Applied Mathematical Sciences, Vol. 4, 2010, no. 53, 2599 - 2608

A Note on Convexity and Starshapedness

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Abstract

We establish sufficient and necessary conditions for a set to be affine or convex. The concepts of weak near convexity and affine starshapedness are studied. A characterization of the (affine) kernel of (affinely) starshaped sets and their radial contraction is discussed.

Mathematics Subject Classification: 53C20, 53C21

Keywords: Convex sets, wn-convex sets, starshaped sets, affine sets, affinely starshaped sets, radial contraction

1 An introduction

Convexity is a basic notion in geometry but also is widely used in other areas of mathematics [8]. Many mathematicians established a criterion for the convexity a domain in the Euclidean space and in other spaces [3, 5, 11]. In this note we establish sufficient and necessary conditions for a set to be affine or convex.

Let A be a subset of the Euclidian space E^n and let x, y be in A. The closed segment joining x and y is denoted by $[xy]$ where $(xy) = [xy] \setminus \{x, y\}$ denotes the open segment joining x and y. The ray starting from x and passing through y is denoted by $\overrightarrow{xy}$ where the straight line determined by x and y is denoted by xy. For points x and y in A, we say that x sees y via A if and only if $[xy]$ is contained in A [4]. A is called *starshaped* if there exists some point $p \in A$ such that p sees each point of A via A [1, 2]. In this case we say that A is starshaped relative to p [1, 2]. The set of all such points p is called kernel of A and is denoted by $\ker A$. A is called *convex* if for each two points p and q in A, p sees q via A. The set $\ker A$ is convex [9, 10].

Definition 1 *[7]A set $A \subset E^n$ is called weakly nearly convex (in brief wn-convex) if for every x, y in A there exists a real number $\alpha \in (0,1)$ such that $\alpha x + (1-\alpha)y$ is in A.*

It is clear from the above definition that a wn-convex set is a midpoint convex set if every $\alpha = \frac{1}{2}$. So, every (midpoint) convex set is wn-convex. The intersection of midpoint convex sets is a midpoint convex set where the intersection of wn-convex set is not generally a wn-convex set.

Example 2 *Let $A = \{x : x \text{ is a rational number in} [0,1]\}$ and let*

$$B = \{0,1\} \cup \{x : x \text{ is an irrational number in} (0,1)\} \tag{1}$$

Then $A \cap B = \{0,1\}$ is not wn-convex.

Theorem 3 *[7]A closed weakly nearly convex set of E^n is convex.*

Definition 4 *Let $A \neq \phi$ be a subset of E^n. The midpoint contraction of A about $p \in A$, denoted by $C_p(A)$, is the set of all points $\frac{1}{2}(p+x), x \in A$ i.e.*

$$C_p(A) = \left\{ y : y = \frac{1}{2}(p+x), x \in A \right\} \tag{2}$$

Definition 5 *Let $A \neq \phi$ be a subset of E^n. For a fixed point $p \in A$ and a fixed real number $\alpha \in (0,1)$, we define the λ-radial contraction of A based at p, denoted by $C_p^\lambda(A)$, by*

$$C_p^\lambda(\bar{x}) = \lambda \bar{x} + (1-\lambda)p, \bar{x} \in A \tag{3}$$

The set $C_p^\lambda(A)$ is called the λ-radial contraction of A based at p. If $\lambda = \frac{1}{2}$, we get the midpoint contraction.

It is clear that the λ-radial contraction map is a bijection onto its image set which has p as its unique fixed point. We will take $p \in E^n$ and the definition still true but in the case $p \notin A$ the map C_p^λ is a translation followed by a contraction of A.

Example 6 *The following figures show two contractions of a set A when $p \in A$ and $p \notin A$.*

The following proposition lists some properties of the λ-contraction map.

