Applied Mathematical Sciences, Vol. 4, 2010, no. 56, 2773 - 2779

Applications of Operational Calculus: Equations for Rectangular Arrays of Four and Five Data

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*Los Alamos National Laboratory is operated by the Los Alamos National Security, LLC for the National Nuclear Security Administration of the U.S. Department of Energy under Contract No. DE-AC52-06NA25396.

Abstract

Selections of equations for four or five data in rectangular array are seldom encountered in textbooks. New methods illustrate polynomial and exponential equations for the two designs. The methods are based on operational calculus and they are easy to apply.

Mathematics Subject Classifications: 65D07, 65D17

Keywords: Interpolation, shifting operator, rectangular data array, operational equations

1. Introduction

The bilinear equation has traditionally been used to represent four numbers in a rectangular array as defined by the vertices of prism face ABDC in Fig. 1. As the name implies, the equation is exact on bilinear numbers. It does not estimate curvature coefficients. An alternative four-point equation is exact on bilinear numbers and their squares [1,2]. Equations for the five-point data array, where the fifth datum is at the center of the design, are seldom elaborated in textbooks. This paper introduces methods that yield new polynomial and exponential interpolating equations for the five-point design. The polynomial types are exact on bilinear numbers as well as on their second and third powers. However, the exponential types appear to be more practical. The equations apply in the –1 .. 1 coordinate system.

2. Cubic equations for the four or five-point rectangle

An operational, cubic equation for the eight-point rectangular prism (also denoted cube) was recently introduced [3]. It is exact on the first, second, third powers of trilinear numbers in prismatic array as in Fig 1. The construction of the equation, and its applications to trial data, are summarized in Ref. [3]. For the sake of brevity, use Eqs. (1)-(3) below instead of Eqs. (6)-(8) in Ref. [3].

nxc = xc – y2x – z2x – x3 (1)
nyc = yc – x2y – z2y – y3 (2)
nzc = zc – x2z – y2z – z3 (3)

Substitute z=(–1) into the operational interpolating equation for the eight-point cube as constructed using Eqs. (1)-(3) and as illustrated in Ref. [3]. The result is a new interpolating equation for the four-point rectangle ABDC in Fig. 1. Call the new equation P4. The notation indicates a polynomial equation for four data in rectangular array. To demonstrate its properties, substitute A=u(1), B=u(3), C=u(7), D=u(9), F=u(1+T), G=u(3+T), H=u(7+T), I=u(9+T) into P4. In these expressions, T represents a constant and u(x) is an operator on its argument x. If u(x) is the linear operator then B and G remain 3 and 3+T, respectively, and likewise for the remaining letters. P4 now reduces to P4=(5+x+3y). If u(x) is the squaring operator then B=9 and $G=(3+T)^2$ and P4 reduces to $P4=(5+x+3y)^2$. If u(x) raises its argument to the third power then B=27 and $G=(3+T)^3$, and P4 reduces to $P4=(5+x+3y)^3$.

In the three cited cases, T disappears on simplifying substituted P4. In other cases, T typically remains in P4. For example, T does not disappear from P4 when u(x) changes the data into their fourth powers as in $(G+T)^4$. When T does not disappear, P4 can be used to represent a five-point rectangle. The equation P4 is now renamed P5(T) to indicate its applicability to the five-point design and its dependence on T. At the center point of the design, x=0 and y=0. If the center point number is u(5)=625, the true value in the present case, the value of T is determined by Eq. (4).

