r)} \wedge \overset{h+1}{\mathcal{H}}_{r+1} \,, \qquad d_h \left(\overset{k}{\mathcal{C}}_{(r,r-1)} \wedge \overset{n}{\mathcal{H}}_r \right) = \{0\} \,, \\
d_v \left(\overset{k}{\mathcal{C}}_{(r,r-1)} \right) \subset \overset{k+1}{\mathcal{C}}_r \,, \qquad d_v \left(\overset{k}{\mathcal{C}}_r \right) \subset \overset{k+1}{\mathcal{C}}_r \,,
\end{gathered}
$$

Proof. From the action of d_h, d_v on functions and local coordinate bases of forms.

$\boxed{QED}$

Chapter 2

Variational sequence

In this chapter, we recall the theory of variational sequences on finite order jet bundles, as was developed by Krupka in [Kru90]. Our main aim is to present a concise summary of the theory in order to introduce the reader to our notation.

Starting from the de Rham exact sheaf sequence on $J_r\boldsymbol{Y}$, we find a natural exact subsequence. This subsequence is not the unique exact and natural one that we might consider; our choice is inspired by the calculus of variations, as it is shown in Appendix. Then we will define the (r–th order) variational sequence to be the quotient of the de Rham sequence on $J_r\boldsymbol{Y}$ by means of the above exact subsequence.

We start by considering the *de Rham exact sequence of sheaves* on $J_r\boldsymbol{Y}$

$$0 \longrightarrow I\!R \longrightarrow \overset{0}{\Lambda}_r \xrightarrow{\ d\ } \overset{1}{\Lambda}_r \xrightarrow{\ d\ } \dots \xrightarrow{\ d\ } \overset{J}{\Lambda}_r \xrightarrow{\ d\ } 0\,,$$

where $J = \dim J_r\boldsymbol{Y}$ (see [Sau89]).

2.1 Contact subsequence

We are able to provide several natural subsequences of the de Rham sequence. For example, natural subsequences of the de Rham sequence arise by considering the ideals generated in $\overset{k}{\Lambda}_r$ by its natural subsheaves $\overset{1}{\mathcal{H}}_{(r,s)}$, $\overset{1}{\mathcal{C}}_{(r,s)}$, ... Not all natural subsequences of the de Rham sequence turn out to be exact. In this subsection, we study an exact natural subsequence of the de Rham sequence, which is of particular importance in the variational calculus, although being defined independently (see the Appendix).

We introduce a new subsheaf of $\overset{k}{\Lambda}_r$. Namely, we set

$$\mathcal{C}\overset{k}{\Lambda}_r = \{\alpha \in \overset{k}{\Lambda}_r \mid (j_r s)^* \alpha = 0 \text{ for every section } s : \boldsymbol{X} \to \boldsymbol{Y}\}\,.$$

The definition of the above subsheaf is clearly inspired by the calculus of variations (see [Kru90, Kru95a, Kru95b, Vit96a] and Appendix).

Lemma 2.1.1. *We have*

$$\mathcal{C}\overset{k}{\Lambda}_r = \ker h \quad \textit{if} \quad 0 \leq k \leq n\,,$$
$$\mathcal{C}\overset{k}{\Lambda}_r = \overset{k}{\Lambda}_r \quad \textit{if} \quad k > n\,.$$

PROOF. Let $\alpha \in \overset{k}{\Lambda}_r$. Then, for any section $s : \boldsymbol{X} \to \boldsymbol{Y}$ we have

$$(j_r s)^* \alpha = (j_{r+1} s)^* h(\alpha)\,,$$

and $\alpha \in \ker h$ implies $\alpha \in \mathcal{C}\overset{k}{\Lambda}_r$. Conversely, suppose $\alpha \in \mathcal{C}\overset{k}{\Lambda}_r$. Then we have

$$(j_{r+1} s)^* h(\alpha) = h(\alpha)_{\lambda_1 \dots \lambda_k} \circ j_{r+1} s \;\; d_1^\lambda \wedge \dots \wedge d_k^\lambda\,,$$

hence $h(\alpha) = 0$.

The first assertion comes from the above identities and $\dim \boldsymbol{X} = n$. QED

We define the subsheaf $\overset{k}{\Theta}_r \subset \overset{k}{\Lambda}_r$ to be the sheaf generated by the presheaf $\ker h + d \ker h$, i.e.

$$\overset{k}{\Theta}_r := \ker h + \overline{d \ker h}\,.$$

Of course, $\ker h$ is a sheaf. We recall that $\overline{d \ker h}$ consists of sections $\alpha \in \overset{k}{\Lambda}_r$ which are of the local type $\alpha = d\beta$, with $\beta \in d \ker h$.

Remark 2.1.1. If $\dim \boldsymbol{X} = 1$ we have two important facts

1. $\ker h = \overset{k}{\mathcal{C}}_{(r,r-1)}$;

2. the above sum turns out to be a direct sum [Kru95a, Vit95]. □

Lemma 2.1.2. *If* $0 \leq k \leq n$*, then* $d \ker h \subset \ker h$*, so that* $\overset{k}{\Theta}_r = \mathcal{C}\overset{k}{\Lambda}_r$.

PROOF. By the above Lemma, if $\alpha \in \ker h$, then for any section $s : \boldsymbol{X} \to \boldsymbol{Y}$ we have $(j_r s)^* \alpha = 0$, hence $(j_r s)^* d\alpha = 0$. So, $d\alpha \in \ker h$. QED

It is clear that $\overset{k}{\Theta}_r$ is a subsheaf of $\overset{k}{\Lambda}_r$. Thus, we say the following natural subsequence

$$0 \longrightarrow \overset{1}{\Theta}_r \xrightarrow{\;d\;} \overset{2}{\Theta}_r \xrightarrow{\;d\;} \dots \xrightarrow{\;d\;} \overset{I}{\Theta}_r \xrightarrow{\;d\;} 0$$

to be the *contact subsequence* of the de Rham sequence. We note that, in general, the sheaves $\overset{k}{\Theta}_r$ are not the sheaves of sections of a vector subbundle of $T^* J_r \boldsymbol{Y}$.

Remark 2.1.2. In general, I depends on the dimension of the fibers of $J_r \boldsymbol{Y} \to \boldsymbol{X}$; its value is given in [Kru90]. □

The following theorem is proved in [Kru90] by means of a contact homotopy formula.

Theorem 2.1.1. *The contact subsequence is exact [Kru90].*

Proposition 2.1.1. *The sheaves $\overset{k}{\Theta}_r$ are soft sheaves [Kru90].*

Proof. We rephrase the proof of Krupka for convenience of the reader by adapting it to our notation.

It can be easily seen that the sheaves $\mathcal{C}\overset{k}{\Lambda}_r$ are soft. Let us consider the short exact sequence

$$0 \to \ker d \to \mathcal{C}\overset{1}{\Lambda}_r \xrightarrow{d} \mathrm{im}\ d \to 0 \ ,$$

From the above Theorem we have $\ker d = 0$, and this is a soft sheaf. Hence $\mathrm{im}\ d = d(\mathcal{C}\overset{1}{\Lambda}_r)$ is soft (see [Wel80]). By induction on k, the equality $\ker d = \mathrm{im}\ d$ on k–forms, and exactness of the sequence

$$0 \to \ker d \to \mathcal{C}\overset{k}{\Lambda}_r \xrightarrow{d} \mathrm{im}\ d \to 0$$

we obtain that each one of the sheaves $d(\mathcal{C}\overset{k}{\Lambda}_r)$ is soft.

Now, let us take into account the exact sheaf sequence

$$0 \to \ker d \xrightarrow{f} \mathcal{C}\overset{k-1}{\Lambda}_r \oplus \mathcal{C}\overset{k}{\Lambda}_r \xrightarrow{g} \overset{k}{\Theta}_r \to 0 \ ,$$

where f, g are the sheaf morphisms given on each tubular neighbourhood $\pi^{1\,-1}_0(\boldsymbol{U})$ (with $\boldsymbol{U} \subset \boldsymbol{Y}$ open subset) as

$$f_{\boldsymbol{U}} : (\ker d)_{\boldsymbol{U}} \to \left(\mathcal{C}\overset{k-1}{\Lambda}_r \oplus \mathcal{C}\overset{k}{\Lambda}_r\right)_{\boldsymbol{U}} : \alpha \mapsto (\alpha, -d\alpha) \ ,$$

$$g_{\boldsymbol{U}} : \left(\mathcal{C}\overset{k-1}{\Lambda}_r \oplus \mathcal{C}\overset{k}{\Lambda}_r\right)_{\boldsymbol{U}} \to (\overset{k}{\Theta}_r)_{\boldsymbol{U}} : (\alpha, \beta) \mapsto d\alpha + \beta \ .$$

$\ker d = \mathrm{im}\ d$ (on $(k-1)$–forms) implies that $\ker d$ is a soft sheaf, and, being $\mathcal{C}\overset{k-1}{\Lambda}_r \oplus \mathcal{C}\overset{k}{\Lambda}_r$ soft, we obtain the result. $\boxed{QED}$

2.2 Variational bicomplex

Here, we introduce a bicomplex by quotienting the de Rham sequence on $J_r\boldsymbol{Y}$ by the contact subsequence. We obtain a new sequence, the variational sequence, which turns out to be exact. In the last part of the section, we describe the relationships between bicomplexes on jet spaces of different orders.