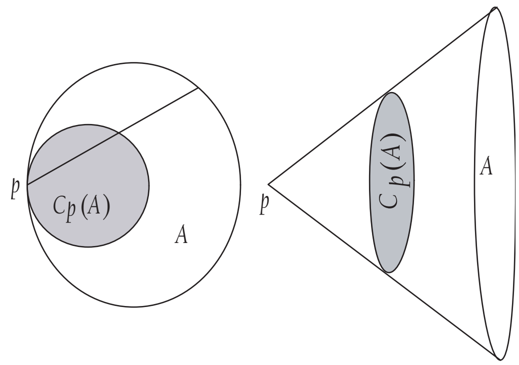

Figure 1: Contractions of a set A when $p \in A$ and $p \notin A$

Proposition 7 *Let A be a non-empty subset of E^n and λ, β be in $(0,1)$ and $p, q \in E^n$. If x, y are in A, then*

1. $C_p^\lambda(x+y) = C_p^\lambda(x) + C_p^\lambda(y)$ if $p = 0$,
2. $C_p^\beta\left(C_p^\lambda(x)\right) = C_p^{\lambda\beta}(x)$,
3. $\alpha\ C_p^\lambda(x) = C_{\alpha p}^\lambda(\alpha x)$, α is a real number,
4. $C_{p+q}^\lambda(x+y) = C_p^\lambda(x) + C_q^\lambda(y)$.

Lemma 8 *Let $\bar{x}, \bar{y} \in A$, with $C_p^\lambda(\bar{x}) = x$ and $C_p^\lambda(\bar{y}) = y$ for some $p \in E^n$ and some $\lambda \in (0,1)$, then $C_p^\lambda([\bar{x}\bar{y}]) = [xy]$.*

Proof. Let $\bar{z} \in [\bar{x}\bar{y}]$, then there is a real number $\alpha \in [0,1]$, such that $\bar{z} = \alpha\bar{x} + (1-\alpha)\,\bar{y}$, and so

$$\begin{aligned} C_p^\lambda(\bar{z}) &= C_p^\lambda(\alpha\bar{x} + (1-\alpha)\,\bar{y}) \\ &= \lambda\alpha\bar{x} + \lambda\,(1-\alpha)\,\bar{y} + (1-\lambda)\,p \\ &= \alpha C_p^\lambda\,(\bar{x}) + (1-\alpha)\,C_p^\lambda\,(\bar{y}) \\ &= \alpha x + (1-\alpha)\,y \\ &= z \end{aligned}$$

where $z \in [xy]$. Hence, $C_p^\lambda([\bar{x}\bar{y}]) = [xy]$. ∎

Moreover, the λ-radial contraction of a straight line $\vec{r} = \vec{a} + \alpha\vec{v}, \alpha \in \Re$ passing through $\vec{a}$ and parallel to $\vec{v}$ is

$$C_p^\lambda(\vec{r}) = C_p^\lambda(\vec{a}) + \alpha\lambda\vec{v} \tag{4}$$

which is a straight line passing through $C_p^\lambda(\vec{a})$ and also parallel to $\vec{v}$ i.e. the radial contraction of a straight line is a straight line parallel to the old one.

2 Radial contraction and convexity

In this section, we get necessary and sufficient conditions for a subset of E^n to be convex. Some results related to starshapedness are derived.

The following theorems(Theorem 9,10) show that λ-radial contraction preserves convexity, midpoint convexity and weak near convexity. Their proofs depend on Lemma 8.

Theorem 9 *Let $A \neq \phi$ be a subset of E^n. Then $C_p^\lambda(A)$ is wn-convex for some $p \in E^n$ and any $\lambda \in (0,1)$ if and only if A is.*

Theorem 10 *Let A be a non-empty subset of E^n, and let $p \in E^n$ and $\lambda \in (0,1)$. Then A is convex if and only if $C_p^\lambda(A)$ is.*

It is clear that if A is convex, $p \in A$, then $C_p^\lambda(A)$ is contained in A. Now, starshapedness is discussed in the following two theorems.

