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Blow-up for a Degenerate and Singular Parabolic System with Nonlocal Sources and Absorptions

Lixin Shi[a1], **Chunlai Mu**[b] **and Furong Li**[a]

a.Department of Mathematics, Sichuan Agriculture University
Yaan Sichuan 625014. P. R. China

b.Department of Mathematics Sichuan University
Chengdu Sichuan 610064. P. R. China

Abstract

This paper deals with the blow-up properties of the solution to the degenerate and singular parabolic system with nonlocal sources, absorptions and homogeneous Dirichlet boundary conditions. The existence of a unique classical nonnegative solution is established and the sufficient conditions for the solution to exist globally or blow up in finite time are obtained. Furthermore, under certain conditions it is proved that the blow-up set of the solution is the whole domain.

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1 Introduction

In this paper, we consider the following degenerate and singular nonlinear reaction-diffusion system with nonlocal sources and absorptions

$$\begin{cases} x^{m_1}u_t - (x^{r_1}u_x)_x = \int_0^a v^{p_1}dx - k_1u^{q_1}, & (x,t)\in(0,a)\times(0,T),\\ x^{m_2}v_t - (x^{r_2}v_x)_x = \int_0^a u^{p_2}dx - k_2v^{q_2}, & (x,t)\in(0,a)\times(0,T),\\ u(0,t)=u(a,t)=v(0,t)=v(a,t)=0, & t\in(0,T),\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x), & x\in[0,a]. \end{cases} \tag{1.1}$$

[1]shilixin19790501@126.com

Where $u_0(x),\ v_0(x) \in C^{2+\alpha}(\overline{D})$ for some $\alpha \in (0,1)$ are nonnegative nontrivial functions. $u_0(0) = u_0(a) = v_0(0) = v_0(a) = 0,\ u_0(x) \geq 0,\ v_0(x) \geq 0$, u_0, v_0 satisfy the compatibility condition, $k_1 > 0,\ k_2 > 0,\ T > 0,\ a > 0,\ r_1, r_2 \in [0,1),\ |m_1|+r_1 \neq 0,\ |m_2| + r_2 \neq 0$ and $p_1 > 1,\ p_2 > 1,\ q_1 > 1,\ q_2 > 1$.

Set $D = (0,a)$ and $\Omega_t = D \times (0,t]$, $\overline{D}$ and $\overline{\Omega}_t$ are their closures respectively. Since $|m_1| + r_1 \neq 0,\ |m_2| + r_2 \neq 0$, the coefficients of $u_t,\ u_x,\ u_{xx}$ and $v_t,\ v_x,\ v_{xx}$ may tend to 0 or ∞ as x tends to 0, we can regard the equations as degenerate and singular.

Floater [7] and Chan et al. [2] investigated the blow-up properties of the following degenerate parabolic problem

$$\begin{cases} x^q u_t - u_{xx} = u^p, & (x,t) \in (0,a) \times (0,T), \\ u(0,t) = u(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,a]. \end{cases} \tag{1.2}$$

Where $q > 0$ and $p > 1$. Under certain conditions on the initial datum $u_0(x)$, Floater [7] proved that the solution $u(x,t)$ of (1.2) blows up at the boundary $x = 0$ for the case $1 < p \leq q + 1$. This contrasts with one of the results in [8], which showed that for the case $q = 0$, the blow-up set of the solution $u(x,t)$ of (1.2) is a proper compact subset of D.

In [4], Chen et al. studied the following degenerate nonlinear reaction-diffusion problem with a nonlocal source

$$\begin{cases} x^q u_t - (x^\gamma u_x)_x = \int_0^a u^p dx, & (x,t) \in (0,a) \times (0,T), \\ u(0,t) = u(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,a]. \end{cases} \tag{1.3}$$

They established the local existence and uniqueness of a classical solution. Under appropriate hypotheses, they also got some sufficient conditions for the global existence and blow-up of a positive solution. Furthermore, under certain conditions, it is proved that the blow-up set of the solution is the whole domain.

In [5], Chen discussed the following degenerate and singular semilinear parabolic problem

$$\begin{cases} u_t - (x^\alpha u_x)_x = \int_0^a f(u(x,t)) dx, & (x,t) \in (0,a) \times (0,T), \\ u(0,t) = u(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,a]. \end{cases} \tag{1.4}$$

They established the local existence and uniqueness of a classical solution. Under appropriate hypotheses, they obtained some sufficient conditions for the global existence and the blow-up of a positive solution.

