An inverse problem of ocean acoustics *†

A.G. Ramm
Mathematics Department, Kansas State University,
Manhattan, KS 66506-2602, USA
ramm@math.ksu.edu

Abstract

Let

$$\Delta u + k^2 n(z) u = -\frac{\delta(r)}{2\pi r} f(z) \text{ in } \mathbb{R}^2 \times [0,1], \tag{1}$$

$$u(x^1, 0) = 0, \quad u'(x^1, 1) = 0, \tag{2}$$

where $u = u(x^1, z)$, $x^1 := (x_1, x_2)$, $r := |x^1|$, $x_3 := z$, $u' = \frac{\partial u}{\partial z}$, $\delta(r)$ is the delta-function, $n(z)$ is the refraction coefficient, which is assumed to be a real-valued integrable function, $k > 0$ is a fixed wavenumber. The solution to (1)-(2) is selected by the limiting absorption principle.

It is proved that if $f(z) = \delta(z-1)$, then $n(z)$ is uniquely determined by the data $u(x^1, 1)$ known $\forall x^1 \in \mathbb{R}^2$. Comments are made concerning the earlier study of a similar problem in the literature.

1 Introduction

In [1] the following inverse problem is studied:

$$[\Delta + k^2 n(z)] u = -\frac{\delta(r)}{2\pi r} f(z), \quad \text{in } \mathbb{R}^2 \times [0,1], \tag{1.1}$$

$$u(x^1, 0) = u'(x^1, 1) = 0, \quad x^1 := (x_1, x_2), \quad x_3 := z, \quad u' := \frac{\partial u}{\partial z}. \tag{1.2}$$

*Key words and phrases: inverse scattering, wave propogation, waveguides,ocean acoustics
†Math subject classification: 35R30

Here $k > 0$ is a fixed wavenumber, $n(z) > 0$ is the refraction coefficient, which is assumed in [1] to be a continuous real-valued function satisfying the condition $0 \le n(z) < 1$, the layer $\mathbb{R}^2 \times [0,1]$ models shallow ocean, $r := |x^1| = \sqrt{x_1^2 + x_2^2}$, $\delta(r)$ is the delta-function, $\frac{\delta(r)}{2\pi r} = \delta(x^1)$, $f(z) \in C^2[0,1]$ is a function satisfying the following conditions [1], p.127:

$$f(0) = f''(0) = f'(1) = 0, \quad f'(0) \neq 0, \quad f(1) \neq 0, \quad f(z) > 0 \text{ in } (0,1). \quad (C)$$

The solution to (1.1)-(1.2) in [1] is required to satisfy some conditions ([1], p. 122, formulas (1.4), (1.8)-(1.10)) of the radiation conditions type.

It is convenient to define the solution as $u(x) = \lim_{\varepsilon \downarrow 0} u_\varepsilon(x)$, that is by the limiting absorption principle. We do not show the dependence on k in $u(x)$ since $k > 0$ is fixed throughout the paper. The function $u_\varepsilon(x)$ is the unique solution to problem (1.1)–(1.2) in which equation (1.1) is replaced by the equation with absorption:

$$[\Delta + k^2 n(z) - i\varepsilon] u_\varepsilon(x) = -\frac{\delta(r)}{2\pi r} f(z), \quad \text{in } \mathbb{R}^2 \times [0,1], \ \varepsilon > 0.$$

One defines the differential operator corresponding to differential expression (1.1) and the boundary conditions (1.2) in $L^2(\mathbb{R}^2 \times [0,1])$ as a selfadjoint operator (for example, as the Friedrichs extension of the symmetric operator with the domain consisting of $H^2(\mathbb{R}^2 \times [0,1])$ functions vanishing near infinity and satisfying conditions (1.2)), and then the function $u_\varepsilon(x)$ is uniquely defined. By H^m we mean the usual Sobolev space. One can prove that the limit of this function $u(x) = \lim_{\varepsilon \downarrow 0} u_\varepsilon(x)$ does exist globally in the weighted space $L^2(\mathbb{R}^2 \times [0,1], \frac{1}{(1+r)^a})$, $a > 1$, and locally in $H^2(\mathbb{R}^2 \times [0,1])$ outside a neighborhood of the set $\{r = 0, 0 \le z \le 1\}$, provided $\lambda_j \neq 0 \ \forall j$, where λ_j are defined in (1.7) below. This limit defines the unique solution to problem (1.1)–(1.2) satisfying the limiting absorption principle if $\lambda_j \neq 0 \ \forall j$. If $f(z) = \delta(z-1)$, where $\delta(z-1)$ is the delta-function, then an analytical formula for $u_\varepsilon(x)$ can be written:

$$u_\varepsilon(x) = \sum_{j=1}^{\infty} \psi_j(z) f_j \frac{1}{2\pi} K_0(r\sqrt{\lambda_j^2 + i\varepsilon}),$$

where $K_0(r)$ is the modified Bessel function (the Macdonald function), and $f_j = \psi_j(1)$ are defined in (1.6) below, and $\psi_j(z)$ and λ_j^2 are defined in formula (1.7) below. This formula can be checked by direct calculation and is obtained by the separation of variables. The known formula $\mathcal{F}^{-1}\frac{1}{\lambda^2+a^2} = \frac{1}{2\pi} K_0(ar)$ was used, and $\mathcal{F}u := \hat{u}$ is the Fourier transform defined above formula (1.3).

From the formula for $u_\varepsilon(x)$, the known asymptotics $K_0(r) = \sqrt{\frac{\pi}{2r}} e^{-r}[1 + O(r^{-1})]$ for large values of r, the boundedness of $|\psi_j(z)|$ as $j \to \infty$ and formula (1.8) below, one can see that the limit of $u_\varepsilon(x)$ as $\varepsilon \to 0$ does exist for any $r > 0$ and $z \in [0,1]$, if and only if $\lambda_j \neq 0$. If $\lambda_j = 0$ for some $j = j_0$, then the limiting absorption principle holds if and only if $f_{j_0} = 0$. If $\lambda_j \neq 0 \ \forall j$, then the limiting absorption principle holds and the solution to problem (1.1)-(1.2) is well defined. If $\lambda_j = 0$ for some $j = j_0$, then we define the solution to problem (1.1)-(1.2) with $f(z) = \delta(z-1)$ by the formula:

$$u(x) = \psi_{j_0}(z)\psi_{j_0}(1)\frac{1}{2\pi}\log(\frac{1}{r}) + \sum_{j=1,\, j\neq j_0}^{\infty} \psi_j(z)\psi_j(1)\frac{1}{2\pi}K_0(r\lambda_j), \quad r := |x^1|.$$

This solution is unique in the class of functions of the form $u(x) = \sum_{j=1}^{\infty} u_j(x^1)\psi_j(z)$, where $\Delta_1 u_j - \lambda_j^2 u_j = -\delta(x^1)$ in $\mathbb{R}^2$, $\Delta_1 w := w_{x_1x_1} + w_{x_2x_2}$, $u_j \in L^2(\mathbb{R}^2)$ if $\lambda_j^2 > 0$; if $\lambda_j^2 < 0$ then u_j satisfies the radiation condition $r^{1/2}(\frac{\partial u_j}{\partial r} - i|\lambda_j|u_j) \to 0$ as $r \to \infty$, uniformly in directions $\frac{x^1}{r}$; and if $\lambda_j^2 = 0$ then $u_j = \frac{1}{2\pi}\log(\frac{1}{r}) + o(1)$ as $r \to \infty$.

The inverse problem (IP) consists of finding $n(z)$ given $g(x^1) := u(x^1, 1)$ and assuming that $f(z) = \delta(z-1)$ in (1.1).

By the cylindrical symmetry one has $g(x^1) = g(r)$.

It is claimed in [1, p. 137] that the above inverse problem has not more than one solution, and a method for finding this solution is proposed. The arguments in [1] are not satisfactory (see Remark 2.1 below, where some of the incorrect statements from [1], which invalidate the approach in [1], are pointed out).

The aim of our paper is to prove that if $f(z) = \delta(z-1)$, then $n(z)$ can be uniquely and constructively determined from the data $g(r)$ known for all $r > 0$. It is an open problem to find all such $f(z)$ for which the IP has at most one solution.

The method we use is developed in [5] (see also [7]). Properties of the operator $\Delta + k^2 n(z)$ in a layer were studied in [6]. In [8] an inverse problem for an inhomogeneous Schrödinger equation on the full axis was investigated.

Let us outline our approach to IP.