Proposition 2.2.1. *The following diagram*

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \overset{1}{\Theta}_r \arrow[r, "d"] \arrow[d] & \overset{2}{\Theta}_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & \overset{I}{\Theta}_r \arrow[r, "d"] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \arrow[r] & 0 \\
0 \arrow[r] & I\!\!R \arrow[r] \arrow[d] & \overset{0}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \overset{1}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \overset{2}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & \overset{I}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \overset{I+1}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & 0 \\
0 \arrow[r] & I\!\!R \arrow[r] \arrow[d] & \overset{0}{\Lambda}_r \arrow[r, "\mathcal{E}_0"] \arrow[d] & \overset{1}{\Lambda}_r/\overset{1}{\Theta}_r \arrow[r, "\mathcal{E}_1"] \arrow[d] & \overset{2}{\Lambda}_r/\overset{2}{\Theta}_r \arrow[r, "\mathcal{E}_2"] \arrow[d] & \dots \arrow[r, "\mathcal{E}_{I-1}"] & \overset{I}{\Lambda}_r/\overset{I}{\Theta}_r \arrow[r, "\mathcal{E}_I"] \arrow[d] & \overset{I+1}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & 0 \\
 & 0 & 0 & 0 & 0 & & 0 & 0 & &
\end{tikzcd}
$$

is a commutative diagram, where rows and columns are exact.

Proof. We have to prove only the exactness of the bottom row of the diagram. But this follows from the exactness of the other rows and of the columns. $\boxed{QED}$

Definition 2.2.1. The above diagram is said to be the r–th order *variational bicomplex* associated with the fibred manifold $\boldsymbol{Y} \to \boldsymbol{X}$ (see [Kru90]).

We say the bottom row of the above diagram to be the r–th order *variational sequence* associated with the fibred manifold $\boldsymbol{Y} \to \boldsymbol{X}$. $\square$

Proposition 2.2.2. *The sheaves* $\overset{k}{\Lambda}_r/\overset{k}{\Theta}_r$ *are soft sheaves (see [Kru90]).*

Proof. In fact, each column is a short exact sheaf sequence in which $\overset{k}{\Theta}_r$ and $\overset{k}{\Lambda}_r$ are soft sheaves (see [Wel80]). $\boxed{QED}$

Corollary 2.2.1. *The variational sequence is a soft resolution of the constant sheaf* $I\!\!R$ *over* $\boldsymbol{Y}$ *[Kru90].*

Proof. In fact, except $I\!\!R$, each one of the sheaves in the sequence is soft [Wel80]. $\boxed{QED}$

The most interesting consequence of the above corollary is the following one (for a proof, see [Wel80]). Let us consider the cochain complex

$$
0 \longrightarrow I\!\!R_{\boldsymbol{Y}} \longrightarrow \left(\overset{0}{\Lambda}_r\right)_{\boldsymbol{Y}} \xrightarrow{d} \left(\overset{1}{\Lambda}_r/\overset{1}{\Theta}_r\right)_{\boldsymbol{Y}} \xrightarrow{\mathcal{E}_1} \dots \xrightarrow{d} \left(\overset{J}{\Lambda}_r\right)_{\boldsymbol{Y}} \xrightarrow{d} 0
$$

and denote by H^k_{VS} the k^{th}–cohomology group of the above cochain complex.

Corollary 2.2.2. *For all $k \geq 0$ there is a natural isomorphism*

$$H^k_{VS} \simeq H^k_{de\ Rham}\boldsymbol{Y}$$

(see [Kru90]).

Proof. In fact, the variational sequence is a soft resolution of $I\!R$, hence the cohomology of the sheaf $I\!R$ is naturally isomorphic to the cohomology of the above cochain complex. Also, the de Rham sequence gives rise to a cochain complex of global sections, whose cohomology is naturally isomorphic to the cohomology of the sheaf $I\!R$ on $\boldsymbol{Y}$. Hence, we have the result by a composition of isomorphisms. (See [Wel80] for more details on the above natural isomorphisms.) $\boxed{QED}$

Finally, we investigate the relationship between variational bicomplexes of different orders. To this purpose, we recall the intrinsic inclusions ($0 \leq s \leq r$)

$$\overset{k}{\Lambda}_s \simeq \pi^{r*}_s \overset{k}{\Lambda}_s \subset \overset{k}{\Lambda}_r \ , \qquad \overset{k}{\Theta}_s \simeq \pi^{r*}_s \overset{k}{\Theta}_s \subset \overset{k}{\Theta}_r \ ,$$

and the isomorphism

$$\left(\overset{k}{\Lambda}_s / \overset{k}{\Theta}_s\right) \simeq \left(\pi^{r*}_s \overset{k}{\Lambda}_s / \pi^{r*}_s \overset{k}{\Theta}_s\right) \ .$$

Lemma 2.2.1. *Let $s \leq r$. Then, the above inclusions induce the injective sheaf morphism (see [Kru90])*

$$\chi^r_s : \left(\overset{k}{\Lambda}_s / \overset{k}{\Theta}_s\right) \to \left(\overset{k}{\Lambda}_r / \overset{k}{\Theta}_r\right) : [\alpha] \mapsto [\pi^{r*}_s \alpha] \ ,$$

where $[\alpha]$ denotes an equivalence class of a form on $J_s\boldsymbol{Y}$.

Proof. The above morphism χ^r_s is well–defined, because

$$[\alpha] = [\beta] \Rightarrow [\pi^{r*}_s \alpha] = [\pi^{r*}_s \beta]$$

due to the above inclusions.

The morphism is injective too. For if $\alpha \in \overset{k}{\Lambda}_s$ and $\beta \in \overset{k}{\Lambda}_s$ such that

$$[\pi^{r*}_s \alpha] = [\pi^{r*}_s \beta] \ ,$$

then, being $\pi^{r*}_s(\alpha - \beta) \in \pi^{r*}_s \overset{k}{\Lambda}_s$, and $\pi^{r*}_s(\alpha - \beta) \in \overset{k}{\Theta}_r$, it must be $\pi^{r*}_s(\alpha - \beta) \in \pi^{r*}_s \overset{k}{\Theta}_s$, hence $[\alpha] = [\beta]$. $\boxed{QED}$

Remark 2.2.1. It is clear that, if $t \leq s \leq r$, then $\chi^r_s \circ \chi^s_t = \chi^r_t$.
We have the commutative diagrams

$$\begin{tikzcd} \overset{k}{\Lambda}_r \arrow[r, "d"] & \overset{k+1}{\Lambda}_r \\ \overset{k}{\Lambda}_s \arrow[r, "d"] \arrow[u, "\pi^{r*}_s"] & \overset{k+1}{\Lambda}_s \arrow[u, "\pi^{r*}_s"'] \end{tikzcd} \qquad \begin{tikzcd} \overset{k}{\Theta}_r \arrow[r, "d"] & \overset{k+1}{\Theta}_r \\ \overset{k}{\Theta}_s \arrow[r, "d"] \arrow[u, "\pi^{r*}_s"] & \overset{k+1}{\Theta}_s \arrow[u, "\pi^{r*}_s"'] \end{tikzcd}$$

hence the following commutative diagram holds

$$\begin{tikzcd} \overset{k}{\Lambda}_r/\overset{k}{\Theta}_r \arrow[r, "\mathcal{E}_k"] & \overset{k+1}{\Lambda}_r/\overset{k+1}{\Theta}_r \\ \overset{k}{\Lambda}_s/\overset{k}{\Theta}_s \arrow[r, "\mathcal{E}_k"] \arrow[u, "\pi^{r*}_s"] & \overset{k+1}{\Lambda}_s/\overset{k+1}{\Theta}_s \arrow[u, "\pi^{r*}_s"'] \end{tikzcd}$$

We can summarise the above commutative diagrams stating the existence of a three–dimensional commutative diagram (which is not exact), whose bi–dimensional slices are the variational bicomplexes of order 1, 2, ... □

Chapter 3

Representation of the variational sequence

In this section we find suitable sheaves of fibred morphisms that are isomorphic to the quotient sheaves of the variational sequence.

As a consequence, we will recover the sheaves of the geometric objects that arise in the variational calculus (like Lagrangians, Euler–Lagrange morphisms, …). Also, we will be able to give an intrinsic formulation of the Helmholtz conditions of local variationality. By the way, one can see that in the infinite–jet formalism one loses information relatively to the order of the jet in which objects really 'live'.

We start by restricting our analysis to the following *short exact subcomplex*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \overset{1}{\Theta}_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & \overset{n+1}{\Theta}_r \arrow[r, "d"] \arrow[d] & d\overset{n+1}{\Theta}_r \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d] & \overset{0}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \overset{1}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & \overset{n+1}{\Lambda}_r \arrow[r, "d"] \arrow[d] & d\overset{n+1}{\Lambda}_r \arrow[r, "d"] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d] & \overset{0}{\Lambda}_r \arrow[r, "\mathcal{E}_0"] \arrow[d] & \overset{1}{\Lambda}_r/\overset{1}{\Theta}_r \arrow[r, "\mathcal{E}_1"] \arrow[d] & \dots \arrow[r, "\mathcal{E}_n"] & \mathcal{E}_n(\overset{I}{\Lambda}_r/\overset{I}{\Theta}_r) \arrow[r, "\mathcal{E}_{n+1}"] \arrow[d] & 0 \arrow[d] & \\
 & 0 & 0 & 0 & & 0 & 0 &
\end{tikzcd}$$

due to the fact that, to our knowledge, if $k \geq n+3$, there is no interpretation of the k^{th}–column of the variational bicomplex in terms of geometric objects of the variational calculus. We say the bottom row of the above bicomplex to be the *short variational sequence*.

3.1 Lagrangian

In this section, we show that the quotient sheaves

$$\overset{1}{\Lambda}_r/\overset{1}{\Theta}_r\,, \dots\,, \overset{n}{\Lambda}_r/\overset{n}{\Theta}_r$$

are isomorphic to certain subsheaves of sheaves of sections of a vector bundle. In this way, we are able to find an explicit expression for the sheaf morphisms $\mathcal{E}_0, \dots \mathcal{E}_{n-1}$.