Theorem 11 *Let A be a nonempty subset of E^n, and let $p \in E^n$ and $\lambda \in (0,1)$. Then $C_p^\lambda(A)$ is a starshaped set if and only if A is a starshaped set.*

Theorem 12 *Let A be a non-empty starshaped subset of E^n. Then $\ker C_p^\lambda(A) = C_p^\lambda(\ker A)$, for any $p \in E^n$ and $\lambda \in (0,1)$.*

Proof. Let $x \in \ker C_p^\lambda(A)$. Then there is $\bar{x}$ in A with $C_p^\lambda(\bar{x}) = x$. Let $\bar{a} \in A$, then $a = C_p^\lambda(\bar{a}) \in C_p^\lambda(A)$ and hence $[xa] \subset C_p^\lambda(A)$, since it is starshaped relative to x. By definition of C_p^λ, we get $[\bar{x}\bar{a}] \subset A$ i.e. $\bar{x} \in \ker A$ and so $x \in C_p^\lambda(\ker A)$ i.e. $\ker C_p^\lambda(A) \subset C_p^\lambda(\ker A)$.

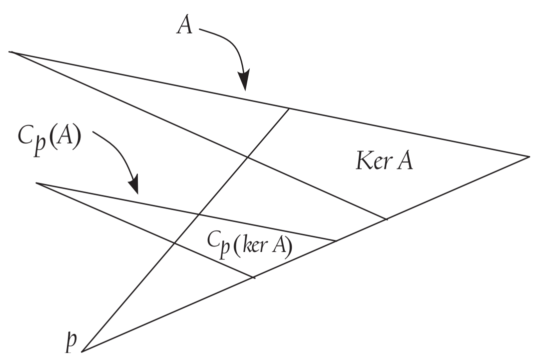

Figure 2: The radial contraction of the kernel

Let $x \in C_p^\lambda(\ker A)$, then there is $\bar{x} \in \ker A$ with $C_p^\lambda(\bar{x}) = x$. Now, let $c \in C_p^\lambda(A)$, then $\bar{c}$ with $c = C_p^\lambda(\bar{c})$ is in A and hence $[\bar{c}\bar{x}] \subset A$. Also, $C_p^\lambda([\bar{c}\bar{x}] = [cx] \subset C_p^\lambda(A)$ i.e. $x \in C_p^\lambda(A)$ i.e. $C_p^\lambda(\ker A) \subset \ker C_p^\lambda(A)$. This completes the proof. ■

Remark 13 *If $p \in \ker A$, we get that $C_p^{\lambda}(\ker A) = \ker C_p^{\lambda}(A) \subset \ker A$ since $\ker A$ is convex.*

The λ-radial contraction of a convex set A where $p \in A$ and $\lambda \in (0,1)$ is a subset of A. The intersection of $C_p^{\lambda}(A)$ and ∂A characterizes the convexity and strict convexity of A.

Example 14 *The following examples(see Figure 3) show the intersection of $C_p^{\lambda}(A)$ and ∂A.*

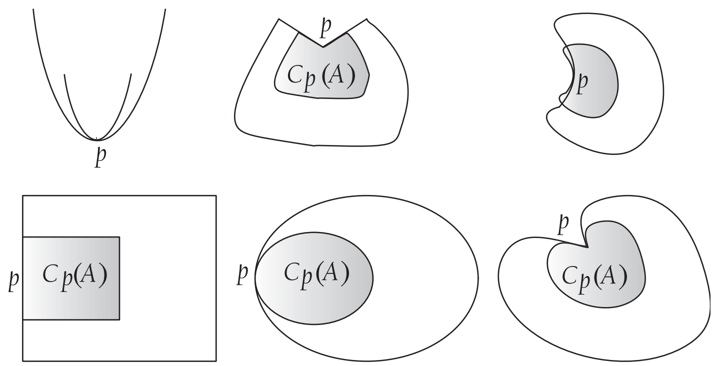

Figure 3: The intersection of $C_p^{\lambda}(A)$ and ∂A

Theorem 15 *Let A be a closed strictly convex set in E^n. Then $C_p^{\lambda}(A) \cap \partial A = \{p\}$, for all $p \in \partial A$, and some $\lambda \in (0,1)$.*

Proof. Suppose that $C_p^{\lambda}(A) \cap \partial A$ contains a point $x \neq p$. Then there exists $\bar{x} \in A$ such that $C_p^{\lambda}(\bar{x}) = x$.

