Graph-Laplacians and Dirac Operators on (Infinite) Graphs and the Calculation of the Connes-Distance-Functional

Manfred Requardt

Institut für Theoretische Physik
Universität Göttingen
Bunsenstrasse 9
37073 Göttingen Germany
(E-mail: requardt@theorie.physik.uni-goettingen.de)

Abstract

We develop a graph-Hilbert-space framework, inspired by non-commutative geometry, on (infinite) graphs and use it to study spectral properies of *graph-Laplacians* and so-called *graph-Dirac-operators*. Putting the various pieces together we define a *spectral triplet* sharing most (if not all, depending on the particular graph model) of the properties of what Connes calls a *spectral triple*. With the help of this scheme we derive an explicit expression for the *Connes-distance function* on general graphs and prove both a variety of apriori estimates for it and calculate it for certain examples of graphs. As a possibly interesting aside, we show that the natural setting of approaching such problems may be the framework of *(non-)linear programming* or *optimization*. We compare our results (arrived at within our particular framework) with the results of other authors and show that the seeming differences depend on the use of different graph-geometries and/or Dirac operators.

1 Introduction

We recently embarked on a programme to reconstruct continuous physics and/or mathematics from an underlying more primordial and basically discrete theory living on the Planck-scale (cf. [1],[2],[3]). As sort of a "spin-off" various problems of a more mathematical and technical flavor emerged which have an interest of their own. *Discrete differential geometric* concepts were dealt with in [1], the theory of *random graphs* was a central theme of [2], topics of *dimension theory* and *fractal geometry* were addressed in [4].

If one wants to recover the usual *(differential) operators* of continuum physics and mathematics by some sort of limiting process from their discrete protoforms, living on a relatively disordered discrete background like, say, a network, one has, in a first step, to study their discrete counterparts. This will be our main theme in the following with particular emphasis on discrete *Laplacians* and *Dirac-operators* on general *graphs*.

We note in passing that *functional analysis on graphs* is both of interest in pure and applied mathematics and also in various fields of (mathematical) physics. For one, discrete systems have an increasing interest of their own or serve as easier to analyse prototypes of their continuum counterparts. To mention a few fields of applications: *graph theory in general, analysis on (discrete) manifolds, lattice or discretized versions of physical models in statistical mechanics and quantum field theory, non-commutative geometry, networks, fractal geometry* etc. From the vast and widely scattered literature we mention (possibly) very few sources which were of relevance for our own motivation or we came across recently (after finishing a first draft): [6] to [12],[5] and [32], paper [13] appeared after finishing our first draft, some more literature like e.g. [14] was pointed out to us by Mueller-Hoissen; the possible relevance of references [15] to [17] were brought to our notice by some unknown referee, [18] was also discovered by us only recently. Last, but not least, there is the vast field of *discretized quantum gravitiy* (see e.g. [19] or [20]). All this shows that the sort of discrete functional analysis we are dealing with in the following, is presently a very active field with a lot of different applications.

We use the graph-Hilbert-space machinery, developed in the first part of our paper to investigate the spectral properties of *graph Laplacians* and *Dirac operators*. In a next step we study and test concepts and ideas, which arose in the framework of *non-commutative geometry*. As we (and others) showed in preceding papers, networks and graphs may (or even should) be understood as examples of *non-commutative* spaces. A currently interesting topic in this field is the investigation of certain *distance functionals* on "nasty" or *non-standard* spaces and their mathematical or physical "naturalness". Graphs carry, on the one hand, a natural *metric structure* given by a *distance function* $d(x, y)$, with x, y two *nodes* of the graph (see the following sections). This fact was already employed by us in e.g. [4] to develop dimensional concepts on graphs. Having Connes' concept

of distance in noncommutative geometry in mind (cf. chapt. VI of [5]), it is a natural question to try to compute it in model systems, which means in our context: arbitrary graphs, and compare it with the already existing notion of graph distance mentioned above. (We note in passing that the calculation of the Connes distance for general graphs turns out to be surprisingly complex and leads to perhaps unexpected connections to fields of mathematics like e.g. *(non-)linear programming* or *optimization*; see the last section).

Therefore, as one of many possible applications we construct a protoform of what Connes calls a *spectral triple*, that is, a Hilbert space structure , a corresponding representation of a certain (function) algebra and a (in our framework) natural candidate for a so-called *Dirac operator* (not to be confused with the ordinary Dirac operator of the Dirac equation), which encodes certain properties of the *graph geometry*. This will be done in section 4.

In the last section, which deals with the distance concept deriving from this spectral triplet (as we like to call it), we will give this notion a closer inspection as far as graphs and similar spaces are concerned. In this connection some recent work should be mentioned, in which Connes' distance function was analyzed in certain simple models like e.g. one-dimensional lattices ([23]-[25]). These papers already show that it is a touchy business to isolate "the" appropriate Dirac operator (after all, different Dirac operators are expected to lead to different geometries!) and that it is perhaps worthwhile to scrutinize the whole topic in a more systematic way. We show in particular that one may choose different Dirac-operators on graphs (or rather, different types of graphs over the same node set) which may lead to different results for e.g. the corresponding Connes-distance.

The problem of finding suitable metrics on "non-standard" spaces is a particularly interesting research topic of its own, presently pursued by quite a few people (see the beautiful paper by Rieffel, [12] and the references mentioned therein). Another earlier source is e.g. [14]. As to this latter paper we would like to remark that, while much of the framework is different from ours, there is, on the other side, a small overlap as far as some technical notions and results are concerned (after an appropriate translation of the respective technical notions and definitions). To give an example: While the definition of Dirac operators is different, some of the operator norms and metrics, being calculated, turn out to be identical to ours. This suggests a more careful comparison of the underlying conceptual ideas which we plan to give elsewhere. We presently extend this investigation of metric structures to *lump spaces* and *probabilistic metric spaces* (see [36] and [37]) in the general context of quantum gravity (cf. also [2]).

Our own approach provides a *systematic* recipe to calculate the *Connes distance* in the most general cases of graphs and exhibits its role as a non-trivial constraint on certain function classes on graphs. We prove various rigorous a priori estimates and show how the constraints have to be dealt with in several examples.

Remark: (For possible reasons of priority) we would like to mention that many of our results can already be found in a (preliminary form) in an earlier draft version ([26]).

2 A brief Survey of Differential Calculus on Graphs

The following is a brief survey of certain concepts and tools needed in the further analysis. While our framework may deviate at various places from the ordinary one, employed in e.g. *algebraic graph theory*, this is mainly done for reasons of greater mathematical flexibility and generality and, on the other side, possible physical applications (a case in point being the analysis of *non-commutative spaces*). Some more motivations are provided in [1] and [2]). We begin with the introduction of some graph theoretical concepts. We would however like to mention, that it is not our intention to cover any appreciable amount of the close interrelationship between graph spectra and graph characteristics (as has e.g. been done in [15]; our main emphasis lies on providing various Hilbert-space-techniques).