Theorem 3.1.1. *Let $k \le n$. Then, the sheaf morphism h yields the isomorphism*

$$I_k : \overset{k}{\Lambda}_r/\overset{k}{\Theta}_r \to \overset{k}{\mathcal{H}}{}^h_{r+1} : [\alpha] \mapsto h(\alpha)\,.$$

PROOF. This is by the fact that, if $k \le n$, then $\overset{k}{\Theta}_r = \ker h$, and to the characterisation of the image of h of Theorem 1.4.1.

$\boxed{QED}$

Corollary 3.1.1. *The sheaf morphisms $\mathcal{E}_0, \dots, \mathcal{E}_{n-1}$ are expressed through the above isomorphisms I_k as*

$$\mathcal{E}_k(h(\alpha)) = \mathcal{E}_k(h(d\alpha))\,.$$

As an example, we have $\mathcal{E}_0 = d_h$. It is easy to compute coordinate expressions for $\mathcal{E}_0, \dots, \mathcal{E}_{n-1}$ via the above Corollary.

Definition 3.1.1. Let us set $\overset{k}{\mathcal{V}}_r := \overset{k}{\mathcal{H}}{}^h_{r+1}$.
We say a section $L \in \overset{n}{\mathcal{V}}_r$ to be a r*–th order generalised Lagrangian.* □

It is worth to note that the sheaf of the r–th order Lagrangians of the standard literature is $\overset{n}{\mathcal{H}}_r$, and that $\overset{n}{\mathcal{H}}_r \subset \overset{n}{\mathcal{H}}{}^h_{r+1}$ (see the Appendix).

3.2 Euler–Lagrange morphism

In this section we will show that the quotient sheaf $\overset{n+1}{\Lambda}{}_r/\overset{n+1}{\Theta}{}_r$ of the variational sequence is isomorphic to certain subsheaves of sheaves of sections of a vector bundle. In this way, we are able to find an explicit coordinate expression for the sheaf morphism $\mathcal{E}_n$.

Throughout this chapter we will adopt the notation

$$\overline{d_h}(\overset{k-n}{\mathcal{C}}{}_r \wedge \overset{n-1}{\mathcal{H}}{}^h_{r+1}) := \overline{d_h(\overset{k-n}{\mathcal{C}}{}_r \wedge \overset{n-1}{\mathcal{H}}{}^h_{r+1})}\,,$$

for evident practical reasons.

It is possible to introduce a first simplification of the quotient sheaves.

Lemma 3.2.1. *If $k > n$, then the restriction of h to the sheaf $\overset{k}{\Theta}_r$ is the surjective presheaf morphism*

$$h : \overset{k}{\Theta}_r \to h(\overline{d \ker h}) \,.$$

Moreover, pull–back yields the natural inclusion

$$h(\overline{d \ker h}) \subset \overline{d_h}(\overset{k-n}{\mathcal{C}}{}_r \wedge \overset{n-1}{\mathcal{H}}{}^h_{r+1}) \subset h(\overset{k}{\Theta}_{r+1}) = h(\overline{d \ker h}) \,,$$

which turns out to be an equality if $\dim \boldsymbol{X} = 1$*.*

PROOF. The first statement is obvious. We have the natural identification $\ker h \simeq v(\ker h)$, which yields

$$h(d \ker h) \simeq h(d_h v(\ker h) + d_v v(\ker h)) \simeq h d_h v(\ker h) \,,$$

due to Proposition 1.5.1. The same Proposition yields the inclusion

$$h d_h v(\ker h) \subset d_h(\overset{k-n}{\mathcal{C}}{}_r \wedge \overset{n-1}{\mathcal{H}}{}^h_{r+1}) \,,$$

hence the inclusions of the statement.

If $\dim \boldsymbol{X} = 1$, then $\ker h = \overset{k}{\mathcal{C}}_{(r,r-1)}$ [Kru95a, Vit95], hence the result. $\boxed{QED}$

Proposition 3.2.1. *Let $k > n$. Then, the projection h induces the natural sheaf isomorphism*

$$\left(\overset{k}{\Lambda}_r / \overset{k}{\Theta}_r\right) \to \left(\overset{k-n}{\mathcal{C}}{}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}\right) \Big/ h(\overline{d \ker h}) : [\alpha] \mapsto [h(\alpha)] \;.$$

PROOF. The map is clearly well defined.

Also, the map is injective, for if $\alpha, \alpha' \in \overset{k}{\Lambda}_r$, then

$$[h(\alpha)] = [h(\alpha')] \Rightarrow h(\alpha - \alpha') = h dp \;,$$

with $p \in \ker h$. Hence

$$\alpha - \alpha' = v(\alpha - \alpha' - dp) + dp \,,$$

where, being $dp \in \overset{k}{\Lambda}_r$ and $\alpha - \alpha' \in \overset{k}{\Lambda}_r$, we have $v(\alpha - \alpha' - dp) \in \overset{k}{\Lambda}_r$. Due to $h \circ v = 0$, we have $[\alpha - \alpha'] = 0$.

Finally, the map is surjective, due to the surjectivity of h. $\boxed{QED}$

Remark 3.2.1. In spite of the apparent complexity of the quotient sheaf $\left(\overset{k-n}{\mathcal{C}}{}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}\right) \big/ h(\overline{d \ker h})$, we notice that it is made with proper subsheaves of the sheaves $\overset{k}{\Lambda}_{r+1}$ and $\overset{k}{\Theta}_{r+1}$. Hence, our search for a natural representative in each equivalence class will be considerably simplified.

$\square$

Remark 3.2.2. Let $0 \leq s \leq r$. Then, the sheaf injection χ^r_s induces the sheaf injection

$$\left(\overset{k-n}{\mathcal{C}}{}_s \wedge \overset{n}{\mathcal{H}}{}^h_{s+1}\right) \big/ h(\overline{d \ker h}) \to \left(\overset{k-n}{\mathcal{C}}{}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}\right) \big/ h(\overline{d \ker h}) \, . \quad \square$$

As for the sheaf $\overset{n+1}{\Lambda}{}_r / \overset{n+1}{\Theta}{}_r$, taking into account the isomorphism of Proposition 3.2.1 and the identification of Lemma 3.2.1, we have two main tasks:

1. to find for all $\alpha \in \overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}$ a natural (and possibly unique, in some sense) n–form $F_\alpha \in h(\overline{d \ker h})$ in such a way that the sheaf morphism

$$I_{n+1} : \left(\overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}\right) \big/ h(\overline{d \ker h}) \to \overset{n+1}{\Lambda}{}_{r+s} : [\alpha] \mapsto \alpha + F_\alpha$$

   is injective (for some $s \in \mathbb{N}$);

2. to characterise the image of the above sheaf morphism, so to obtain a sheaf of sections of a vector bundle that is isomorphic to $\overset{n+1}{\Lambda}{}_r / \overset{n+1}{\Theta}{}_r$.

The above first problem can be solved by means of a result by Kolář [Kol83]. To proceed further, we need some notation. On the domain of any chart, we set

$$\omega := \sum_{\lambda_1, \ldots, \lambda_n = 1}^{n} d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_n} = n! d^1 \wedge \ldots \wedge d^n \, ,$$

$$\omega_\lambda := i_{\partial_\lambda} \omega \, , \qquad \omega_{\lambda\mu} := i_{\partial_\mu}(\omega_\lambda) \, .$$

We have

$$\sum_{\mu=1}^{n} d^\mu \wedge \omega_\lambda = \omega \, .$$

If $\boldsymbol{U} \subset \boldsymbol{Y}$ is a coordinate open subset and $f \in \left(\overset{0}{\Lambda}_r\right)_{\boldsymbol{U}}$, then we set, by induction

$$J_\lambda f := (д_{r+1})_\lambda f \, , \qquad J_{\underline{p}+\lambda} f := J_\lambda J_{\underline{p}} f \, ;$$

analogously, we denote by $L_{J_{\underline{p}}}$ the iterated Lie derivative. We have the characterisation

$$J_{\underline{p}} f \circ j_{r+|\underline{p}|} s = \partial_{\underline{p}} (f \circ j_r s) \,,$$

A Leibnitz' rule holds (see [Sau89]); if $g \in \left(\overset{0}{\Lambda}_r\right)_{\boldsymbol{U}}$, then we have

$$J_{\underline{p}}(fg) = \sum_{\underline{q}+\underline{t}=\underline{p}} \frac{\underline{p}!}{\underline{q}!\underline{t}!} J_{\underline{q}} f \, J_{\underline{t}} g \,.$$

Let $u : \boldsymbol{Y} \to V\boldsymbol{Y}$ be a vertical vector field with coordinate expression $u = u^i \partial_i$. Then, the coordinate expression of the prolongation $u_r : J_r\boldsymbol{Y} \to V J_r \boldsymbol{Y}$ is $u_r = J_{\underline{p}} u^i \partial_i^{\underline{p}}$.

Theorem 3.2.1. *(First variation formula for higher–order variational calculus [Kol83]) Let* $\alpha \in \overset{1}{\Lambda}_r \wedge \overset{n}{\mathcal{H}}_r \simeq \overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}_r$. *Then there is a unique pair of sheaf morphisms*

$$E_\alpha \in \overset{1}{\mathcal{C}}_{(2r,0)} \wedge \overset{n}{\mathcal{H}}_{2r} \,, \qquad F_\alpha \in \overset{1}{\mathcal{C}}_{(2r,r)} \wedge \overset{n}{\mathcal{H}}_{2r} \,,$$

such that

i. $(\pi^{2r}_r)^* \alpha = E_\alpha - F_\alpha$;

ii. F_α *is locally of the form* $F_\alpha = d_h p_\alpha$, *with* $p_\alpha \in \overset{1}{\mathcal{C}}_{(2r-1,r-1)} \wedge \overset{n}{\mathcal{H}}_{2r}$.