P5(T) – 625 = 0 (4)

Eq. (4) is satisfied by T~(–24.563) and T~9.563. These numbers can be substituted into P5(T) yielding two third-degree polynomial models of the five-point rectangle with vertex data $[A,B,C,D]=[1^4,3^4,7^4,9^4]$ and the introduced datum 5^4 at the center point of face ABDC in Fig. 1. The two interpolating equations, with rounded coefficients, are Eqs. (5) and (6). R represents a "response" or an interpolated number.

$$R = -59.04x^3 - 506.2x^2y - 1447xy^2 - 1378y^3 + 178.5x^2 + 1020xy + 1458y^2 + 2566x + 4104y + 625 \quad (5)$$

$$R = 46.65x^3 + 400x^2y + 1143xy^2 + 1089y^3 + 178.5x^2 + 1020xy + 1458y^2 - 129.7x + 731.1y + 625 \quad (6)$$

The illustration has been a success with respect to the limited objective of rendering one or more polynomial equations for interpolating a five-point array. Both equations reproduce the original data. They also estimate the number assigned to the center point but that is not critical unless the assigned number is a measurement. That is, the fifth point can be chosen so as to change the properties of the interpolating surface that passes through four measurements in a rectangular array. The locus and the number at the arbitrary point can be changed if Eq. (4) still renders real roots for T, and provided the resulting surface passes through the four measurements in the rectangular array.

In a larger sense, however, the method is a disappointment because of its limitations. The interpolating equations generated by this method can predict unjustified extrema within the experimental space. Extrema predicted by equations rendered by the polynomial method seldom reflect true properties of an experimental space defined by only four data in a rectangular array. There may be no easy method for choosing among candidate equations without ancillary information. More laboratory work may solve this problem but it means increased costs. The method has limited applications. Not all data in the four-point array can be treated by the illustrated approach.

The described method renders a better performance if the experimental data are the numbers $A=1^{(3/2)}$, $B=3^{(3/2)}$, $C=7^{(3/2)}$, $D=9^{(3/2)}$. In this case, the center point number is $5^{(3/2)}$ so cubic Eq. (7) applies to the five-point rectangle. The sum of the squares of the deviations of Eq. (7) from the true surface is about 0.0225.

$$R = -0.0023034x^3 - 0.020205x^2y - 0.059079xy^2 - 0.057582y^3 + 0.18312x^2 + 1.0709xy + 1.5656y^2 + 3.2304x + 9.9088y + 11.180 \quad (7)$$

3. Exponential equations for the four- or five-point rectangle

The preceding polynomial approach generated models with extrema on one or more edges of the experimental space. Exponential representations seldom have that disadvantage. Exponential equations for the 8-point cube can be generated by means of Eqs. (14)-(26) in Ref. [4]. The procedure for illustrating four- or five-point interpolating equations is similar to the method described in Section 2. Attention is focused on rectangle ABDC by setting z=(–1) in Eq. (14) of Ref. [4]. The trial data are w(1), w(3), w(7), w(9) at vertices A .. D and w(1+TT), w(3+TT), w(7+TT), w(9+TT) at vertices F .. I of the eight-point cube, respectively. (The notation is changed to avoid confusion with the symbols in Eqs. (14)-(26) in Ref. [4].) These numbers and z=(–1) are substituted into Eq. (14) in Ref. [4]. The new equation is denoted E5(TT) to indicate that it is an exponential equation for the five-point rectangle and that it is a function of TT.

The first acceptance criterion for an exponential equation for the four-point, rectangular space is the same as before: the equation should reproduce the data at the vertices of the rectangle and it should estimate the number assigned to the center point.

In the first polynomial example above, the trial data were the fourth powers of monotonic numbers at the vertices of face ABDC of the cube. Use the same approach in the exponential method: assume $w(5)=5^4=625$ at the center point of the four-point array. The center point coordinates are (x,y)=(0,0) so substitute these numbers into Eq. (8). Now solve substituted Eq. (8) for TT, if that is possible. In some cases, Eq. (8) has no real root for TT. In that event, the center point assignment can be changed or the method can be abandoned. In other cases, more than one root for TT may exist. In such cases, an extra measurement, or other information, may be necessary to decide which candidate equation is preferred.

