PERFECT AND PARTIAL HEDGING FOR SWING GAME OPTIONS IN DISCRETE TIME.

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Abstract. The paper introduces and studies hedging for game (Israeli) style extension of swing options considered as multiple exercise derivatives. Assuming that the underlying security can be traded without restrictions we derive a formula for valuation of multiple exercise options via classical hedging arguments. Introducing the notion of the shortfall risk for such options we study also partial hedging which leads to minimization of this risk.

1. Introduction

Swing contracts emerging in energy and commodity markets (see [1] and [4]) are often modeled by multiple exercising of American style options which leads to multiple stopping problems (see, for instance, [6], [2] and [8]). Most closely such models describe options consisting of a package of claims or rights which can be exercised in a prescribed (or in any) order with some restrictions such as a delay time between successive exercises. Observe that peculiarities of multiple exercise options are due only to restrictions such as an order of exercises and a delay time between them since without restrictions the above claims or rights could be considered as separate options which should be dealt with independently.

Attempts to valuate swing options in multiple exercises models are usually reduced to maximizing the total expected gain of the buyer which is the expected payoff in the corresponding multiple stopping problem deviating from what now became classical and generally accepted methodology of pricing derivatives via hedging and replicating arguments. This digression is sometimes explained by difficulties in using an underlying commodity in a hedging portfolio in view of the high cost of storage, for instance, in the case of electricity. We will not discuss here in depth practical possibilities of hedging in energy markets but only observe that the seller of a swing option could, for instance, use for hedging certain securities linked to a corresponding commodity (electricity, gas, oil etc.) index. Another instrument which can be used for hedging is an appropriate basket of stocks of major companies in the corresponding branch whose profit depends in a computable way from the price of commodity in question. Though such indirect hedging may seem to be not very precise it may still be helpful taking into account that all duable mathematical models of financial markets cannot describe them precisely and are used

Date: July 15, 2009.
2000 *Mathematics Subject Classification.* Primary: 91B28 Secondary: 60G40, 91B30.
Key words and phrases. hedging, multiple exercise derivatives, game options, shortfall risk.
Partially supported by the ISF grant no. 130/06.

usually only as an auxiliary tool. Another theoretical but may be not very realistic in practice possibility is to buy from (and sell to) power stations an extra capacity for electricity production instead of storing electricity itself and use it as the underlying risky security for a hedging portfolio. We observe also that multiple exercise options may appear in their own rights when an investor wants to buy or sell an underlying security in several instalments at times of his choosing. Anyway, the study of hedging for multiple exercise options is sufficiently motivated from the financial point of view and it leads to interesting mathematical problems. In this paper we assume that the underlying security can be used for construction of a hedging portfolio without restrictions as in the usual theory of derivatives and, moreover, we will deal here with the more general game (Israeli) option (contingent claim) setup when both the buyer (holder) and the seller (writer) of the option can exercise or cancell, respectively, the claims (or rights) in a given order but as in [5] each cancellation entails a penalty payment by the seller. This required us, in particular, to extend Dynkin's games machinery to the multiple stopping setup.

In this paper a discrete time swing (multi stopping) game option is a contract between its seller and the buyer which allows to the seller to cancel (or terminate) and to the buyer to exercise L specific claims or rights in a particular order. Such contract is determined given $2L$ payoff processes $X_i(n) \geq Y_i(n) \geq 0$, $n = 0, 1, ...,$ $i = 1, 2, ..., L$ adapted to a filtration $\mathcal{F}_n$, $n \geq 0$ generated by the stock (underlying risky security) S_n, $n \geq 0$ evolution. If the buyer exercises the k-th claim $k \leq L$ at the time n then the seller pays to him the amount $Y_k(n)$ but if the latter cancels the claim k at the time n before the buyer he has to pay to the buyer the amount $X_k(n)$ and the difference $\delta_k(n) = X_k(n) - Y_k(n)$ is viewed as the cancelation penalty. In addition, we require a delay of one unit of time between successive exercises and cancellations. Observe that unlike some other papers (cf. [2]) we allow payoffs depending on the exercise number so, for instance, our options may change from call to put and vice versa after different exercises.

The first goal of this paper is to develop a mathematical theory for pricing of swing game options. The standard definition of the fair price of a derivative security in a complete market is the minimal initial capital needed to create a (perfect) hedging portfolio, and so we have to start with a precise definition of a perfect hedge. Observe that a natural definition of a perfect hedge in a multi exercise framework is not a straightforward extension of a standard one and it has certain peculiarities. Namely, the seller of the option does not know in advance when the buyer will exercise the $(j-1)$-th claim but his hedging strategy of the j-th claim should depend on this (random) time and on the capital he is left with in the portfolio after the $(j-1)$-payoff. Thus, in addition to the usual dependence on the stock evolution a perfect hedge of the j-th claim should depend on the past behavior of both seller and the buyer of the option. Actually, an optimal portfolio allocation depends also on the payoff processes of the future claims. The construction of hedging strategies in the multiple exercise setup requires a nontrivial additional iterative procedure in contrast to the 1-exercise case where perfect hedging strategies are obtained directly from the martingale representation. Several papers dealt with mathematical analysis of swing American options (see, for instance, [2] and [8]) but none of these papers defined explicitly what is a perfect hedge and what is the option price. In [8] the authors studied a specific type of swing American options but they treated the problem from the buyer point of view which in general is not interested in hedging but only on a stopping strategy which will provide him a maximal profit. In [2] the authors studied an optimal multi stopping problem for continuous time models but they did not explained why the value of the above problem under the martingale measure in a complete market is the option price. In this paper we define the notion of a perfect hedge for swing game options which generalize swing American options, prove that in the binomial Cox-Ross-Rubinstein (CRR) market the option price V^* is equal to the value of the multi stopping Dynkin game with discounted payoffs under the unique martingale measure and provide a dynamical programming algorithm which allows to compute both this value and a corresponding perfect hedge. Similar results can be obtained for the continuous time Black–Scholes market with the stock price evolving according to the geometric Brownian motion but in this paper we restrict ourselves to the discrete time setup.

Our second goal is to study hedging with risk for swing game options. In real market conditions a writer of an option may not be willing for various reasons to tie in a hedging portfolio the full initial capital required for a perfect hedge. In this case the seller is ready to accept a risk that his portfolio value will be less than his obligation to pay and he will need additional funds to fulfil the contract, i.e. the writer must add money to his portfolio from other sources. In our setup the writer is allowed to add money to his portfolio only at moments when the contract is exercised. The shortfall risk is defined as the expectation with respect to the market probability measure of the total sum that the seller added from other sources. We will show that for any initial capital $x < V^*$ there exists a hedge which minimizes the shortfall risk and this hedge can be computed by a dynamical programming algorithm. Observe that the existence of a hedge minimizing the shortfall risk is not known in the continuous time even for usual (one stopping) game options (see [3]). Hedging with risk was not studied before for swing options of any type.

In Section 2 we define explicitly the notions of perfect and partial hedges (the latter, for the shortfall risk case). Relying on these we define the option price and the shortfall risk. Then we state Theorem 2.4 which yields the option price together with the corresponding perfect hedge. Next, we formulate Theorem 2.7 which for a given initial capital provides the shortfall risk and the corresponding optimal hedge together with the dynamical programming algorithm for their computation. In Section 3 we derive auxiliary lemmas needed in the proof, introduce the concept of multi stopping Dynkin game and prove existence of a saddle point for this game. Section 4 and Section 5 are devoted to the proofs of Theorem 2.4 and Theorem 2.7, respectively.

2. Preliminaries and main results

Let $\Omega = \{1, -1\}^N$ be the space of finite sequences $\omega = (\omega_1, \omega_2, ..., \omega_N)$; $\omega_i \in \{1, -1\}$ with the product probability $P = \{p, 1-p\}^N$, $p > 0$. Consider the binomial model of a financial market which is active at times $n = 0, 1, ..., N < \infty$ and it consists of a savings account B_n with an interest rate r which without loss of generality (by discounting) we assume to be zero, i.e.

$$B_n = B_0 > 0, \tag{2.1}$$

and of a stock whose price at time n equals

$$S_n = S_0 \prod_{i=1}^{n} (1 + \rho_i), \quad S_0 > 0 \tag{2.2}$$

where $\rho_i(\omega_1, \omega_2, ..., \omega_N) = \frac{a+b}{2} + \frac{b-a}{2}\omega_i$ and $-1 < a < 0 < b$. Thus ρ_i, $i = 1, ..., N$ form a sequence of independent identically distributed (i.i.d.) random variables on the probability space (Ω, P) taking values b and a with probabilities p and $1-p$, respectively. Recall, that the binomial CRR model is complete (see [9]) and S_n, $n \geq 0$ is a martingale with respect to the filtration $\mathcal{F}_n = \sigma\{\rho_k, k \leq n\}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and the unique martingale measure is given by $\tilde{P} = \{\tilde{p}, 1-\tilde{p}\}^N$ where $\tilde{p} = \frac{a}{a-b}$.

We consider a swing option of the game type which has the i-th payoff, $i \geq 1$ having the form

$$H^{(i)}(m, n) = X_i(m)\mathbb{I}_{m<n} + Y_i(n)\mathbb{I}_{n\leq m}, \ \ \forall m, n \tag{2.3}$$

where $X_i(n), Y_i(n)$ are $\mathcal{F}_n$-adapted and $0 \leq Y_i(n) \leq X_i(n) < \infty$. Thus for any i, n there exist functions $f_n^{(i)}, g_n^{(i)} : \{a, b\}^n \to \mathbb{R}_+$ such that

$$Y_i(n) = f_n^{(i)}(\rho_1, ..., \rho_n), \ \ X_i(n) = g_n^{(i)}(\rho_1, ..., \rho_n). \tag{2.4}$$

For any $1 \leq i \leq L-1$ let C_i be the set of all pairs $((a_1, ..., a_i), (d_1, ..., d_i)) \in \{0, ..., N\}^i \times \{0, 1\}^i$ such that $a_{j+1} \geq N \wedge (a_j + 1)$ for any $j < i$. Such sequences represent the history of payoffs up to the i-th one in the following way. If $a_j = k$ and $d_j = 1$ then the seller canceled the j-th claim at the moment k and if $d_j = 0$ then the buyer exercised the j-th claim at the moment k (maybe together with the seller). For $n \geq 1$ denote by Γ_n the set of all stopping times with respect to the filtration $\{\mathcal{F}_n\}_{n=0}^N$ with values from n to N and set $\Gamma = \Gamma_0$.

Definition 2.1. *A stopping strategy is a sequence $s = (s_1, ..., s_L)$ such that $s_1 \in \Gamma$ is a stopping time and for $i > 1$, $s_i : C_{i-1} \to \Gamma$ is a map which satisfies $s_i((a_1, ..., a_{i-1}), (d_1, ..., d_{i-1})) \in \Gamma_{N\wedge(1+a_{i-1})}$.*

In other words for the i-th payoff both the seller and the buyer choose stopping times taking into account the history of payoffs so far. Denote by $\mathcal{S}$ the set of all stopping strategies and define the map $F : \mathcal{S} \times \mathcal{S} \to \Gamma^L \times \Gamma^L$ by $F(s, b) = ((\sigma_1, ..., \sigma_L), (\tau_1, ..., \tau_L))$ where $\sigma_1 = s_1$, $\tau_1 = b_1$ and for $i > 1$,

$$\begin{aligned} \sigma_i &= s_i((\sigma_1 \wedge \tau_1, ..., \sigma_{i-1} \wedge \tau_{i-1}), (\mathbb{I}_{\sigma_1<\tau_1}, ..., \mathbb{I}_{\sigma_{i-1}<\tau_{i-1}})) \ \text{and} \\ \tau_i &= b_i((\sigma_1 \wedge \tau_1, ..., \sigma_{i-1} \wedge \tau_{i-1}), (\mathbb{I}_{\sigma_1<\tau_1}, ..., \mathbb{I}_{\sigma_{i-1}<\tau_{i-1}})). \end{aligned} \tag{2.5}$$

Set

$$c_k(s, b) = \sum_{i=1}^{L} \mathbb{I}_{\sigma_i\wedge\tau_i\leq k} \tag{2.6}$$

which is a random variable equal to the number of payoffs until the moment k.

For swing options the notion of a self financing portfolio involves not only allocation of capital between stocks and the bank account but also payoffs at exercise times. At the time k the writer's decision how much money to invest in stocks (while depositing the remaining money into a bank account) depends not only on his present portfolio value but also on the current claim. Denote by Ξ the set of functions on the (finite) probability space Ω.

Definition 2.2. *A portfolio strategy with an initial capital $x > 0$ is a pair $\pi = (x, \gamma)$ where $\gamma : \{0, ..., N-1\} \times \{1, ..., L\} \times \mathbb{R} \to \Xi$ is a map such that $\gamma(k, i, y)$ is an $\mathcal{F}_k$-measurable random variable which represents the number of stocks which the seller buy at the moment k provided that the current claim has the number i and the present portfolio value is y. At the same time the sum $y - \gamma(k, i, y)S_k$ is deposited to* *the bank account of the portfolio. We call a portfolio strategy* $\pi = (x, \gamma)$ *admissible if for any* $y \geq 0$,

$$-\frac{y}{S_k b} \leq \gamma(k,i,y) \leq -\frac{y}{S_k a}. \tag{2.7}$$

For any $y \geq 0$ *denote* $K(y) = [-\frac{y}{b}, -\frac{y}{a}]$.