Take the Fourier transorm of (1.1)-(1.2) with respect to x^1 and let

$$v := v(z,\lambda) := \hat{u} := \int_{\mathbb{R}^2} u(x^1, z)e^{ix^1\cdot\zeta}dx^1, \quad |\zeta| := \lambda, \quad \zeta \in \mathbb{R}^2,$$

and

$$G(\lambda) := \hat{g}(r).$$

Then

$$\ell v := v'' - \lambda^2 v + q(z)v = -f(z), \quad q(z) := k^2 n(z), \quad v = v(z,\lambda), \tag{1.3}$$

$$v(0,\lambda) = v'(1,\lambda) = 0, \tag{1.4}$$

$$v(1,\lambda) = G(\lambda). \tag{1.5}$$

IP: The inverse problem is: given $G(\lambda)$, for all $\lambda > 0$ and a fixed $f(z) = \delta(z-1)$, find $q(z)$.

The solution to (1.3)-(1.4) is:

$$v(z,\lambda)=\sum_{j=1}^{\infty}\frac{\psi_j(z)f_j}{\lambda^2+\lambda_j^2},\quad f_j:=(f,\psi_j):=\int_0^1 f(z)\psi_j(z)dz, \tag{1.6}$$

where $\psi_j(z)$ are the real-valued normalized eigenfunctions of the operator $L:=-\frac{d^2}{dz^2}-q(z)$:

$$L\psi_j=\lambda_j^2\psi_j,\quad \psi_j(0)=\psi_j'(1)=0,\quad ||\psi_j(z)||=1. \tag{1.7}$$

We can choose the eigenfunctions $\psi_j(z)$ real-valued since the function $q(z)=k^2n(z)$ is assumed real-valued. One can check that all the eigenvalues are simple, that is, there is just one eigenfunction ψ_j corresponding to the eigenvalue λ_j^2 (up to a constant factor, which for real-valued normalized eigenfunctions can be either 1 or -1).

It is known (see e.g. [4. p.71]) that

$$\lambda_j^2=\pi^2(j-\frac{1}{2})^2[1+O(\frac{1}{j^2})] \text{ as } j\to+\infty. \tag{1.8}$$

The data can be written as

$$G(\lambda)=\sum_{j=1}^{\infty}\frac{\psi_j(1)f_j}{\lambda^2+\lambda_j^2}, \tag{1.9}$$

where f_j are defined in (1.6). The series (1.9) converges absolutely and uniformly on compact sets of the complex plane λ outside the union of small discs centered at the points $\pm i\lambda_j$. Thus, $G(\lambda)$ is a meromorphic function on the whole complex λ-plane with simple poles at the points $\pm i\lambda_j$. Its residue at $\lambda=i\lambda_j$ equals $\frac{\psi_j(1)f_j}{2i\lambda_j}$.

If $f(z)=\delta(z-1)$, then $f_j=\psi_j(1)\neq 0\ \forall j=1,2,.....$, (see section 2 for a proof of the inequality $\psi_j(1)\neq 0\ \forall j=1,2,.....$,) and the data (1.9) determine uniquely the set

$$\{\lambda_j^2,\quad \psi_j^2(1)\}_{j=1,2,...} \tag{1.10}$$

In section 2 we prove the basic result:

Theorem 1.1. *If $f(z)=\delta(z-1)$ then the data (1.5) determine $q(z)\in L^1(0,1)$ uniquely.*

An algorithm for calculation of $q(z)$ from the data is described in section 2.

Remark 1.2. *The proof and the conclusion of Theorem 1.1 remain valid for other boundary conditions, for example, $u'(x^1,0)=u(x^1,1)=0$ with the data $u(x^1,0)$ known for all $x^1\in\mathbb{R}^2$.*

2 Proofs: uniqueness theorem and inversion algorithm

Proof of Theorem 1.1. The data (1.9) with $f(z) = \delta(z-1)$, that is, with $f_j = \psi_j(1)$, determine uniquely $\{\lambda_j^2\}_{j=1,2,\dots}$ since $\pm i\lambda_j$ are the poles of the meromorphic function $G(\lambda)$ which is uniquely determined for all $\lambda \in \mathbb{C}$ by its values for all $\lambda > 0$ (in fact, by its values at any infinite sequence of $\lambda > 0$ which has a finite limit point on the real axis). The residues $\psi_j^2(1)$ of $G(\lambda)$ at $\lambda = i\lambda_j$ are also uniquely determined.

