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AN INEQUALITY FOR THE NORM OF A POLYNOMIAL FACTOR

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ABSTRACT. Let $p(z)$ be a monic polynomial of degree n, with complex coefficients, and let $q(z)$ be its monic factor. We prove an asymptotically sharp inequality of the form $\|q\|_E \le C^n \|p\|_E$, where $\|\cdot\|_E$ denotes the sup norm on a compact set E in the plane. The best constant C_E in this inequality is found by potential theoretic methods. We also consider applications of the general result to the cases of a disk and a segment.

1. Introduction

Let $p(z)$ be a monic polynomial of degree n, with complex coefficients. Suppose that $p(z)$ has a monic factor $q(z)$, so that

$$p(z) = q(z)\, r(z),$$

where $r(z)$ is also a monic polynomial. Define the uniform (sup) norm on a compact set E in the complex plane $\mathbb{C}$ by

$$\|f\|_E := \sup_{z\in E} |f(z)|. \tag{1.1}$$

We study the inequalities of the following form

$$\|q\|_E \le C^n \, \|p\|_E, \quad \deg p = n, \tag{1.2}$$

where the main problem is to find the best (the smallest) constant C_E, such that (1.2) is valid for *any* monic polynomial $p(z)$ and *any* monic factor $q(z)$.

In the case $E = \overline{D}$, where $D := \{z : |z| < 1\}$, the inequality (1.2) was considered in a series of papers by Mignotte [9], Granville [7] and Glesser [6], who obtained a number of improvements on the upper bound for $C_{\overline{D}}$. D. W. Boyd [3] made the final step here, by proving that

$$\|q\|_{\overline{D}} \le \beta^n \|p\|_{\overline{D}}, \tag{1.3}$$

with

$$\beta := \exp\left(\frac{1}{\pi}\int_0^{2\pi/3} \log\left(2\cos\frac{t}{2}\right) dt\right). \tag{1.4}$$

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The constant $\beta = C_{\overline{D}}$ is asymptotically sharp, as $n \to \infty$, and it can also be expressed in a different way, using Mahler's measure. This problem is of importance in designing algorithms for factoring polynomials with integer coefficients over integers. We refer to [5] and [8] for more information on the connection with symbolic computations.

A further development related to (1.2) for $E = [-a, a],\ a > 0$, was suggested by P. B. Borwein in [1] (see Theorems 2 and 5 there or see Section 5.3 in [2]). In particular, Borwein proved that if $\deg q = m$ then

$$|q(-a)| \le \|p\|_{[-a,a]} a^{m-n} 2^{n-1} \prod_{k=1}^{m} \left(1 + \cos \frac{2k-1}{2n}\pi\right), \tag{1.5}$$

where the bound is attained for a monic Chebyshev polynomial of degree n on $[-a, a]$ and a factor q. He also showed that, for $E = [-2, 2]$, the constant in the above inequality satisfies

$$\begin{aligned} &\limsup_{n\to\infty} \left(2^{m-1} \prod_{k=1}^{m} \left(1 + \cos \frac{2k-1}{2n}\pi\right)\right)^{1/n} \\ \le\ &\lim_{n\to\infty} \left(2^{[2n/3]-1} \prod_{k=1}^{[2n/3]} \left(1 + \cos \frac{2k-1}{2n}\pi\right)\right)^{1/n} \\ =\ &\exp\left(\int_0^{2/3} \log\left(2 + 2\cos \pi x\right) dx\right) = 1.9081\ldots, \end{aligned}$$

which hints that

$$C_{[-2,2]} = \exp\left(\int_0^{2/3} \log\left(2 + 2\cos \pi x\right) dx\right) = 1.9081\ldots. \tag{1.6}$$

We find the asymptotically best constant C_E in (1.2) for a rather arbitrary compact set E. The general result is then applied to the cases of a disk and a line segment, so that we recover (1.3)-(1.4) and confirm (1.6).

2. Results

Our solution of the above problem is based on certain ideas from the logarithmic potential theory (cf. [12] or [13]). Let $\operatorname{cap}(E)$ be the *logarithmic capacity* of a compact set $E \subset \mathbb{C}$. For E with $\operatorname{cap}(E) > 0$, denote the *equilibrium measure* of E (in the sense of the logarithmic potential theory) by μ_E. We remark that μ_E is a positive unit Borel measure supported on E, $\operatorname{supp} \mu_E \subset E$ (see [13, p. 55]).