Notice that if the portfolio value at the moment k is $y \geq 0$ then the portfolio value at the moment $k+1$ before the payoffs (if there are any payoffs at this time) is given by $y + \gamma(k,i,y)S_k(\frac{S_{k+1}}{S_k} - 1)$ where i is the number of the next payoff. In view of independency of $\frac{S_{k+1}}{S_k} - 1$ and $\gamma(k,i,y)S_k$ we conclude that the inequality (2.7) is equivalent to the inequality $y + \gamma(k,i,y)S_k(\frac{S_{k+1}}{S_k} - 1) \geq 0$, i.e. the portfolio value at the moment $k+1$ before the payoffs is nonnegative. Denote by $\mathcal{A}(x)$ be the set of all *admissible* portfolio strategies with an initial capital $x > 0$. Denote $\mathcal{A} = \bigcup_{x>0} \mathcal{A}(x)$. Let $\pi = (x, \gamma)$ be a portfolio strategy and $s, b \in \mathcal{S}$. Set $((\sigma_1, ..., \sigma_L), (\tau_1, ..., \tau_L)) = F(s,b)$ and $c_k = c_k(s,b)$. The portfolio value at the moment k after the payoffs (if there are any payoffs at this moment) is given by

$$\begin{aligned} V_0^{(\pi,s,b)} &= x - H^{(1)}(\sigma_1,\tau_1)\mathbb{I}_{\sigma_1\wedge\tau_1=0} \quad \text{and for } k>0, \\ V_k^{(\pi,s,b)} &= V_{k-1}^{(\pi,s,b)} + \mathbb{I}_{c_{k-1}<L}[\gamma(k-1, c_{k-1}+1, V_{k-1}^{(\pi,s,b)})(S_k - S_{k-1}) - \\ &\qquad \textstyle\sum_{i=1}^L H^{(i)}(\sigma_i,\tau_i)\mathbb{I}_{\sigma_i\wedge\tau_i=k}]. \end{aligned} \tag{2.8}$$

Definition 2.3. *A perfect hedge is a pair* (π, s) *which consists of a portfolio strategy and a stopping strategy such that* $V_k^{(\pi,s,b)} \geq 0$ *for any* $b \in \mathcal{S}$ *and* $k \leq N$.

Observe that if (π, s) is a perfect hedge then without loss of generality we can assume that π is an *admissible* portfolio strategy and throughout this paper we will consider only *admissible* portfolio strategies. As usual, the option price V^* is defined as the infimum of $V \geq 0$ such that there exists a perfect hedge with an initial capital V.

The following theorem provides a dynamical programming algorithm for computation of both the option price and the corresponding perfect hedge.

Theorem 2.4. *Denote by* $\tilde{E}$ *the expectation with respect to the unique martingale measure* $\tilde{P}$. *For any* $n \leq N$ *set*

$$X_n^{(1)} = X_L(n),\ Y_n^{(1)} = Y_L(n) \text{ and } V_n^{(1)} = \min_{\sigma\in\Gamma_n}\max_{\tau\in\Gamma_n} \tilde{E}(H^{(L)}(\sigma,\tau)|\mathcal{F}_n) \tag{2.9}$$

and for $1 < k \leq L$,

$$\begin{aligned} X_n^{(k)} &= X_{L-k+1}(n) + \tilde{E}(V_{(n+1)\wedge N}^{(k-1)}|\mathcal{F}_n), \\ Y_n^{(k)} &= Y_{L-k+1}(n) + \tilde{E}(V_{(n+1)\wedge N}^{(k-1)}|\mathcal{F}_n) \text{ and} \\ V_n^{(k)} &= \min\nolimits_{\sigma\in\Gamma_n}\max\nolimits_{\tau\in\Gamma_n} \tilde{E}(X_\sigma^{(k)}\mathbb{I}_{\sigma<\tau} + Y_\tau^{(k)}\mathbb{I}_{\sigma\geq\tau}|\mathcal{F}_n). \end{aligned} \tag{2.10}$$

Then

$$V^* = V_0^{(L)} = \min_{s\in\mathcal{S}}\max_{b\in\mathcal{S}} G(s,b) \tag{2.11}$$

where $G(s,b) = \tilde{E}\sum_{i=1}^L H^{(i)}(\sigma_i,\tau_i)$ *and* $((\sigma_1, ..., \sigma_L), (\tau_1, ..., \tau_L)) = F(s,b)$. *Furthermore, the stopping strategies* $s^* = (s_1^*, ..., s_L^*) \in S$ *and* $b = (b_1^*, ..., b_L^*)$ *given* *by*

$$
\begin{aligned}
(2.12)\qquad & s_1^*=N\wedge\min\{k|X_k^{(L)}=V_k^{(L)}\},\ b_1^*=\min\{k|Y_k^{(L)}=V_k^{(L)}\},\\
& s_i^*((a_1,...,a_{i-1}),(d_1,...,d_{i-1}))=N\wedge\min\{k>a_{i-1}|\\
& X_k^{(L-i+1)}=V_k^{(L-i+1)}\},\ b_i^*((a_1,...,a_{i-1}),(d_1,...,d_{i-1}))\\
& =N\wedge\min\{k>a_{i-1}|Y_k^{(L-i+1)}=V_k^{(L-i+1)}\},\ i>1
\end{aligned}
$$

satisfy

$$
(2.13)\qquad G(s^*,b)\le G(s^*,b^*)\le G(s,b^*)\ \textit{for all}\ s,b
$$

and there exists a portfolio strategy $\pi^*\in\mathcal{A}(V_0^{(L)})$ *such that* (π^*,s^*) *is a perfect hedge.*

Next, consider an option seller whose initial capital is x, which is less than the option price, i.e. $x<V^*$. In this case the seller must (in order to fulfill his obligation to the buyer) add money to his portfolio from other sources. In our setup the seller is allowed to add money to his portfolio only at times when the contract is exercised. We also require that after the addition of money by the seller the portfolio value must be positive.

Definition 2.5. *An infusion of capital is a map* $I:\{0,...,N\}\times\{1,...,L\}\times\mathbb{R}\to\Xi$ *such that* $I(k,j,y)\ge(-y)^+$ *is* $\mathcal{F}_k$*-measurable,* $I(k,L,y)=(-y)^+$ *for any* k*, and for any* $j<L$*,* $I(N,j,y)=\big((\sum_{i=j+1}^{L}Y_i(N))-y\big)^+$*. The set of such maps will be denoted by* $\mathcal{I}$*.*

Thus $I(k,j,y)$ is the amount that the seller adds to his portfolio after the j-th payoff payed at the moment k and the portfolio value after this payment is y. When $k=N$ or $j=L$ then clearly $I(k,j,y)$ is the minimal amount which the seller should add in order to fulfill his obligation to the buyer. Observe that when $k=N$ one infusion of capital to the seller's portfolio is already sufficient in order to fulfill his obligations even if there are additional payoffs at this moment, so we conclude that at each step that the contract is exercised there is no more than one infusion of capital. A hedge with an initial capital $x<V^*$ is a triple $(\pi,\mathcal{I},s)\in\mathcal{A}(x)\times I\times S$ which consists of an *admissible* portfolio strategy with an initial capital x, infusion of capital and a stopping strategy. Let (π,I,s) be a hedge and $b\in\mathcal{S}$ be a stopping strategy for the buyer. Set $((\sigma_1,...,\sigma_L),(\tau_1,...,\tau_L))=F(s,b)$ and $c_k=c_k(s,b)$. Define the stochastic processes $\{W_k^{(\pi,I,s,b)}\}_{k=0}^N$ and $\{V_k^{(\pi,I,s,b)}\}_{k=0}^N$ by

$$
\begin{aligned}
(2.14)\qquad & W_0^{(\pi,I,s,b)}=x,\ V_0^{(\pi,I,s,b)}=x-\mathbb{I}_{\sigma_1\wedge\tau_1=0}\big(H^{(1)}(\sigma_1,\tau_1)-\\
& I(0,1,x-H^{(1)}(\sigma_1,\tau_1))\big)\ \text{and for}\ k>0,\\
& W_k^{(\pi,I,s,b)}=V_{k-1}^{(\pi,I,s,b)}+\mathbb{I}_{c_{k-1}<L}\gamma(k-1,c_{k-1}+1,V_{k-1}^{(\pi,I,s,b)})(S_k-S_{k-1}),\\
& V_k^{(\pi,I,s,b)}=W_k^{(\pi,I,s,b)}-\mathbb{I}_{c_{k-1}<L}\mathbb{I}_{\sigma_{c_{k-1}+1}\wedge\tau_{c_{k-1}+1}=k}\times\\
& \big(H^{(c_{k-1}+1)}(\sigma_{c_{k-1}+1},\tau_{c_{k-1}+1})+\mathbb{I}_{k=N}\textstyle\sum_{i=c_{k-1}+2}^{L}Y_i(N)\\
& -I(k,c_{k-1}+1,W_k^{(\pi,I,s,b)}-H^{(c_{k-1}+1)}(\sigma_{c_{k-1}+1},\tau_{c_{k-1}+1}))\big).
\end{aligned}
$$

Observe that if the contract was not exercised at a moment k then $W_k^{(\pi,I,s,b)}=V_k^{(\pi,I,s,b)}$ is the portfolio value at this moment. If the contract was exercised at a moment k then $W_k^{(\pi,I,s,b)}$ and $V_k^{(\pi,I,s,b)}$ are the portfolio values before and after the payoff, respectively. Thus the total infusion of capital that made by the seller is given by

$$C(\pi, I, s, b) = \sum_{i=1}^{(c_{N-1}+1)\wedge L} I(\sigma_i \wedge \tau_i, i, W^{(\pi,I,s,b)}_{\sigma_i\wedge\tau_i} - H^{(i)}(\sigma_1, \tau_i)). \tag{2.15}$$

Definition 2.6. *Given a hedge* $(\pi, I, s) \in \mathcal{A} \times \mathcal{I} \times S$ *the shortfall risk for it is defined by*

$$R(\pi, I, s) = \max_{b\in\mathcal{S}} EC(\pi, I, s, b) \tag{2.16}$$

which is the maximal expectation with respect to the market probability measure P *of the total infusion of capital. The shortfall risk for the intitial capital* x *is defined by*

$$R(x) = \inf_{(\pi,I,s)\in\mathcal{A}(x)\times\mathcal{I}\times S} R(\pi, I, s). \tag{2.17}$$

The following result asserts for any initial capital x there exists a hedge $(\pi, I, s) \in \mathcal{A}(x) \times \mathcal{I} \times S$ which minimizes the shortfall risk and both the risk and the optimal hedge can be obtained recurrently.

Theorem 2.7. *Define a sequence of functions* $J_k : \mathbb{R}_+ \times \{0, ..., L\} \times \{a, b\}^k \to \mathbb{R}_+$, $0 \le k \le N$ *by the following formulas*

$$J_N(y, j, u_1, ..., u_N) = ((\textstyle\sum_{i=L-j+1}^{L} f_N^{(i)}(u_1, ..., u_N)) - y)^+, \; j > 0, \tag{2.18}$$
$$J_k(y, 0, u_1, ..., u_k) = 0, \; 0 \le k \le N$$

and for $k < N$ *and* $j > 0$,

$$\begin{aligned} &J_k(y, j, u_1, ..., u_k) = \\ &\min\Bigg(\inf_{z\ge(g_k^{(L-j+1)}(u_1,...,u_k)-y)^+} \inf_{\alpha\in K(y+z-g_k^{(L-j+1)}(u_1,...,u_k))} \\ &\big(z + pJ_{k+1}(y + z - g_k^{(L-j+1)}(u_1, ..., u_k) + b\alpha, j-1, u_1, ..., u_k, b) + \\ &(1-p)J_{k+1}(y + z - g_k^{(L-j+1)}(u_1, ..., u_k) + a\alpha, j-1, u_1, ..., u_k, a)\big), \\ &\max\Bigg(\inf_{z\ge(f_k^{(L-j+1)}(u_1,...,u_k)-y)^+} \inf_{\alpha\in K(y+z-f_k^{(L-j+1)}(u_1,...,u_k))} \\ &\big(z + pJ_{k+1}(y + z - f_k^{(L-j+1)}(u_1, ..., u_k) + b\alpha, j-1, u_1, ..., u_k, b) + \\ &(1-p)J_{k+1}(y + z - f_k^{(L-j+1)}(u_1, ..., u_k) + a\alpha, j-, u_1, ..., u_k, a)\big), \\ &\inf_{\alpha\in K(y)} \big(pJ_{k+1}(y + b\alpha, j, u_1, ..., u_k, b) + \\ &(1-p)J_{k+1}(y + a\alpha, j, u_1, ..., u_k, a)\big)\Bigg)\Bigg). \end{aligned} \tag{2.19}$$

Then the shortfall risk for an initial capital x *is given by*

$$R(x) = J_0(x, L). \tag{2.20}$$

Furthermore, the hedge $(\tilde\pi = (x, \tilde\gamma), \tilde I, \tilde s) \in \mathcal{A}(x) \times \mathcal{I} \times S$ *given by the formulas (5.34), (5.37) and (5.46) satisfies*

$$R(\tilde\pi, \tilde I, \tilde s) = R(x). \tag{2.21}$$

Not surprisingly the formulas above and their proof are quite technical and complex since already for one stopping game options the corresponding recurrent formulas for the shortfall risk in [3] and their proof are rather complicated. Our method extends the approach of [3] by relying on the dynamical programming algorithm for Dynkin's games with appropriately modified payoff processes.

Remark 2.8. *Some applications may require a more general setup where the first payoff is as before but the i-th payoff for $i > 1$ depends also on the first time when the i-th claim can be exercised, i.e. the i-th payoff depends on the time of the $(i-1)$-th payoff. The first payoff is exactly as in formula (2.3). For $i > 1$ we set*

$$\forall m, n \geq k \quad H^{(i,k)}(m,n) = X_{i,k}(m)\mathbb{I}_{m<n} + Y_{i,k}(n)\mathbb{I}_{n\leq m} \tag{2.22}$$

which is the i-th payoff if the seller cancells at time m and the buyer exercises at time n provided the i-th claim can be exercised only starting from the time k. Here $X_{i,k}(n), Y_{i,k}(n)$ are $\mathcal{F}_n$-adapted stochastic processes and $0 \leq Y_{i,k}(n) \leq X_{i,k}(n) < \infty$. Definition 2.2 of a portfolio strategy $\pi = (x, \gamma)$ with an initial capital x should be also modified so that $\gamma = \gamma(k, m, i, y)$ is an $\mathcal{F}_k$-measurable random variable which represents the number of stocks which the seller buy at the moment m provided that the current claim which started at the time $k \leq m$ has the number i and the present portfolio value is y. The definitions of perfect and partial hedges are the same as above. Then we can obtain corresponding generalizations of Theorems 2.4 and 2.7 whose proofs proceed similarly to the proof in Sections 4–5 but require an induction in an additional parameter which represents the time of the previous payoff. Since the notations in this case are quite unwieldy and the argument is longer but does not contain additional ideas we will not deal with this generalization here.

3. Auxiliary lemmas

The following lemma is a well known result about Dynkin games (see [7]) which will be used for proving Theorems 2.4 and 2.7.