In [12], Zhou et al. discussed the following degenerate parabolic problem with nonlocal sources

$$\begin{cases} x^{m_1}u_t - (x^{r_1}u_x)_x = \int_0^a v^{p_1}dx, & (x,t) \in (0,a) \times (0,T), \\ x^{m_2}v_t - (x^{r_2}v_x)_x = \int_0^a u^{p_2}dx, & (x,t) \in (0,a) \times (0,T), \\ u(0,t) = u(a,t) = v(0,t) = v(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in [0,a]. \end{cases} \tag{1.6}$$

The existence of a unique classical nonnegative solution is established and the sufficient conditions for the solution to exist globally or blow up in finite time are obtained. Furthermore, under certain conditions, they proved that the blow-up set of the solution is the whole domain.

In [9], Liu et al. considered the following degenerate parabolic problem with a nonlocal source and an absorption

$$\begin{cases} u_t - x^{\gamma}(u^m)_{xx} = \int_0^a u^p dx - ku^q, & (x,t) \in (0,a) \times (0,\infty), \\ u(0,t) = u(a,t) = 0, & t \in (0,\infty), \\ u(x,0) = u_0(x), & x \in [0,a]. \end{cases} \tag{1.7}$$

They proved that the positive solution of (1.7) blow up in finite time if u_0 is large enough. Furthermore, under certain conditions, they proved that the blow-up set of the solution is the whole domain.

we obtain our main results as follows.

Theorem 1.1 *There exists some t_0 ($< T$) such that problem (1.1) has a unique nonnegative solution $(u(x,t), v(x,t)) \in (C(\overline{\Omega}_{t_0}) \cap C^{2,1}(\Omega_{t_0}))^2$.*

Theorem 1.2 *Let T be the supreme over to for which there is a unique nonnegative solution $(u(x,t), v(x,t)) \in (C(\overline{\Omega}_{t_0}) \cap C^{2,1}(\Omega_{t_0}))^2$ of (1.1). Then (1.1) has a unique nonnegative solution $(u(x,t), v(x,t)) \in (C([0,a] \times [0,T)) \cap C^{2,1}((0,a) \times (0,T)))^2$. If $T < +\infty$, then*

$$\limsup_{t \to T} \max_{x \in [0,a]} (|u(x,t)| + |v(x,t)|) = +\infty.$$

In order to get the blow up results,we would assume $m_1 > r_1 - 1,\ m_2 > r_2 - 1$.

First, the solution of the following boundary value problem

$$-(x^{r_1}\psi'(x))' = 1,\ x \in (0,a); \quad \psi(0) = \psi(a) = 0$$

is given by

$$\psi(x) = \frac{a^{2-r_1}}{2-r_1}(\frac{x}{a})^{1-r_1}(1-\frac{x}{a}) \quad (a).$$

Analogously, the solution of the following boundary value problem

$$-(x^{r_2}\varphi'(x))' = 1,\ x \in (0,a); \quad \varphi(0) = \varphi(a) = 0$$

is given by

$$\varphi(x) = \frac{a^{2-r_2}}{2-r_2}(\frac{x}{a})^{1-r_2}(1-\frac{x}{a}) \quad (b).$$

Let

$$\begin{cases} a_1 = a_2^{p_1}[a^{(2-r_2)p_1+1}B(p_1(1-r_2)+1, p_1+1)]/(2-r_2)^{p_1}, \\ a_2 = a_1^{p_2}[a^{(2-r_1)p_2+1}B(p_2(1-r_1)+1, p_2+1)]/(2-r_1)^{p_2}. \end{cases}$$

Where $B(l,m)$ is a Beta function defined by $B(l,m) = \int_0^1 x^{l-1}(1-x)^{m-1}dx$. Then we have the following result.