Definition 2.1 (Simple Locally Finite (Un)directed Graph)

1. *We write the* simple *graph as* $G := (\mathcal{V}, E)$ *where* $\mathcal{V}$ *is the countable set of nodes* $\{n_i\}$ *(or vertices) and* E *the set of bonds (edges). The graph is called simple if there do not exist elementary* loops *and* multiple edges*, in other words: each existing bond connects two different nodes and there exists at most one bond between two nodes. (We could of course also discuss more general graphs). Furthermore, for simplicity, we assume the graph to be connected, i.e. two arbitrary nodes can be connected by a sequence of consecutive bonds called an* edge sequence *or* walk*. A minimal edge sequence, that is one with each intermediate node occurring only once, is called a* path *(note that these definitions may change from author to author).*

2. *We assume the graph to be* locally finite*, that is, each node is incident with only a finite number of bonds. Sometimes it is useful to make the stronger assumption that this* vertex degree*,* v_i*, (number of bonds being incident with* n_i*), is globally bounded away from* ∞*.*

3. *One can give the edges both an* orientation *and a* direction *(these two, in our view, slightly different geometric concepts are frequently intermixed in the literature). In our context we adopt the following convention: If two nodes* n_i, n_k *are connected by a bond, we interpret this as follows: There exists a* directed bond*,* d_{ik}*, pointing from* n_i *to* n_k *and a* directed bond*,* d_{ki}*,*

pointing in the opposite direction. In an algebraic sense, which will become clear below (for more details see also [1]), we call their superposition

$$b_{ik} := d_{ik} - d_{ki} = -b_{ki} \tag{1}$$

the corresponding oriented bond *(for obvious reasons; the directions are fixed while the orientation can change its sign). In a sense the above reflects the equivalence of an undirected graph with a* directed multi-graph *having two directed bonds pointing in opposite directions for each undirected bond.*

This way of algebraic implementation of geometric structures allows us to treat in principle all kinds of graphs on essentially the same footing. That is, it also applies to, say, graphs with only one directed edge being existent between two nodes. On the other side, an orientation should exist also for undirected graphs. As an aside, we remark that, on the one side, our generators, b_{ik}, correspond to the oriented pairs of nodes, (i, k) in e.g. [15], on the other hand, our d_{ik} correspond to the oriented pairs, (i, k), in [14]. One sees from this that the conventions are far from being unique and that a certain unification may be desirable.

We now take the elementary building blocks $\{n_i\}$ and $\{d_{ik}\}$ as basis elements of a certain hierarchy of vector spaces over, say, $\mathbb{C}$ with scalar product

$$(n_i|n_k) = \delta_{ik} \quad (d_{ik}|d_{lm}) = \delta_{il} \cdot \delta_{km} \tag{2}$$

Definition 2.2 (Vertex-, Edge-Space) *The vector spaces (or modules) C_0, C_1^a (a for antisymmetric) and C_1 consist of the finite sums*

$$f := \sum f_i n_i \quad g := \sum g_{ik} d_{ik} \quad \textit{with} \quad g_{ik} = -g_{ki} \quad \textit{and} \quad g' := \sum g_{ik} d_{ik} \tag{3}$$

f_i, g_{ik} ranging over a certain given field like e.g. $\mathbb{C}$ or ring like e.g. $\mathbb{Z}$ in case of a module. Evidently we have $C_1^a \subset C_1$.

These spaces can be easily completed to *Hilbert spaces* by assuming

$$\sum |f_i|^2 < \infty \quad \sum |g_{ik}|^2 < \infty \tag{4}$$

if one chooses e.g. the field $\mathbb{C}$ (see the next section). Furthermore, one can continue this row of vector spaces in ways which are common practice in, say, *algebraic topology* (see [1] sections 3.1 and 3.2). In this context they are frequently called *chain complexes* (see also [18]). Evidently the above vector spaces could as well be viewed as *discrete function spaces* over the *node-, bond set* with n_i, d_{ik} now representing the elementary *indicator functions.*

In the same spirit we can now introduce two linear maps between C_0, C_1 called for obvious reasons *boundary-* and *coboundary map.* On the basis elements they act as follows:

Definition 2.3 ((Co)boundary Operator)

$$\delta : \; d_{ik} \to n_k \quad hence \quad b_{ik} \to n_k - n_i \tag{5}$$

$$d : \; n_i \to \sum_k (d_{ki} - d_{ik}) = \sum_k b_{ki} \tag{6}$$

and linearly extended. That is, δ maps the directed bonds d_{ik} onto the terminal node and b_{ik} onto its (oriented) boundary*, while d maps the node n_i onto the sum of the* ingoing *directed bonds minus the sum of the* outgoing *directed bonds or on the sum of* oriented ingoing *bonds b_{ki}.*

The following results show, that these definitions lead in fact to a kind of *discrete differential calculus* on C_0, C_1.

Observation 2.4 (Discrete Differential Forms) *From the above it follows that*

$$df = d(\sum f_i n_i) = \sum_{k,i} (f_k - f_i) d_{ik} \tag{7}$$

Combining now the operators δ and d, we can construct, what is called the *canonical graph Laplacian.* On the vertex space it reads:

Observation 2.5 (Graph Laplacian)

$$\delta df = -\sum_i (\sum_k f_k - v_i \cdot f_i) n_i = -\sum_i (\sum_k (f_k - f_i)) n_i =: -\Delta f \tag{8}$$

where v_i denotes the node degree *or* valency *defined above and the k-sum extends over the nodes adjacent to n_i.*

Note that there exist several variants in the literature (see e.g. [15] or [9]). Furthermore, many mathematicians employ a different sign-convention. We stick in the following to the convention being in use in the mathematical-physcis literature where $-\Delta$ is the positive(!) operator.

This *graph Laplacian* is intimately connected with yet another important object, employed by graph-theorists, i.e. the *adjacency matrix* of a graph.

Definition 2.6 (Adjacency Matrix) *The entries a_{ik} of the* adjacency matrix *A have the value one if the nodes n_i, n_k are connected by a bond and are zero elsewhere. If the graph is* undirected *(but orientable; the case we mainly discuss), the relation between n_i, n_k is* symmetric*, i.e.*

$$a_{ik} = 1 \quad \Rightarrow \quad a_{ki} = 1 \quad etc. \tag{9}$$

This has the obvious consequence that in case the graph is simple *and* undirected*, A is a symmetric matrix with zero diagonal elements.*

Remark: More general A's occur if more general graphs are admitted (e.g. general multigraphs).