Lemma 3.1. *Let $\{X_n, Y_n \geq 0\}_{n=0}^N$ be two adapted stochastic processes. Set*

$$R(m,n) = \mathbb{I}_{m<n}X_m + \mathbb{I}_{m\geq n}Y_n$$

and define the stochastic process $\{V_n\}_{n=0}^N$ by

$$V_N = Y_N, \text{ and for } n < N$$
$$V_n = Y_n\mathbb{I}_{Y_n > X_n} + \min(X_n, max(Y_n, E(V_{n+1}|\mathcal{F}_n)))\mathbb{I}_{Y_n \leq X_n}.$$

Then

$$V_n = \text{ess-inf}_{\sigma\in\Gamma_n}\text{ess-sup}_{\tau\in\Gamma_n}E(R(\sigma,\tau)|\mathcal{F}_n).$$

Moreover, for any stopping time $\theta \in \Gamma$ the stopping times

$$\sigma_\theta = \min\{k \geq \theta | X_k \leq V_k\} \wedge N \text{ and } \tau_\theta = \min\{k \geq \theta | Y_k = V_k\}$$

satisfy

$$E(R(\sigma_\theta, \tau)|\mathcal{F}_\theta) \leq V_\theta \leq E(R(\sigma, \tau_\theta)|\mathcal{F}_\theta)$$

for any stopping times $\sigma, \tau \geq \theta$. Furthermore, for the filtration $\{\mathcal{F}_{(\theta+k)\wedge N}\}_{k=0}^N$ the processes $\{V_{\sigma_\theta\wedge(\theta+k)\wedge N}\}_{k=0}^N$, $\{V_{\tau_\theta\wedge(\theta+k)\wedge N}\}_{k=0}^N$ and $V_{\sigma_\theta\wedge\tau_\theta\wedge(\theta+k)\wedge N}$, are supermartingale, submartingale and martingale, respectively.

Next, we derive auxiliary results which will be used for proving Theorem 2.4. First, we generalize Dynkin games to the multi stopping setup and show that also in this case there is a saddle point, i.e., in particular, the multi stopping Dynkin game has a value. Note that the following results about multi stopping Dynkin's games are valid for any probability space with a discrete finite filtration for which we use the same notations as before. The main result concerning multi stopping Dynkin's games is the following.

Proposition 3.2. *For any* $s, b \in \mathcal{S}$,

$$G(s^*, b) \le G(s^*, b^*) \le G(s, b^*) \tag{3.1}$$

where s^* *and* b^* *are the same as in (2.12).*

The above statement is, actually, a part of Theorem 2.4 (see (2.12)) but since it holds true in a wider setting we give it separately. Observe also that the above result is correct for different definitions of strategies. For instance, we could take s_i to be dependent only on the last time a_{i-1} but in order to be consistent we provide the argument only for the strategies set $\mathcal{S}$. In fact, it is easy to see that in the proof we just use the assumption $\sigma_i, \tau_i \ge (\sigma_{i-1} \wedge \tau_{i-1} + 1) \wedge N$. Before we pass to the proof of Proposition 3.2 we shall derive the following key lemma.

Lemma 3.3. *For* $s, b \in \mathcal{S}$ *set*

$$F(s^*, b) = \big((\sigma_1^*, ..., \sigma_L^*), (\tau_1, ..., \tau_L)\big) \text{ and } F(s, b^*) = \big((\sigma_1, ..., \sigma_L), (\tau_1^*, ..., \tau_L^*)\big).$$

For every $0 \le n \le N$ *put*

$$X_n^{(0)} = Y_n^{(0)} = V_n^{(0)} = 0$$

and for any $0 \le i \le L$ *define*

$$R^{(i)}(\sigma, \tau) = \mathbb{I}_{\sigma < \tau} X_\sigma^{(i)} + \mathbb{I}_{\sigma \ge \tau} Y_\tau^{(i)}.$$

Then

$$\begin{aligned} E(R^{(i-1)}(\sigma^*_{L-i+2}, \tau_{L-i+2}) &+ H^{(L-i+1)}(\sigma^*_{L-i+1}, \tau_{L-i+1})) \\ &\le E(R^{(i)}(\sigma^*_{L-i+1}, \tau_{L-i+1})) \quad \text{and} \end{aligned} \tag{3.2}$$

$$\begin{aligned} E(R^{(i-1)}(\sigma_{L-i+2}, \tau^*_{L-i+2}) &+ H^{(L-i+1)}(\sigma_{L-i+1}, \tau^*_{L-i+1})) \\ &\ge E(R^{(i)}(\sigma_{L-i+1}, \tau^*_{L-i+1})). \end{aligned} \tag{3.3}$$

Proof. We shall give only the proof of inequality (3.2) since (3.3) can be proven in a similar way. Set $\eta_i = (\sigma_i^* \wedge \tau_i + 1) \wedge N$ then we obtain from the definition that

$$\begin{aligned} R^{(i)}(\sigma^*_{L-i+1}, \tau_{L-i+1}) &= \mathbb{I}_{\{\sigma^*_{L-i+1} < \tau_{L-i+1}\}} X^{(i)}_{\sigma^*_{L-i+1} \wedge \tau_{L-i+1}} + \mathbb{I}_{\{\sigma^*_{L-i+1} \ge \tau_{L-i+1}\}} \\ \times Y^{(i)}_{\sigma^*_{L-i+1} \wedge \tau_{L-i+1}} &= \mathbb{I}_{\{\sigma^*_{L-i+1} < \tau_{L-i+1}\}} \big(X_{L-i+1}(\sigma^*_{L-i+1} \wedge \tau_{L-i+1}) \\ &+ E(V^{(i-1)}_{\eta_{L-i+1}} | \mathcal{F}_{\sigma^*_{L-i+1} \wedge \tau_{L-i+1}})\big) + \mathbb{I}_{\{\sigma^*_{L-i+1} \ge \tau_{L-i+1}\}} \\ &\times \big(Y_{L-i+1}(\sigma^*_{L-i+1} \wedge \tau_{L-i+1}) + E(V^{(i-1)}_{\eta_{L-i+1}} | \mathcal{F}_{\sigma^*_{L-i+1} \wedge \tau_{L-i+1}})\big) \\ &= H^{(L-i+1)}(\sigma^*_{L-i+1}, \tau_{L-i+1}) + E(V^{(i-1)}_{\eta_{L-i+1}} | \mathcal{F}_{\sigma^*_{L-i+1} \wedge \tau_{L-i+1}}), \end{aligned}$$

and so

$$\begin{aligned} &E(R^{(i)}(\sigma^*_{L-i+1}, \tau_{L-i+1})) \\ &= E(H^{(L-i+1)}(\sigma^*_{L-i+1}, \tau_{L-i+1})) + E(V^{(i-1)}_{\eta_{L-i+1}}). \end{aligned} \tag{3.4}$$

On the other hand,

$$R^{(i-1)}(\sigma^*_{L-i+2},\tau_{L-i+2})=\mathbb{I}_{\{\sigma^*_{L-i+2}<\tau_{L-i+2}\}}X^{(i-1)}_{\sigma^*_{L-i+2}}+\mathbb{I}_{\{\sigma^*_{L-i+2}\geq\tau_{L-i+2}\}}Y^{(i-1)}_{\tau_{L-i+2}}$$
$$\leq\mathbb{I}_{\{\sigma^*_{L-i+2}<\tau_{L-i+2}\}}V^{(i-1)}_{\sigma^*_{L-i+2}}+\mathbb{I}_{\{\sigma^*_{L-i+2}\geq\tau_{L-i+2}\}}V^{(i-1)}_{\tau_{L-i+2}}=V^{(i-1)}_{\sigma^*_{L-i+2}\wedge\tau_{L-i+2}}$$

which holds true by the definition of σ^*_{L-i+2} and the fact that $Y^{(i)}_n\leq V^{(i)}_n$ for every $0\leq n\leq N$ and $1\leq i\leq L$. Applying the last inequality in Lemma 3.1 with $\theta=\eta_{L-i+1}$ we obtain that

$$E\big(R^{(i-1)}(\sigma^*_{L-i+2},\tau_{L-i+2})\big)\leq E(V^{(i-1)}_{\eta_{L-i+1}}). \tag{3.5}$$

Now (3.2) follows from (3.4) and (3.5). □

Observe that in the special case $s=s^*$ and $b=b^*$ if

$$\big((\sigma^*_1,...,\sigma^*_L),(\tau^*_1,...,\tau^*_L)\big)=F(s^*,b^*)$$

then inequalities (3.2) and (3.3) become equalities and

$$\begin{aligned}E(R^{(i-1)}(\sigma^*_{L-i+2},\tau^*_{L-i+2})+H^{(L-i+1)}(\sigma^*_{L-i+1},\tau^*_{L-i+1}))\\=E(R^{(i)}(\sigma^*_{L-i+1},\tau^*_{L-i+1}))\end{aligned} \tag{3.6}$$

for every $1<i\leq L$.

Proof of Proposition 3.2. For $b\in\mathcal{S}$ let

$$F(s^*,b)=\big((\sigma_1(s^*,b),...,\sigma_L(s^*,b)),(\tau_1(s^*,b),...,\tau_L(s^*,b))\big)\ \text{and}$$
$$F(s^*,b^*)=\big((\sigma_1(s^*,b^*),...,\sigma_L(s^*,b^*)),(\tau_1(s^*,b^*),..,\tau_L(s^*,b^*))\big).$$

We shall prove only the left hand side of (3.1) while its right hand side follows in the same way. By Lemma 3.3 we see that for every $1<i\leq L$,

$$E(R^{(i-1)}(\sigma_{L-i+2}(s^*,b),\tau_{L-i+2}(s^*,b))+\textstyle\sum_{j=1}^{L-i+1}H^{(j)}(\sigma_j(s^*,b),\tau_j(s^*,b)))$$
$$\leq E(R^{(i)}(\sigma_{L-i+1}(s^*,b),\tau_{L-i+1}(s^*,b))+\textstyle\sum_{j=1}^{L-i}H^{(j)}(\sigma_j(s^*,b),\tau_j(s^*,b)))$$

and for (s^*,b^*),

$$E(R^{(i-1)}(\sigma_{L-i+2}(s^*,b^*),\tau_{L-i+2}(s^*,b^*))+\textstyle\sum_{j=1}^{L-i+1}H^{(j)}(\sigma_j(s^*,b^*),\tau_j(s^*,b^*)))$$
$$=E(R^{(i)}(\sigma_{L-i+1}(s^*,b^*),\tau_{L-i+1}(s^*,b^*))+\textstyle\sum_{j=1}^{L-i}H^{(j)}(\sigma_j(s^*,b^*),\tau_j(s^*,b^*))).$$

By induction it follows that

$$G(s^*,b)=E(\sum_{j=1}^{L}H^{(j)}(\sigma_j(s^*,b),\tau_j(s^*,b)))\leq E(R^{(L)}(\sigma_1(s^*,b),\tau_1(s^*,b))) \tag{3.7}$$

and for (s^*,b^*),

$$G(s^*,b^*)=E(R^{(L)}(\sigma_1(s^*,b^*),\tau_1(s^*,b^*)))=V^{(L)}_0 \tag{3.8}$$

where the last term is the value of the usual (one stopping) Dynkin game . Observe that from the definition of s^*,b^* for every $b\in\mathcal{S}$ the inequality

$$E(R^{(L)}(\sigma_1(s^*,b),\tau_1(s^*,b)))\leq E(R^{(L)}(\sigma_1(s^*,b^*),\tau_1(s^*,b^*)))=V^{(L)}_0 \tag{3.9}$$

is just the saddle point property of the usual Dynkin game. From (3.6), (3.7) and (3.8) it follows that

$$G(s^*,b) \le E(R^{(L)}(\sigma_1(s^*,b),\tau_1(s^*,b)))$$
$$\le E(R^{(L)}(\sigma_1(s^*,b^*),\tau_1(s^*,b^*))) = G(s^*,b^*) = V_0^{(L)}.$$

□

As a consequence we obtain

Corollary 3.4. *The multi stopping Dynkin game possess a saddle point* $< s^*, b^* >$, *and so it has a value which is equal to* $G(s^*,b^*)$.

In the remaining part of this section we derive auxiliary lemmas which will be used for the proof of Theorem 2.7.