Now, we have the following cases:

1. $\bar{x} \in \partial A$: in this case the pointsp, x and $\bar{x}$ are all in ∂A and hence the segment $(p\bar{x})$ contains $x \in \partial A$. This contradicts the strict convexity of A.

2. $\bar{x} \notin \partial A$: then the segment $(p\bar{x})$ cuts A tangentially otherwise A is not convex. Construct the cone with vertex at p and base contained in $B(\bar{x}, \delta) \subset A$, where $B(\bar{x}, \delta)$ is a sufficiently small ball with center at $\bar{x}$ and radius δ. This cone is not contained in A which contradicts the convexity of A as shown in Figure(4).

The above discussion shows that such point does not exist and hence we get $C_p^{\lambda}(A) \cap \partial A = \{p\}$, for all $p \in \partial A$. ■

The converse of the above theorem is also true. To prove this converse and other results, we need the following lemma.

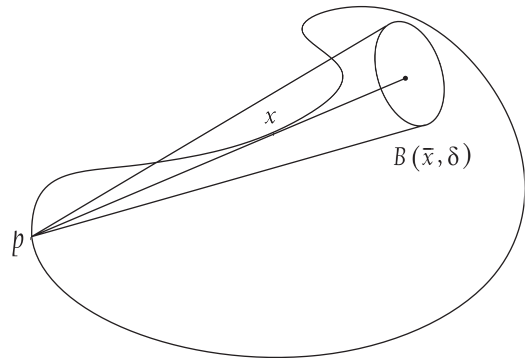

Figure 4: A nonconvex case

Lemma 16 *Let A be a closed subset of E^n. If $x, y \in A$ and (xy) is not contained in A, then there exist $p, q \in [xy] \cap A$ such that $(pq) \cap A = \phi$.*

Proof. Since A and $[xy]$ are closed sets, then $[xy] \cap A$ is closed. Suppose that such points do not exist. Then for every pair of points $p, q \in [xy] \cap A$, there is r in $(pq) \cap A \subset [xy] \cap A$ i.e. $[xy] \cap A$ is *wn*-convex and hence convex (by Theorem 3). This shows that $[xy]$ is contained in A and completes the proof. ■

Remark 17 *It is clear that the above two points p, q are elements in ∂A, otherwise the segment (pq) intersects A.*

Theorem 18 *Let A be a closed subset of E^n. If $C_p^\lambda(A) \subset A$ for all $p \in \partial A$, and some $\lambda \in (0, 1)$, then A is convex.*

Proof. Suppose that A is not convex. Then there are x, y in A such that $[xy]$ is not contained in A. By the above lemma, we get $p, q \in A$ such that $(pq) \cap A = \phi$. Construct the λ-radial contraction of A based on p, then the image of q by C_p^λ is in (pq) and hence is not contained in A. This contradiction completes the proof. ■

Remark 19

1. *In the above theorem, if we replace $C_p^\lambda(A)$ by $C_p(A)$ we will get the same result.*
2. *If, in addition, $C_p^\lambda(A) \cap \partial A = \{p\}$ for every $p \in \partial A$, then A is strictly convex.*

Theorem 20 *Let A be a closed starshaped set in E^n. Then* ker *A is precisely the set of points of A that see ∂A via A.*