Proof. The proof is carried on by induction. We set, in a coordinate neighbourhood,

$$\alpha = \alpha^{\underline{p}}_i \, \vartheta^i_{\underline{p}} \wedge \omega \,, \qquad p_\alpha = p^{\underline{q}\lambda}_i \, \vartheta^i_{\underline{q}} \wedge \omega_\lambda \,, \qquad E_\alpha = E_i \, \vartheta^i \wedge \omega \,,$$

where $0 \leq |\underline{p}| \leq r$ and $0 \leq |\underline{q}| \leq r-1$, hence we have

$$d_h p_\alpha = -J_\lambda p^{\underline{q}\lambda}_i \, \vartheta^i_{\underline{q}} \wedge \omega - p^{\underline{q}\lambda}_i \, \vartheta^i_{\underline{q}+\lambda} \wedge \omega \,.$$

The requirements on E_α and on p_α yield the vanishing of some components of the sum $\alpha + d_h p_\alpha$, hence a system of linear equations which has a unique pair of local solutions E_α and $d_h p_\alpha$. In particular, we have

$$E_\alpha = (-1)^{|\underline{p}|} J_{\underline{p}} \alpha^{\underline{p}}_i \, \vartheta^i \wedge \omega \,, \qquad 0 \leq |\underline{p}| \leq r \,. \tag{3.1}$$

The uniqueness ensures that E_α and $d_h p_\alpha$ are intrinsically characterised, hence they yield two sections E_α and F_α which fulfill the requirement of the statement; in particular, they have the same domain of definition as α. $\boxed{QED}$

Remark 3.2.3. As it is proved in [Kol83], to any F_α there always exists a section $p_\alpha \in \overset{1}{\mathcal{C}}_{(2r-1,r-1)} \wedge \overset{n}{\mathcal{H}}_{2r}$ (with the same domain as α) such that F_α. But in general such a p_α is not unique. In fact [Kol83], by adding to p_α the horizontal differential of a suitable $n-1$–form we obtain another form which fulfills the conditions of the statement. Anyway, we have some particular cases where a form p_α can be uniquely determined.

1. Suppose that $\dim \boldsymbol{X} = 1$. Then, it can be easily proved [AnTh92, Kru95a, Vit95] that $d_h p_\alpha = 0$ implies $p_\alpha = 0$, so that p_α is uniquely determined.

2. Suppose that $r = 1$. Then, one can easily realise that there does not exists a $n-1$–form such that its horizontal differential is a section of $\overset{1}{\mathcal{C}}_{(2r-1,r-1)} \wedge \overset{n-1}{\mathcal{H}}_{2r-1}$, so that p_α is uniquely determined. In this case, we can say even more. In fact, we are able to determine p_α from α by means of the natural sheaf morphism
$$p : \overset{1}{\mathcal{C}}_1 \wedge \overset{n}{\mathcal{H}}_1 \to \overset{1}{\mathcal{C}}_{(1,0)} \wedge \overset{n-1}{\mathcal{H}}_1 \,.$$
which fulfills
$$p(d_h\beta) = -\beta \qquad\qquad \forall\beta \in \overset{1}{\mathcal{C}}_{(1,0)} \wedge \overset{n-1}{\mathcal{H}}_1 \,.$$
The above morphism was introduced in different forms by several authors [Kol93, MaMo83b, Sau89, Vit95]. If
$$\alpha = \alpha_i \, d^i \wedge \omega + \alpha_i^\lambda \, d^i_\lambda \wedge \omega \,,$$
then we have the coordinate expression
$$p(\alpha) = \alpha_i^\lambda \, \vartheta^i \wedge \omega_\lambda \,.$$

3. In the case $r = 2$ we are able to characterise a unique p_α by means of an additional requirement. There is a natural morphism
$$s : \overset{1}{\mathcal{C}}_{(3,1)} \wedge \overset{n-1}{\mathcal{H}}_3 \to \overset{1}{\mathcal{C}}_{(3,0)} \wedge \overset{n-2}{\mathcal{H}}_3$$
where, if $p \in \overset{1}{\mathcal{C}}_{(3,1)} \wedge \overset{n-1}{\mathcal{H}}_3$ has the coordinate expression
$$p = p_i{}^\mu \, \vartheta^i \wedge \omega_\mu + p_i^{\lambda\mu} \, \vartheta^i_\lambda \wedge \omega_\mu \,,$$
then we have
$$s(p) = p_i^{\lambda\mu} \, \vartheta^i \wedge \omega_{\mu\lambda} \,.$$
It is easily proved that there exists a unique morphism $p_\alpha \in \overset{1}{\mathcal{C}}_{(3,1)} \wedge \overset{n-1}{\mathcal{H}}_3$ such that $s(p_\alpha) = 0$. Such a morphism is called *quasisymmetric*. This result has been shown in [Kol83]. In particular, if we have the coordinate expression
$$\alpha = \alpha_i \, \vartheta^i \wedge \omega + \alpha_i^\lambda \, \vartheta^i_\lambda \wedge \omega + \alpha_i^{\lambda+\mu} \, \vartheta^i_{\lambda+\mu} \wedge \omega \,,$$
then we have
$$p_\alpha = (\alpha_i^\lambda - J_\mu \alpha_i^{\mu+\lambda}) \, \vartheta^i \wedge \omega_\lambda + \alpha_i^{\mu+\lambda} \, \vartheta^i_\mu \wedge \omega_\lambda \,.$$

4. In the case $r \geq 3$ we have no natural ways to select a form p_α. In [Kol83] a sufficient condition to the uniqueness of p_α is given. Namely, let us denote with $T^*_s\boldsymbol{X}$ the s–th order cotangent bundle. Let $\Gamma : T^*\boldsymbol{X} \to T^*_{r-1}\boldsymbol{X}$ be a section which is also a linear morphism. Then, there exists a unique form $p_\alpha[\Gamma]$. Note that, if $r = 2$, then we have the natural choice $\Gamma = \mathrm{id}$, which yields the natural form of the previous item.

□

Remark 3.2.4. The choice of the subsheaf

$$\overset{1}{\mathcal{C}}_{(2r,0)} \wedge \overset{n}{\mathcal{H}}_{2r} \subset \overset{1}{\mathcal{C}}_{(2r,2r-1)} \wedge \overset{n}{\mathcal{H}}_{2r}$$

will provide representatives E_α of sections of the quotient sheaf $\overset{2}{\Lambda}_1/\overset{2}{\Theta}_1$ with a minimal number of components. □

Remark 3.2.5. The section $E_\alpha \in \overset{1}{\mathcal{C}}_{(2r,0)} \wedge \overset{n}{\mathcal{H}}_{2r}$ has a peculiar structure with respect to the derivative coordinates of order greater than r. In fact, if we assign to the variables $y^i_{\underline{p}}$ with $|\underline{p}| = r + s$ the weight s, then it is easily seen that E_α is a polynomial with weighted degree r with respect to $y^i_{\underline{p}}$, with $|\underline{p}| > r$. This kind of structure was first introduced and studied in [KoMo90]. □

Corollary 3.2.1. *Let $\alpha \in \overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1} \subset \overset{1}{\Lambda}_{r+1} \wedge \overset{n}{\mathcal{H}}_{r+1}$. Then E_α and p_α are sections of the following subsheaves*

$$E_\alpha \in \overset{1}{\mathcal{C}}_{(2r,0)} \wedge \overset{n}{\mathcal{H}}{}^h_{2r+1} \,, \qquad p_\alpha \in \overset{1}{\mathcal{C}}_{(2r,r-1)} \wedge \overset{n-1}{\mathcal{H}}{}^h_{2r} \,.$$

Proof. This depends on the form of the system

$$\alpha = E_\alpha - d_h p_\alpha \,,$$

and on the fact that the form α takes values into the vector bundle $T^*J_r\boldsymbol{Y} \wedge T^*\boldsymbol{X}$, even if it depends on $J_{r+1}\boldsymbol{Y}$. QED

Remark 3.2.6. In the case $\alpha \in \overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}$, the section E_α has an additional feature with respect to the polynomial structure of Remark 3.2.5. In fact, the coefficients of the polynomial are polynomials of (standard) degree n with respect to the coordinates $y^i_{\underline{p}}$, with $|\underline{p}| = r + 1$. □

Theorem 3.2.2. *Let $q \in \overset{1}{\mathcal{C}}_{(2r-1,r-1)} \wedge \overset{n-1}{\mathcal{H}}_{2r-1}$. Then we have*

$$d_h p_{d_h q} = -d_h q \,,$$

hence $E_{d_h q} = 0$.

PROOF. In fact, by recalling the proof of the above Theorem, we find that the system

$$d_h q = E_{d_h q} - d_h p_{d_h q}$$

has the unique solutions $d_h p_{d_h q} = -d_h q$ and $E_{d_h q} = 0$. $\boxed{QED}$

Remark 3.2.7. The above Theorem is the geometric interpretation of the well–known fact that 'the Euler–Lagrange morphism annihilates divergencies' (see also [Tra96]). □

Proposition 3.2.2. *The sheaf morphism*

$$\overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1} \to \overset{n+1}{\Lambda}{}_{2r+1} : \alpha \mapsto \alpha + F_\alpha$$

induces the injective sheaf morphism

$$I_{n+1} : \left(\overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1} \right) \Big/ h(\overline{d \ker h}) \to \overset{n+1}{\Lambda}{}_{2r+1} : [\alpha] \mapsto \alpha + F_\alpha$$

PROOF. We make use of the injective morphism χ^s_r of Remark 3.2.2 and of Lemma 3.2.1. The morphism I_{n+1} is well–defined, due to Corollary 3.2.1 and to the fact that, if $\alpha, \beta \in \overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}$ such that $\beta = \alpha + F$, where F is of the local form $F = d_h q$ with $d_h q \in d \ker h$ then

$$\beta + F_\beta = \alpha + F + F_\alpha + F_F\,,$$

where $F_F = -F$ by the uniqueness in Theorem 3.2.1.