Definition 3.5. *A function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a piecewise linear function vanishing at* ∞ *if there exists a natural number* n*, such that*

$$\psi(y) = \sum_{i=1}^{n} \mathbb{I}_{[a_i,b_i)}(c_i y + d_i) \tag{3.10}$$

where $c_1,...,c_n,d_1,...,d_n \in \mathbb{R}$ *and* $\{[a_i,b_i)\}_{i=1}^n$ *is a sequence of disjoint finite intervals.*

Lemma 3.6. *Let* $A \ge 0$ *and* $\psi_1,\psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ *be continuous, decreasing and piecewise linear functions vanishing at* ∞. *Define* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *and* $\psi_A : \mathbb{R} \to \mathbb{R}_+$ *by*

$$\psi(y) = \min\nolimits_{\lambda\in K(y)} \big(p\psi_1(y+b\lambda) + (1-p)\psi_2(y+a\lambda)\big)$$
$$\text{and} \quad \psi_A(y) = \inf\nolimits_{z\ge(A-y)^+} \big(z + \psi(y+z-A)\big).$$

Then ψ *and* ψ_A *are continuous, decreasing and piecewise linear functions vanishing at* ∞. *Furthermore, there exists* $u \ge (A-y)^+$ *such that*

$$\psi_A(y) = u + \psi(y+u-A). \tag{3.11}$$

Proof. From Lemma 3.3 in [3] it follows that $\psi(y)$ is a decreasing continuous function. Let us show that $\psi(y)$ is a piecewise linear function vanishing at ∞. Since $0 \in K(y)$ then

$$\psi(y) \le p\psi_1(y) + (1-p)\psi_2(y) \le \max(\psi_1(y),\psi_2(y)). \tag{3.12}$$

There exists a natural number n such that

$$\psi_i(y) = \sum_{j=1}^{n} \mathbb{I}_{[a_j,b_j)}(c_j^{(i)} y + d_j^{(i)}), \ i = 1,2 \tag{3.13}$$

where $c_j^{(i)}, d_j^{(i)} \in \mathbb{R}$ and $\{[a_i,b_i)\}_{i=1}^n$ is a sequence of disjoint finite intervals. Fix y and define the function $\phi_y(\lambda) = p\psi_1(y+b\lambda) + (1-p)\psi_2(y+a\lambda)$. From (3.13) it follows that there exists

$$\lambda \in \{-\frac{y}{b}, -\frac{y}{a}\} \cup \{\frac{a_j - y}{b}, \frac{b_j - y}{b}, \frac{a_j - y}{a}\frac{b_j - y}{b}\}_{j=1}^{n}. \tag{3.14}$$

such that $\psi(y) = \phi_y(\lambda)$. Thus, there exists a finite sequence of real numbers $u_1,...,u_m,v_1,...,v_m$ such that for any y,

$$\psi(y) = u_i y + v_i \tag{3.15}$$

for some i (which depends on y). This together with (3.12) and the fact that $\psi(y)$ is a continuous function gives that $\psi(y)$ is a piecewise linear function vanishing at ∞. Next, we deal with $\psi_A(y)$. Observe that $\psi_A(y) \leq \psi(0) + (A-y)^+$. Thus

$$\psi_A(y) = \inf_{(A-y)^+ \leq z \leq (A-y)^+ + \psi(0)} \big(z + \psi(y+z-A)\big) \tag{3.16}$$

and (3.11) follows from the fact that ψ is continuous. Choose $y_1 < y_2$. Since $\psi(y)$ is a decreasing function then

(3.17)

$$\psi_A(y_2) \leq \inf_{z \geq (A-y_1)^+} \big(z + \psi(y_2+z-A)\big) \leq \inf_{z \geq (A-y_1)^+} \big(z + \psi(y_1+z-A)\big) = \psi_A(y_1).$$

Thus $\psi_A(y)$ is a decreasing function. Now we want to prove continuity. Choose $\epsilon > 0$. Since $\psi(y)$ is a continuous piecewise linear function vanishing at ∞ then there exists a $\delta_1 > 0$ such that

$$|y_1 - y_2| < \delta_1 \Rightarrow |\psi(y_1) - \psi(y_2)| < \epsilon. \tag{3.18}$$

Set $\delta = \min(\epsilon, \delta_1)$. We will show that

$$|y_1 - y_2| < \frac{\delta}{2} \Rightarrow |\psi_A(y_1) - \psi_A(y_2)| \leq 2\epsilon \tag{3.19}$$

assuming without loss of generality that $y_1 < y_2$. There exists $u \geq (A-y_2)^+$ such that

$$\psi_A(y_2) = u + \psi(y_2 + u - A). \tag{3.20}$$

If $u \geq (A-y_1)^+$ then using (3.18,)

$$\psi_A(y_1) - \psi_A(y_2) \leq u + \psi(y_1 + u - A) - (u + \psi(y_2 + u - A)) \leq \epsilon. \tag{3.21}$$

If $u < (A-y_1)^+$ then $|u - (A-y_1)^+| \leq (A-y_1)^+ - (A-y_2)^+ \leq \frac{\delta}{2}$ and $|(y_1 + (A - y_1)^+ - A) - (y_2 + u - A)| \leq \delta$. Thus from (3.18) it follows that

$$\psi_A(y_1) - \psi_A(y_2) \leq (A-y_1)^+ + \psi(y_1 + (A-y_1)^+ - A) - (u + \psi(y_2 + u - A)) \leq 2\epsilon. \tag{3.22}$$

By (3.21) and (3.22) we obtain (3.19) and conclude that $\psi_A(y)$ is a continuous function. Next, let

$$\psi_(y) = \sum_{i=1}^{k} \mathbb{I}_{[\alpha_i, \beta_i)}(w_i y + x_i) \tag{3.23}$$

where k is a natural number, $w_i, x_i \in \mathbb{R}$ and $\{[\alpha_i, \beta_i)\}_{i=1}^k$ is a sequence of disjoint finite intervals. Fix y and define the function $\phi_{A,y}(z) = z + \psi(y+z-A)$. From (3.16) and (3.23) it follows that there exists

$$z \in \{(A-y)^+\} \cup \{\alpha_i + A - y, \beta_i + A - y\}_{i=1}^k$$

such that $\psi_A(y) = \phi_{A,y}(z)$. Hence, as before we see that there exists a finite sequence of real numbers $U_1, ..., U_M, V_1, ..., V_M$ such that for any y,

$$\psi_A(y) = U_i y + V_i$$

for some i which depends on y. This together with (3.16) and the fact that $\psi_A(y)$ is a continuous function gives that $\psi_A(y)$ is a piecewise linear function vanishing at ∞. □

Lemma 3.7. *For any $0 \leq k \leq N$ and $0 \leq j \leq L$ and $u_1, ..., u_k \in \{a, b\}$ the function $J_k(\cdot, j, u_1, ..., u_k)$ is continuous, decreasing, piecewise linear and vanishing at ∞.*

Proof. We will use backward induction in k. For $k = N$ the statement follows from (2.18). Suppose the statement is correct for $k = n + 1$ and prove it for $k = n$. Fix $j > 0$ (for $j = 0$ the statement is clear) and $u_1, ..., u_n \in \{a, b\}$. Set $\psi_1^{(i)}(y) = J_{n+1}(y, i, u_1, ..., u_n, b)$ and $\psi_2^{(i)}(y) = J_{n+1}(y, i, u_1, ..., u_n, a)$. From the induction hypothesis it follows that $\psi_1^{(i)}, \psi_2^{(i)}$ are continuous, decreasing and piecewise linear functions vanishing at ∞. Thus, applying Lemma 3.1 to the functions $\psi_1^{(j-1)}(y), \psi_2^{(j-1)}(y)$ and $A = g_n^{(L-j+1)}(u_1, ..., u_n)$ we obtain that the first term in (2.19) is a continuous, decreasing and piecewise linear function vanishing at ∞ (with respect to y). Similarly we obtain that the second term in (2.19) is a continuous, decreasing and a piecewise linear function vanishing at ∞. Using Lemma 3.1 for the functions $\psi_1^{(j)}(y), \psi_2^{(j)}(y)$ we see that the third term in (2.19) is a continuous, decreasing and a piecewise linear function vanishing at ∞. Thus $J_n(\cdot, j, u_1, ..., u_n)$ is a continuous, decreasing and piecewise linear function vanishing at ∞ completing the proof. □

4. Hedging and fair price

In this section we prove Theorem 2.4 starting with the following observation.

Lemma 4.1. *Assume Y_k, V_{k+1} are random variables which are respectively $\mathcal{F}_k$ and $\mathcal{F}_{k+1}$ measurable. Assume that $Y_k \geq \tilde{E}(V_{k+1}|\mathcal{F}_k)$. Then there exist a $\mathcal{F}_k$-measurable random variable γ_k such that*

$$Y_k + \gamma_k(S_k - S_{k+1}) \geq V_{k+1}. \tag{4.1}$$

Proof. Set $V_k = \tilde{E}(V_{k+1}|\mathcal{F}_k)$. Then by the martingale representation theorem in the binomial model (see, for instance [9]) there exists a $\mathcal{F}_k$-measurable random variable γ_k such that

$$V_{k+1} = V_k + \gamma_k(S_k - S_{k+1})$$

and (4.1) follows. □

Next, we define a special portfolio strategy $\pi^* = (x^*, \gamma^*)$ setting $x^* = G(s^*, b^*) = V_0^{(L)}$ and taking $\gamma^*(k, i, y)$ to be the random variable γ_k from Lemma 4.1 with respect to $Y_k = y$ and $V_{k+1} = V_{k+1}^{(L-i+1)}\mathbb{I}_{\{y \geq \tilde{E}(V_{k+1}^{(L-i+1)}|\mathcal{F}_k)\}}$. Note that if $y \geq \tilde{E}(V_{k+1}^{(L-i+1)}|\mathcal{F}_k)$ then by Lemma 4.1,

$$y + \gamma^*(k, i, y)(S_{k+1} - S_k) \geq V_{k+1}^{(L-i+1)}. \tag{4.2}$$

Now we obtain.

Lemma 4.2. *The pair (π^*, s^*) is a perfect hedge.*

Proof. Let $b \in \mathcal{S}$ be any stopping strategy. Set $F(s^*, b) = ((\sigma_1, ..., \sigma_L), (\tau_1, ..., \tau_L))$. In order to derive that the pair (π^*, S^*) is a perfect hedge we have to show that for every $0 \leq k \leq N$,

$$V_k^{(\pi^*, s^*, b)} \geq 0.$$

In fact, we shall see that for every $0 \leq k \leq L$,

$$V_k^{(\pi^*, s^*, b)} \geq \tilde{E}(V_{k+1}^{(L-c_k)}|\mathcal{F}_k) \tag{4.3}$$

where $c_k = \sum_{i=1}^{L} \mathbb{I}_{\{\sigma_i \wedge \tau_i \le k\}}$. Since c_k is measurable with respect to the σ-algebra $\mathcal{F}_k$ the inequality (4.3) is a consequence of the following inequalties

$$V_k^{(\pi^*,s^*,b)} \mathbb{I}_{c_k=i} \ge \tilde{E}(V_{k+1}^{(L-i)} | \mathcal{F}_k) \mathbb{I}_{c_k=i}, \quad 1 \le i \le L.$$

For every $1 \le i \le L$ the above inequality will be proved by induction in k. For $k = 0$ we may have either $c_0 = 0$ or $c_0 = 1$ where the second event occurs when either the writer or the holder exercised the first claim at the time $k = 0$. If $c_0 = 0$ then by (2.8),

$$V_0^{(\pi^*,s^*,b)} = x^* = V_0^{(L)}.$$

Since $V_{\sigma_1^* \wedge k}^{(L)}$ is a supermartingale with respect to $\{\mathcal{F}_k\}_{k=0}^N$ and $1 \le \sigma_1^* \wedge \tau_1 \le \sigma_1^*$ it follows that on the event $c_0 = 0$, which is $\mathcal{F}_0$ measurable, we have

$$V_0^{(L)} \ge \tilde{E}(V_{\sigma_1^* \wedge 1}^{(L)}) = \tilde{E}(V_1^{(L)}).$$

If $c_0 = 1$ we obtain

$$V_0^{(\pi^*,s^*,b)} = V_0^{(L)} - H^{(1)}(\sigma_1^*, \tau_1) \ge \big(X_{\sigma_1^*}^{(L)} - X_1(\sigma_1^*)\big) \mathbb{I}_{\{\sigma_1^* < \tau_1\}}$$
$$+ \big(Y_{\tau_1}^{(L)} - Y_1(\tau_1)\big) \mathbb{I}_{\{\sigma_1^* \ge \tau_1\}} = \tilde{E}(V_{\sigma_1^* \wedge \tau_1 + 1}^{(L-1)} | \mathcal{F}_{\sigma_1^* \wedge \tau_1}) = \tilde{E}(V_1^{(L-1)})$$

where the first equality is (2.8), the inequality is derived from the definition of the stopping time σ_1^* and the fact that $V^{(L)} \ge Y^{(L)}$ and the last equalities follow from the definitions of $X^{(L)}$ and $Y^{(L)}$ and the fact that $\sigma_1^* \wedge \tau_1 = 0$ when $c_0 = 1$.

Next, let $0 < k \le N$. Assume, first, that $c_k = i < L$. Then by the definition of c_k it follows that $\sigma_i^* \wedge \tau_i \le k$. Similarly to the case $k = 0$ we may have either $\sigma_i^* \wedge \tau_i < k$ or $\sigma_i^* \wedge \tau_i = k$. If $\sigma_i^* \wedge \tau_i < k$ then $c_{k-1} = i$ and so by (2.8),

$$V_k^{(\pi^*,s^*,b)} = V_{k-1}^{(\pi^*,s^*,b)} + \gamma^*(k-1, i+1, V_{k-1}^{(\pi^*,s^*,b)})(S_k - S_{k-1}).$$

where the equality holds on the $\mathcal{F}_k$ event $\sigma_i^* \wedge \tau_i < k$. By the induction hypothesis we obtain on this event that

$$V_{k-1}^{(\pi^*,s^*,b)} \ge \tilde{E}(V_k^{(L-i)} | \mathcal{F}_{k-1}).$$

By (4.2) it follows that

$$V_k^{(\pi^*,s^*,b)} = V_{k-1}^{(\pi^*,s^*,b)} + \gamma^*(k-1, i+1, V_{k-1}^{\pi^*,s^*,b})(S_k - S_{k-1}) \ge V_k^{(L-i)}.$$

Since $c_k = i$ the definition of c_k yields that $\sigma_{i+1}^* \ge \sigma_{i+1}^* \wedge \tau_{i+1} \ge k+1$, and so from the supermartingale property of $V_{\sigma_{i+1}^* \wedge l}^{(L-i)}$ for $l \ge k+1 \ge \sigma_i^* \wedge \tau_i^* + 1$ we obtain

$$V_k^{(L-i)} \ge \tilde{E}(V_{\sigma_{i+1}^* \wedge k+1}^{(L-i)} | \mathcal{F}_k) = \tilde{E}(V_{k+1}^{(L-i)} | \mathcal{F}_k).$$

Now consider the $\mathcal{F}_k$ event $\sigma_i^* \wedge \tau_i = k$. Then $c_{k-1} = i - 1$ and (2.8) becomes

$$V_k^{(\pi^*,s^*,b)} = V_{k-1}^{(\pi^*,s^*,b)} + \gamma^*(k-1, i, V_{k-1}^{(\pi^*,s^*,b)})(S_k - S_{k-1}) - H(\sigma_i^*, \tau_i).$$

Since $c_{k-1} = i - 1$ the induction hypothesis yields that

$$V_{k-1}^{(\pi^*,s^*,b)} \ge \tilde{E}(V_k^{(L-i+1)} | \mathcal{F}_{k-1}),$$

and so from the definition of $\gamma^*(k-1, i, y)$ we obtain that

$$V_{k-1}^{(\pi^*,s^*,b)} + \gamma^*(k-1, i, V_{k-1}^{(\pi^*,s^*,b)})(S_k - S_{k-1}) - H^{(i)}(\sigma_i^*, \tau_i)$$
$$\ge V_k^{(L-i+1)} - H^{(i)}(\sigma_i^*, \tau_i) = V_{\sigma_i^* \wedge \tau_i}^{(L-i+1)} - H^{(i)}(\sigma_i^*, \tau_i).$$

From the definition of σ_i^*, the fact that $V^{(i)} \geq Y^{(i)}$ and the definition of $X^{(i)}, Y^{(i)}$ it follows that

$$\begin{aligned} V^{(L-i+1)}_{\sigma_i^* \wedge \tau_i} - H^{(i)}(\sigma_i^*, \tau_i) \geq & \left(X^{(L-i+1)}_{\sigma_i^*} - X_i(\sigma_i^*)\right)\mathbb{I}_{\{\sigma_i^* < \tau_i\}} \\ & + \left(Y^{(L-i+1)}_{\tau_i} - Y_i(\tau_i)\right)\mathbb{I}_{\{\sigma_i^* \geq \tau_i\}} \\ = & \tilde{E}(V^{(L-i)}_{\sigma_i^* \wedge \tau_i + 1} | \mathcal{F}_{\sigma_i^* \wedge \tau_i}) = \tilde{E}(V^{(L-i)}_{k+1} | \mathcal{F}_k). \end{aligned}$$

We are left only with the event $c_k = L$. On this event the inequality (4.3) is reduced to

$$V_k^{(\pi^*, s^*, b)} \geq 0.$$

If $\sigma_L^* \wedge \tau_L = k$ then the proof is the same as above in the case $\sigma_i^* \wedge \tau_i = k$ for $i < L$. In the case $\sigma_L^* \wedge \tau_L < k$ there are no claims left to exercise or cancel, and so by the definition of γ^* we see that the portfolio value will stay nonnegative till the time N. □

Next, we show that $x^* = V_0^{(L)}$ is the minimal initial capital for a perfect hedge.