Let us show that:

i) $\psi_j(1) \neq 0 \quad \forall j = 1, 2, \dots$

ii) The set (1.10) determines $q(z) \in L^1(0,1)$ uniquely.

Let us prove i):

If $\psi_j(1) = 0$ then equation (1.7) and the Cauchy data $\psi_j(1) = \psi_j'(1) = 0$ imply that $\psi_j(z) \equiv 0$ which is impossible since $\| \psi_j(z) \| = 1$, where $\| u \|^2 := \int_0^1 |u|^2 dx$.

Let us prove ii):

It is sufficient to prove that the set (1.10) determines the norming constants

$$\alpha_j := \| \Psi_j(z) \|^2$$

and therefore the set

$$\{\lambda_j^2, \alpha_j\}_{j=1,2,\dots},$$

where the eigenvalues λ_j^2 are defined in (1.7), $\Psi_j = \Psi(z, \lambda_j)$, $\psi_j(z) := \frac{\Psi(z,\lambda_j)}{\|\Psi_j\|}$,

$$-\Psi'' - s^2\Psi - q(z)\Psi = 0, \quad \Psi(0,s) = 0, \quad \Psi'(0,s) = 1, \tag{2.1}$$

and λ_j are the zeros of the equation

$$\Psi'(1,s) = 0, \quad s = \lambda_j, \ j = 1, 2, \dots\dots \tag{2.2}$$

The function $\Psi'(1,s)$ is an entire function of $\nu = s^2$ of order $\frac{1}{2}$, so that (see [2]):

$$\Psi'(1,s) = \gamma \prod_{j=1}^{\infty} \left(1 - \frac{s^2}{\lambda_j^2}\right), \quad \gamma = const. \tag{2.3}$$

From the Hadamard factorization theorem for entire functions of order < 1 formula (2.3) follows but the constant factor γ remains undetermined. This factor is determined by the data $\{\lambda_j^2\}_{\forall j}$ because the main term of the asymptotics of function (2.3) for large positive s is $\cos(s)$, and the result in [4], p.243, (see Claim 1 below) implies that the constant γ in formula (2.3) can be computed explicitly:

$$\gamma = \prod_{j=1}^{\infty} \frac{\lambda_j^2}{(\lambda_j^0)^2}, \tag{2.3'}$$

where λ_j^0 are the roots of the equation $\cos(s) = 0$, $\lambda_j^0 = \frac{(2j-1)\pi}{2}$, $j = 1, 2, ...$, and the infinite product in (2.3') converges because of (1.8).

A simple derivation of (2.3'), independent of the result formulated in Claim 1 below, is based on the formula:

$$1 = \lim_{y\to+\infty} \frac{\Psi'(1, iy)}{\cos(iy)} = \gamma \prod_{j=1}^{\infty} \frac{(\lambda_j^0)^2}{\lambda_j^2}.$$

For convenience of the reader let us formulate the result from [4], p.243, which yields formula (2.3') as well:

Claim 1: The function $w(\lambda)$ admits the representation

$$w(\lambda) = \cos(\lambda) - B\frac{\sin(\lambda)}{\lambda} + \frac{h(\lambda)}{\lambda},$$

where $B = const$, $h(\lambda) = \int_0^1 H(t)\sin(\lambda t)dt$, and $H(t) \in L^2(0, 1)$ if and only if

$$w(\lambda) = \prod_{j=1}^{\infty} \frac{\lambda_j^2 - \lambda^2}{(\lambda_j^0)^2},$$

where $\lambda_j = \lambda_j^0 - \frac{B}{j} + \frac{\beta_j}{j}$, β_j are some numbers satisfying the condition: $\sum_{j=1}^{\infty} |\beta_j|^2 < \infty$, λ_j are the roots of the even function $w(\lambda)$ and $\lambda_j^0 = (j - \frac{1}{2})\pi$, $j = 1, 2, ...$, are the positive roots of $\cos(\lambda)$.