$$E5(TT) - 625 = 0 \tag{8}$$

In the example using the fourth powers of integers, the numerical values of TT are about 1.079 and –5.872. Substitute one of these numbers into E5(TT). Then insert the other one. Two interpolating equations for the five-point rectangle derive from this process. They are Eqs. (9) and (10), respectively. Their coefficients have been rounded. Neither equation predicts extrema on the boundaries of the rectangle. Both equations reproduce the four corner-point data. At (x,y)=(0,0) the estimate rendered by both equations is about 625. In the present case, the exponential method is preferred. It is easy to use and it has rendered two simple interpolating equations that satisfy the requirements of the problem.

$$R4 = -42.59 - 32.46(1.587)^{(x+1)} - 9.107(5.706)^{(y+1)} + 85.16(1.587)^{(x+1)}(5.706)^{(y+1)} \quad (9)$$

$$R4 = -101.0 - 14.43(3.029)^{(x+1)} + 92.22(4.649)^{(y+1)} + 24.22(3.029)^{(x+1)}(4.649)^{(y+1)} \quad (10)$$

Experimenters may prefer the exponential approach because it can render extrema-free surfaces by which to model laboratory data. In some cases, however, the perceived versatility of the exponential method is an artifact that is not to be trusted. An example helps to make this clear. Laboratory measurements are inevitably contaminated with errors. If experimental data appear to follow a strict fourth-power (40000/10000) law, they typically appear to follow laws with other powers such as (40001/10000) and (39999/10000). That remark is true as far as most laboratory results permit it to be verified. The experimenter may not know the difference between the exponents but the exponential method may know the difference.

The illustrated approach rendered Eqs. (9) and (10) when the trial data followed a precise fourth power law. One of them, Eq. (9), was based on the approximation TT~1.07881369. Now let the trial data be the trial integers raised to the power (39999)/(10000). That is a non-integer exponent, 3.9999. The described approach now yields TT~1.07879842, nearly. Let the exponent of the trial data be changed to another non-integer power, (40001)/(10000), or 4.0001. The same approach now yields TT~1.07882896, nearly. In both of the non-integer cases, the values of TT have changed modestly but the other root near TT~(–5.872) has disappeared. As far as we know, that value of TT is an artifact of an attempt to model error-ridden data by a precise, integer-power law. Laboratory data seldom adhere to equations with precise integer exponents. A test based on computational results has eliminated Eq. (10). In this example, the test suggests Eq. (9) is the preferred representation of the data.

Four or five laboratory data in a rectangular array sometimes generate several exponential interpolating equations. In these cases, a robustness test may be useful. Let typical errors be introduced into the data and choose the equation that is most resistant to the changes. That work is an inconvenience. It is part of the price of generating interpolating equations that do not depend on straight lines but generally do not introduce adverse effects such as spurious boundary extrema, and also maintain the flexibility that an adjustable center point provides. The final test is the usual one: does the modeling equation render predictions that correspond satisfactorily to laboratory results? That equation may not be the most robust one but the test has helped to illuminate its weaknesses.

The preceding sections have illustrated operational methods for four or five data in a rectangular array. Other operational methods for four- and five-point rectangular arrays are illustrated in Refs. [1,2,5,6,7]. For the most part, they are easy to apply. The shifting operator offers a variety of methods for four- or five-point rectangles just as it offers a variety of methods for eight- and nine-point rectangular prisms. The bilinear and trilinear equations have the advantage of tradition but they represent only one of several possible approaches. Five-point rectangular arrays can be advantageous for modeling purposes if economics permit the extra experiment. The literature citations illustrate applications of the shifting operator to geometry.

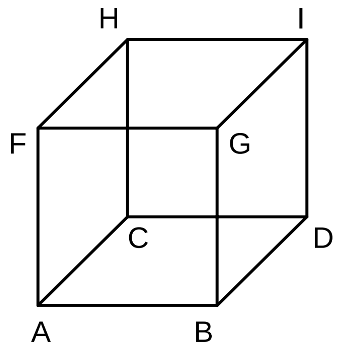

Fig. 1. The eight-point rectangular prism

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Received: March, 2010