We have to prove that the morphism is injective. Suppose that

$$\beta + F_\beta = \alpha + F_\alpha\ .$$

Hence $\beta - \alpha = F_\alpha - F_\beta$, so $[\beta - \alpha] = 0$. $\boxed{QED}$

The final step is to characterise the image of I_{n+1}.

Theorem 3.2.3. *We have the sheaf isomorphism*

$$I_{n+1} : \overset{n+1}{\Lambda}{}_r / \overset{n+1}{\Theta}{}_r \to \overset{n+1}{\mathcal{V}}{}_r,$$

where

$$\overset{n+1}{\mathcal{V}}{}_r := \left(\overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1} + \overline{d_h}(\overset{1}{\mathcal{C}}_{(2r,r-1)} \wedge \overset{n-1}{\mathcal{H}}{}_{2r}) \right) \cap \left(\overset{1}{\mathcal{C}}_{(2r+1,0)} \wedge \overset{n}{\mathcal{H}}_{2r+1} \right)\ .$$

PROOF. It comes from the isomorphism of Proposition 3.2.1, the injective morphism I_{n+1} and the characterisation of the image of I_{n+1} provided by Theorem 3.2.1. $\boxed{QED}$

Now, we can evaluate $\mathcal{E}_n$ by means of the isomorphisms I_n, I_{n+1}.

Theorem 3.2.4. *Let* $\alpha \in \overset{n}{\mathcal{V}}_r$. *Then,* $\mathcal{E}_n(\alpha) \in \overset{n+1}{\mathcal{V}}_r$ *coincides with the standard higher–order Euler–Lagrange morphism [Fer83, FeFr82, GaMu82, Kol83, Kru83, Sau89] associated with the generalised* r*–th order Lagrangian* α*, regarded as a standard* $r+1$*–th order Lagrangian.*

Proof. In fact, Theorem 3.2.1 yields the standard higher–order Euler–Lagrange morphism. Moreover, we have the inclusions

$$\overset{n}{\mathcal{V}}_r \subset \overset{n}{\mathcal{H}}_{r+1} \subset \overset{n}{\mathcal{V}}_{r+1} \,.$$

The result now is immediate, due to the commutativity of the inclusion of the bicomplex of order r into the bicomplex of order $r+1$ (Remark 2.2.1). $\boxed{QED}$

Definition 3.2.1. Let $\alpha \in \overset{n+1}{\Lambda}_r$.

We say $E_{h(\alpha)} \in \overset{n+1}{\mathcal{V}}_r$ to be the *generalised* r*–th order Euler–Lagrange morphism* associated with α.

We say $p_{h(\alpha)}$ to be a *generalised* r *th order momentum* associated with $E_{h(\alpha)}$.

We say $\mathcal{E}_n$ to be the *generalised* r*–th order Euler–Lagrange operator.* □

Remark 3.2.8. It is of fundamental importance to note that some theories which are based upon polynomial $(r+1)$–th order horizontal Lagrangians can be seen also as r–th order theories using a non–horizontal Lagrangian (see Appendix). And it is worth to point out that most of second–order horizontal Lagrangians known in physics are affine. □

3.3 Helmholtz morphism

In this section we will devote ourselves to a description of the presheaf

$$\mathcal{E}_{n+1}\left(\overset{n+1}{\mathcal{V}}_r\right) \simeq \mathcal{E}_{n+1}\left(\overset{n+1}{\Lambda}_r / \overset{n+1}{\Theta}_r\right) = \left(d\overset{n+1}{\Lambda}_r / d\overset{n+1}{\Theta}_r\right) \,.$$

In particular, we will find an isomorphism of this presheaf with a subpresheaf of a sheaf of sections of a vector bundle. Hence, we will be able to provide an explicit expression for the map $\mathcal{E}_{n+1}$. This will yield an intrinsic geometric object whose vanishing is equivalent to the Helmholtz conditions of local variationality.

Let $E \in \overset{n+1}{\mathcal{V}}_r$. In order to evaluate the expression of $\mathcal{E}_{n+1}(E)$, it is very difficult to find a $n+1$–form $\alpha \in \overset{n+1}{\Lambda}_s$ such that $I_{n+1}([h(\alpha)]) = E$. So, it is difficult in concrete cases to use the commutativity of the diagram in order to compute $\mathcal{E}_{n+1}$.

Hence, the most convenient way to reach our task is to use Theorem 3.2.1 together with the isomorphism $\overset{n+1}{\Lambda}{}_r/\overset{n+1}{\Theta}{}_r \to \overset{n+1}{\mathcal{V}}{}_r$ in order to simplify the analysis of the sheaf $\mathcal{E}_{n+1}\left(\overset{n+1}{\Lambda}{}_r/\overset{n+1}{\Theta}{}_r\right)$.

Lemma 3.3.1. *We have the natural injection*

$$\left(d\overset{n+1}{\Lambda}{}_r/d\overset{n+1}{\Theta}{}_r\right) \to \left(\overset{2}{\mathcal{C}}_{2r+1} \wedge \overset{n}{\mathcal{H}}{}^h_{2r+2}\right) \Big/ h(\overline{d\ker h}) : [d\alpha] \mapsto [dE_{h(\alpha)}] \,.$$

Proof. It is a direct consequence of the decomposition

$$\alpha = E_{h(\alpha)} - d_h p_{h(\alpha)} + v(\alpha)\,.$$

together with $dd_h = -d_h d_v$. $\boxed{QED}$

Hence, we search for natural representatives of the classes of the image of

$$d\overset{n+1}{\mathcal{V}}{}_r \subset \overset{1}{\Lambda}_{2r+1} \wedge \overset{1}{\mathcal{C}}_{(2r+1,0)} \wedge \overset{n}{\mathcal{H}}_{2r+1} \simeq \overset{1}{\mathcal{C}}_{2r+1} \wedge \overset{1}{\mathcal{C}}_{(2r+1,0)} \wedge \overset{n}{\mathcal{H}}_{2r+1} \subset \overset{2}{\mathcal{C}}_{2r+1} \wedge \overset{n}{\mathcal{H}}{}^h_{2r+2}$$

into the quotient $\left(\overset{2}{\mathcal{C}}_{2r+1} \wedge \overset{n}{\mathcal{H}}{}^h_{2r+2}\right) \Big/ h(\overline{d\ker h})$; we denote this image by $[d\overset{n+1}{\mathcal{V}}{}_r]$.

Our task is the following one: *to characterise a unique representative of every equivalence class of $[d\overset{n+1}{\mathcal{V}}{}_r]$ by means of the higher–order Euler–Lagrange morphism.*

First of all, we need a technical Lemma.

Lemma 3.3.2. *Let $\beta \in \overset{1}{\mathcal{C}}_s \wedge \overset{1}{\mathcal{C}}_{(s,0)} \wedge \overset{n}{\mathcal{H}}_s$. Suppose that the coordinate expression of β is*

$$\beta = \beta^{\underline{p}}_{i\,j}\vartheta^i_{\underline{p}} \wedge \vartheta^j \wedge \omega\,, \qquad 0 \le |\underline{p}| \le s\,.$$

Let $u : \boldsymbol{Y} \to V\boldsymbol{Y}$ be a vertical vector field, with coordinate expression $u = u^i\partial_i$, and set

$$\hat{\beta} := i_{u_s}\beta\,.$$

Then we have $E_{\hat{\beta}} = e_j\vartheta^j \wedge \omega$, with

$$e_j \;=\; J_{\underline{p}}u^i\left(\beta^{\underline{p}}_{i\,j} - \sum_{|\underline{q}|=0}^{k-|\underline{p}|}(-1)^{|\underline{p}+\underline{q}|}\frac{(\underline{p}+\underline{q})!}{\underline{p}!\,\underline{q}!}J_{\underline{q}}\beta^{\underline{p}+\underline{q}}_{j\ \ i}\right)\,,$$

where $0 \le |\underline{p}| \le s$.

Proof. It follows from the coordinate expression of $E_{\hat{\beta}}$ and the Leibnitz'rule for $J_{\underline{p}}$. $\boxed{QED}$

Lemma 3.3.3. *Let $\beta \in \overset{1}{\mathcal{C}}_s \wedge \overset{1}{\mathcal{C}}_{(s,0)} \wedge \overset{n}{\mathcal{H}}_s$. Then, there is a unique*

$$\tilde{H}_\beta \in \overset{1}{\mathcal{C}}_{(2s,s)} \otimes \overset{1}{\mathcal{C}}_{(2s,0)} \wedge \overset{n}{\mathcal{H}}_{2s}$$

such that, for all $u : \boldsymbol{Y} \to V\boldsymbol{Y}$,

$$E_{\hat{\beta}} = C^1_1 \left(u_{2s} \otimes \tilde{H}_\beta \right) ,$$

where $\hat{\beta} := i_{u_s}\beta$, and C^1_1 stands for tensor contraction.

