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Estimation of Reliability Based on Pareto Distribution

Naser Odat

Jadara University, Dept. of Mathematics
P.O. Box (733) Postal Code 21110, Jordan
n-odt@yahoo.com

Abstract

This paper deal with estimation of $p(x > y)$ when x and y are tow independent pareto distribution . the maximum likelihood and its asymptotical distribution is obtain.

Keywords: Pareto, maximum likelihood, asymptotic distribution

1 Introduction

The maximum likelihood estimate of , where x and y have bivariate exponential distribution has been considered by Awad et al, (1981). Church and Harris (1970), Downtwon (1973), Govidarajulu (1967), Wood Ward and Kelley (1977), and Owen, Grawelland Hanson (1977) considered the estimation of , when x and y are normally distributed. Similar problem for the multivariate normal distribution has been considered by Gupta and Gupta (1990), Kelley and Schucany (1976), Sath and shah (1981), Tong (1977), considered the estimation of. Under the assumption that x and y are independent exponential random variable, the testing of hypothesis and interval estimation of the reliability parameters in strength model involving two parameter exponential distribution has been considered by Krishnamoorthy et al (2006).

There has been continuous interest in the problem of estimating the probability that one random variable exceeds another, that is, $(x > y)$, where x and y are independent random variable , the parameter R is referred to as the reliability parameter . this problem arises in the classical stress-strength reliability where one is interested in assessing the problem of the times the random strength x of a component exceed the stress Y to which the component is subjected if $x \leq y$ then either the component fails or the system that uses the component may malfunction. Weerahandi and Johnson(1992) presented a rocket-motor experiment data where X represent the chaber burst strength and Y represents the operation pressure, these authers propsed inferential procedures for $p(x > y)$ assuming that x and y are independent normal random variable.

In this paper we consider the problem of estimation of the reliability $R(\theta_1, \theta_2) = p(x > y)$, based on $x_1, \ldots ., x_n \sim iid$, where x is the strength with $p.d.f$ $f(x) = \theta_1 x^{-(\theta_1+1))}$, $1 \leq x < \infty$.
And $y_1 \ldots \ldots ., y_m$ where y is the stress with $p.d.f$ $f(x) = \alpha_2 y^{-(\theta_2+1))}$, $1 \leq y < \infty$.

2 Main result

Since $x \sim par(\theta_1)$ and $y \sim par(\theta_2)$ where x and y are independent then

$$R(\theta_1, \theta_2) = p(x > y) = \iint \theta_1 \theta_2 x^{-(\theta_1+1)} y^{-(\theta_2+1)} dxdy = \frac{\theta_1}{\theta_1 + \theta_2 + 3}.$$

Now to compute the MLE of R ,first we obtain the MLE of θ_1 and θ_2.
Suppose that $X_1, X_2, \ldots ., X_n$ is a random sample from $par(\theta_1)$ and $Y_1, Y_2, \ldots , Y_m$ is a random sample from $par(\theta_2)$, there for the log likely hood function is.

$$\ln L = n \ln \theta_1 + m \ln \theta_2 - (\theta_1 + 1) \ln \sum x_i - (\theta_2 + 1) \ln \sum y_i;$$

$$\frac{d \ln l}{d\theta_1} = \frac{n}{\theta_1} - \ln \sum x_i = 0;$$

$$\frac{d \ln l}{d\theta_1} = \frac{m}{\theta_1} - \ln \sum y_i = 0.$$

Then the standard estimations of θ_1 and θ_2 are $\frac{n}{\ln \sum x_i}$ and $\frac{m}{\ln \sum y_i}$ respectively by using the maximum likely hood method.
Where

$$\hat{R}(\theta_1, \theta_2) = \frac{\hat{\theta}_1}{\hat{\theta}_1 + \hat{\theta}_2 + 3},$$

then

$$\hat{R}(\theta_1, \theta_2) = \frac{\frac{n}{\ln \sum x_i}}{\frac{n}{\ln \sum x_i} + \frac{m}{\ln \sum y_i} + 3}.$$

Theorem1. AS $n \to \infty$,$m \to \infty$ and $\dfrac{n}{m} \to p$,
then

$$[(\hat{\theta}_1 - \theta_1), (\hat{\theta}_2 - \theta_2)] \to N(0, I^{-1}(\beta)),$$

where $I(\beta)$ is the fisher information matrix, i.e

$$\begin{aligned} I(\beta) &= \begin{bmatrix} E\left(\frac{\partial^2 l}{\partial \theta_1^2}\right) & E\left(\frac{\partial^2 l}{\partial \theta_1 \partial \theta_2}\right) \\ E\left(\frac{\partial^2 l}{\partial \theta_1 \partial \theta_2}\right) & E\left(\frac{\partial^2 l}{\partial \theta_2^2}\right) \end{bmatrix} \\ &= -\begin{bmatrix} \frac{-n}{\theta_1^2} & 0 \\ 0 & \frac{-m}{\theta_2^2} \end{bmatrix}. \end{aligned}$$

Thus

$$I^{-1}(\beta) = \begin{bmatrix} \frac{\theta_1^2}{n} & 0 \\ 0 & \frac{\theta_2^2}{m} \end{bmatrix}.$$

Proof. The proof follows by expanding the derivative of the log likelihood function using Taylor series and using the central limit theorem.
Since β is unknown in (2.1), $I^{-1}(\beta)$ is estimated by $I^{-1}(\hat{\beta})$ and this can be used to obtain the asymptotic confidence intervals of θ_1 and θ_2.