Theorem 1.3 *Let $(u(x,t), v(x,t))$ be the solution of (1.1). If $u_0(x) \le a_1\psi(x)$, $v_0(x) \le a_2\varphi(x)$, where $\psi(x), \varphi(x)$ are defined as $((a),(b))$, then $(u(x,t), v(x,t))$ exists globally.*

Theorem 1.4 *Let $(u(x,t), v(x,t))$ be the solution of (1.1), $\psi(x), \varphi(x)$ are defined as $((a),(b))$.*
(i) If $p_1p_2 > q_1q_2$, then the nonnegative solutions of (1.1) blow up in finite time for large initial data .
(ii) If $p_1p_2 = q_1q_2$, $\int_0^a \varphi^{p_1}(x)dx \ge k_1\psi^{q_1}(x)$ and $\int_0^a \psi^{p_2}(x)dx \ge k_2\varphi^{q_2}(x)$, then the solutions blow up in finite time for large initial data.

This paper is organized as follows: in the next section, we show the existence of a unique classical solution. In section 3, we give some criteria for the solution $(u(x,t), v(x,t))$ to exist globally or blow up in finite time, and lastly, we discuss the blow-up set.

2 Local existence

In order to prove the existence of a unique positive solution to (1.1), we start with the following comparison principle.

Lemma 2.1 *Let $b_1(x,t)$, $b_2(x,t)$, $c_1(t,x)$, $c_2(t,x)$ are continuous functions defined on $[0,a]\times[0,r]$ for any $r \in (0,T)$. $b_1(x,t)$, $b_2(x,t)$ are nonnegative, and $(u(x,t), v(x,t)) \in$*

$(C(\overline{\Omega}_r) \cap C^{2,1}(\Omega_r))^2$ *satisfy*

$$\begin{cases} x^{m_1}u_t - (x^{r_1}u_x)_x \geq \int_0^a b_1(x,t)v(x,t)dx + c_1 u, & (x,t) \in (0,a) \times (0,r], \\ x^{m_2}v_t - (x^{r_2}v_x)_x \geq \int_0^a b_2(x,t)u(x,t)dx + c_2 v, & (x,t) \in (0,a) \times (0,r], \\ u(0,t) \geq 0,\ u(a,t) \geq 0,\ v(0,t) \geq 0,\ v(a,t) \geq 0, & t \in (0,r], \\ u(x,0) \geq 0,\ v(x,0) \geq 0, & x \in [0,a]. \end{cases} \tag{2.1}$$

Then, $u(x,t) \geq 0,\ v(x,t) \geq 0$ *on* $[0,a] \times [0,T)$.

Proof. At first, similar to the Lemma 2.1 in [11], by using Lemma 2.2.1 in [10], we can easily obtain the following conclusion:
If $W(x,t)$ and $Z(x,t) \in C(\overline{\Omega}_r) \cap C^{2,1}(\Omega_r)$ satisfy

$$\begin{cases} x^{m_1}W_t - (x^{r_1}W_x)_x \geq \int_0^a b_1(x,t)Z(x,t)dx + c_1 W, & (x,t) \in (0,a) \times (0,r], \\ x^{m_2}Z_t - (x^{r_2}Z_x)_x \geq \int_0^a b_2(x,t)W(x,t)dx + c_2 Z, & (x,t) \in (0,a) \times (0,r], \\ W(0,t) > 0,\ W(a,t) \geq 0,\ Z(0,t) > 0,\ Z(a,t) \geq 0, & t \in (0,r], \\ W(x,0) \geq 0,\ Z(x,0) \geq 0, & x \in [0,a]. \end{cases} \tag{2.2}$$

Then, $W(x,t) > 0,\ Z(x,t) > 0,\ (x,t) \in (0,a) \times (0,r]$.

Next let $r_1' \in (r_1, 1),\ r_2' \in (r_2, 1)$ are positive constants and

$$W(x,t) = u(x,t) + \eta(1 + x^{r_1' - r_1})e^{ct},\ Z(x,t) = v(x,t) + \eta(1 + x^{r_2' - r_2})e^{ct},$$

where $\eta > 0$ is sufficiently small and c is a positive constant to be determined. Then $W(x,t) > 0$, $Z(x,t) > 0$ on the parabolic boundary of Ω_r, and in $(0,a) \times (0,r]$, we have