Proof. Let $\boldsymbol{U} \subset \boldsymbol{Y}$ be an open coordinate subset, and suppose that we have the expression on $\boldsymbol{U}$

$$\beta = \beta^{\underline{p}}_{i\,j} \vartheta^i_{\underline{p}} \wedge \vartheta^j \wedge \omega \,, \qquad 0 \leq |\underline{p}| \leq s \,.$$

Then we have the coordinate expression

$$E_{\hat{\beta}} = J_{\underline{p}} u^i \left(\beta^{\underline{p}}_{i\,j} - \sum_{|\underline{q}|=0}^{s-|\underline{p}|} (-1)^{|\underline{p}+\underline{q}|} \frac{(\underline{p}+\underline{q})!}{\underline{p}!\underline{q}!} J_{\underline{q}} \beta^{\underline{p}+\underline{q}}_{j\ \ i} \right) \vartheta^j \wedge \omega \,.$$

Let us set

$$\tilde{H}_\beta[\boldsymbol{U}] := \left(\beta^{\underline{p}}_{i\,j} - \sum_{|\underline{q}|=0}^{s-|\underline{p}|} (-1)^{|\underline{p}+\underline{q}|} \frac{(\underline{p}+\underline{q})!}{\underline{p}!\underline{q}!} J_{\underline{q}} \beta^{\underline{p}+\underline{q}}_{j\ \ i} \right) \vartheta^i_{\underline{p}} \otimes \vartheta^j \wedge \omega \,.$$

Then, by the arbitrariness of u, $\tilde{H}_\beta[\boldsymbol{U}]$ is the unique morphism fulfilling the conditions of the statement on $\boldsymbol{U}$.

If $\boldsymbol{V} \subset \boldsymbol{Y}$ is another open coordinate subset and $\boldsymbol{U} \cap \boldsymbol{V} \neq \emptyset$, then, by uniqueness, we have $\tilde{H}_\beta[\boldsymbol{U}]|_{\boldsymbol{U}\cap\boldsymbol{V}} = \tilde{H}_\beta[\boldsymbol{V}]|_{\boldsymbol{U}\cap\boldsymbol{V}}$. Hence, we obtain the result by setting $\tilde{H}_\beta|_{\boldsymbol{U}} := \tilde{H}_\beta[\boldsymbol{U}]$ on any coordinate open subset $\boldsymbol{U} \subset \boldsymbol{Y}$. $\boxed{QED}$

Theorem 3.3.1. *(Generalised second variation formula).*
Let $\beta \in \overset{1}{\mathcal{C}}_s \wedge \overset{1}{\mathcal{C}}_{(s,0)} \wedge \overset{n}{\mathcal{H}}_s$. Then, there is a unique pair of sheaf morphisms

$$H_\beta \in \overset{1}{\mathcal{C}}_{(2s,s)} \wedge \overset{1}{\mathcal{C}}_{(2s,0)} \wedge \overset{n}{\mathcal{H}}_{2s} \,, \quad G_\beta \in \overset{2}{\mathcal{C}}_{(2s,s)} \wedge \overset{n}{\mathcal{H}}_{2s} \,,$$

such that

i. $\pi^{2s*}_s \beta = H_\beta - G_\beta$

ii. $H_\beta = 1/2\, A(\tilde{H}_\beta)$, where A is the antisymmetrisation map.

Moreover, G_β is locally of the type $G_\beta = d_h q_\beta$, where $q_\beta \in \overset{2}{\mathcal{C}}_{2s-1} \wedge \overset{n-1}{\mathcal{H}}_{2s-1}$, hence $[\beta] = [H_\beta]$.

Proof. It is clear that G_β is uniquely determined by β and the choice $H_\beta = 1/2\, A(\tilde{H}_\beta)$.

Moreover, it can be easily seen [Sau89] by induction on $|\underline{p}|$ that, on a coordinate open subset $\boldsymbol{U} \subset \boldsymbol{Y}$, we have

$$\beta = \beta^{\underline{p}}_{i\,j} \vartheta^i_{\underline{p}} \wedge \vartheta^j \wedge \omega = \beta^{\underline{p}}_{i\,j} L_{\underline{p}}(\vartheta^i) \wedge \vartheta^j \wedge \omega = (-1)^{|\underline{p}|} \vartheta^i \wedge L_{\underline{p}}(\beta^{\underline{p}}_{i\,j} \vartheta^j) \wedge \omega + 2 d_h q_\beta \,,$$

which yields the thesis by the Leibnitz' rule, the injective morphism χ^s_r of remark 2.2.1, and the inclusions (3.2.1) (a similar local result can be found in [Bau82, Kru90]).

$\boxed{QED}$

Remark 3.3.1. In general, the section q_β is not uniquely characterised. But, if $\dim \boldsymbol{X} = 1$, then there exists a unique q_β fulfilling the conditions of the statement of the above theorem. □

Corollary 3.3.1. *The presheaf* $\mathcal{E}_{n+1}\left(\overset{n+1}{\mathcal{V}}{}_r\right)$ *is isomorphic to the image of the injective morphism*

$$I_{n+2} : \left(d \overset{n+1}{\Lambda}{}_r / d \overset{n+1}{\Theta}{}_r \right) \to \overset{1}{\mathcal{C}}_{4r+1} \wedge \overset{1}{\mathcal{C}}_{(4r+1,0)} \wedge \overset{n}{\mathcal{H}}_{4r+1} : [d\alpha] \mapsto H_{dE_{h(\alpha)}} \ .$$

Proof. I_{n+2} is well defined, because, recalling Lemma 3.3.1, if we add a suitable form G of the local type $G = d_h q$ to $dE_{h(\alpha)}$, the uniqueness of the decomposition of the generalised second variation formula (see also the above Corollary) yields $H_G = 0$. Moreover, I_{n+2} is valued into

$$\overset{1}{\mathcal{C}}_{4r+1} \wedge \overset{1}{\mathcal{C}}_{(4r+1,0)} \wedge \overset{n}{\mathcal{H}}_{4r+1} \subset \overset{1}{\mathcal{C}}_{4r+2} \wedge \overset{1}{\mathcal{C}}_{(4r+2,0)} \wedge \overset{n}{\mathcal{H}}_{4r+2}$$

due to the coordinate expression of $E_{h(\alpha)}$ and $H_{dE_{h(\alpha)}}$; more precisely, being $E_{h(\alpha)}$ affine with respect to the highest order derivatives, such derivatives disappear from the coefficient of $dE_{h(\alpha)}$ which produces the higher order coefficient of $H_{dE_{h(\alpha)}}$. The injectivity of I_{n+2} follows from Lemma 3.3.1 and the above Corollary, because if $dE_{h(\alpha)}$ and $dE_{h(\beta)}$ fulfill $H_{dE_{h(\alpha)}} = H_{dE_{h(\beta)}}$, then we have

$$dE_{h(\alpha)} - dE_{h(\beta)} = G_{dE_{h(\beta)}} - G_{dE_{h(\alpha)}} \,,$$

hence $[dE_{h(\alpha)} - dE_{h(\beta)}] = 0$. $\boxed{QED}$

Remark 3.3.2. Unlike the Euler–Lagrange morphism, the Helmholtz morphism is not characterised as being a section of a particular subsheaf. Anyway, the vanishing of $[d\alpha]$ is completely equivalent to the vanishing of $H_{d\alpha}$. See also [And86, GiMa90] for a derivation of the Helmholtz conditions as Euler–Lagrange equations. Also, it is evident that the vanishing of $H_{d\alpha}$ is a weaker condition than the vanishing of $d\alpha$. □

Corollary 3.3.2. *The sheaf morphism $\mathcal{E}_{n+1}$ can be expressed via I_{n+1} and I_{n+2} by*

$$\mathcal{E}_{n+1} : \overset{n+1}{\mathcal{V}}_r \to \overset{1}{\mathcal{C}}_{4r+1} \wedge \overset{1}{\mathcal{C}}_{(4r+1,0)} \wedge \overset{n}{\mathcal{H}}_{4r+1} : E \mapsto H_{dE} \;.$$

Moreover, if the coordinate expression of E is $E = E_j \vartheta^j \wedge \omega$, then the coordinate expression of $\mathcal{E}_{n+1}(E)$ is

$$\mathcal{E}_{n+1}(E) \;= \frac{1}{2}\left(\partial_i^{\underline{p}} E_j - \sum_{|\underline{q}|=0}^{2r+1-|\underline{p}|} (-1)^{|\underline{p}+\underline{q}|} \, \frac{(\underline{p}+\underline{q})!}{\underline{p}!\underline{q}!} \, J_{\underline{q}} \partial_j^{\underline{p}+\underline{q}} E_i \right) \vartheta^i_{\underline{p}} \wedge \vartheta^j \wedge \omega \,.$$

Definition 3.3.1. Let $\alpha \in \overset{n+1}{\Lambda}_r$.

We say $H_{dE_{h(\alpha)}}$ to be the *generalised r–th order Helmholtz morphism.*

We say $q_{dE_{h(\alpha)}}$ to be a *generalised r–th order momentum* associated to the Helmholtz morphism.

We say $\mathcal{E}_{n+1}$ to be the *generalised r–th order Helmholtz operator.* □

Remark 3.3.3. In this section we have obtained an intrinsic Helmholtz morphism that is associated to each first–order generalised Euler–Lagrange morphism via the sheaf morphism $\mathcal{E}_{n+1}$. The vanishing of the Helmholtz morphism is completely equivalent to the standard local Helmholtz conditions (see, for example, [AnDu80, And86, Bau82, GiMa90, Kru90, LaTu77, Ton69]).

As a by–product, to each first–order generalised Euler–Lagrange morphism $E \in \overset{n+1}{\mathcal{V}}_r$ we find a unique intrinsic contact two–form G_{dE}, where $G_{dE} = d_h q_{dE}$ locally; q plays a role analogous to that of p. □

3.4 Inverse problems

In this section, we show that the results of the above sections together with the exactness of the variational sequence yield the solution for two important inverse problems: the minimal order variationally trivial Lagrangians and the minimal order Lagrangian corresponding to a locally variational Euler–Lagrange morphism. As for trivial Lagrangians, our result agrees with the local result of [Gri99b, KrMu99].

We can summarise the results of the above sections in the following theorem.