Observation 2.7 *: With our definition of Δ it holds:*

$$\Delta = A - V \tag{10}$$

where V is the diagonal degree matrix*, having v_i as diagonal entries.*

(Note that the other sign-convention would lead to $\Delta = V - A$).

Proof: As we have not yet introduced the full Hilbert space machinery (which we will introduce in the next section), the proof has to be understood, for the time being, in an algebraic way. We then have:

$$Af = A(\sum f_i n_i) = \sum_i f_i (\sum_{k-i} n_k) = \sum_i (\sum_{k-i} f_k) n_i \tag{11}$$

$$Vf = \sum_i (v_i f_i) n_i \tag{12}$$

hence the result. □

(Here and in the following we use the abbreviation $k - i$ if the nodes n_k, n_i are connected by a bond, the summation always extending over the first variable).

As we already remarked above, our approach to functional analysis on graphs is perhaps a little bit different compared with the usual one. It seems therefore to be appropriate to exhibit some of the conceptual differences as compared to the more traditional framework, as e.g. developed in the beautiful monographs [8] or [9], by briefly discussing the following (however only minor) point. In general, the graphs under discussion may be directed or undirected. In the traditional approach the edges are typically independently labelled of the nodes and the corresponding edge space, denoted in this case for the time being by $\hat{C}_1$, is built over this edge set. In contrast to that habit we found it useful to label the occurring edges as d_{ik}, b_{ik} with $b_{ik} = -b_{ki}$ which leads in our view to a more flexible discrete calculus and, among other things, to a natural *Dirac operator* on graphs (see below).

More or less related to our operators d, d^* (d^* the adjoint of d; see the next section) are now the so-called *incidence matrix*, B, and its adjoint in the traditional approach which relate the edges with the nodes. To do this, the edges are given an adhoc orientation, denoting one vertex arbitrarily as *initial point*, the other as *end point*. With the n labelled vertices, n_i and m labelled edges, e_j (B_{ij}) has the entries

$$\begin{cases} +1 & \text{if } n_i \text{ is the endpoint of } e_j \\ -1 & \text{if } n_i \text{ is the initial point of } e_j \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

Evidently B is a mapping from $\hat{C}_1$ to C_0 and maps an edge to the respective difference of end vertex and initial vertex. By the same token, the transpose, B^t is defined as a map from C_0 to $\hat{C}_1$ and one gets:

$$B \cdot B^t = V - A \tag{14}$$

Note that the above introduced adhoc orientation does not enter in any end result; on the other hand, our approach is not based on such a contingent structure.

3 Some Spectral Analysis and Operator Theory on (Infinite) Graphs

After these preliminary remarks we now enter the heart of the matter. Our first task consists of endowing a general graph with a natural Hilbert space structure on which the various operators constructed in the following can operate. (The following analysis is done on *undirected* graphs, but could be extended to more general but less symmetric situations).

Definition 3.1 (Hilbert Space) *As indicated in the previous section, we extend C_0, C_1^a, C_1 to the respective Hilbert spaces $H_0, H_1^a \subset H_1$ of sequences over the respective ON-bases $\{n_i\}$, $\{d_{ik}\}$, that is $(n_i|n_k) = \delta_{i,k}$, $(d_{ik}|d_{i'k'}) = \delta_{ii'}\delta_{kk'}$.*

As H^a, H we take the direct sums:

$$H^a := H_0 \oplus H_1^a \subset H := H_0 \oplus H_1 \tag{15}$$

Note that members of H_1^a can be written

$$\sum g_{ik} d_{ik} = 1/2 \sum g_{ik} d_{ik} + 1/2 \sum g_{ki} d_{ki} = 1/2 \sum g_{ik}(d_{ik} - d_{ki}) = 1/2 \sum g_{ik} b_{ik} \tag{16}$$

Obviously H^a is a subspace of H and we have

$$(b_{ik}|b_{ik}) = 2 \tag{17}$$

i.e. the b_{ik} are not(!) normalized if the d_{ik} are. We could of course enforce this but then a factor two would enter elsewhere.

With these definitions it is now possible to define the maps d, δ as true operators between these Hilbert (sub)spaces. To avoid domain problems we assume from now on that the *node degree* $v(n_i)$ is *uniformly bounded* on the graph G, i.e.

$$v_i \leq v_{max} < \infty \tag{18}$$

Observation 3.2 *We have the following relations*

$$d:\ H_0 \to H_1^a \subset H_1, \quad \delta : H_1 \to H_0 \tag{19}$$

$d_{1,2}$ *with*

$$d_{1,2}:\ n_i \to \sum d_{ki}\ ,\ \sum d_{ik} \tag{20}$$

respectively and linearly extended, are linear operators from $H_0 \to H_1$ *and we have*

$$d = d_1 - d_2 \tag{21}$$

Similarly we may define $\delta =: \delta_1$ *and* δ_2 *via:*

$$\delta_{1,2}:\ d_{ik} \to n_k,\, n_i \tag{22}$$

It is remarkable (but actually not surprising) that $v_i \leq v_{max}$ implies that all the above operators are *bounded* (in contrast to their continuous counterparts, which are typically unbounded). Taking this for granted at the moment, there are no domain problems and a straightforward analysis yields the following relations:

Observation 3.3

1. *The adjoint* d^* *of* d *with respect to the spaces* H_0, H_1^a *is* 2δ
2. *On the other side we have for the natural extension of* d, δ *to the larger space* H_1 *(cf. the definitions in Observation 3.2):*

$$\delta_1 = (d_1)^*\ ,\ \delta_2 = (d_2)^* \tag{23}$$

hence

$$(\delta_1 - \delta_2) = (d_1 - d_2)^* = d^* \neq 2\delta = 2\delta_1 \tag{24}$$

3. *Furthermore we have*

$$d_1^* \cdot d_1 = \delta_1 \cdot d_1 = d_2^* \cdot d_2 = V:\ n_i \to v_i n_i \tag{25}$$

$$d_1^* \cdot d_2 = \delta_1 \cdot d_2 = d_2^* \cdot d_1 = \delta_2 \cdot d_1 = A:\ n_i \to \sum_{k-i} n_k \tag{26}$$

Similar geometric properties of the graph are encoded in the products coming in reversed order.

That and how d, d^* encode some more geometric information about the graph can be seen from the following domain- and range-properties (for corresponding results in the more traditional approach see also [8],p.24ff).

Theorem 3.4 *Let the graph be connected and finite, $|\mathcal{V}| = n$, then*

$$dim(Rg(d^*)) = n - 1 \tag{27}$$

$$dim(Ker(d^*)) = \sum_i v_i - (n-1) \tag{28}$$

With $dim(H_1) = \sum_i v_i$, $dim(H_1^a) = 1/2 \cdot dim(H_1)$ we have

$$codim(Ker(d^*)) = dim(Rg(d)) = n - 1 \tag{29}$$

We see that both $Rg(d^*)$ and $Rg(d)$ have the same dimension $(n-1)$.