$$\begin{aligned} & x^{m_1}W_t - (x^{r_1}W_x)_x - \int_0^a b_1(x,t)Z(x,t)dx - c_1 W \\ & \geq \eta e^{ct}[cx^{m_1} + (r_1' - r_1)(1 - r_1')/x^{2-r_1'} - (aM_1 + C_1)(1 + a^{\widetilde{r'} - \widetilde{r}})], \\ & x^{m_2}Z_t - (x^{r_2}Z_x)_x - \int_0^a b_2(x,t)W(x,t)dx - c_2 Z \\ & \geq \eta e^{ct}[cx^{m_2} + (r_2' - r_2)(1 - r_2')/x^{2-r_2'} - (aM_2 + C_2)(1 + a^{\widetilde{r'} - \widetilde{r}})]. \end{aligned}$$

$$M_1 = \max_{(x,t)\in[0,a]\times[0,r]} b_1(x,t), M_2 = \max_{(x,t)\in[0,a]\times[0,r]} b_2(x,t),$$

$$C_1 = \max_{(x,t)\in[0,a]\times[0,r]} c_1(x,t), C_2 = \max_{(x,t)\in[0,a]\times[0,r]} c_2(x,t).$$

$$1 + a^{\widetilde{r'} - \widetilde{r}} = \max(1 + a^{r_1' - r_1}, 1 + a^{r_2' - r_2})$$

We will prove the above inequalities are nonnegative in three cases.
Case 1. When

$$(aM_1 + C_1)(1 + a^{\widetilde{r'} - \widetilde{r}}) \leq (r_1' - r_1)(1 - r_1')/a^{2-r_1'},$$

and

$$(aM_2 + C_2)(1 + a^{\widetilde{r'} - \widetilde{r}}) \le (r_2' - r_2)(1 - r_2')/a^{2 - r_2'}$$

It is obvious that

$$x^{m_1} W_t - (x^{r_1} W_x)_x - \int_0^a b_1(x,t) Z(x,t) dx - c_1 W \ge 0,$$

and

$$x^{m_2} Z_t - (x^{r_2} Z_x)_x - \int_0^a b_2(x,t) W(x,t) dx - c_2 Z \ge 0.$$

Case 2. If

$$(aM_1 + C_1)(1 + a^{\widetilde{r'} - \widetilde{r}}) > (r_1' - r_1)(1 - r_1')/a^{2 - r_1'},$$

and

$$(aM_2 + C_2)(1 + a^{\widetilde{r'} - \widetilde{r}}) > (r_2' - r_2)(1 - r_2')/a^{2 - r_2'}$$

Let x_0 and y_0 be the root of the algebraic equations

$$\begin{cases} (aM_1 + C_1)(1 + a^{\widetilde{r'} - \widetilde{r}}) = (r_1' - r_1)(1 - r_1')/x^{2 - r_1'}, \\ (aM_2 + C_2)(1 + a^{\widetilde{r'} - \widetilde{r}}) = (r_2' - r_2)(1 - r_2')/y^{2 - r_2'}. \end{cases}$$

and $D_1,\ D_2 > 0$ are sufficient large such that

$$D_1 > \begin{cases} (aM_1 + C_1)(1 + a^{\widetilde{r'} - \widetilde{r}})/x_0^{m_1}, & for\ m_1 \ge 0, \\ (aM_1 + C_1)(1 + a^{\widetilde{r'} - \widetilde{r}})/a^{m_1}, & for\ m_1 < 0. \end{cases}$$

$$D_2 > \begin{cases} (aM_2 + C_2)(1 + a^{\widetilde{r'} - \widetilde{r}})/y_0^{m_2}, & for\ m_2 \ge 0, \\ (aM_2 + C_2)(1 + a^{\widetilde{r'} - \widetilde{r}})/a^{m_2}, & for\ m_2 < 0. \end{cases}$$

Set $c = \max\{D_1, D_2\}$, then we have

$$\begin{aligned} & x^{m_1} W_t - (x^{r_1} W_x)_x - \textstyle\int_0^a b_1(x,t) Z(x,t) dx - c_1 W \\ & \ge \begin{cases} \eta e^{ct}[(r_1' - r_1)(1 - r_1')/x^{2 - r_1'} - (aM_1 + C_1)(1 + a^{\widetilde{r'} - \widetilde{r}})], & for\ x \le x_0, \\ \eta e^{ct}[c x^{m_1} - (aM_1 + C_1)(1 + a^{\widetilde{r'} - \widetilde{r}})], & for\ x > x_0, \end{cases} \\ & \ge 0. \end{aligned}$$