Lemma 4.3. *Assume that the pair $(\pi, s) = ((x, \gamma), s)$ is a perfect hedge. Then*

$$x \geq x^* = V_0^{(L)}.$$

Proof. Let b^* be the stopping strategy for the buyer defined in (2.12) and set

$$F(s, b^*) = ((\sigma_1, ..., \sigma_L), (\tau_1^*, ..., \tau_L^*)).$$

We want to show that

$$V_k^{(\pi, s, b^*)} \geq \tilde{E}(V^{(L-c_k)}_{(k+1)\wedge N} | \mathcal{F}_k) \tag{4.4}$$

where c_k is computed with respect to (s, b^*). Recall that for every $0 \leq k \leq N$ the function c_k is $\mathcal{F}_k$ measurable and since inequality (4.4) is between $\mathcal{F}_k$ measurable functions we can prove (4.4) separately on the events $c_k = i$.

The inequality (4.4) will be proved by the backward induction in k. When $c_k = L$ the right hand side of (4.4) is zero and the definition of a perfect hedge yields that the left hand side of (4.4) is non negative, hence (4.4) is true in these cases. Next, assume that $c_k = i$ where $0 \leq i \leq L-1$ (thus $k < N$). We split the proof into two events $c_{k+1} = i$ and $c_{k+1} = i+1$. In the second event the $(i+1)$-th claim was exercised or canceled at the time $k+1$.

We begin with the event $c_{k+1} = i$ (thus $k < N-1$). From the induction hypothesis it follows that

$$V_{k+1}^{(\pi, s, b^*)} \geq \tilde{E}(V^{(L-i)}_{k+2} | \mathcal{F}_{k+1}) = \tilde{E}(V^{(L-i)}_{\tau^*_{i+1} \wedge (k+2)} | \mathcal{F}_{k+1}) \geq V^{(L-i)}_{\tau^*_{i+1} \wedge (k+1)}.$$

The equality here holds true since $\tau^*_{i+1} \geq \tau^*_{i+1} \wedge \sigma_{i+1} \geq k+2 > k+1 > \sigma_i \wedge \tau_i^*$ when $c_{k+1} = c_k = i$ and the last inequality follows from the submartingale property of $V^{(L-i)}_{\tau^*_{i+1} \wedge l}$ for $l > \sigma_i \wedge \tau_i^*$. Since $c_i = k$ we have from (2.8) that

$$V_{k+1}^{\pi, s, b^*} = V_k^{(\pi, s, b^*)} + \gamma(k, i+1, V_k^{(\pi, s, b^*)})(S_{k+1} - S_k).$$

Since S_k is a martingale with respect to $\tilde{P}$ then using this equality and taking the conditional expectation with respect to $\mathcal{F}_k$ in the above inequality we obtain

$$V_k^{(\pi, s, b^*)} \geq \tilde{E}(V^{(L-i)}_{\tau^*_{i+1} \wedge (k+1)} | \mathcal{F}_k) = \tilde{E}(V^{(L-i)}_{k+1} | \mathcal{F}_k).$$

Next, assume that $c_{k+1} = i+1$ which together with the assumption $c_k = i$ yields that $\sigma_{i+1} \wedge \tau^*_{i+1} = k+1$. By the induction hypothesis it follows that

$$V^{(\pi,s,b^*)}_{k+1} \geq \tilde{E}(V^{(L-i-1)}_{k+2} | \mathcal{F}_{k+1}) = \tilde{E}(V^{(L-i-1)}_{\sigma_i \wedge \tau^*_i + 1} | \mathcal{F}_{\sigma_i \wedge \tau^*_i}),$$

and so

$$V^{(\pi,s,b^*)}_{k+1} + H^{(i+1)}(\sigma_{i+1}, \tau^*_{i+1}) \geq \tilde{E}(V^{(L-i-1)}_{\sigma_i \wedge \tau^*_i + 1} | \mathcal{F}_{\sigma_i \wedge \tau^*_i}) + H^{(i+1)}(\sigma_{i+1}, \tau^*_{i+1})$$
$$= X^{(L-i)}_{\sigma_{i+1}} \mathbb{I}_{\{\sigma_{i+1} < \tau^*_{i+1}\}} + Y^{(L-i)}_{\tau^*_{i+1}} \mathbb{I}_{\{\sigma_{i+1} \geq \tau^*_{i+1}\}} \geq V^{(L-i)}_{\sigma_{i+1} \wedge \tau^*_{i+1}} = V^{(L-i)}_{k+1}$$

where the second inequality holds true since $X^{(L-i)} \geq V^{(L-i)}$ and in view of the definition of the stopping time τ^*_{i+1}. On the event $c_k = i$ and $c_{k+1} = i+1$ the equality (2.8) becomes

$$V^{(\pi,s,b^*)}_{k+1} + H^{(i+1)}(\sigma_{i+1}, \tau^*_{i+1}) = V^{(\pi,s,b^*)}_k + \gamma(k, i+1, V^{(\pi,s,b^*)}_k)(S_{k+1} - S_k)$$

and taking the conditional expectation of the above inequality with respect to the sigma algebra $\mathcal{F}_k$ we obtain that

$$V^{(\pi,s,b^*)}_k \geq \tilde{E}(V^{(L-i)}_{k+1} | \mathcal{F}_k)$$

completing the proof of (4.4). As a special case of (4.4) for $k = 0$ it follows that

$$V^{(\pi,s,b^*)}_0 \geq \tilde{E}(V^{(L-c_0)}_1).$$

If $c_0 = 0$ then $\tau^*_1 \geq \sigma_1 \wedge \tau^*_1 \geq 1$ and since $V^{(L)}_{\tau^*_1 \wedge l}$, $l \geq 0$ is a submartingale we see that

$$V^{(\pi,s,b^*)}_0 \geq \tilde{E}(V^{(L-c_0)}_1) = \tilde{E}(V^{(L)}_{\tau^*_1 \wedge 1}) \geq V^{(L)}_0 = x^*.$$

If $c_0 = 1$ then $\sigma_1 \wedge \tau^*_1 = 0$, and so

$$x - H(\sigma_1, \tau^*_1) = V^{(\pi,s,b^*)}_0 \geq \tilde{E}(V^{(L-1)}_1)$$

which can also be written in the form

$$x \geq \tilde{E}(V^{(L-1)}_{\sigma_1 \wedge \tau^*_1 + 1}) + H(\sigma_1, \tau^*_1)$$
$$= X^{(L)}_{\sigma_1} \mathbb{I}_{\{\sigma_1 < \tau^*_1\}} + Y^{(L)}_{\tau^*_1} \mathbb{I}_{\{\sigma_1 \geq \tau^*_1\}} \geq V^{(L)}_{\sigma_1 \wedge \tau^*_1} = V^{(L)}_0 = x^*$$

or in short

$$x \geq x^* = V^{(L)}_0.$$

□

We can now prove the main theorem of this section.

Proof of Theorem 2.4. From Lemma 4.2 and the definition of the fair price V^* we obtain that

$$V^{(L)}_0 = x^* \geq V^*.$$

On the other hand, Lemma 4.3 yields that

$$x^* \leq V^*.$$

By Proposition 3.1,

$$G(b, s^*) \leq G(b^*, s^*) = V^{(L)}_0 \leq G(b^*, s)$$

for any pair of stopping strategies $b, s \in \mathcal{S}$ which gives (2.13) and collecting together the above inequalities we obtain (2.11). Since $\pi^* = (V^{(L)}_0, \gamma^*)$ we it follows that

$\pi^* \in \mathcal{A}(V_0^{(L)})$ and by Lemma 3.3 the pair (π^*, s^*) is a perfect hedge completing the proof of Theorem 2.4. □

5. Shortfall risk and its hedging

In this section we derive Theorem 2.7 whose proof is quite technical but the main idea is to apply Lemma 3.1 to Dynkin's games with appropriately constructed payoff processes which via Lemma 5.1 below enables us to produce a hedge for the shortfall risk whose optimality is established by means of Lemmas 5.2 and 5.4 below.

For any $I \in \mathcal{I}$ set

$$(5.1)\qquad Z^{(I)}(y,k,j,u_1,...,u_k) = y - f_k^{(L-j+1)}(u_1,...,u_k) + I(k, L-j+1, y - f_k^{(L-j+1)}(u_1,...,u_k)) \text{ and } \tilde{Z}^{(I)}(y,k,j,u_1,...,u_k) = y - g_k^{(L-j+1)}(u_1,...,u_k) + I(k, L-j+1, y - g_k^{(L-j+1)}(u_1,...,u_k)).$$

Observe that if at the moment k the seller pays his $(L-j+1)$-th payoff and this is his first payoff at this moment (at $k = N$ more than one payoff can occur) then his portfolio value after this payoff is either $Z^{(I)}(y,k,j,\rho_1,...,\rho_k)$ or $\tilde{Z}^{(I)}(y,k,j,\rho_1,...,\rho_k)$ in the case of an exercise or a cancellation, respectively, provided an infusion of capital before the payoff is y (where ρ_i is the same as in (2.2)). Next, for any $\pi = (x,\gamma) \in \mathcal{A}(x)$ and $I \in \mathcal{I}$ define

$$(5.2)\qquad U^{(\pi,I)}(y,k,j,u_1,...,u_{k+1}) = Z^{(I)}(y,k,j,u_1,...,u_k) + \mathbb{I}_{j>1} \times\gamma(k, L-j+2, Z^{(I)}(y,k,j,u_1,...,u_k)) S_0 u_{k+1} \prod_{i=1}^k (1+u_i) \text{ and } \tilde{U}^{(\pi,I)}(y,k,j,u_1,...,u_{k+1}) = \tilde{Z}^{(I)}(y,k,j,u_1,...,u_k) + \mathbb{I}_{j>1} \times\gamma(k, L-j+2, \tilde{Z}^{(I)}(y,k,j,u_1,...,u_k)) S_0 u_{k+1} \prod_{i=1}^k (1+u_i).$$

Note that if at the moment $k < N$ the seller pays his $(L-j+1)$-th payoff then his portfolio value at the time $k+1$ before any payoffs is either $U^{(\pi,I)}(y,k,j,\rho_1,...,\rho_{k+1})$ or $\tilde{U}^{(\pi,I)}(y,k,j,\rho_1,...,\rho_{k+1})$ in the case of an exercise or a cancellation, respectively, at the time k provided that his portfolio value before payoffs was y. Finally, for any $(\pi, I) \in \mathcal{A}\times\mathcal{I}$ define a sequence of functions $J_k^{(\pi,I)} : \mathbb{R}_+ \times \{0,...,L\} \times \{a,b\}^k \to \mathbb{R}_+$, $0 \le k \le N$ setting, first,

$$(5.3)\qquad J_N^{(\pi,I)}(y,j,u_1,...,u_N) = ((\sum_{i=L-j+1}^L f_N^{(i)}(u_1,...,u_N)) - y)^+,\ j > 0,\quad J_k^{(\pi,I)}(y,0,u_1,...,u_k) = 0,\ 0 \le k \le N.$$

Next, for $k < N$ and $j > 0$ set

$$(5.4)\qquad J_k^{(\pi,I)}(y,j,u_1,...,u_k) = I(k, L-j+1, y - f_k^{(L-j+1)}(u_1,...,u_k)) +pJ_{k+1}^{(\pi,I)}(U^{(\pi,I)}(y,k,j,u_1,...,u_k,b), j-1, u_1,...,u_k,b) +(1-p)J_{k+1}^{(\pi,I)}(U^{(\pi,I)}(y,k,j,u_1,...,u_k,a), j-1, u_1,...,u_k,a)$$

if

$$
\begin{aligned}
(5.5) \qquad & I(k, L-j+1, y - f_k^{(L-j+1)}(u_1,...,u_k)) \\
&+pJ_{k+1}^{(\pi,I)}(U^{(\pi,I)}(y,k,j,u_1,...,u_k,b), j-1, u_1,...,u_k,b) \\
&+(1-p)J_{k+1}^{(\pi,I)}(U^{(\pi,I)}(y,k,j,u_1,...,u_k,a), j-1, u_1,...,u_k,a) \\
&\geq I(k, L-j+1, y - g_k^{(L-j+1)}(u_1,...,u_k)) \\
&+pJ_{k+1}^{(\pi,I)}(\tilde{U}^{(\pi,I)}(y,k,j,u_1,...,u_k,b), j-1, u_1,...,u_k,b) \\
&+(1-p)J_{k+1}^{(\pi,I)}(\tilde{U}^{(\pi,I)}(y,k,j,u_1,...,u_k,a), j-1, u_1,...,u_k,a)
\end{aligned}
$$

and

$$
\begin{aligned}
(5.6) \qquad J_k^{(\pi,I)}(y,j,u_1,...,u_k) = \min\Bigg(& I(k, L-j+1, y - g_k^{(L-j+1)}(u_1,...,u_k)) \\
&+pJ_{k+1}^{(\pi,I)}(\tilde{U}^{(\pi,I)}(y,k,j,u_1,...,u_k,b), j-1, u_1,...,u_k,b) \\
&+(1-p)J_{k+1}^{(\pi,I)}(\tilde{U}^{(\pi,I)}(y,k,j,u_1,...,u_k,a), j-1, u_1,...,u_k,a), \\
&\max\Bigg(I(k, L-j+1, y - f_k^{(L-j+1)}(u_1,...,u_k)) \\
&+pJ_{k+1}^{(\pi,I)}(U^{(\pi,I)}(y,k,j,u_1,...,u_k,b), j-1, u_1,...,u_k,b) \\
&+(1-p)J_{k+1}^{(\pi,I)}(U^{(\pi,I)}(y,k,j,u_1,...,u_k,a), j-1, u_1,...,u_k,a), \\
&pJ_{k+1}^{(\pi,I)}(y + \gamma(k, L-j+1, y)S_0 b \textstyle\prod_{i=1}^k (1+u_i), j, u_1,...,u_k,b) \\
&+(1-p)J_{k+1}^{(\pi,I)}(y + \gamma(k, L-j+1, y)S_0 a \textstyle\prod_{i=1}^k (1+u_i), j, u_1,...,u_k,a)\Bigg)\Bigg)
\end{aligned}
$$

if the inequality in (5.5) does not hold true.