Remark 3.5 *In case the graph has, say, c components, the above results are altered in an obvious way; we have for example*

$$dim(Rg(d^*)) = n - c \tag{30}$$

Proof: we first state the general result for bounded operators

$$Rg(T^*) = Ker(T)^\perp \tag{31}$$

we then have for $T = d$

$$f \in Ker(d) \Rightarrow 0 = d(f) = \sum_{ik} (f_k - f_i) d_{ik} \tag{32}$$

As the d_{ik} are linearly independent this entails $f_k = f_i$ for the pairs $(i.k)$ which occur in the sum. Since the graph is connected we have $f_k = f_i = const$ for all nodes, hence $dim(Ker(d)) = 1$ and is spanned by $\sum_i n_i$. this proves the first item.

In a similar way we proceed for d^*.

$$0 = d^*(g) = \sum_i (\sum_k (g_{ki} - g_{ik})) n_i \Rightarrow \sum_k (g_{ki} - g_{ik}) = 0 \text{ for all nodes } n_i \tag{33}$$

In H_1 g_{ik}, g_{ki} can be independently chosen. We have n linear equations, which are, however, not independent. There is, in fact, exactly one apriori constraint of the form

$$\sum_i (\sum_k (g_{ik} - g_{ki})) = 0 \tag{34}$$

Hence, the above yields exactly $n-1$ independent linear equations for the $\sum v_i$ coefficients. This implies that the subspace, so defined, has dimension $\sum v_i - (n-1)$. This proves items two and three. □

Observation 3.6 *In the literature $Ker(d^*)$ is called (for obvious reasons) the* cycle subspace *(cf e.g. [8]). On the antisymmetric subspace H_1^a we have $d^* = 2\delta$ and $\delta(b_{ik}) = n_k - n_i$. Choosing now a* cycle*, given by its sequence of consecutive vertices $n_{i_1}, \ldots, n_{i_k}; n_{i_{k+1}} := n_{i_1}$, we have*

$$d^*(\sum b_{i_l i_{l+1}}) = 2\sum(n_{i_{l+1}} - n_{i_l}) = 0 \tag{35}$$

that is, vectors of this kind lie in the kernel of d^*

We will now provide quantitative lower and upper bounds for the respective norms of the occurring operators. For d we have:

$$d:\ H_0 \ni \sum_i f_i n_i \to \sum_i f_i(\sum_{k-i} b_{ki}) = \sum_{ik}(f_k - f_i)d_{ik} \tag{36}$$

and it follows for the norm of the rhs:

$$\begin{aligned}\|rhs\|^2 = \sum_{ik}|(f_k - f_i)|^2 = \sum_i v_i \cdot |f_i|^2 + \sum_k v_k \cdot |f_k|^2 - \sum_{i\neq k}(\overline{f_k}f_i + \overline{f_i}f_k) \\ = 2\cdot\sum_i v_i|f_i|^2 - 2\cdot\sum_{i\neq k}\overline{f_k}f_i\end{aligned} \tag{37}$$

The last expression can hence be written:

$$\|df\|^2 = 2((f|Vf) - (f|Af)) = (f| - 2\Delta f) \tag{38}$$

and shows the close relationship of the norm of d with the *expectation values* of the *adjacency* and *degree matrix* respectively the *graph Laplacian.* That is, norm estimates for, say, d, derive in a natural manner from the corresponding estimates for A or $-\Delta$. It follows from the above that we have:

Observation 3.7

$$\|df\|^2 = (f|d^*df) = (f| - 2\Delta f) \tag{39}$$

i.e.

$$d^*d\ =\ -2\Delta \quad \textit{hence} \quad \|d\|^2 = \sup_{\|f\|=1}(f| - 2\Delta f) = \| - 2\Delta\| \tag{40}$$

and

$$0 < \sup_{\|f\|=1}(f| - 2\Delta f) \leq 2v_{max} + 2\sup_{\|f\|=1}|< f|Af >| \tag{41}$$

i.e.

$$\| - \Delta\| \leq v_{max} + \|A\| \tag{42}$$

We want to note that we are exclusively using the *operator norm* also for matrices (in contrast to most of the matrix literature), which is also called the *spectral norm*. It is unique in so far as it coincides with the so-called *spectral radius* (cf. e.g. [27] or [28]), that is

$$\|A\| := \sup\{|\lambda|;\, \lambda \in spectr(A)\} \tag{43}$$

We now provide upper and lower bounds for the operator norm of the adjacency matrix, A, both in the finite- and infinite-dimensional case. In [29] we estimated the upper bound by a method being different from the following calculation which, being based on *form-estimates*, is much more direct. The previous proof was based on the so-called *Gerschgorin-inequality* for finite matrices and a not entirely straightforward extension to the infinite dimensional case. We expect this upper bound to be well-known (cf. e.g. [15], Theorem 2.8 – *Lemma of Gabber-Galil* – which is slightly more general). We do not know whether this is also the case for the lower bound. As such lower bounds are frequently less straightforward to derive and since, as a byproduct, we develop several potentially useful techniques, we give our own proof of the lower bound below.

Theorem 3.8 (Norm of A) *With the adjacency matrix A finite or infinite and a finite v_{max} we have the following result (a certain fixed labelling of the nodes being assumed):*

$$\limsup\, n^{-1} \cdot \sum_{i=1}^{n} v_i \leq \|A\| = \sup\{|\lambda|;\, \lambda \in \ spectr(A)\} \leq v_{max} \tag{44}$$

Proof: In order to prove the upper bound we use a form-estimate directly for the infinite case. We have

$$|(x, Ax)| = |\sum \overline{x_j} a^{ji} x_i| \leq \sum_{a^{ji} \neq 0} |x_j| \cdot |x_i| \tag{45}$$

with $a^{ji} = 1$ or 0. Note that, due to the symmetry of A, each term, $|x_j| \cdot |x_i|$ occurs twice in the above sum. With

$$2|x_j| \cdot |x_i| \leq |x_j|^2 + |x_i|^2 \tag{46}$$

we have

$$\text{rhs of (45)} \leq v_{max} \cdot |x|^2 \tag{47}$$

and hence

$$|(x, Ax)| \leq v_{max}(x, x) \tag{48}$$

Strictly speaking we have the norm-bound up to now only proved for the above quadratic form. The well-known *Riesz-lemma* associates the form with a unique bounded operator which is the adjacency matrix we started with. This proves the first estimate.