$$\begin{aligned} & x^{m_2} Z_t - (x^{r_2} Z_x)_x - \textstyle\int_0^a b_2(x,t) W(x,t) dx - c_2 Z \\ & \ge \begin{cases} \eta e^{ct}[(r_2' - r_2)(1 - r_2')/x^{2 - r_2'} - (aM_2 + C_2)(1 + a^{\widetilde{r'} - \widetilde{r}})], & for\ x \le y_0, \\ \eta e^{ct}[c x^{m_2} - (aM_2 + C_2)(1 + a^{\widetilde{r'} - \widetilde{r}})], & for\ x > y_0, \end{cases} \\ & \ge 0. \end{aligned}$$

Case 3. When

$$(aM_1+C_1)(1+a^{\widetilde{r'}-\widetilde{r}})\le(r_1'-r_1)(1-r_1')/a^{2-r_1'},$$

and

$$(aM_2+C_2)(1+a^{\widetilde{r'}-\widetilde{r}})>(r_2'-r_2)(1-r_2')/a^{2-r_2'},$$

or,

$$(aM_1+C_1)(1+a^{\widetilde{r'}-\widetilde{r}})>(r_1'-r_1)(1-r_1')/a^{2-r_1'},$$

and

$$(aM_2+C_2)(1+a^{\widetilde{r'}-\widetilde{r}})\le(r_2'-r_2)(1-r_2')/a^{2-r_2'},$$

Combining Case 1 with Case 2, it is easy to prove

$$x^{m_1}W_t-(x^{r_1}W_x)_x-\int_0^a b_1(x,t)Z(x,t)dx-C_1W\ge 0,$$

and

$$x^{m_2}Z_t-(x^{r_2}Z_x)_x-\int_0^a b_2(x,t)W(x,t)dx-c_2Z\ge 0,$$

so we omit the proof here.
From the above three cases, we know that $W(x,t)>0,\ Z(x,t)>0$ on $[0,a]\times[0,r]$. Letting $\eta\to 0^+$, we have $u(x,t)\ge 0,\ v(x,t)\ge 0$ on $[0,a]\times[0,r]$. By the arbitrariness of $r\in(0,T)$, we complete the proof of Lemma 2.1. □

Obviously, $(\underline{u},\underline{v})=(0,0)$ is a subsolution of (1.1), we need to construct a supersolution.

Lemma 2.2 *There exists a positive constant* t_0 ($t_0<T$) *such that the problem (1.1) has a supersolution* $(h_1(x,t),h_2(x,t))\in(C(\overline{\Omega}_{t_0})\cap C^{2,1}(\Omega_{t_0}))^2$.

Proof. The proof is similar to the Lemma 2.2 in [4, 12].□

To show the existence of the classical solution $(u(x,t),v(x,t))$ of (1.1), let us introduce a cut-off function $\rho(x)$. By Dunford and Schwartz [6] [pp. 1640], there exists a nondecreasing $\rho(x)\in C^3(R)$ such that $\rho(x)=0$ if $x\le 0$ and $\rho(x)=1$ if $x\ge 1$. Let $0<\delta<\min\{\frac{1-r_1}{2-r_1}a,\frac{1-r_2}{2-r_2}a\}$,

$$\rho_\delta(x)=\begin{cases}0, & x\le\delta,\\ \rho(\frac{x}{\delta}-1), & \delta<x<2\delta,\\ 1, & x\ge 2\delta,\end{cases}$$

and $u_{0\delta}(x) = \rho_\delta(x)u_0(x),\ v_{0\delta}(x) = \rho_\delta(x)v_0(x)$. We note that

$$\frac{\partial u_{0\delta}(x)}{\partial\delta} = \begin{cases} 0, & x \le \delta, \\ -\frac{x}{\delta^2}\rho'(\frac{x}{\delta}-1)u_0(x), & \delta < x < 2\delta, \\ 0, & x \ge 2\delta. \end{cases}$$

$$\frac{\partial v_{0\delta}(x)}{\partial\delta} = \begin{cases} 0, & x \le \delta, \\ -\frac{x}{\delta^2}\rho'(\frac{x}{\delta}-1)v_0(x), & \delta < x < 2\delta, \\ 0, & x \ge 2\delta. \end{cases}$$

Since ρ is nondecreasing, we have $\frac{\partial u_{0\delta}(x)}{\partial\delta} \le 0,\ \frac{\partial v_{0\delta}(x)}{\partial\delta} \le 0$. From $0 \le \rho(x) \le 1$, we have $u_0(x) \ge u_{0\delta}(x),\ v_0(x) \ge v_{0\delta}(x)$ and $\lim_{\delta\to 0} u_{0\delta}(x) = u_0(x),\ \lim_{\delta\to 0} v_{0\delta}(x) = v_0(x)$.