For any $j \geq 1$ and $k \leq N$ consider the set $\mathcal{S}_k^{(j)}$ of sequences $s = (s_1, ..., s_j)$ such that $s_1 \in \Gamma_k$ and for $i > 1$, $s_i : C_{i-1} \to \Gamma$ is a map which satisfy

$$s_i((a_1,...,a_{i-1}),(d_1,...,d_{i-1})) \in \Gamma_{N \wedge (1+a_{i-1})}.$$

Next, define a map $F : \mathcal{S}_k^{(j)} \times \mathcal{S}_k^{(j)} \to \Gamma^j \times \Gamma^j$ by

$$F(s,b) = ((\sigma_1,...,\sigma_j),(\tau_1,...,\tau_j))$$

in the same way as in (2.5). Fix $(\pi, I) \in \mathcal{A} \times \mathcal{I}$, $j \geq m \geq 1$, $k \leq N$ and $y \geq 0$. Consider a swing option which starts at the time k where the initial capital of the seller equal y, the number of remaining payoffs is m and it starts from the $(L-j+1)$-th claim. Let $z = (a,d) = ((a_1,...,a_m),(d_1,...,d_m)) \in C_m$ be a sequence which represents the history of the payoffs. Set $c_n = c_n(z) = L - j + \sum_{i=1}^m \mathbb{I}_{a_i \leq n}$.

Define the stochastic processes $\{W_n^{(y,\pi,I,k,j,z)}\}_{n=k}^N$ and $\{V_n^{(y,\pi,I,k,j,z)}\}_{n=k}^N$ by

$$
\begin{aligned}
(5.7) \qquad & W_k^{(y,\pi,I,k,j,z)} = y,\ V_k^{(y,\pi,I,k,j,z)} = W_k^{(y,\pi,I,k,j,z)} - \mathbb{I}_{a_1=k} \\
&\times\Big(\mathbb{I}_{d_1=1}X_{L-j+1}(k) + \mathbb{I}_{d_1=0}Y_{L-j+1}(k) + \mathbb{I}_{k=N}\textstyle\sum_{i=L-j+2}^L Y_i(N) - I(k, L-j+1, \\
&W_k^{(y,\pi,I,k,j,z)} - \mathbb{I}_{d_1=1}X_{L-j+1}(k) - \mathbb{I}_{d_1=0}Y_{L-j+1}(k))\Big) \quad \text{and for } n>k, \\
&V_n^{(y,\pi,I,k,j,z)} = W_{n-1}^{(y,\pi,I,k,j,z)} + \mathbb{I}_{c_{n-1}<L}\gamma(n-1, c_{n-1}+1, \\
&W_{n-1}^{(y,\pi,I,k,j,z)})(S_n - S_{n-1}), W_n^{(y,\pi,I,k,j,z)} = V_n^{(y,\pi,I,k,j,z)} - \mathbb{I}_{c_{n-1}<L}\mathbb{I}_{a_{c_{n-1}+1}=n} \\
&\times\Big(X_{c_{n-1}+1}(n)\mathbb{I}_{d_{c_{n-1}+1}=1} + Y_{c_{n-1}+1}(n)\mathbb{I}_{d_{c_{n-1}+1}=0} + \mathbb{I}_{n=N}\textstyle\sum_{i=c_{n-1}+2}^L Y_i(N) \\
&-I(n, c_{n-1}+1, V_n^{(y,\pi,I,k,j,z)} - X_{c_{n-1}+1}(n)\mathbb{I}_{d_{c_{n-1}+1}=1} - Y_{c_{n-1}+1}(n)\mathbb{I}_{d_{c_{n-1}+1}=0})\Big).
\end{aligned}
$$

Similarly to (2.14) we conclude (under the conditions that were described above) that if the contract was not exercised at a moment n then $W_n^{(y,\pi,I,k,j,z)} = V_n^{(y,\pi,I,k,j,z)}$ is the portfolio value at this moment . If the contract was exercised at the moment n then $W_n^{(y,\pi,I,k,j,z)}$ and $V_n^{(y,\pi,I,k,j,z)}$ are the portfolio values before and after the payoff, respectively. For the case $m = 0$ (no history of payoffs) we define the stochastic processes $\{W_n^{(y,\pi,I,k,j)}\}_{n=k}^N$ by

$$
\begin{aligned}
(5.8) \qquad & W_k^{(y,\pi,I,k,j)} = y \text{ and for } n>k, \\
& W_n^{(y,\pi,I,k,j)} = W_{n-1}^{(y,\pi,I,k,j)} + \gamma(n-1, L-j+1, W_{n-1}^{(y,\pi,I,k,j)})(S_n - S_{n-1}).
\end{aligned}
$$

Clearly, $W_n^{(y,\pi,I,k,j)}$ is the portfolio value if no payoffs were made until the moment n. Let $s \in \mathcal{S}_k^{(j)}$ and $b \in \mathcal{S}_k^{(j)}$ be stopping strategies of the seller and the buyer, respectively. Set $((\sigma_1,...,\sigma_j),(\tau_1,...,\tau_j)) = F(s,b)$, $a_i = \sigma_i \wedge \tau_i$, $d_i = \mathbb{I}_{\sigma_i<\tau_i}$ and $z = ((a_1,...,a_i),(d_1,...,d_i))$. Define

$$
\begin{aligned}
(5.9) \qquad & W_n^{(y,\pi,I,k,j,s,b)}(\omega) = W_n^{(y,\pi,I,k,j,z(\omega))}(\omega) \text{ and} \\
& V_n^{(y,\pi,I,k,j,s,b)}(\omega) = V_n^{(y,\pi,I,k,j,z(\omega))}(\omega).
\end{aligned}
$$

Similarly to (2.15) the total infusion of capital is given by

$$
(5.10) \quad C(y,\pi,I,k,j,s,b) = \sum_{i=1}^{\alpha\wedge j} I(\sigma_i\wedge\tau_i, i+L-j, W_{\sigma_i\wedge\tau_i}^{(y,\pi,I,k,j,s,b)} - H^{(L-j+i)}(\sigma_i,\tau_i))
$$

where $\alpha = 1 + \sum_{i=1}^j \mathbb{I}_{\sigma_i\wedge\tau_i<N}$. Thus for any $(\pi, I) \in \mathcal{A}\times\mathcal{I}$, $j \geq 1$, $k \leq N$, $s, b \in \mathcal{S}_k^{(j)}$ and $y \geq 0$ we have the following definition for the shortfall risk

$$
\begin{aligned}
(5.11) \qquad & R(y,\pi,I,k,j,s,b) = E(C(y,\pi,I,k,j,s,b)|\mathcal{F}_k), \\
& R(y,\pi,I,k,j,s) = \max_{b\in\mathcal{S}_k^{(j)}} R(y,\pi,I,k,j,s,b), \\
& R(y,\pi,I,k,j) = \min_{s\in\mathcal{S}_k^{(j)}} R(y,\pi,I,k,j,s).
\end{aligned}
$$

Next, we define stopping strategies which will turn out to be optimal. Let $(\pi, I) \in \mathcal{A}\times\mathcal{I}$, $j \geq 1$, $k \leq N$ and $y \geq 0$. Define $\tilde{s}(y,\pi,I,k,j) = (\tilde{s}_1,...,\tilde{s}_j) \in \mathcal{S}_k^{(j)}$ and

$\tilde{b}(y,\pi,I,k,j)=(\tilde{b}_1,...,\tilde{b}_j)\in\mathcal{S}_k^{(j)}$ by

$$\tilde{s}_1=N\wedge\min\Big\{n\geq k|J_n^{(\pi,I)}(W_n^{(y,\pi,I,k,j)},j,\rho_1,...,\rho_n) \geq I(n,L-j+1,W_n^{(y,\pi,I,k,j)}-X_{L-j+i}(n)) +E(J_{n+1}^{(\pi,I)}(\tilde{U}^{(\pi,I)}(W_n^{(y,\pi,I,k,j)},n,j,\rho_1,...,\rho_{n+1}),j-1,\rho_1,...,\rho_{n+1})|\mathcal{F}_n)\Big\}, \tag{5.12}$$

$$\tilde{b}_1=N\wedge\min\Big\{n\geq k|J_n^{(\pi,I)}(W_n^{(y,\pi,I,k,j)},j,\rho_1,...,\rho_n) =I(n,L-j+1,W_n^{(y,\pi,I,k,j)}-Y_{L-j+1}(n)) +E(J_{n+1}^{(\pi,I)}(U^{(\pi,I)}(W_n^{(y,\pi,I,k,j)},n,j,\rho_1,...,\rho_{n+1}),j-1,\rho_1,...,\rho_{n+1})|\mathcal{F}_n)\Big\}.$$

For $i>1$ let $z=(a,d)=((a_1,...,a_{i-1}),(d_1,...,d_{i-1}))\in C_{i-1}$ and define

$$\tilde{s}_i(z)=N\wedge\min\Big\{n>a_{i-1}|J_n^{(\pi,I)}(W_n^{(y,\pi,I,k,j,z)},j-i+1,\rho_1,...,\rho_n) \geq I(n,L-j+i,W_n^{(y,\pi,I,k,j,z)}-X_{L-j+i}(n)) +E(J_{n+1}^{(\pi,I)}(\tilde{U}^{(\pi,I)}(W_n^{(y,\pi,I,k,j,z)},n,j-i+1,\rho_1,...,\rho_{n+1}),j-i,\rho_1,...,\rho_{n+1})|\mathcal{F}_n)\Big\}, \tag{5.13}$$

$$\tilde{b}_i(z)=N\wedge\min\Big\{n>a_{i-1}|J_n^{(\pi,I)}(W_n^{(y,\pi,I,k,j,z)},j-i+1,\rho_1,...,\rho_n) =I(n,L-j+i,W_n^{(y,\pi,I,k,j,z)}-Y_{L-j+i}(n)) +E(J_{n+1}^{(\pi,I)}(U^{(\pi,I)}(W_n^{(y,\pi,I,k,j,z)},n,j-i+1,\rho_1,...,\rho_{n+1}),j-i,\rho_1,...,\rho_{n+1})|\mathcal{F}_n)\Big\}.$$

The following two lemmas will be crucial for the proof of Theorem 2.7.

Lemma 5.1. *Let $\pi,I\in\mathcal{A}\times\mathcal{I}$, $n\leq N$, $j\geq 1$, and $y\geq 0$. Define the stochastic processes $\{A_k\}_{k=n}^N$ and $\{D_k\}_{k=n}^N$ by*

$$A_N=D_N=(\textstyle\sum_{q=L-j+1}^L Y_q(N)-W_N^{(y,\pi,I,n,j)})^{(+)} \text{ and for } k<N, \tag{5.14}$$

$$A_k=I(k,L-j+1,W_k^{(y,\pi,I,n,j)}-Y_{L-j+1}(k)) +E(J_{k+1}^{(\pi,I)}(U^{(\pi,I)}(W_k^{(y,\pi,I,n,j)},k,j,\rho_1,...,\rho_{k+1}),j-1,\rho_1,...,\rho_{k+1})|\mathcal{F}_k),$$

$$D_k=I(k,L-j+1,W_k^{(y,\pi,I,n,j)}-X_{L-j+1}(k)) +E(J_{k+1}^{(\pi,I)}(\tilde{U}^{(\pi,I)}(W_k^{(y,\pi,I,n,j)},k,j,\rho_1,...,\rho_{k+1}),j-1,\rho_1,...,\rho_{k+1})|\mathcal{F}_k).$$

Set

$$V_k=\min_{\sigma\in\Gamma_k}\max_{\tau\in\Gamma_k}E(D_\sigma\mathbb{I}_{\sigma<\tau}+A_\tau\mathbb{I}_{\tau\leq\sigma}|\mathcal{F}_k). \tag{5.15}$$

Then for any $k\geq n$,

$$V_k=J_k^{(\pi,I)}(W_k^{(y,\pi,I,n,j)},k,j,\rho_1,...,\rho_k). \tag{5.16}$$

Furthermore, the stopping times

$$\tilde{\sigma}=\tilde{s}_1=\tilde{s}(y,\pi,I,n,j)_1 \text{ and } \tilde{\tau}=\tilde{b}_1=\tilde{b}(y,\pi,I,n,j)_1 \tag{5.17}$$

given by (5.12) with $\tilde{s}(y,\pi,I,k,j)=(\tilde{s}_1,...,\tilde{s}_j)$ and $\tilde{b}(y,\pi,I,k,j)=(\tilde{b}_1,...,\tilde{b}_j)$ satisfy

$$E(D_{\tilde{\sigma}}\mathbb{I}_{\tilde{\sigma}<\tau}+A_\tau\mathbb{I}_{\tilde{\sigma}\geq\tau}|\mathcal{F}_n)\leq V_n\leq E(D_\sigma\mathbb{I}_{\sigma<\tilde{\tau}}+A_{\tilde{\tau}}\mathbb{I}_{\sigma\geq\tilde{\tau}}|\mathcal{F}_n) \tag{5.18}$$

for any $\sigma,\tau\in\Gamma_n$.