To prove the lower bound, we label the nodes or the corresponding orthonormal basis by $(e_1, e_2, \dots)$, and introduce the respective adjacency matrices on the corresponding subspaces, X_n, belonging to the *induced subgraphs*, G_n, spanned by $(e_1, \dots, e_n)$. We choose a normalized vector, x_n, in X_n with all its entries being $n^{-1/2}$. We then have

$$\|A\| = \sup_{\|x\|=1} |(x, Ax)| \geq |(x_n, Ax_n)| = |(x_n, A_n x_n)| = n^{-1} \sum_1^n v_i \tag{49}$$

This proves the theorem.

Lemma 3.9 *The adjacency matrices, A_n, converge strongly to A and we have in particular $\|A_n\| \nearrow \|A\|$.*

Remark 3.10 *To prove strong convergence of operators is of some relevance for the limit behavior of* spectral properties *of the operators A_n, A. That is (cf. e.g. [30] section VIII.7), we have in that case (A_n, A selfadjoint and uniformly bounded) $A_n \to A$ in* strong resolvent sense, *which implies that the spectrum of the limit operator, A, cannot suddenly expand, i.e.*

$$\lambda \in spec(A) \Rightarrow \exists\ \lambda_n \in spec(A_n) \text{ with } \lambda_n \to \lambda \tag{50}$$

and for $a, b \notin spec_{pp}(A)$

$$P_{(a,b)}(A_n) \to P_{(a,b)}(A) \text{ strongly} \tag{51}$$

Proof of the Lemma: Strong convergence can be proved as follows. $A - A_n$ has the matrix representation

$$A - A_n = \begin{pmatrix} 0 & B_n \\ B_n^t & C_n \end{pmatrix} \tag{52}$$

with

$$(A - A_n)x = \begin{pmatrix} B_n x'_n \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ B_n^t x_n \end{pmatrix} + \begin{pmatrix} 0 \\ C_n x'_n \end{pmatrix} \tag{53}$$

where

$$x = \begin{pmatrix} x_n \\ x'_n \end{pmatrix} \quad \text{with} \quad x_n = \sum_1^n x_i e_i\ ,\ x'_n = \sum_{n+1}^\infty x_i e_i \tag{54}$$

(and the B_n not to be confused with the incidence matrices of section 2) Multiplying from the left with x we easily establish weak convergence since

$$(x|(A - A_n)x) = (x_n|B_n x'_n) + (x'_n|B_n^t x_n) + (x'_n|C_n x'_n) \tag{55}$$

and with $n \to \infty$ all the terms on the rhs go to zero, as $\|x'_n\| \to 0$ for $n \to \infty$ since $\|x\| < \infty$ and B_n, B_n^t are again uniformly bounded.

To show strong convergence the critical term is $B_n^t x_n$. B_n^t maps the vector $x_n \in X_n$ into $X'_n = X \ominus X_n$, X_n, X'_n living on the node sets $V_n, V - V_n$. As $v_{max} < \infty$ we can find for each given n a finite, minimal m_n so that all bonds beginning at nodes of V_n end in V_{m_n}, in other words:

$$\forall\ n\ \exists\ m_n \geq n \text{ with } B_n^t x_n \in X_{m_n} \ominus X_n \tag{56}$$

or

$$B_{m_n}^t x_{m_n} = B_{m_n}^t (x_{m_n} - x_n) \tag{57}$$

as $B_{m_n}^t x_n = 0$ by construction.

The B_n^t are uniformly bounded and $\|x_{m_n} - x_n\| \to 0$ for $n \to \infty$, hence

$$\|B_{m_n}^t x_{m_n}\| \to 0 \text{ with } n \to \infty \tag{58}$$

Each $l \in \mathbb{N}$ lies between some m_n and $m_{(n+1)}$ and we have

$$B_l^t x_l = B_l^t (x_n + (x_l - x_n)) = B_l^t (x_l - x_n) \tag{59}$$

as $B_l^t x_n = 0$ for all $l \geq m_n \geq n$.

$$l \to \infty \Rightarrow m_n \to \infty \Rightarrow n \to \infty \text{ hence } \|x_l - x_n\| \to 0 \tag{60}$$

which shows that

$$s - \lim(B_l^t x_l) = s - \lim(B_l^t x) = 0 \tag{61}$$

□

Remark 3.11 *A slightly simpler but perhaps less instructive proof can be given by exploiting the already established weak convergence together with special properties of A_n, B_n etc., yielding*

$$\lim_n ((A - A_n)x|(A - A_n)x) = -\lim_n (A_n x_n|B_n x'_n) = 0 \tag{62}$$

To prove the monotone convergence of $\|A_n\|$ to $\|A\|$, we proceed as follows. For the principal minors we have

$$A_n = P_n A P_n \quad \text{and} \quad A_n = P_n A_m P_n \tag{63}$$

with P_n projecting on the subspace spanned by $e_1, \ldots, e_n$ and $m \geq n$. Hence

$$\|A_n\| \leq \|A\| \quad \text{and} \quad \|A_n\| \leq \|A_m\| \tag{64}$$

as $\|P_n\| = 1$. From this we see that $\|A_n\|$ is monotonely increasing with $n \to \infty$ and uniformly bounded by $\|A\|$. In other words:

$$\|A_n\| \to a \leq \|A\| \tag{65}$$

The equality of a and $\|A\|$ follows then immediately from the strong convergence of A_n towards A. This proves the above lemma.

To test the effectiveness of the upper and lower bounds derived above, we apply them to a non-trivial model recently discussed in [31], i.e. the infinite *binary tree* with *root* n_0 where v_0 is two and v_i equals three for $i \neq 0$. The authors show (among other things) that the spectrum consists of the interval $[-2\sqrt{2}, 2\sqrt{2}]$, i.e. $\|A\| = 2\sqrt{2}$. v_{max} is three, we have to calculate $\limsup 1/n \cdot \sum_1^n v_i$. For simplicity we choose a subsequence so that $n := n(N)$ with N denoting the N-th level (consisting of 2^N nodes) of the tree starting from the root n_0. Note that in the corresponding *induced subgraph* G_N the *boundary nodes* sitting in the N-th level have only *node degree* one with respect to G_N but three viewed as nodes in the full tree.

We then have

$$n = \sum_{k=1}^{N} 2^k \ , \ \sum_{i=0}^{n(N)} v_i = 2 + 3 \cdot \sum_{k=1}^{N-1} 2^k + 2^N = 3 \cdot \sum_{k=0}^{N} 2^k - 2 \cdot 2^N - 1 \tag{66}$$

Hence

$$\lim_{n(N)} 1/n(N) \sum_{i=0}^{n(N)} = 3 - 2 \lim_{N} (\sum_{0}^{N} 2^{k-N})^{-1} = 2 \tag{67}$$

That is, our genral estimate imply $2 \leq \|A\| \leq 3$, which is not so bad.