Proof. Let B be the set of all points of A that see ∂A via A. It is clear that $\ker A \subset B$. Now, we want to prove that $B \subset \ker A$. Let $x \in B$, and $y \in A$. If $[xy]$ is not contained in A, then by Lemma 16 there exist p, q in ∂A such that $(pq) \cap A = \phi$ and hence x does not see one of the points p and q. This contradicts the definition of B and the proof is complete. ∎

Theorem 21 *A closed starshaped set is convex if and only if $\partial A \subset \ker A$.*

Proof. If A is convex then $\ker A = A \supset \partial A$. Now, let $\partial A \subset \ker A$, we want to prove that A is convex. Suppose that A is not. Then there exists x, y in A such that $[xy]$ is not contained in A. Since A is closed, then by lemma(16), there exist p, q in $[xy] \cap A$ such that $(pq) \cap A = \phi$ and hence p does not see q via A which contradicts the fact that $\partial A \subset \ker A$ and so A is convex. ∎

3 Affine and affinely starshaped sets

Definition 22 *A set A is said to be affine if $x, y \in A$ implies $\lambda x + (1 - \lambda) y \in A$ for all real numbers λ i.e. for each pair of distinct points of A, the straight xy is in A[9].*

Definition 23 *A set A is said to be affinely starshaped if there exists $p \in A$ such that $\lambda x + (1 - \lambda) p \in A$ for every $x \in A$ and $\lambda \succeq 0$ i.e. the ray $\overrightarrow{px}$ is in A. In this case we say that p affinely sees x via A, and the set of all such points p is called affine kernel of A, and is denoted by Aker A.*

The following remark lists some properties of affine and affinely starshaped sets.

Remark 24

1. *The only affine set with nonempty interior is E^n itself.*
2. *Affine sets are closed [6, 9].*
3. *Every affine set is convex and is affinely starshaped relative to all of its points.*
4. *Every affine starshaped set is starshaped. Moreover, the affine kernel is contained in the kernel(i.e. Aker $A \subset \ker A$).*
5. *The affine sets in E^n are precisely the totally geodesic submanifolds (singletons, lines, planes,...).*
6. *The intersection of two affine sets is affine.*

The kernel of a starshaped set is convex. The following theorem presents a similar result for affinely starshaped sets.

Lemma 25 *[9]Let x, y, z be three distinct points, and suppose $u \in (xy)$. Then if $v \in [zu]$, there exists a point $w \in [zy]$ such that $v \in [xw]$.*

Theorem 26 *Let $A \subset E^n$ be an affinely starshaped set. Then Aker A is an affine set.*

Proof. Let x, y be in *Aker A*. Let $p \in xy$ and $z \in A$, we want to prove that $\overrightarrow{pz}$ is a subset of A. Let $w \in \overrightarrow{pz}$, then we have the following cases.

1. $p \in (xy)$, $w \in (pz)$: Since A is affine, then $(yz) \subset A$. By the above lemma, we find $m \in (yz)$ such that $w \in (xm)$. Also, $x \in Aker\ A$, then $w \in (xm) \subset A$ i.e. $w \in A$(see Figure 5, case 1).

2. $p \in (xy)$, $w \notin [pz]$: Since A is affine, then $\overrightarrow{xz} \subset A$. By the above lemma, we get $m \in (yw)$ such that $m \in \overrightarrow{xz} \subset A$. But $y \in Aker\ A$ and so $\overrightarrow{ym} \subset A$ i.e. $w \in A$(see Figure 5, case 2).

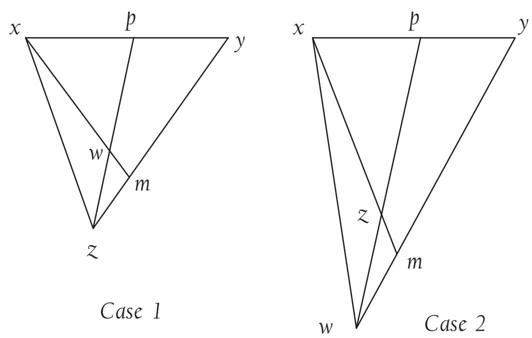

Figure 5: Affine kernel

Similarly, by using the same lemma, we can prove the rest two cases($p \notin [xy]$, $w \in (pz)$, and $p \notin [xy]$, $w \notin [pz]$) and get $w \in A$, i.e., $\overrightarrow{pz} \subset A$ and hence $xy \subset Aker\ A$. Then *Aker A* is affine. ■

In the following, we try to get sufficient conditions for a set to be affine. Without loss of generality, we will consider only the midpoint contraction.