Proof. Fix $\pi, I \in \mathcal{A} \times \mathcal{I}$, and $j \geq 1$. We will use backward induction on n. For $n = N$ the statement is obvious since all the terms in (5.16) and (5.18) are equal to $((\sum_{i=L-j+1}^{L} f_N^{(i)}(\rho_1, ..., \rho_N)) - y)^+$. Suppose that the assertion holds true for $n+1, ..., N$ and prove it for n. Fix $y \geq 0$ and $n \leq k < N$ (for $k = N$ the statement is obvious). Fix $m > k$ and denote $Z_m = W_m^{(y,\pi,I,n,j)}$. For any $i \geq m$ we have $W_i^{(Z_m,\pi,I,m,j)} = W_i^{(y,\pi,I,n,j)}$, and so

$$A_N = (\textstyle\sum_{q=L-j+1}^{L} Y_q(N) - W_N^{(Z_m,\pi,I,m,j)})^{(+)} \text{ and for } m \leq i < N, \tag{5.19}$$
$$A_i = I(i, L-j+1, W_i^{(Z_m,\pi,I,m,j)} - Y_{L-j+1}(i))$$
$$+E(J_{i+1}^{(\pi,I)}(U^{(\pi,I)}(W_i^{(Z_m,\pi,I,m,j)}, i, j, \rho_1, ..., \rho_{i+1}), j-1, \rho_1, ..., \rho_{i+1})|\mathcal{F}_i),$$
$$D_i = I(i, L-j+1, W_i^{(Z_m,\pi,I,m,j)} - X_{L-j+1}(i))$$
$$+E(J_{i+1}^{(\pi,I)}(\tilde{U}^{(\pi,I)}(W_i^{(Z_m,\pi,I,m,j)}, i, j, \rho_1, ..., \rho_{i+1}), j-1, \rho_1, ..., \rho_{i+1})|\mathcal{F}_i).$$

Since Z_m is $\mathcal{F}_m$-measurable then using the induction hypothesis for $m > k \geq n$ (with Z_m in place of y) we obtain that for any $m > k$,

$$V_m = J_m^{(\pi,I)}(Z_m, j, \rho_1, ..., \rho_m). \tag{5.20}$$

Thus

$$E(V_{k+1}|\mathcal{F}_k) = pJ_{k+1}^{(\pi,I)}(W_k^{(y,\pi,I,n,j)} + \gamma(k, L-j+1, W_k^{(y,\pi,I,n,j)}) \tag{5.21}$$
$$\times S_0 b \textstyle\prod_{i=1}^{k}(1+\rho_i), j, \rho_1, ..., \rho_k, b) + (1-p)J_{k+1}^{(\pi,I)}(W_k^{(y,\pi,I,n,j)}$$
$$+\gamma(k, L-j+1, W_k^{(y,\pi,I,n,j)})S_0 a \textstyle\prod_{i=1}^{k}(1+\rho_i), j, \rho_1, ..., \rho_k, a).$$

Using Lemma 3.1 for the processes $\{A_i\}_{i=k}^N$ and $\{D_i\}_{i=k}^N$ together with (5.3)-(5.6) and (5.21) we obtain that for any $k \geq n$,

$$V_k = J_k^{(\pi,I)}(W_k^{(y,\pi,I,n,j)}, 1, \rho_k, ..., \rho_k). \tag{5.22}$$

From (5.12) and (5.22) it follows that

$$\tilde{\sigma} = N \wedge \min\{i \geq n | V_i \geq D_i\}, \ \tilde{\tau} = N \wedge \min\{i \geq n | V_i = A_i\}. \tag{5.23}$$

Thus applying Lemma 3.1 to the processes $\{A_i\}_{i=n}^N$ and $\{D_i\}_{i=n}^N$ we obtain (5.18). $\square$

Lemma 5.2. *For any $\pi, I \in \mathcal{A} \times \mathcal{I}$, $n \leq N$, $j \geq 1$, $s, b \in \mathcal{S}_n^{(j)}$ and $y \geq 0$,*

$$R(y, \pi, I, n, j, \tilde{s}(y, \pi, I, n, j), b) \leq R(y, \pi, I, n, j) \tag{5.24}$$
$$= J_n^{(\pi,I)}(y, j, \rho_1, ..., \rho_n) \leq R(y, \pi, I, n, j, s, \tilde{b}(y, \pi, I, n, j)).$$

Proof. Fix $\pi, I \in \mathcal{A} \times \mathcal{I}$. We will use the backward induction in n. For $n = N$ the statement is obvious since all the terms are equal to $((\sum_{i=L-j+1}^{L} f_N^{(i)}(\rho_1, ..., \rho_N)) - y)^+$. Suppose that the assertion is correct for $n+1, ..., N$ and let us prove it for n. For $j > 1$, $n \leq k_1 < N$ and $k_2 \in \{0, 1\}$ define the map $Q^{(k_1,k_2)} : \mathcal{S}_n^{(j)} \to \mathcal{S}_{k_1+1}^{(j-1)}$ by $Q^{(k_1,k_2)}(s_1, ..., s_{i+1}) = (s'_1, ..., s'_i)$ where

$$s'_1 = s_2(k_1, k_2) \text{ and for } m > 1, \tag{5.25}$$
$$s'_m((a_1, ..., a_{m-1}), (d_1, ..., d_{m-1}))$$
$$= s_{m+1}((k_1, a_1, ..., a_{m-1}), (k_2, d_1, ..., d_{m-1})).$$

For any $j \geq 1$ and $y \geq 0$ set $\tilde{s} = \tilde{s}(y, \pi, I, n, j)$. From (5.12)-(5.13) it follows that for any $j > 1$ the stopping strategy $\tilde{s}^{(k_1,k_2)} = Q^{(k_1,k_2)}(\tilde{s})$ satisfies

$$\tilde{s}^{(k_1,k_2)} = \tilde{s}(W_{k_1+1}^{(y,\pi,I,n,j,(k_1,k_2))}, \pi, I, k_1+1, j-1). \tag{5.26}$$

Thus by the induction hypothesis we obtain that for any $n \leq k_1 < N$, $k_2 \in \{0,1\}$, $j > 1$ and $b' \in \mathcal{S}_{k_1+1}^{(j)}$,

$$R(W_{k_1+1}^{(y,\pi,I,n,j,(k_1,k_2))}, \pi, I, k_1+1, j-1, \tilde{s}^{(k_1,k_2)}, b') \leq J_{k_1+1}^{(\pi,I)}(W_{k_1+1}^{(y,\pi,I,n,j,(k_1,k_2))}, j-1, \rho_1, ..., \rho_{k_1+1}). \tag{5.27}$$

Fix $j \geq 1$, $y \geq 0$ and let $b \in \mathcal{S}_n^{(j)}$. Set $F(\tilde{s}, b) = ((\sigma_1, ..., \sigma_j), (\tau_1, ..., \tau_j))$, $A = \{\sigma_1 < \tau_1\}$ and $z = (\sigma_1 \wedge \tau_1, \mathbb{I}_A)$. If $j > 1$ denote also $\tilde{s}' = \tilde{s}^{(\sigma_1 \wedge \tau_1, \mathbb{I}_A)}$ and $b' = \mathbb{I}_{\sigma_1 \wedge \tau_1 < N} Q^{(\sigma_1 \wedge \tau_1, \mathbb{I}_A)}(b) + N\mathbb{I}_{\sigma_1 \wedge \tau_1 = N}$. In this case it follows from (5.10) that

$$C(y, \pi, I, n, j, s, b) = \mathbb{I}_{j>1} \mathbb{I}_{\sigma_1 \wedge \tau_1 < N} C(W_{\sigma_1 \wedge \tau_1 + 1}^{(y,\pi,I,n,j,z)}, \pi, I, \sigma_1 \wedge \tau_1 + 1, j-1, \tilde{s}', b')$$
$$+ I(\sigma_1 \wedge \tau_1, L-j+1, W_{\sigma_1 \wedge \tau_1}^{(y,\pi,I,n,j)} - H^{(L-j+1)}(\sigma_1, \tau_1)).$$

This together with (5.27) gives

$$R(y, \pi, I, n, j, \tilde{s}, b) = \mathbb{I}_{j>1} E\Big(E\big(\mathbb{I}_{\sigma_1 \wedge \tau_1 < N} C(W_{\sigma_1 \wedge \tau_1 + 1}^{(y,\pi,I,n,j,z)}, \pi, I, \tag{5.28}$$
$$\sigma_1 \wedge \tau_1 + 1, j-1, \tilde{s}', b')|\mathcal{F}_{\sigma_1 \wedge \tau_1 + 1}\big)|\mathcal{F}_n\Big) + E\big(I(\sigma_1 \wedge \tau_1, L-j+1,$$
$$W_{\sigma_1 \wedge \tau_1}^{(y,\pi,I,n,j)} - H^{(L-j+1)}(\sigma_1, \tau_1))|\mathcal{F}_n\big)$$
$$= \mathbb{I}_{j>1} \times E\big(\mathbb{I}_{\sigma_1 \wedge \tau_1 < N} R(W_{\sigma_1 \wedge \tau_1 + 1}^{(y,\pi,I,n,j,z)}, \pi, I, \sigma_1 \wedge \tau_1 + 1, j-1, \tilde{s}', b')|\mathcal{F}_n\big)$$
$$+ E\big(I(\sigma_1 \wedge \tau_1, L-j+1, W_{\sigma_1 \wedge \tau_1}^{(y,\pi,I,n,j)} - H^{(L-j+1)}(\sigma_1, \tau_1))|\mathcal{F}_n\big)$$
$$\leq \mathbb{I}_{j>1} E\big(\mathbb{I}_{\sigma_1 \wedge \tau_1 < N} J_{\sigma_1 \wedge \tau_1 + 1}^{(\pi,I)}(W_{\sigma_1 \wedge \tau_1 + 1}^{(y,\pi,I,n,j,z)}, j-1, \rho_1, ..., \rho_{\sigma_1 \wedge \tau_1 + 1})|\mathcal{F}_n\big)$$
$$+ E\big(I(\sigma_1 \wedge \tau_1, L-j+1, W_{\sigma_1 \wedge \tau_1}^{(y,\pi,I,n,j)} - H^{(L-j+1)}(\sigma_1, \tau_1))|\mathcal{F}_n\big).$$

Define the stochastic processes $\{A_k\}_{k=n}^N$ and $\{D_k\}_{k=n}^N$ by

$$A_N = D_N = (\textstyle\sum_{q=L-j+1}^L Y_q(N) - W_N^{(y,\pi,I,n,j)})^{(+)} \text{ and for } k < N, \tag{5.29}$$
$$A_k = I(k, L-j+1, W_k^{(y,\pi,I,n,j)} - Y_{L-j+1}(k))$$
$$+ E(J_{k+1}^{(\pi,I)}(U^{(\pi,I)}(W_k^{(y,\pi,I,n,j)}, k, j, \rho_1, ..., \rho_{k+1}), j-1, \rho_1, ..., \rho_{k+1})|\mathcal{F}_k),$$
$$D_k = I(k, L-j+1, W_k^{(y,\pi,I,n,j)} - X_{L-j+1}(k))$$
$$+ E(J_{k+1}^{(\pi,I)}(\tilde{U}^{(\pi,I)}(W_k^{(y,\pi,I,n,j)}, k, j, \rho_1, ..., \rho_{k+1}), j-1, \rho_1, ..., \rho_{k+1})|\mathcal{F}_k).$$

Observe that for any $\sigma, \tau \in \Gamma_n$,

$$D_\sigma \mathbb{I}_{\sigma < \tau} + A_\tau \mathbb{I}_{\tau \leq \sigma} = I(\sigma \wedge \tau, L-j+1, W_{\sigma \wedge \tau}^{(y,\pi,I,n,j)} \tag{5.30}$$
$$-H^{(L-j+1)}(\sigma, \tau)) + E\big(\mathbb{I}_{\sigma \wedge \tau < N} J_{\sigma \wedge \tau + 1}^{(\pi,I)}(W_{\sigma \wedge \tau + 1}^{(y,\pi,I,n,j,z')}, j-1, \rho_1, ..., \rho_{\sigma \wedge \tau + 1})|\mathcal{F}_{\sigma \wedge \tau}\big)$$

where $z' = (\sigma \wedge \tau, \mathbb{I}_{\sigma < \tau})$. Thus

$$E(D_\sigma \mathbb{I}_{\sigma < \tau} + A_\tau \mathbb{I}_{\tau \leq \sigma}|\mathcal{F}_n) = E\big(I(\sigma \wedge \tau, L-j+1, W_{\sigma \wedge \tau}^{(y,\pi,I,n,j)} \tag{5.31}$$
$$-H^{(L-j+1)}(\sigma, \tau))|\mathcal{F}_n\big) + E\big(\mathbb{I}_{\sigma \wedge \tau < N} J_{\sigma \wedge \tau + 1}^{(\pi,I)}(W_{\sigma \wedge \tau + 1}^{(y,\pi,I,n,j,z')}$$
$$, j-1, \rho_1, ..., \rho_{\sigma \wedge \tau + 1})|\mathcal{F}_n\big).$$

Since $\sigma_1 = \tilde{s}_1 = \tilde{s}(y,\pi,I,n,j)_1$ then from (5.28), (5.31) and Lemma 5.1 it follows that for any $b \in \mathcal{S}_n^{(j)}$,

$$R(y,\pi,I,n,j,\tilde{s}(y,\pi,I,n,j),b) \le E(D_{\sigma_1}\mathbb{I}_{\sigma_1<\tau_1} + A_{\tau_1}\mathbb{I}_{\tau_1\le\sigma_1}|\mathcal{F}_n) \le J_n^{(\pi,I)}(y,j,\rho_1,...,\rho_n). \tag{5.32}$$

In a similar way we obtain that for any $s \in \mathcal{S}_n^{(j)}$,

$$R(y,\pi,I,n,j,s,\tilde{b}(y,\pi,I,n,j)) \ge J_n^{(\pi,I)}(y,j,\rho_1,...,\rho_n) \tag{5.33}$$

completing the proof. □

In the final step we use Lemmas 3.6 and 3.7 and Lemmas 5.2 in order to construct an optimal hedge.