Let $D_\delta = (\delta, a),\ w_\delta = D_\delta \times (0, t_0]$, $\overline{D}_\delta$ and $\overline{w}_\delta$ be their respective closures, and $S_\delta = \{\delta,\ a\} \times (0, t_0]$. We consider the following regularized problem

$$\begin{cases} x^{m_1}u_{\delta t} - (x^{r_1}u_{\delta x})_x = \int_\delta^a v_\delta^{p_1}dx - k_1 u_\delta^{q_1}, & (x,t) \in w_\delta, \\ x^{m_2}v_{\delta t} - (x^{r_2}v_{\delta x})_x = \int_\delta^a u_\delta^{p_2}dx - k_2 v_\delta^{q_2}, & (x,t) \in w_\delta, \\ u_\delta(\delta,t) = u_\delta(a,t) = v_\delta(\delta,t) = v_\delta(a,t) = 0, & t \in (0,t_0], \\ u_\delta(x,0) = u_{0\delta}(x),\ v_\delta(x,0) = v_{0\delta}(x), & x \in \overline{D}_\delta. \end{cases} \tag{2.5}$$

By using Schauder's fixed point theorem, we have

Theorem 2.3 *Problem (2.5) admits a unique nonnegative solution* $(u_\delta, v_\delta) \in (C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{w}_\delta))^2$. *Moreover,* $0 \le u_\delta \le h_1(x,t),\ 0 \le v_\delta \le h_2(x,t),\ (x,t) \in \overline{w}_\delta$, *where* $h_1(x,t),\ h_2(x,t)$ *are given by Lemma 2.2.*

Proof. The proof is similar to the Theorem 2.3 in [4, 12].□

Proof theorem 1.1. The proof is similar to the Theorem 2.4 in [4, 12].□

Proof theorem 1.2. By using Lemma 2.1, there exists at most one nonnegative solution of (1.1). Similar to the Theorem 2.5 in Floater [7], we obtain the result.

3 Blow-up of solution

In this section, we give some global existence and blow-up result of the solution of (1.1).

Proof of theorem 1.3. Let $\overline{u} = a_1\psi(x)$, $\overline{v} = a_2\varphi(x)$, then we have

$$
\begin{aligned}
&x^{q_1}\overline{u}_t(x,t) - (x^{r_1}\overline{u}_x(x,t))_x = -(x^{r_1}a_1\psi'(x))' = a_1 \\
&= a_2^{p_1}[a^{(2-r_2)p_1+1}B(p_1(1-r_2)+1, p_1+1)/(2-r_2)^{p_1}] \\
&= \int_0^a (a_2\varphi)^{p_1}dx = \int_0^a \overline{v}^{p_1}(x,t)dx \geq \int_0^a \overline{v}^{p_1}(x,t)dx - k_1\overline{u}^{q_1}, && (x,t) \in (0,a)\times(0,T), \\
&x^{q_2}\overline{v}_t(x,t) - (x^{r_2}\overline{v}_x(x,t))_x = \int_0^a \overline{u}^{p_2}(x,t)dx \geq \int_0^a \overline{u}^{p_2}(x,t)dx - k_2\overline{v}^{q_2}, && (x,t) \in (0,a)\times(0,T), \\
&\overline{u}(0,t) = \overline{u}(a,t) = \overline{v}(0,t) = \overline{v}(a,t) = 0, && t \in (0,T), \\
&\overline{u}(x,0) = a_1\psi(x) \geq u_0(x), \quad \overline{v}(x,0) = a_2\varphi(x) \geq v_0(x), && x \in [0,a].
\end{aligned}
$$