Theorem 2. As $n \to \infty$ and $m \to \infty$ so that $\dfrac{n}{m} \to p$ then

$$\sqrt{n}(\hat{R} - R) \to N(0, A),$$

where

$$A = \frac{1}{u(\theta_1 + \theta_2)^4}\left[\frac{n\theta_2^2}{\theta_1^2} + \frac{m\theta_1^2}{\theta_1^2}\right],$$

$$u = \frac{nm}{\theta_1^2\theta_2^2}.$$

Proof. It is follows using Theorem 1, confidence interval for $\hat{R}$:
to compute the confidence interval of R, the variance of R need to be estimate. We will use the empirical fisher information matrix and the maximum likelihood estimate of θ_1 and θ_2 to estimate A Where

$$\hat{A} = \frac{1}{\hat{u}(\hat{\theta}_1 + \hat{\theta}_2)^4} \left[\frac{n\hat{\theta}_2^2}{\hat{\theta}_1^2} + \frac{m\hat{\theta}_1^2}{\hat{\theta}_1^2} \right],$$

$$\hat{u} = \frac{nm}{\hat{\theta}_1^2 \hat{\theta}_2^2}.$$

Thus, $\frac{(\hat{R}-R)}{\sqrt{Var(\hat{R})}} \backsim N(0,1)$ asymptotically.

This result yield an approximate confidence interval for R as

$$\hat{R} \pm Z_{1-\alpha} \sqrt{\widehat{Var}\left(\hat{R}\right)}.$$

Instead of approximation $\hat{R}$ by normal distribution , Mukherjee and Maiji (1998) also considered some normalizing transformation $g(\hat{R})$,whose approximation variance can be obtained by delta method as

$$Varg(\hat{R}) = (g'(R))^2 \, \widehat{Var}\left(\hat{R}\right).$$

Mukherjee and Maiji (1998) considered the following two transformation
1) logit transformation:
let

$$g\left(\hat{R}\right) = \ln\left(\frac{\hat{R}}{1-\hat{R}}\right),$$

with $g'(\hat{R}) = \frac{1}{\hat{R}(1-\hat{R})}$ the $1-\alpha$ confedence interval for $g(R)$ is given by

$$\ln\left(\frac{\hat{R}}{1-\hat{R}}\right) \pm Z_{1-\frac{\alpha}{2}} \frac{\sqrt{\widehat{Var}\left(\hat{R}\right)}}{\hat{R}\left(1-\hat{R}\right)}.$$

2) Arc sine transformation:
let

$$g(\hat{R}) = \sin^{-1}\left(\sqrt{\hat{R}}\right),$$

noting that

$$g'(\hat{R}) = \frac{1}{2\sqrt{\hat{R}\left(1-\hat{R}\right)}}.$$

A $(1-\alpha)$ confidence interval for $g(R)$ is given by

$$\sin^{-1}\left(\sqrt{\hat{R}}\right) \pm Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\widehat{Var}\left(\hat{R}\right)}{4\hat{R}\left(1-\hat{R}\right)}}.$$

References

[1] Awad et.al.,*Some inference results in in the bivariate exponential model.* Communications in statistics – Theory and methods, 10(1981), 2515-2524.

[2] Church and Harris, *The estimation of reliability from stress strength relationships.* Technometrics, 12(1970), 49-54 .

[3] Downtwon, The estimation of in the normal case. Technometrics, 15(1973), 551-558.

[4] Govinara Julu, *Two sided confidence limits for based on normal samples of x and y.* sankhya B,29(1967)35-40.

[5] Wood and Kelley,*Minimum variance unbiased estimation of in the normal case.* Technometric, 19(1977),95-98.

[6] Owen, et.al., *Non-parametric upper confidence bounds for and confidence for when x and y are normal.* Journal of the American statistical association, 59(1977), 906-924.

[7] Kelly and Schucany, *Efficient estimation of in the exponential case.* Technometric, 17(1976), 359-360.

[8] Gupta and Gupta, *Estimation of in the multivariate normal case.* Statistics,1(1990),91-97.

[9] Sath and Shah, *On estimation for the exponential distribution.* Communication in statistics-theory and method, 10(1981),39-47.

[10] Tong, *On the estimation for the exponential families.* IEEE transaction of reliability,26(1977),54-56.

[11] mukharjee,s.p, Maiti,s.s Stress-strength reliability case.Frontiers in Reliabilty,4(1998),231-248.Singapore world scientific.

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