Theorem 2.1. *Let $E \subset \mathbb{C}$ be a compact set, $\operatorname{cap}(E) > 0$. Then the best constant C_E in (1.2) is given by*

$$C_E = \frac{\max_{u\in\partial E} \exp\left(\int_{|z-u|\ge 1} \log|z-u|\, d\mu_E(z)\right)}{\operatorname{cap}(E)}. \tag{2.1}$$

Furthermore, if E is regular then

$$C_E = \max_{u\in\partial E} \exp\left(-\int_{|z-u|\le 1} \log|z-u|\, d\mu_E(z)\right). \tag{2.2}$$

The above notion of regularity is to be understood in the sense of the exterior Dirichlet problem (cf. [13, p. 7]). Note that the condition $\mathrm{cap}(E) > 0$ is usually satisfied for all applications, as it only fails for very *thin* sets (see [13, pp. 63-66]), e.g., finite sets in the plane. But if E consists of finitely many points then the inequality (1.2) cannot be true for a polynomial $p(z)$ with zeros at every point of E and for its linear factors $q(z)$. On the other hand, Theorem 2.1 is applicable to any compact set with a connected component consisting of more than one point (cf. [13, p. 56]).

One can readily see from (1.2) or (2.1) that the best constant C_E is invariant under the rigid motions of the set E in the plane. Therefore we consider applications of Theorem 2.1 to the family of disks $D_r := \{z : |z| < r\}$, which are centered at the origin, and to the family of segments $[-a, a],\ a > 0$.

Corollary 2.2. *Let D_r be a disk of radius r. Then the best constant $C_{\overline{D_r}}$, for $E = \overline{D_r}$, is given by*

$$C_{\overline{D_r}} = \begin{cases} \dfrac{1}{r}, & 0 < r \le 1/2, \\ \dfrac{1}{r} \exp\left(\dfrac{1}{\pi} \displaystyle\int_0^{\pi - 2 \arcsin \frac{1}{2r}} \log\left(2r \cos \dfrac{x}{2}\right) dx \right), & r > 1/2. \end{cases} \tag{2.3}$$

Note that (1.3)-(1.4) immediately follow from (2.3) for $r = 1$. The graph of $C_{\overline{D_r}}$, as a function of r, is in Figure 1.

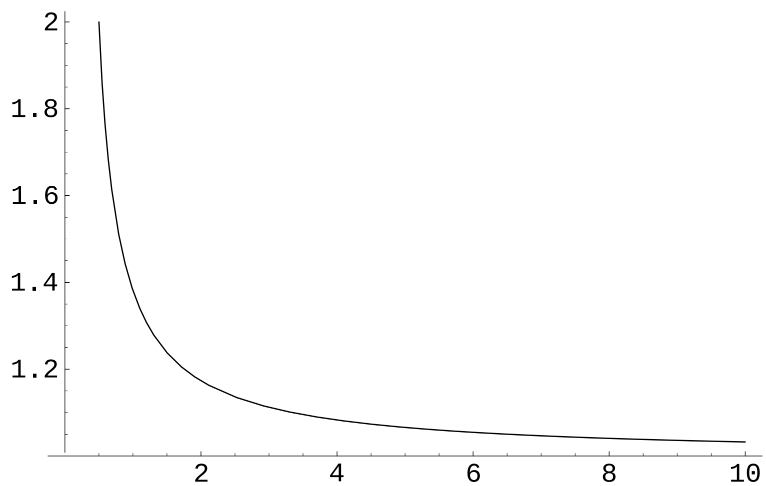

FIGURE 1. $C_{\overline{D_r}}$ as a function of r.

Corollary 2.3. *If $E = [-a, a],\ a > 0$, then*

$$C_{[-a,a]} = \begin{cases} \dfrac{2}{a}, & 0 < a \le 1/2, \\ \dfrac{2}{a} \exp\left(\displaystyle\int_{1-a}^{a} \dfrac{\log(t+a)}{\pi\sqrt{a^2 - t^2}}\, dt \right), & a > 1/2. \end{cases} \tag{2.4}$$

Observe that (2.4), with $a = 2$, implies (1.6) by the change of variable $t = 2\cos \pi x$. We include the graph of $C_{[-a,a]}$, as a function of a, in Figure 2.