4 The Spectral Triplet on a general (undirected) Graph

Note what we said at the beginning about our restriction to *undirected* graphs (made, however, only for convenience!). Furthermore our Dirac operator intertwines node-vectors and bond-vectors while in other examples it maps node- to node-functions. Our bond-functions have (in some sense) the character of *cotangential-vectors*, while in other approaches derivatives of functions are interpreted as *tangent-vectors*. In our view, the latter formalism is effective only in certain classes of highly regular models (like e.g. lattices) where one has kind of

global directions and will become cumbersome for general graphs. We developed this latter approach a little bit in section 3.3 of [1] and showed how these cotangent and tangent vectors can be mapped into each other.

The Hilbert space under discussion in the following is

$$H = H_0 \oplus H_1 \tag{68}$$

The *natural representation* of the function algebra $\mathcal{F}$

$$\{f; f \in \mathcal{C}_0, \sup_i |f_i| < \infty\} \tag{69}$$

on H by bounded operators is given by:

$$H_0 : \ f \cdot f' = \sum f_i f'_i \cdot n_i \text{ for } f' \in H_0 \tag{70}$$

$$H_1 : \ f \cdot \sum g_{ik} d_{ik} := \sum f_i g_{ik} d_{ik} \tag{71}$$

From previous work ([1]) we know that $\mathcal{C}_1$ carries also a right-module structure, given by:

$$\sum g_{ik} d_{ik} \cdot f := \sum g_{ik} f_k \cdot d_{ik} \tag{72}$$

(For convenience we do not distinguish notationally between elements of $\mathcal{F}$ and their Hilbert space representations).

An important object in various areas of modern analysis on manifolds or in Connes' approach to noncommutative geometry is the so-called *Dirac operator D* (or rather, a certain version or variant of its classical counterpart; for the wider context see e.g. [5] or [32] to [34]). As D we will take in our context the operator:

$$D := \begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix} \tag{73}$$

acting on

$$H = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix} \tag{74}$$

with

$$d^* = (\delta_1 - \delta_2) \tag{75}$$

Note however, that there may exist in general several possibilities to choose such an operator. On the other hand, we consider our personal choice to be very natural from a geometrical point of view.

Lemma 4.1 *There exists in our scheme a natural chirality- or grading operator,* χ *and an antilinear involution,* J*. given by*

$$\chi := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{76}$$

with

$$[\chi, \mathcal{F}] = 0 \quad \chi \cdot D + D \cdot \chi = 0 \tag{77}$$

and

$$J : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} \tag{78}$$

so that

$$J \cdot f \cdot J = \overline{f} \tag{79}$$

These are some of the ingredients which establish what Connes calls a *spectral triple* (cf. e.g. [21] or [22]). We do not want, however, to introduce the full machinery at the moment as our scheme has an independent geometric meaning of its own. So, being careful, we call in the following these structures simply *spectral triplets* (we were kindly warned by B.Iochum to be more careful with this concept; note the observation below about the non-compactness of the inverse of such Dirac-operators on infinite graphs with a uniformly bounded vertex degree).

Definition 4.2 (Spectral Triplets) *As* spectral triplet *on a general (undirected) graph we take*

$$(H, \mathcal{F}, D) \tag{80}$$

At this point we would like to remark the following. In our general framework we restricted ourselves, mostly for (possibly subjective) aesthetic reasons – the mathematics tends to be more transparent – to *undirected graphs* and a total Hilbert space being the direct sum of the *node space* (a function space) and the *bond space* (sort of cotangent vectors). A was then selfadjoint and a Dirac operator emerged naturally as kind of a square root of the Laplacian.

On the other side, if one studies simple models as e.g. in [23] to [25], other choices are possible. In [23],[24], where the one-dimensional lattice was studied, the *symmetric difference operator* was taken as Dirac operator. In [25] the one-dimensional lattice was assumed to be directed (i.e. only $d_{i,i+1}$ were present) and the Dirac operator was defined as a certain self adjoint "doubling" of the (one-sided, i.e. non-symmetric) adjacency matrix. This latter model would fit in our general approach if we had included more general graphs. All these Dirac

operators are different and it is hence no wonder that they lead to different consequences (see below). It is our opinion that, in the end, an appropriate choice has to be dictated by physical intuition. Nevertheless, this apparent non-uniqueness should be studied more carefully.

As can be seen from the above, the connection with the graph Laplacian is relatively close since:

$$D^2 = \begin{pmatrix} d^*d & 0 \\ 0 & dd^* \end{pmatrix} \tag{81}$$

and

$$d^*d = -2\Delta \tag{82}$$

dd^* is the corresponding object on H_1. (In the vector analysis of the continuum the two entries correspond to divgrad , graddiv respectively).

Observation 4.3 *Note that all our operators are bounded, the Hilbert space is (in general) infinite dimensional, hence there is no chance to have e.g.* $(D-z)^{-1}$ *or* $(D^2-z)^{-1}$ *compact. At the moment we are sceptical whether this latter phenomenon dissappears generically if the vertex degree is allowed to become infinite. There are some results on spectra of random graphs which seem to have a certain bearing on this problem.*

We now calculate the commutator $[D, f]$ applied to an element $f' \in H_0$:

$$(d \cdot f)f' = \sum_{ik}(f_k f'_k - f_i f'_i)d_{ik} \tag{83}$$

$$(f \cdot d)f' = \sum_{ik} f_i(f'_k - f'_i)d_{ik} \tag{84}$$

hence

$$[D, f]f' = \sum_{ik}(f_k - f_i)f'_k d_{ik} \tag{85}$$

On the other side the right-module structure allows us to define df as an operator on H_0 via:

$$df \cdot f' = (\sum_{ik}(f_k - f_i)d_{ik}) \cdot (\sum_k f'_k n_k) = \sum_{ik}(f_k - f_i)f'_k d_{ik} \tag{86}$$

In a next step we define df as operator on H_1 which is not as natural as on H_0. We define:

$$df|_{H_1} : \; d_{ik} \to (f_i - f_k)n_k \tag{87}$$

and linearly extended. A short calculation shows

$$df|_{H_1} = -(d\bar{f}|_{H_0})^* = [d^*, f] \tag{88}$$

This then has the following desirable consequence:

Observation 4.4 *With the above definitions the representation of df on H is given by*

$$df|_H = \begin{pmatrix} 0 & df|_{H_1} \\ df|_{H_0} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -(d\bar{f}|_{H_0})^* \\ df|_{H_0} & 0 \end{pmatrix} \tag{89}$$

and it immediately follows

$$df|_H = \begin{pmatrix} 0 & [d^*, f] \\ [d, f] & 0 \end{pmatrix} = [D, f] \tag{90}$$