Theorem 27 *If $p \in E^n$, A is affine then $C_p(A)$ is affine. Moreover, $C_p(A) = A$ for all $p \in A$.*

Proof. Suppose that A is affine. Let x, y be in $C_p(A)$, then there are $\bar{x}, \bar{y}$ in A such that $C_p(\bar{x}) = x$ and $C_p(\bar{y}) = y$. Now, let $z \in xy$, then there is a real number α such that $z = \alpha x + (1-\alpha)\, y = \frac{1}{2}\left(p + \alpha\bar{x} + (1-\alpha)\,\bar{y}\right)$. Since A is affine, then the point $\bar{z} = \alpha\bar{x} + (1-\alpha)\,\bar{y}$ is in $\bar{x}\bar{y} \subset A$ and hence $z = C_p(\bar{z}) \in C_p(A)$. So $C_p(A)$ is affine.

It is clear that $C_p(A) \subset A$ since A is affine and $p \in A$. Let $\bar{x} \in A$, then $p\bar{x}$ is in A i.e. for any real number α, the point $\alpha p + (1-\alpha)\,\bar{x}$ is in A. Let $\alpha = -1$, then $-p + 2x$ is in A. Therefore, $C_p(-p+2x) = x$ is in $C_p(A)$. Thus $C_p(A) = A$. ■

Similar proof will lead to the converse of the above theorem and hence we get the following theorem.

Theorem 28 *Let A be a subset of E^n. Then A is affine if and only if $C_p(A)$ is.*

Theorem 29 *Let A be a convex subset of E^n. If $C_p(A) = A$ for any $p \in A$, then A is affine.*

Proof. Let $x, y \in A$. Since A is convex, then $[xy] \subset A$ and $[xy] \subset C_x\,(A) = A$. Therefore, there exists $y_1 = 2y - x$ satisfying $C_x\,(y_1) = y$ i.e. $y_1 \in A$ and so $[xy_1] \subset A = C_x\,(A)$. Again, there exists $y_2 = 2y_1 - x$ satisfying $C_x\,(y_2) = y_1$ i.e. $y_2 \in A$ and so $[xy_2] \subset A = C_x\,(A)$. By repeating this process we get that $\overrightarrow{xy} \subset A$. Similarly, $\overrightarrow{yx} \subset A$ and hence A is affine. ■

Corollary 30 *Let A be an affine subset of E^n. Then $C_p(A) \cap A = \phi$, for each $p \notin A$.*

Proof. First, $p \notin C_p(A)$ since $C_p(\bar{x}) = p$ implies $\bar{x} = p$. Suppose that $C_p(A) \cap A \neq \phi$ i.e. there is $\bar{x} \in C_p(A) \cap A$. Then $C_p(\bar{x}) = y = \frac{1}{2}\,(p + \bar{x})$ is in $C_p(A)$ i.e. $\bar{x}$ and $y = \frac{1}{2}\,(p + \bar{x})$ are in $C_p(A)$ which by Theorem 27 is affine. Hence $\bar{x}y$ is in $C_p(A)$ and so p is in $C_p(A)$ which is a contradiction. This contradiction completes the proof. ■

Theorem 31 *Let A be an affinely starshaped subset of E^n. Then $C_p(A)$ is affinely starshaped. Moreover, $C_p(A \ker\ A) = A \ker\ C_p(A)$.*

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Received: March, 2010