That is to say $(\overline{u}(x,t), \overline{v}(x,t))$=$(a_1\psi(x), a_2\varphi(x))$ is a supersolution of (1.1). By Theorem 2.5, $T = +\infty$, i.e, $(u(x,t), v(x,t))$ exists globally. The proof of Theorem 3.1 is complete. □

Proof of theorem 1.4. (i) Set $(\underline{u}, \underline{v}) = (\frac{\psi(x)}{(T_1-t)^{\alpha}}, \frac{\varphi(x)}{(T_1-t)^{\beta}})$, where $\alpha, \beta, T_1 > 0$ are to be determined, then we have

$$
\begin{cases}
L_1\underline{u} = x^{m_1}\underline{u}_t - (x^{r_1}\underline{u}_x)_x - \int_0^a \underline{v}^{p_1}dx + k_1\underline{u}^{q_1} \\
\qquad = \alpha x^{m_1}\psi(x)(T_1-t)^{-\alpha-1} - (T_1-t)^{-\alpha} - (T_1-t)^{-p_1\beta}\int_0^a \varphi^{p_1}dx + k_1(T_1-t)^{-q_1\alpha}\psi^{q_1}(x) \\
L_2\underline{v} = x^{m_2}\underline{v}_t - (x^{r_2}\underline{v}_x)_x - \int_0^a \underline{u}^{p_2}dx + k_2\underline{v}^{q_2} \\
\qquad = \beta x^{m_2}\varphi(x)(T_1-t)^{-\beta-1} - (T_1-t)^{-\beta} - (T_1-t)^{-p_2\alpha}\int_0^a \psi^{p_2}dx + k_2(T_1-t)^{-q_2\beta}\varphi^{q_2}(x)
\end{cases}
$$

Since $p_1p_2 > q_1q_2$ we choose α, β satisfy

$$\max(p_1 - \frac{1}{\beta}, \frac{p_1}{q_2}) > \frac{\alpha}{\beta} > \max(\frac{q_1}{p_2}, \frac{\beta+1}{p_2\beta}).$$

hence

$$q_1\alpha < \max(\alpha+1, p_1\beta), q_2\beta < \max(\beta+1, p_2\alpha)$$

We have $L_1\underline{u} \leq 0, L_2\underline{v} \leq 0$ in $(0,a)\times(0,T_1)$,with $T_1 - t$ small. If the initial data are large that $u_0(x) \geq \frac{\psi(x)}{T_1^{\alpha}}, v_0(x) \geq \frac{\varphi(x)}{T_1^{\beta}}$, then $(\underline{u}, \underline{v})$ is a blow up sub-solution to (1.1).□

(ii) Let α, β satisfies

$$p_1 - \frac{1}{\beta} \geq \frac{p_1}{q_2} = \frac{\alpha}{\beta} = \frac{q_1}{p_2} \geq \frac{\beta+1}{p_2\beta}.$$

Construct $(\underline{u}, \underline{v}) = (\frac{\psi(x)}{(T_1-t)^{\alpha}}, \frac{\varphi(x)}{(T_1-t)^{\beta}})$ with $\int_0^a \varphi^{p_1}(x)dx \geq k_1\psi^{q_1}(x), \int_0^a \psi^{p_2}(x)dx \geq k_2\varphi^{q_2}(x)$. By taking T_1 properly small, $(\underline{u}, \underline{v})$ is the sub-solution of (1.1).□

Remark: Since the system (1.1) is completely coupled, we know that if the solution (u, v) blows up in finite time, then u and v blow up simultaneously.

Global blow-up.

We discuss the global blow-up in two special cases.

Case 1. $m_1 > 0$, $r_1 = 0$ or $m_2 > 0$, $r_2 = 0$.

Chan et al. [1][3] proved that there exists Green's function $G(x, \xi, t-\tau)$ associated with the operator $L = x^{m_1}\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$ with the first boundary condition, and obtained the following Lemmas

Lemma 3.1 *(a) For $t > \tau$, $G(x, \xi, t-\tau)$ is continuous for $(x, t, \xi, \tau) \in ([0, a] \times (0, T]) \times ((0, a] \times [0, T))$.*
(b) For each fixed $(\xi, \tau) \in (0, a] \times [0, T)$, $G_t(x, \xi, t-\tau) \in C([0, a] \times (\tau, T])$.
(c) In $\{(x, t, \xi, \tau) : x$ and ξ are in $(0, a)$, $T \geq t > \tau \geq 0\}$, $G(x, \xi, t-\tau)$ is positive.