Theorem 3.4.1. *The r–th order short variational sequence is isomorphic to the exact sequence*

$$0 \longrightarrow I\!R \longrightarrow \overset{0}{\Lambda}_r \xrightarrow{\mathcal{E}_0} \overset{1}{\mathcal{V}}_r \xrightarrow{\mathcal{E}_1} \dots$$
$$\dots \xrightarrow{\mathcal{E}_{n-1}} \overset{n}{\mathcal{V}}_r \xrightarrow{\mathcal{E}_n} \overset{n+1}{\mathcal{V}}_r \xrightarrow{\mathcal{E}_{n+1}} \mathcal{E}_{n+1}\left(\overset{n+1}{\mathcal{V}}_r\right) \xrightarrow{\mathcal{E}_{n+2}} 0 \;,$$

We have two main consequences of the exactness of the above sequence.

Corollary 3.4.1. *Let $L \in (\overset{n}{\mathcal{V}}_r)_{\boldsymbol{Y}}$ such that $\mathcal{E}_n(L) = 0$. Then, for any $y \in \boldsymbol{Y}$ there exist an open neighbourhood $\boldsymbol{U} \subset \boldsymbol{Y}$ of y and a section $T \in (\overset{n-1}{\mathcal{V}}_r)_{\boldsymbol{U}}$ such that $\mathcal{E}_{n-1}(T) = L$. If $H^n_{de\ Rham}\boldsymbol{Y} = 0$, then we can choose $\boldsymbol{U} = \boldsymbol{Y}$.*

Proof. The first statement comes from the definition of exactness for a sheaf sequence. The second statement comes from the abstract de Rham theorem; in fact, the (long) variational sequence is a (soft) resolution of the constant sheaf $\mathbb{R}$ (see [Kru90, Wel80]). $\boxed{QED}$

Definition 3.4.1. Let $L \in (\overset{n}{\mathcal{V}}_r)_{\boldsymbol{Y}}$ such that $\mathcal{E}_n(L) = 0$. We say L to be a variationally trivial r–th order (generalised) Lagrangian.

□

Remark 3.4.1. If $L \in \overset{n}{\mathcal{V}}_r$ is variationally trivial, then L is (locally) of the form $L = \mathcal{E}_{n-1}(h(\alpha)) = d_h\alpha$, with $\alpha \in \overset{n-1}{\Lambda}_r$.

We stress that a similar result is obtained in [Kru93], but with a computational proof. □

As for $(\overset{n+1}{\mathcal{V}}_r)_{\boldsymbol{Y}}$, we have a result which is analogous to the above corollary, and justifies the following definition.

Definition 3.4.2. Let $E \in (\overset{n+1}{\mathcal{V}}_r)_{\boldsymbol{Y}}$. If $\mathcal{E}_{n+1}(E) = 0$, then we say E to be a locally variational (generalised) r–th order Euler–Lagrange morphism. □

So, to any locally variational Euler–Lagrange morphism there exists a local Lagrangian whose associated Euler–Lagrange morphism (locally) coincides with the given one. This is a well–known fact in the theory of infinite order Lagrangian sequences, but the novelty provided by our approach is the *minimality* of the order of the local Lagrangian. In fact, we have the following obvious proposition.

Proposition 3.4.1. *Let $E \in (\overset{n+1}{\mathcal{V}}_r)_{\boldsymbol{Y}}$ such that $E \notin (\overset{n+1}{\mathcal{V}}_{r-1})_{\boldsymbol{Y}}$. Let E be locally variational. Then, for any (local) Lagrangian $L \in \overset{n}{\mathcal{V}}_r$ of E, we have $L \notin \overset{n}{\mathcal{V}}_{r-1}$.*

Remark 3.4.2. In the literature there are similar results [AnDu80, And86, AnTh92], but proofs are done by computations. The finite order variational sequence provides a structural answer to the minimal order Lagrangian problem. □

Remark 3.4.3. We stress that a minimal order Lagrangian $L \in \overset{n}{\mathcal{V}}_r$ for a locally variational Euler–Lagrange morphism $E \in \overset{n+1}{\mathcal{V}}_r$ can be *explicitly* computed.

Namely, we pick an $\alpha \in \overset{n+2}{\Lambda}_r$ corresponding to the Euler–Lagrange morphism (i.e. , $I_{n+1}(h(\alpha)) = E$), and apply the contact homotopy operator (which is just the restriction of the Poincaré's homotopy operator to $\overset{n+2}{\Theta}_r$) to the closed form $d\alpha \in \overset{n+2}{\Theta}_r$, finding $\beta \in \overset{n+1}{\Theta}_r$ such that $d\beta = d\alpha$. By using once again using the (standard) homotopy operator we find $\gamma \in \overset{n}{\Lambda}_r$ such that $d\gamma = \beta - \alpha$: $L := I_n(\gamma)$ is the minimal order Lagrangian.

We recall that the well–known Volterra–Vainberg method for finding a Lagrangian for E yields a $(2r+1)$–th order Lagrangian. □

Appendix: calculus of variations

In this Appendix we give the intrinsic geometrical setting for the calculus of variations in Lagrangian mechanics ([GoSt73, Kru73, Tul75, Gar74, FeFr82, GaMu82, Fer83, Kol83, MaMo83b, Cos94]). The aim is to justify the choice of the contact subsequence in the variational bicomplex, and to give an interpretation of the results of paper.

Suppose that a section $L \in \overset{n}{\mathcal{H}}_r$ is given. Then the *action* of an r–th Lagrangian L on a section $s : \boldsymbol{I} \to \boldsymbol{Y}$ ($\boldsymbol{I}$ is an orientable open subset of $\boldsymbol{X}$ with compact closure and regular boundary) is defined to be the real number

$$\int_{\boldsymbol{I}} (j_r s)^* L \ .$$

A vertical vector field $u : \boldsymbol{Y} \to V\boldsymbol{Y}$ defined on $\pi^{-1}(\boldsymbol{I})$ and vanishing on $\pi^{-1}(\partial \boldsymbol{I})$ is said to be a *variation field*.

A section $s : \boldsymbol{I} \to \boldsymbol{Y}$ is said to be *critical* if, for each variation field with flow ϕ_p, we have

$$\delta \int_{\boldsymbol{I}} (J_r\phi_p \circ j_r s)^* L = 0 \ ,$$

where δ is the variational derivative with respect to the parameter p, and $J_r\phi_p : J_r\boldsymbol{Y} \to J_r\boldsymbol{Y}$ is first jet prolongation of the morphism ϕ_p (see [MaMo83a]).

The derivative δ commutes with $\int_{\boldsymbol{I}}$, so that the above condition is equivalent to

$$\int_{\boldsymbol{I}} (j_r s)^* \mathrm{L}_{u_r} L = 0$$

for each variation field u, where $u_r : J_r\boldsymbol{Y} \to VJ_r\boldsymbol{Y}$ is the r–th jet prolongation of u (see the first section), and L_{u_r} stands for the Lie derivative.

Using the splitting of Proposition 3.2.1 (or, equivalently, adding the form p_{dL} to L) together with $\mathrm{L}_{u_r} L = i_{u_r} dL$ and the Stokes' theorem, we find that the above equation is equivalent to

$$\int_{\boldsymbol{I}} (j_{2r}s)^*(i_u E_{dL}) = 0$$

for each variation field u. Finally, by virtue of the fundamental lemma of calculus of variations, the above condition is equivalent to

$$(j_{2r}s)^* E_{dL} = 0 \ ,$$

or, that is the same, $E_{dL} \circ j_{2r}s = 0$.

Remark 3.4.4. The reason of the choice of the sheaf $\overset{k}{\Theta}_r$ (for $O \leq k \leq n$) as the first non–trivial sheaf of the contact subsequence is now clear: for $k = n$ $\overset{n}{\Theta}_r$ is made by forms which does not contribute to the action.

As for the sheaf $\overset{n+1}{\Theta}_r$, it is easily seen that this is precisely the sheaf of forms that give no contribution to the above last integral when added to E_{dL}.

Analogously, a 'second variation' of on Euler–Lagrange type operator can be defined (see [Tak79]); the sheaf $\overset{n+2}{\Theta}_r$ is the sheaf of forms that give no contribution to the integral of this second variation. □

Remark 3.4.5. Given $\alpha \in \overset{n}{\Lambda}_r$, we can extend the definition of action of a first–order (generalised) Lagrangian α on a section $s : \boldsymbol{I} \to \boldsymbol{Y}$ as $\int_{\boldsymbol{I}} (j_r s)^* \alpha$. By means of a pull–back on $J_{r+1}\boldsymbol{Y}$, we obtain the equivalent action $\int_{\boldsymbol{I}} (j_{r+1}s)^* h(\alpha)$, being $(j_{r+1}s)^* v(\alpha) = 0$, and we have $h(\alpha) \in \overset{n}{\mathcal{V}}_r$. This explains how the r–th order variational sequence generalises the r–th order variational calculus (see [Kru90, Kru95a, Kru95b]). □

Bibliography

[And86] I. M. Anderson: *Aspects of the Inverse Problem to the Calculus of Variations*, in Proc. Conf. Diff. Geom. and its Appl., Brno, 1986.

[And92] I. M. Anderson: *Introduction to the variational bicomplex*, Contemporary Mathematics, American Math. Soc., **132**, 1992, 51–73.

[AnDu80] I. M. Anderson, T. Duchamp: *On the existence of global variational principles*, Amer. Math. J., **102** (1980), 781–868.

[AnTh92] I. M. Anderson, G. Thompson: The Inverse Problem of the Calculus of Variations for Ordinary Differential Equations, Memoirs of the Am. Math. Soc., **98**, n. 473 (1992).

[Bau82] M. Bauderon: *Le problème inverse du calcul des variations*, Ann. de l'I.H.P., **36**, n. 2 (1982), 159–179.

[Bau85] M. Bauderon: *Differential Geometry and Lagrangian Formalism in the Calculus of Variations*, in Lecture Notes in Pure and Applied Mathematics n. 100: Differential Geometry, Calculus of Variations and their Applications, Marcel Dekker, New York, 1985, p. 67.