The equality

$$\prod_{j=1}^{\infty} \frac{\lambda_j^2 - \lambda^2}{(\lambda_j^0)^2} = \gamma \prod_{j=1}^{\infty} \left(1 - \frac{\lambda^2}{\lambda_j^2}\right), \qquad (2.3")$$

where γ is defined in (2.3'), is easy to prove: if w is the left-hand side and v the right-hand side of the above equality, then w and v are entire functions of λ, the infinite products converge absolutely, $\frac{\lambda_j^2-\lambda^2}{(\lambda_j^0)^2} = \frac{\lambda_j^2}{(\lambda_j^0)^2}\left(1 - \frac{\lambda^2}{\lambda_j^2}\right)$, and taking the infinite product and using (2.3'), one concludes that $\frac{w}{v} = 1$, as claimed.

In fact, one can establish formula (2.3") and prove that γ in (2.3") is defined by (2.3') without assuming a priori that (2.3') holds and without using Claim 1. The following assumption suffices for the proof of (2.3"):

i) $\lambda_j^2 = (\lambda_j^0)^2 + O(1)$, $(\lambda_j^0)^2 = \pi^2(j - \frac{1}{2})^2$.

Indeed, if i) holds then both sides of (2.3") are entire functions with the same set of zeros and their ratio is a constant. This constant equals to 1 if there is a sequence of points at which this ratio converges to 1. Using the known formula: $\cos(\lambda) = \prod_{j=1}^{\infty} \frac{(\lambda_j^0)^2-\lambda^2}{(\lambda_j^0)^2}$, and the assumption i) one checks easily that the ratio of the left- and right-hand sides of (2.3") tends to 1 along the positive imaginary semiaxis. Thus, we have proved formulas (2.3)-(2.3') without reference to Claim 1.

The above claim is used with $w(s) = \Psi'(1,s)$ in our paper. The fact that $\Psi'(1,s)$ admits the representation required in the claim is checked by means of the formula for $\Psi'(1,s)$ in terms of the transformation operator: $\Psi(z,s) = \frac{\sin(sz)}{s} + \int_0^z K(z,t)\frac{\sin(st)}{s}dt$, and the properties of the kernel $K(z,t)$ are studied in [4]. Thus, $\Psi'(1,s) = \cos(s) + \frac{K(1,1)\sin(s)}{s} + \int_0^1 K_z(1,t)\frac{\sin(st)}{s}dt$. This is the representation of $\Psi'(1,s) := w(s)$ used in Claim 1.

Let us derive a formula for $\alpha_j :=\| \Psi_j \|^2$. Denote $\dot{\Psi} := \frac{d\Psi}{d\nu}$, differentiate (2.1), with s^2 replaced by ν, with respect to ν and get:

$$-\dot{\Psi}'' - \nu\dot{\Psi} - q\dot{\Psi} = \Psi. \tag{2.4}$$

Since $q(z)$ is assumed real-valued, one may assume ψ real-valued. Multiply (2.4) by Ψ and (2.1) by $\dot{\Psi}$, subtract and integrate over $(0,1)$ to get

$$0 < \alpha_j := \int_0^1 \Psi_j^2 dz = \left(\Psi_j'\dot{\Psi}_j - \Psi_j\dot{\Psi}_j'\right)\Big|_0^1 = -\Psi_j(1)\dot{\Psi}_j'(1), \tag{2.5}$$

where the boundary conditions $\Psi_j(0) = \Psi_j'(1) = \dot{\Psi}_j(0) = 0$ were used.

From (2.3) with $s^2 = \nu$ one finds the numbers $b_j := \dot{\Psi}_j'(1)$:

$$b_j = \gamma\frac{d}{d\nu}\prod_{j'=1}^{\infty}\left(1 - \frac{\nu}{\lambda_{j'}^2}\right)\Big|_{\nu=\lambda_j^2} = -\frac{\gamma}{\lambda_j^2}\prod_{j'\neq j}\left(1 - \frac{\lambda_j^2}{\lambda_{j'}^2}\right). \tag{2.6}$$

Claim 2: The data $\psi_j^2(1) = \frac{\Psi_j^2(1)}{\alpha_j} := t_j$, *where* $\alpha_j :=\| \Psi_j(z) \|^2$, *and equation (2.5) determine uniquely* α_j.

Indeed, the numbers b_j are the known numbers from formula (2.6). Denote by $t_j := \psi_j^2(1)$ the quantities known from the data (1.10). Then it follows from (2.5) that $\alpha_j^2 = t_j\alpha_j b_j^2$, so that

$$\alpha_j = t_j b_j^2. \tag{2.7}$$

Claim 2 is proved.