Lemma 3.2 *For fixed $x_0 \in (0, a]$, given any $x \in (0, a)$ and any finite time T, there exist positive constants C_1 (depending on x and T) and C_2 (depending on T) such that*

$$\int_0^a G(x, \xi, t)d\xi > C_1, \ \int_0^a G(x_0, \xi, t)d\xi < C_2 \quad for\ 0 \leq t \leq T.$$

Now we give the global blow-up result

Theorem 3.3 *Under the assumption of Case 1, if the solution of (1.1) blows up at the point $x_0 \in (0, a)$, then the blow-up set of the solution of (1.1) is $[0, a]$.*

Proof. From the remark, we know that u and v blow up simultaneously if the solution (u, v) blows up in finite time. Without loss of generality, we assume $m_1 > 0$, $r_1 = 0$ and $u(x, t)$ blows up in finite time T. By Green's second identity we have

$$u(x,t) = \int_0^a \xi^{m_1} G(x, \xi, t) u_0(\xi) d\xi + \int_0^t \int_0^a G(x, \xi, t-\tau)[\int_0^a v^{p_1}(y, \tau)dy - k_1 u^{q_1}(\xi, \tau)]d\xi d\tau, \tag{3.7}$$

for any $(x, t) \in (0, a) \times (0, T)$. According to the conditions, $u(x, t)$ blows up at $x = x_0$, then $\limsup_{t\to T} u(x_0, t) = +\infty$. By (3.7) and Lemma 3.4, we have

$$\begin{aligned} u(x_0, t) &= \textstyle\int_0^a \xi^{q_1} G(x_0, \xi, t) u_0(\xi) d\xi + \int_0^t \int_0^a G(x_0, \xi, \tau) \int_0^a v^{p_1}(y, t-\tau) dy d\xi d\tau \\ &\leq C_2 a^{q_1} \max_{x\in[0,a]} u_0(x) + C_2 \textstyle\int_0^t \int_0^a v^{p_1}(y, t-\tau) dy d\tau. \end{aligned}$$

Since $\limsup_{t\to T} u(x_0, t) = +\infty$, we have

$$\lim_{t\to T} \int_0^t \int_0^a v^{p_1}(y, t-\tau) dy d\tau = +\infty. \tag{3.8}$$

On the other hand, for any $x \in (0, a)$, suppose there exist an $M > 0$, such that $u(x, t) \leq M$ we have

$$\begin{aligned} u(x, t) &\geq \textstyle\int_0^a \xi^{q_1} G(x, \xi, t) u_0(\xi) d\xi + C_1 \int_0^t [\int_0^a v^{p_1}(y, t-\tau) dy - k_1 u^{q_1}](\xi, \tau) d\tau \\ &\geq C_1 \textstyle\int_0^t \int_0^a v^{p_1}(y, t-\tau) dy d\tau - k_1 C_1 M^{q_1}, \quad t \in (0, T). \end{aligned}$$

It follows from the above inequality and (3.8) that

$$\limsup_{t\to T} u(x,t) = +\infty.$$

For any $\tilde{x} \in \{0, a\}$, we can choose a sequence $\{(x_n, t_n)\}$ such that $(x_n, t_n) \to (\tilde{x}, T)$ $(n \to +\infty)$ and

$$\lim_{n\to\infty} u(x_n, t_n) = +\infty.$$

Thus the blow-up set is the whole domain $[0, a]$, and we complete the proof of Theorem 3.5. □

Case 2. $m_1 = 0,\ 0 \le r_1 < 1$, or $m_2 = 0,\ 0 \le r_2 < 1$.

We will prove the blow-up set is the whole domain under the following assumption (H) There exists M $(0 < M < +\infty)$ such that $(x^{r_1} u_{0x}(x))_x \le M$ or $(x^{r_2} v_{0x}(x))_x \le M$ in $(0, a)$.

Theorem 3.4 *Under the assumptions of (H) and Case 2, if the solution of (1.1) blows up at the point $x_0 \in (0, a)$, then the blow-up set of the solution of (1.1) is $[0, a]$.*

Proof. The proof is similar to [5, 12]. □

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