Definition 5.3. *Let $D \subset \mathbb{R}$ be an interval of the form $[a,b]$ or $[a,\infty)$, H be a set and $f : D \times H \to \mathbb{R}$ such that $f(\cdot,h)$ is a continuous function which has a minimum on D. Define the function $argmin_f : H \to D$ by $argmin_f(h) = \min\{y \in D | f(y,h) = \min_{z\in K} f(z,h)\}$.*

Lemmas 3.6 and 3.7 enable us to consider the following functions. Define $\tilde{\gamma} : \{0,...,N-1\} \times \{1,...,L\} \times \mathbb{R} \to \Xi$ by

$$\tilde{\gamma}(k,j,y) = \frac{argmin_f(k,j,\rho_1,...,\rho_k)}{S_0\prod_{i=1}^k(1+\rho_i)} \tag{5.34}$$

where $f : K(y) \times \{0,...,N-1\} \times \{1,...,L\} \times \{a,b\}^k \to \mathbb{R}$ is given by

$$f(\alpha,k,j,u_1,...,u_k) = pJ_{k+1}(y+b\alpha, L-j+1, u_1,...,u_k,b) + (1-p)J_{k+1}(y+a\alpha, L-j+1, u_1,...,u_k,a). \tag{5.35}$$

Also define $\tilde{I} : \{0,...,N\} \times \{1,...,L\} \times \mathbb{R} \to \Xi$ by

$$\tilde{I}(N,j,y) = ((\sum_{i=j+1}^{L} Y_i(N) - y)^+, \ \tilde{I}(k,L,y) = (-y)^+. \tag{5.36}$$

Then for $k < N$ and $j < L$,

$$\tilde{I}(k,j,y) = argmin_g(k,j,\rho_1,...,\rho_k) \tag{5.37}$$

where $g : [-y^+,\infty) \times \{0,...,N-1\} \times \{1,...,L-1\} \times \{a,b\}^k \to \mathbb{R}$ is given by

$$g(z,k,j,u_1,...,u_k) = z + \min_{\alpha\in K(y+z)}\big(pJ_{k+1}(y+z+b\alpha, L-j, u_1,...,u_k,b) + (1-p)J_{k+1}(y+z+a\alpha, L-j, u_1,...,u_k,a)\big). \tag{5.38}$$

Clearly $\tilde{I} \in \mathcal{I}$. For any initial capital x consider the portfolio strategy $\tilde{\pi} = (x,\tilde{\gamma})$. Observe that $\tilde{\gamma}$ satisfies (2.7), and so $\tilde{\pi} \in \mathcal{A}(x)$.

Lemma 5.4. *For any $k \le N$ and $(\pi,I) \in \mathcal{A}(x) \times \mathcal{I}$*

$$J_k(y,j,\rho_1,...,\rho_k) = J_k^{(\tilde{\pi},\tilde{I})}(y,j,\rho_1,...,\rho_k) \le J_k^{(\pi,I)}(y,j,\rho_1,...,\rho_k). \tag{5.39}$$

Proof. We will use the backward induction. Fix $\pi = (x,\gamma) \in \mathcal{A}(x)$ and $I \in \mathcal{I}$. For $k = N$ the statement is obvious. Suppose the assertion holds true for $n+1$ and prove it for n. For $j=0$ the statement is clear. Fix $j\geq 1$. From the induction hypothesis and the definition of $\tilde\gamma,\tilde I$ we obtain that

$$
\begin{aligned}
(5.40)\qquad & \inf_{\alpha\in K(y)}\big(pJ_{n+1}(y+b\alpha,j,\rho_1,...,\rho_n,b)\\
&+(1-p)J_{n+1}(y+a\alpha,j,\rho_1,...,\rho_n,a)\big)\\
&=pJ_{n+1}\big(y+\tilde\gamma(n,L-j+1,y)S_0b\textstyle\prod_{i=1}^n(1+\rho_i),j,u_1,...,u_n,b\big)\\
&+(1-p)J_{n+1}\big(y+\tilde\gamma(n,L-j+1,y)S_0a\textstyle\prod_{i=1}^n(1+\rho_i),j,u_1,...,u_n,a\big)\\
&=pJ^{(\tilde\pi,\tilde I)}_{n+1}\big(y+\tilde\gamma(n,L-j+1,y)S_0b\textstyle\prod_{i=1}^n(1+\rho_i),j,\rho_1,...,\rho_n,b\big)\\
&+(1-p)J^{(\tilde\pi,\tilde I)}_{n+1}\big(y+\tilde\gamma(n,L-j+1,y)S_0\textstyle\prod_{i=1}^n(1+\rho_i),j,\rho_1,...,\rho_n,a\big).
\end{aligned}
$$

From the induction hypothesis and the fact that γ satisfies (2.7) it follows that

$$
\begin{aligned}
(5.41)\qquad & \inf_{\alpha\in K(y)}\big(pJ_{n+1}(y+b\alpha,j,\rho_1,...,\rho_n,b)\\
&+(1-p)J_{n+1}(y+a\alpha,j,\rho_1,...,\rho_n,a)\big)\\
&\leq \inf_{\alpha\in K(y)}\big(pJ^{(\pi,I)}_{n+1}(y+b\alpha,j,\rho_1,...,\rho_n,b)\\
&+(1-p)J^{(\pi,I)}_{n+1}(y+a\alpha,j,\rho_1,...,\rho_n,a)\big)\\
&\leq pJ^{(\pi,I)}_{n+1}(y+\gamma(n,L-j+1,y)S_0b\textstyle\prod_{i=1}^n(1+\rho_i),j,\rho_1,...,\rho_n,b)\\
&+(1-p)J^{(\pi,I)}_{n+1}\big(y+\gamma(n,L-j+1,y)S_0a\textstyle\prod_{i=1}^n(1+\rho_i),j,\rho_1,...,\rho_n,a\big).
\end{aligned}
$$

From the induction hypothesis and the definition of $\tilde\gamma,\tilde I$ we obtain

$$
\begin{aligned}
(5.42)\qquad & \inf_{z\geq(g_n^{(L-j+1)}(\rho_1,...,\rho_n)-y)^+}\inf_{\alpha\in K(y+z-g_n^{(L-j+1)}(\rho_1,...,\rho_n))}\\
&\Big(z+pJ_{n+1}\big(y+z-g_n^{(L-j+1)}(\rho_1,...,\rho_n)+b\alpha,j-1,\rho_n,...,\rho_n,b\big)\\
&+(1-p)J_{n+1}\big(y+z-g_n^{(L-j+1)}(\rho_1,...,\rho_n)+a\alpha,j-1,\rho_1,...,\rho_n,a\big)\Big)\\
&=\tilde I\big(n,L-j+1,y-g_n^{(L-j+1)}(\rho_1,...,\rho_n)\big)\\
&+pJ_{n+1}\big(\tilde U^{(\tilde\pi,\tilde I)}(y,n,j,\rho_1,...,\rho_n,b),j-1,\rho_1,...,\rho_n,b\big)\\
&+(1-p)J_{n+1}\big(\tilde U^{(\tilde\pi,\tilde I)}(y,n,j,\rho_1,...,\rho_n,a),j-1,\rho_1,...,\rho_n,a\big)\\
&=\tilde I\big(n,L-j+1,y-g_n^{(L-j+1)}(\rho_1,...,\rho_n)\big)+\\
&pJ^{(\tilde\pi,\tilde I)}_{n+1}\big(\tilde U^{(\tilde\pi,\tilde I)}(y,n,j,\rho_1,...,\rho_n,b),j-1,\rho_1,...,\rho_n,b\big)\\
&+(1-p)J^{(\tilde\pi,\tilde I)}_{n+1}\big(\tilde U^{(\tilde\pi,\tilde I)}(y,n,j,\rho_1,...,\rho_n,a),j-1,\rho_1,...,\rho_n,a\big).
\end{aligned}
$$

Using that γ satisfies (2.7) and $I(\cdot,\cdot,u) \geq (-u)^+$ it follows by the induction hypothesis that

$$\begin{aligned}
(5.43)\qquad & \inf_{z\geq (g_n^{(L-j+1)}(\rho_1,...,\rho_n)-y)^+} \inf_{\alpha\in K(y+z-g_n^{(L-j+1)}(\rho_1,...,\rho_n))} \\
& \Big(z + pJ_{n+1}\big(y+z-g_n^{(L-j+1)}(\rho_1,...,\rho_n)+b\alpha, j-1, \rho_n,...,\rho_n, b\big) \\
& +(1-p)J_{n+1}\big(y+z-g_n^{(L-j+1)}(\rho_1,...,\rho_n)+a\alpha, j-1, \rho_1,...,\rho_n, a\big)\Big) \\
& \leq I\big(n, L-j+1, y-g_n^{(L-j+1)}(\rho_1,...,\rho_n)\big) \\
& +pJ_{n+1}\big(\tilde{U}^{(\pi,I)}(y,n,j,\rho_1,...,\rho_n,b), j-1, \rho_1,...,\rho_n, b\big) \\
& +(1-p)J_{n+1}\big(\tilde{U}^{(\pi,I)}(y,n,j,\rho_1,...,\rho_n,a), j-1, \rho_1,...,\rho_n, a\big) \\
& \leq I\big(n, L-j+1, y-g_n^{(L-j+1)}(\rho_1,...,\rho_n)\big) \\
& +pJ_{n+1}^{(\pi,I)}\big(\tilde{U}^{(\pi,I)}(y,n,j,\rho_1,...,\rho_n,b), j-1, \rho_1,...,\rho_n, b\big) \\
& +(1-p)J_{n+1}^{(\pi,I)}\big(\tilde{U}^{(\pi,I)}(y,n,j,\rho_1,...,\rho_n,a), j-1, \rho_1,...,\rho_n, a\big).
\end{aligned}$$

In a similar way we obtain

$$\begin{aligned}
(5.44)\qquad & \inf_{z\geq (f_n^{(L-j+1)}(\rho_1,...,\rho_n)-y)^+} \inf_{\alpha\in K(y+z-f_n^{(L-j+1)}(\rho_1,...,\rho_n))} \\
& \Big(z + pJ_{n+1}\big(y+z-f_n^{(L-j+1)}(\rho_1,...,\rho_n)+b\alpha, j-1, \rho_n,...,\rho_n, b\big) \\
& +(1-p)J_{n+1}\big(y+z-f_n^{(L-j+1)}(\rho_1,...,\rho_n)+a\alpha, j-1, \rho_1,...,\rho_n, a\big)\Big) \\
& = \tilde{I}\big(n, L-j+1, y-f_n^{(L-j+1)}(\rho_1,...,\rho_n)\big) \\
& +pJ_{n+1}\big(U^{(\tilde{\pi},\tilde{I})}(y,n,j,\rho_1,...,\rho_n,b), j-1, \rho_1,...,\rho_n, b\big) \\
& +(1-p)J_{n+1}\big(U^{(\tilde{\pi},\tilde{I})}(y,n,j,\rho_1,...,\rho_n,a), j-1, \rho_1,...,\rho_n, a\big) \\
& = \tilde{I}\big(n, L-j+1, y-f_n^{(L-j+1)}(\rho_1,...,\rho_n)\big) \\
& +pJ_{n+1}^{(\tilde{\pi},\tilde{I})}\big(U^{(\tilde{\pi},\tilde{I})}(y,n,j,\rho_1,...,\rho_n,b), j-1, \rho_1,...,\rho_n, b\big) \\
& +(1-p)J_{n+1}^{(\tilde{\pi},\tilde{I})}\big(U^{(\tilde{\pi},\tilde{I})}(y,n,j,\rho_1,...,\rho_n,a), j-1, \rho_1,...,\rho_n, a\big)
\end{aligned}$$

and

$$\inf_{z\geq(f_n^{(L-j+1)}(\rho_1,...,\rho_n)-y)^+}\inf_{\alpha\in K(y+z-f_n^{(L-j+1)}(\rho_1,...,\rho_n))} \tag{5.45}$$
$$\Big(z+pJ_{n+1}\big(y+z-f_n^{(L-j+1)}(\rho_1,...,\rho_n)+b\alpha,j-1,\rho_n,...,\rho_n,b\big)$$
$$+(1-p)J_{n+1}\big(y+z-f_n^{(L-j+1)}(\rho_1,...,\rho_n)+a\alpha,j-1,\rho_1,...,\rho_n,a\big)\Big)$$
$$\leq I\big(n,L-j+1,y-f_n^{(L-j+1)}(\rho_1,...,\rho_n)\big)$$
$$+pJ_{n+1}\big(U^{(\pi,I)}(y,n,j,\rho_1,...,\rho_n,b),j-1,\rho_1,...,\rho_n,b\big)$$
$$+(1-p)J_{n+1}\big(U^{(\pi,I)}(y,n,j,\rho_1,...,\rho_n,a),j-1,\rho_1,...,\rho_n,a\big)$$
$$\leq I\big(n,L-j+1,y-f_n^{(L-j+1)}(\rho_1,...,\rho_n)\big)$$
$$+pJ_{n+1}^{(\pi,I)}\big(U^{(\pi,I)}(y,n,j,\rho_1,...,\rho_n,b),j-1,\rho_1,...,\rho_n,b\big)$$
$$+(1-p)J_{n+1}^{(\pi,I)}\big(U^{(\pi,I)}(y,n,j,\rho_1,...,\rho_n,a),j-1,\rho_1,...,\rho_n,a\big).$$

Now, (5.39) follows from (5.40)–(5.45). □

Finally, fix an initial capital $x\geq 0$ and let $\tilde{\pi}=(x,\tilde{\gamma})$. Set

$$\tilde{s}=\tilde{s}(x,\tilde{\pi},\tilde{I},0,L). \tag{5.46}$$

Using Lemmas 5.2 and 5.4 (for $j=L$ and $n=0$) we obtain that for any $\pi,I,s\in\mathcal{A}(x)\times\mathcal{I}\times S$,

$$R(\tilde{\pi},\tilde{I},\tilde{s})=J_0^{(\tilde{\pi},\tilde{I})}(x,L)=J_0(x,L)\leq J_0^{(\pi,I)}(x,L)=R(\pi,I,s).$$

Thus

$$R(\tilde{\pi},\tilde{I},\tilde{s})=R(x)=J_0^{(\pi,I)}(x,L).$$

completing the proof of Theorem 2.7. □

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