5 The Connes-Distance Function on Graphs

From the general theory we know that:

$$\|T\| = \|T^*\| \tag{91}$$

Hence

Lemma 5.1

$$\|[d, f]\| = \|[d, \bar{f}]\| = \|[d^*, f]\| \tag{92}$$

and

$$\|[D, f]\| = \|[d, f]\| \tag{93}$$

Proof: The left part of (92) is shown below and is a consequence of formula (99); the right identity follows from (91). With

$$X := \begin{pmatrix} x \\ y \end{pmatrix} \tag{94}$$

and $T_1 := [d, f]$, $T_2 := [d^*, f]$, the norm of $[D, f]$ is:

$$\|[D, f]\|^2 = \sup\{\|T_1 x\|^2 + \|T_2 y\|^2;\ \|x\|^2 + \|y\|^2 = 1\} \tag{95}$$

Normalizing now x, y to $\|x\| = \|y\| = 1$ and representing a general normalized vector X as:

$$X = \lambda x + \mu y\ ,\ \lambda, \mu > 0 \text{ and } \lambda^2 + \mu^2 = 1 \tag{96}$$

we get:

$$\|[D, f]\|^2 = \sup\{\lambda^2 \|T_1 x\|^2 + \mu^2 \|T_2 y\|^2; \|x\| = \|y\| = 1, \lambda^2 + \mu^2 = 1\} \tag{97}$$

where now x, y can be varied independently of λ, μ in their respective admissible sets, hence:

$$\|[D,f]\|^2 = \sup\{\lambda^2\|T_1\|^2 + \mu^2\|T_2\|^2\} = \|T_1\|^2 \quad \square \tag{98}$$

(as a consequence of equation (92)).

It follows that in calculating $\|[D,f]\|$ one can restrict oneself to the easier to handle $\|[d,f]\|$. For the latter expression we then get from the above ($x \in H_0$):

$$\|df \cdot x\|^2 = \sum_i (\sum_{k=1}^{v_i} |f_i - f_k|^2 \cdot |x_i|^2 \tag{99}$$

Abbreviating

$$\sum_{k=1}^{v_i} |f_k - f_i|^2 =: a_i \geq 0 \tag{100}$$

and calling the supremum over i a_s, it follows:

$$\|df \cdot x\|^2 = a_s \cdot (\sum_i a_i/a_s \cdot |x_i|^2) \leq a_s \tag{101}$$

for $\|x\|^2 = \sum_i |x_i|^2 = 1$.

On the other side, choosing an appropriate sequence of normalized basis vectors e_ν so that the corresponding a_ν converge to a_s we get:

$$\|df \cdot e_\nu\|^2 \to a_s \tag{102}$$

We hence have

Theorem 5.2

$$\|[D,f]\| = \sup_i (\sum_{k=1}^{v_i} |f_k - f_i|^2)^{1/2} \tag{103}$$

The *Connes-distance functional* between two nodes, n, n', is now defined as follows:

Definition 5.3 (Connes-distance function)

$$dist_C(n,n') := \sup\{|f_{n'} - f_n|; \|[D,f]\| = \|df\| \leq 1\} \tag{104}$$

Remark 5.4 *It is easy to prove that this defines a metric on the graph.*

Corollary 5.5 *It is sufficient to vary only over the set* $\{f; \|df\| = 1\}$.

Proof: This follows from

$$|f_k - f_i| = c \cdot |f_k/c - f_i/c| \; ; \; c = \|df\| \tag{105}$$

and

$$\|d(f/c)\| = c^{-1}\|df\| = 1 \tag{106}$$

with $c \leq 1$ in our case. □

It turns out to be a nontrivial task (in general) to calculate this distance on an arbitrary graph as the above constraint is quite subtle . The underlying reason is that the constraint is, in some sense, inherently *non-local*. As f is a function, $f_{n'} - f_n$ has to be the same independently of the path connecting n' and n. On the other side, in a typical optimization process one deals with the individual jumps, $f_k - f_i$, along some path. It is then not at all clear that these special choices can be extended to a global function without violating the overall constraint on the expression in theorem 5.2. Nevertheless we think the above closed form is a solid starting point for the calculation of $dist_C$ on various classes of graphs or lattices. We discuss two examples below but refrain at this place from a more complete treatment, adding only some observations concerning the relation to the ordinary (combinatorial) distance function introduced in the beginning of the paper.

Having an admissible function f so that $\sup_i(\sum_{k=1}^{v_i} |f_k - f_i|^2)^{1/2} \leq 1$, this implies that, taking a *minimal path* γ from, say, n to n', the jumps $|f_{\nu+1} - f_\nu|$ between neighboring nodes along the path have to fulfill:

$$|f_{\nu+1} - f_\nu| \leq 1 \tag{107}$$

and are typically strictly smaller than 1 as long as there are not a sufficient number of "zero-jumps" ending at the same node.

On the other side the Connes distance would only become identical to the ordinary distance $d(n, n')$ if there exist a sequence of admissible node functions with all these jumps approaching the value 1 along such a path, which is however impossible in general as can be seen from the structure of the constraint on the expression in theorem 5.2 . Only in this case one may have a chance to get:

$$|\sum_\gamma (f_{\nu+1} - f_\nu)| \to \sum_\gamma 1 = length(\gamma) \tag{108}$$

We express this observation in the following way

Observation 5.6 (Connes-distance) *One has within our general scheme the following inequality*

$$dist_C(n, n') \leq d(n, n') \tag{109}$$

A fortiori one can prove that $dist_C$ between two nodes in an arbitrary graph is even smaller than or equal to the corresponding Connes-distance taken with respect to the (one-dimensional) sub-graph formed by a minimal path between these nodes, i.e.

$$dist_C(n, n') \leq dist_C(min.path)(n, n') \tag{110}$$

The simple reason is that one has more admissible functions at ones disposal for a subgraph, hence the supremum may become larger. This latter distance, on the other side, can be rigorously calculated (see Example 2 below) and is for non-neighboring nodes markedly smaller than the ordinary distance.

Corollary 5.7 *The last inequality implies also that with G' an induced subgraph of G it holds ($n, n' \in V' \subset V$):*

$$dist_C(n, n'; G') \leq dist_C(n, n'; G) \tag{111}$$

We remarked above that the calculation of the Connes distance on graphs is to a large part a continuation problem for admissible functions, defined on subgraphs. Then the following question poses itself.

Problem 5.8 *For what classes of graphs and/or subgraphs do we have an equality in the above corollary?*

Remark 5.9 *Equality can e.g. be achieved for trees. Other results in this direction are in preparation.*

These general results should be contrasted with the results in [23] to [25]. Choosing e.g. the symmetric difference operator as Dirac operator in the case of the one-dimensional lattice the authors got in [23, 24] a distance which is *strictly greater* than the ordinary distance but their choice does not fulfill the above natural constraint given in Theorem 5.2. Note in particular that our operator d is a map from node- to bond-functions which is not the case in the other examples. In [25] the authors employed a symmetric doubling of the upper half of our symmetric adjacency matrix as Dirac operator. In the case of the one-dimensional (directed) lattice this then leads (so to say) to only one (directed) bond per node and makes the optimization process quite simple, hence leading to the ordinary distance which would have been also the case in our general scheme had we admitted directed graphs. We conjecture however that for more general graphs a relation related to the one given in Theorem5.2 would enforce the Connes-distance to be again strictly smaller than the ordinary distance for non-neighboring points. This is however an interesting point and we plan to discuss generalisations of our framework and more general examples elsewhere.