Thus, the data (1.10) determine $\alpha_j =\| \Psi_j \|^2$ uniquely and analytically by the above formula, and consequently $q(z)$ is uniquely determined by the following known theorem (see for example, [3]):

The spectral function of the operator L determines $q(z)$ uniquely.

The spectral function $\rho(\lambda)$ of the operator L is defined by the formula (see [3, formula (10.5)]):

$$\rho(\lambda) = \sum_{\lambda_j^2 < \lambda}\frac{1}{\alpha_j}. \tag{2.8}$$

The Gelfand-Levitan algorithm [3] allows one to reconstruct analytically $q(z)$ from the spectral function $\rho(\lambda)$ and therefore from the data (1.10), since, as we have proved already, these data determine the spectral function $\rho(\lambda)$ uniquely.

Theorem 1.1 is proved. □

Let us describe an algorithm for calculation of $q(z)$ from the data $g(x^1)$:

Step 1: Calculate $G(\lambda)$, the Fourier transform of $g(x^1)$. Given $G(\lambda)$, find its poles $\pm i\lambda_j$, and consequently the numbers λ_j; then find its residues, and consequently the numbers $\psi_j(1)f_j$.

Step 2: Calculate the function (2.3), and the constant γ by formulas (2.3) and (2.3'). Calculate the numbers b_j by formula (2.6) and α_j by formula (2.7). Calculate the spectral function $\rho(\lambda)$ by formula (2.8).

Step 3: Use the known Gel'fand-Levitan algorithm (see [3]-[5]) to calculate $q(z)$ from $\rho(\lambda)$.

This completes the description of the inversion algorithm for IP.

Remark 2.1. *There are inaccuracies in [1]. We point out two of these, of which the first invalidates the approach in [1].*

In [1, p.128, line 2] the α_n are not the same as α_n in formula [1, (3.3)]. If one uses α_n from formula [1, (3.3)], then one has to use in [1, p.128, line 2] the coefficients $\alpha_n\phi_n(h)$, according to formula [1, (1.5)]. In [1] h is the width of the layer, which we took to be $h = 1$ in our paper without loss of generality. However, the numbers $\phi_n(h)$ are not known in the inverse problem, since the coefficient $n(z)$ is not known. Therefore formula [1, (3.9)] is incorrect. This invalidates the approach in [1].

In [1, p.128] a negative decreasing sequence of real numbers a_n is defined by equation (3.1), which we give for $h = 1$:

$$k\sqrt{1-a_n^2} = (n+\frac{1}{2})\pi + O(\frac{1}{n}) \quad (*).$$

Such a sequence does not exist: if $a_n < 0$ and a_n has a finite limit then the right-hand side of $(*)$ cannot grow to infinity, and if $a_n \to -\infty$, then the left-hand side of $(*)$ cannot stay positive for large n, and therefore cannot be equal to the right-hand side of $(*)$.

References

[1] Gilbert R., Xu Y., An inverse problem for harmonic acoustis in stratified ocean. J. Math Anal. Appl., 176, (1993), 121-137.

[2] Levin B.Ya., **Zeros of entire functions**, Amer. Math. Soc., Providence, 1964.

[3] Levitan B.M., **Inverse Sturm-Liouville problems**, VNU Press, Utrecht, The Netherlands, 1987.

[4] Marchenko V. **Sturm-Liouville operators and applications**, Birkhäuser, Basel, 1986.

[5] Ramm A.G., **Multidimensional inverse scattering problems**, Longman/Wiley, New York, 1992, pp.1-385. Russian translation of the expanded monograph, Mir Publishers, Moscow, 1994, pp.1-496.

[6] Ramm A.G., G. Makrakis, Scattering by obstacles in acoustic waveguides, In the book: Spectral and scattering theory, Plenum, New York, 1998 (ed. A.G.Ramm), pp.89-110.

[7] Ramm A.G., Inverse problem for an inhomogeneous Schrödinger equation, Jour. Math. Phys, 40, N8, (1999), 3876-3880.

[8] Weder R., **Spectral and scattering theory for wave propagation in perturbed stratified media**, Springer Verlag, New York, 1991.