[Cos94] P. G. Costantini: *On The Geometrical Structure of Euler–Lagrange Equations*, Ann. Mat. Pura ed Applicata, **167**, n. 4 (1994), 389–402.

[Cra81] M. Crampin: *On the differential geometry of the Euler–Lagrange equations, and the inverse problem of Lagrangian dynamics*, J. Phys. A: Math. Gen., **14** (1981), 2567–2575.

[DeTu80] P. Dedecker, W. M. Tulczyjew: *Spectral sequences and the inverse problem of the calculus of variations*, Internat. Coll. on Diff. Geom. Methods in Math. Phys., Aix–en–Provence, 1979; Lecture Notes in Mathematics, n. 836, Springer–Verlag, Berlin, 1980, 498–503.

[Fer83] M. Ferraris: *Fibered Connections and the global Poincaré–Cartan Form in Higher Order Calculus of Variations*, Proceed. of the Conf. on Diff. Geom. and its Appl., Part II, University J. E. Purkyne, Brno, 1984, 61–91.

[FeFr82] M. Ferraris, M. Francaviglia: *On the Global Structure of Lagrangian and Hamiltonian Formalism in Higher Order Calculus of Variations*, Proceed. of the Int. Meet. on Geometry and Physics, Pitagora Editrice, Bologna, 1983, 12–15.

[FrPaVi99] M. Francaviglia, M. Palese, R. Vitolo: *Symmetries in finite order variational sequences*, preprint 1999, to appear in Czech. Math. J..

[FrPaVi99b] M. Francaviglia, M. Palese, R. Vitolo: *Superpotentials in variational sequences*, Proc. of the VII Conf. on Diff. Geom. and its Appl., Brno, 1998.

[Gar74] P. L. Garcia: *The Poincaré–Cartan Invariant in the Calculus of Variations*, Symposia Mathematica, **14** (1974), 219–246.

[GaMu82] P. L. Garcia, J. Munoz: *On the geometrical structure of higher order variational calculus*, Proc. of the IUTAM–ISIMM Symp. on Modern Developments in Anal. Mech., Torino, 1982, Tecnoprint, Bologna, 1983, 127–147.

[GiMa90] G. Giachetta, L. Mangiarotti: *Gauge–Invariant and Covariant Operators in Gauge Theories*, Int. J. Theoret. Phys. **29**, n. 7 (1990), 789–804.

[GoSt73] H. Goldschmidt, S. Sternberg: *The Hamilton–Cartan formalism in the calculus of variations*, Ann. Inst. Fourier, Grenoble, **23**, n. 1 (1973), 203–267.

[Gre78] W. Greub: *Multilinear Algebra*, Springer–Verlag, 1978.

[Gri99a] D. R. Grigore: *The Variational Sequence on Finite Jet Bundle Extensions and the Lagrangian Formalism*, Diff. Geom. and Appl. **10** (1999), 43–77.

[Gri99b] D. R. Grigore: *Variationally trivial Lagrangians and Locally Variational Differential Equations of Arbitrary Order*, Diff. Geom. and Appl. **10** (1999), 79–105.

[Kas99] J. Kašparova: *A representation of the first order variational sequence in field theory*, Proc. of the VII Conf. on Diff. Geom. and its Appl., Brno (1999), 493–502.

[Kol83] I. Kolář: *A geometrical version of the higher order Hamilton formalism in fibred manifolds*, Jour. Geom. Phys., **1**, n. 2 (1984), 127–137.

[Kol93] I. Kolář: *Natural operators related with the variational calculus*, in Proc. of the Conf. Opava on Diff. Geom. and its Appl., 1992, Silesian Univ., Opava, 1993, 461–472.

[KoMo90] I. Kolář, M. Modugno: *On the algebraic structure on the jet prolongations of fibred manifolds*, Chech. Math. Jour. **40** (115), 1990.

[KoVi98] I. Kolar, R. Vitolo: *On the Helmholtz operator for Euler morphism*, submitted to Ann. di Mat. Pura ed Appl.

[Kru73] D. Krupka: *Some geometric aspects of the calculus of variations in fibred manifolds*, Folia Fac. Sci. Nat. UJEP Brunensis, Brno University, **14**, 1973.

[Kru83] D. Krupka: *Lepagean forms in higher order variational theory*, in Proceedings IUTAM–ISIMM Symposium on Modern Developments in Analytical Mechanics, Atti della Accademia delle Scienze di Torino, 1983, 197–238.

[Kru90] D. Krupka: *Variational sequences on finite order jet spaces*, in Proceedings of the Conf. on Diff. Geom. and its Appl., World Scientific, New York, 1990, 236–254.

[Kru93] D. Krupka: *Topics in the calculus of variations: finite order variational sequences*, Proc. Diff. Geom. and its Appl., Opava (Czech Republic) (1993), 473–495.

[Kru95a] D. Krupka: Variational sequences in mechanics, Univ. of Opava, 1995.

[Kru95b] D. Krupka: *The contact ideal*, Diff. Geom. and its Appl., **5** (1995), 257–276.

[KrMu99] D. Krupka, J. Musilova: *Trivial Lagrangians in Field Theory*, Diff. Geom. and Appl. **9** (1998), 293–505.

[Kup80] B. A. Kuperschmidt: *Geometry of jet bundles and the structure of Lagrangian and Hamiltonian formalism*, in Lecture Notes in Math. n. 775: Geometric Methods in Mathematical Physics, Springer, Berlin (1980) 162–218.

[LaTu77] B. Lawruk, W. M. Tulczyjew: *Criteria for Partial Differential Equations to Be Euler–Lagrange Equations*, Jour. of Diff. Equat., **24** (1977), 211–255.

[MaMo83a] L. Mangiarotti, M. Modugno: *Fibered Spaces, Jet Spaces and Connections for Field Theories*, in Proceed. of the Int. Meet. on Geometry and Physics, Pitagora Editrice, Bologna, 1983, 135–165.

[MaMo83b] L. Mangiarotti, M. Modugno: *Some results on the calculus of variations on jet spaces*, Ann. Inst. H. Poincaré, **39**, n. 1 (1983), 29–43.

[Mus95] J. Musilová: *Variational sequence in higher order mechanics*, Proc. of the V Conf. on Diff. Geom. and its Appl. Brno (1995), 611–624.

[MuKr99] J. Musilová, M. Krbek: *A note to the representation of the variational sequence in mechanics*, Proc. of the VI Conf. on Diff. Geom. and its Appl. Brno (1998), 511–523.

[OlSh78] P. J. Olver, C. Shakiban: *A Resolution of the Euler Operator*, Proc. Am. Math. Soc., **69** (1978), 223–229.

[Olv86] P. J. Olver: Applications of Lie Groups to Differential Equations, Springer, New York, 1986.

[Sau89] D. J. Saunders: The Geometry of Jet Bundles, Cambridge Univ. Press, 1989.

[Ste95] J. Štefánek: *A representation of the variational sequence in higher order mechanics*, Proceedings of the V Conference on Differential Geometry and its Applications (Brno, 1995).

[Tak79] F. Takens: *A global version of the inverse problem of the calculus of variations*, J. Diff. Geom., **14** (1979), 543–562.

[Ton69] E. Tonti: *Variational formulation of nonlinear differential equations (I) and (II)*, Acad. Roy. Belg. C (V) **55** (1969) 137–165 e 262–278.

[Tra96] A. Trautman: *A metaphysical remark on variational principles*, Acta Phys. Pol. B, XX (1996).

[Tul75] W. M. Tulczyjew: *Sur la différentielle de Lagrange*, C. R. Acad. Sc. Paris, série A, **280** (1975), 1295–1298.

[Tul77] W. M. Tulczyjew: *The Lagrange Complex*, Bull. Soc. Math. France, **105** (1977), 419–431.

[Tul80] W. M. Tulczyjew: *The Euler–Lagrange Resolution*, Internat. Coll. on Diff. Geom. Methods in Math. Phys., Aix–en–Provence, 1979; Lecture Notes in Mathematics, n. 836, Springer–Verlag, Berlin, 1980, 22–48.

[Vit95] R. Vitolo: *Some aspects of variational sequences in mechanics*, Proceedings of the V Conference on Differential Geometry and its Applications (Brno, 1995).

[Vit96a] R. Vitolo: *Bicomplessi lagrangiani ed applicazioni alla meccanica relativistica classica e quantistica*, PhD Thesis, Florence, 1996.

[Vit98a] R. Vitolo: *Finite order variational bicomplexes*, Math. Proc. of the Camb. Phil. Soc., **125** (1998), 321–333.

[Vit98b] R. Vitolo: *A new infinite order formulation of variational sequences*, Arch. Math. Un. Brunensis, 4, n.34, 1998.

[Vit99a] R. Vitolo: *On different geometric formulations of Lagrangian formalism*, Diff. Geom. and its Appl. **10** n. 3, 1999.

[Vit99b] R. Vitolo: *The finite order $\mathcal{C}$–spectral sequence*, preprint, 1999.

[Vin77] A. M. Vinogradov: *On the algebro–geometric foundations of Lagrangian field theory*, Soviet Math. Dokl., **18** (1977), 1200–1204.

[Vin78] A. M. Vinogradov: *A spectral sequence associated with a non–linear differential equation, and algebro–geometric foundations of Lagrangian field theory with constraints*, Soviet Math. Dokl., **19** (1978), 144–148.

[Vin84] A. M. Vinogradov: *The $\mathcal{C}$–Spectral Sequence, Lagrangian Formalism and Conservation Laws I and II*, Journal of Mathematical Analysis and Applications, **100**, n. 1 (1984).

[Wel80] R. O. Wells: Differential Analysis on Complex Manifolds, GTM, n. 65, Springer–Verlag, Berlin, 1980.