We want to close this paper with the discussion of two examples. The first one is a simple warm-up exercise, the second one is the one-dimensional lattice discussed also by the other authors mentioned above (treated however within their respective schemes) and is not so simple. The technique used in approaching the second problem may be interseting in general. While we solved it starting, so to speak, from first principles, the real mathematical context, to which the strategy is belonging, is the field of *(non-)linear programming* or *optimization* (see e.g. [35] or any other related textbook). This can be inferred from the structure of our constraint on the expression in theorem 5.2. This means that the techniques developed in this field may be of use in solving such quite intricate problems.

Example 1: The square with vertices and edges:

$$x_1 - x_2 - x_3 - x_4 - x_1 \tag{112}$$

Let us calculate the Connes-distance between x_1 and x_3. As the sup is taken over functions(!) the summation over elementary jumps is (or rather: has to be) pathindependent (this is in fact both a subtle and crucial constraint for practical calculations). It is an easy exercise to see that the *sup* can be found in the class where the two paths between x_1, x_3 have the *valuations* $(1 \geq a \geq 0)$:

$$x_1 - x_2 : a \ , \ x_2 - x_3 : (1 - a^2)^{1/2} \tag{113}$$

$$x_1 - x_4 : (1 - a^2)^{1/2} \ , \ x_4 - x_3 : a \tag{114}$$

Hence one has to find $\sup_{0 \leq a \leq 1}(a + \sqrt{1 - a^2})$. Setting the derivative with respect to a to zero one gets $a = \sqrt{1/2}$. Hence:

Example 5.10 (Connes-distance on a square)

$$dist_C(x_1, x_3) = \sqrt{2} < 2 = d(x_1, x_3) \tag{115}$$

Example 2: The undirected one-dimensional lattice:

The nodes are numbered by $\mathbb{Z}$. We want to calculate $dist_C(0, n)$ within our general framework. The calculation will be done in two main steps. In the first part we make the (in principle quite complicated) optimization process more accessible. For the sake of brevity we state without proof that it is sufficient to discuss real monotonely increasing functions with

$$f(k) = \begin{cases} f(0) & \text{for } k \leq 0 \\ f(n) & \text{for } k \geq n \end{cases} \tag{116}$$

and we write

$$f(k) = f(0) + \sum_{i=1}^{k} h_i \quad \text{for} \quad 0 \leq k \leq n \; h_i \geq 0 \tag{117}$$

The above optimization process then reads:

Observation 5.11 *Find* $\sup \sum_{i=1}^{n} h_i$ *under the constraint*

$$h_1^2 \leq 1, h_2^2 + h_1^2 \leq 1, \ldots, h_n^2 + h_{n-1}^2 \leq 1, h_n^2 \leq 1 \tag{118}$$

The simplifying idea is now the following. Let $h := (h_i)_{i=1}^n$ be an admissible sequence with *all* $h_{i+1}^2 + h_i^2 < 1$. We can then find another admissible sequence h' with

$$\sum h_i' > \sum h_i \tag{119}$$

Hence the supremum cannot be taken on the interior. We conclude that at least some $h_{i+1}^2 + h_i^2$ have to be one. There is then a minimal i for which this holds. We can convince ourselves that the process can now be repeated for the substring ending at $i+1$. Repeating the argument we can fill up all the entries up to place $i+1$ with the condition $h_{l+1}^2 + h_l^2 = 1$ and proceeding now upwards we end up with

Lemma 5.12 *The above supremum is assumed within the subset*

$$h_1^2 \leq 1, h_1^2 + h_2^2 = 1, \ldots, h_{n-1}^2 + h_n^2 = 1, h_n^2 \leq 1 \tag{120}$$

This concludes the first step.

In the second step we calculate $\sup |f(0) - f(n)|$ on this restricted set. From the above we now have the constraint:

$$h_1^2 \leq 1, h_2^2 = 1 - h_1^2, h_3^2 = h_1^2, h_4^2 = 1 - h_1^2, \ldots, h_n^2 = 1 - h_1^2 \text{ or } h_1^2 \tag{121}$$

depending on n being even or uneven. This yields

$$\sup |f(0) - f(n)| = \begin{cases} 1 & \text{for } n = 1 \\ (n/2) \cdot \sup(h_1 + \sqrt{1 - h_1^2}) = (n/2) \cdot \sqrt{2} & \text{for } n \text{ even} \\ \sup([n/2] \cdot (h_1 + \sqrt{1 - h_1^2}) + h_1) & \text{for } n \text{ uneven} \end{cases} \tag{122}$$

In the even case the rhs can be written as $\sqrt{n^2/2} = \sqrt{[n^2/2]}$. In the uneven case we get by differentiating the rhs and setting it to zero:

$$h_1^{max} = A_n / \sqrt{1 + A_n^2}\,, \;\; \sqrt{1 - (h_1^{max})^2} = 1/\sqrt{1 + A_n^2} \tag{123}$$

with $A_n = 1 + 1/[n/2]$. We see that for increasing n both terms approach $1/\sqrt{2}$, the result in the even case. Furthermore we see that the distance is monotonely increasing with n as should be the case for a distance. This yields in the uneven case

$$dist_C(0,n) = \frac{([n/2]+1)A_n + [n/2]}{\sqrt{1+A_n^2}} \tag{124}$$

which is a little bit nasty. Both expressions can however be written in a more elegant and unified way (this was a conjecture by W.Kunhardt, inferred from numerical examples). For n uneven a short calculation yields

$$[n^2/2] = (n^2-1)/2 = 1/2 \cdot (n-1)(n+1) = 2[n/2]([n/2]+1) \tag{125}$$

(with the *floor-,ceiling*-notation the expressions would become even more elegant). With the help of the latter formula the rhs in (124) can be transformed into

$$rhs \text{ of } (124) = \sqrt{[n^2/2]+1} \tag{126}$$

Conclusion 5.13 *For the one-dimensional undirected lattice we have*

$$dist_C(0,n) = \begin{cases} \sqrt{[n^2/2]} & \text{for } n \text{ even} \\ \sqrt{[n^2/2]+1} & \text{for } n \text{ uneven} \end{cases} \tag{127}$$

Remark: With the help of the methods, introduced above, we can now estimate or rigorously calculate the Connes-distance for other classes of graphs.

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