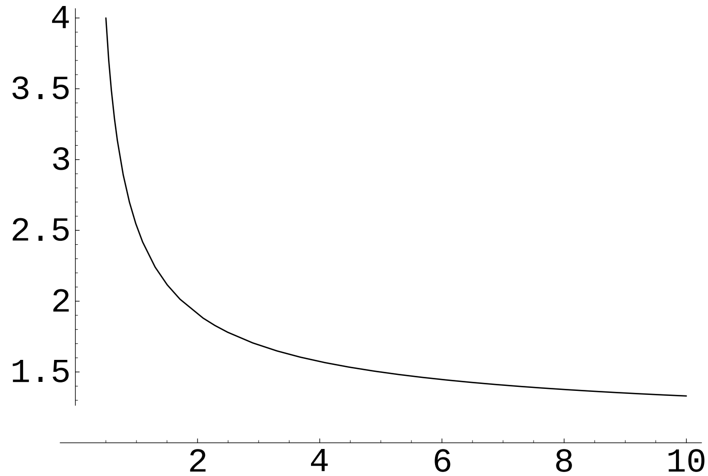

FIGURE 2. $C_{[-a,a]}$ as a function of a.

We now state two general consequences of Theorem 2.1. They explain some interesting features of C_E, which the reader may have noticed in Corollaries 2.2 and 2.3. Let

$$\operatorname{diam}(E) := \max_{z,\zeta \in E} |z - \zeta|$$

be the Euclidean diameter of E.

Corollary 2.4. *Suppose that* $\operatorname{cap}(E) > 0$. *If* $\operatorname{diam}(E) \le 1$ *then*

$$C_E = \frac{1}{\operatorname{cap}(E)}. \tag{2.5}$$

It is well known that $\operatorname{cap}(D_r) = r$ and $\operatorname{cap}([-a,a]) = a/2$ (see [12, p. 135]), which clarifies the first lines of (2.3) and (2.4) by (2.5).

The next Corollary shows how the constant C_E behaves under dilations of the set E. Let αE be the dilation of E with a factor $\alpha > 0$.

Corollary 2.5. *If E is regular then*

$$\lim_{\alpha \to +\infty} C_{\alpha E} = 1. \tag{2.6}$$

Thus Figures 1 and 2 clearly illustrate (2.6).

We remark that one can deduce inequalities of the type (1.2), for various L_p norms, from Theorem 2.1, by using relations between L_p and L_∞ norms of polynomials on E (see, e.g., [11]).

3. PROOFS

Proof of Theorem 2.1. The proof of this result is based on the ideas of [3] and [10]. For $u \in \mathbb{C}$, consider a function

$$\rho_u(z) := \max(|z - u|, 1), \quad z \in \mathbb{C}.$$

One can immediately see that $\log \rho_u(z)$ is a subharmonic function in $z \in \mathbb{C}$, which has the following integral representation (see [12, p. 29]):

$$\log \rho_u(z) = \int \log |z-t| \, d\lambda_u(t), \quad z \in \mathbb{C}, \tag{3.1}$$

where $d\lambda_u(u+e^{i\theta}) = d\theta/(2\pi)$ is the normalized angular measure on $|t-u| = 1$.

Let $u \in \partial E$ be such that

$$\|q\|_E = |q(u)|.$$

If $z_k, \ k = 1, \ldots, m$, are the zeros of $q(z)$, counted according to multiplicities, then

$$\begin{aligned} \log \|q\|_E &= \sum_{k=1}^{m} \log |u - z_k| \le \sum_{k=1}^{m} \log \rho_u(z_k) \\ &= \sum_{k=1}^{m} \int \log |z_k - t| \, d\lambda_u(t) = \int \log |p(t)| \, d\lambda_u(t), \end{aligned} \tag{3.2}$$

by (3.1).

We use the well known Bernstein-Walsh lemma about the growth of a polynomial outside of the set E (see [12, p. 156], for example):
Let $E \subset \mathbb{C}$ be a compact set, $\text{cap}(E) > 0$, with the unbounded component of $\overline{\mathbb{C}} \setminus E$ denoted by Ω. Then, for any polynomial $p(z)$ of degree n, we have

$$|p(z)| \le \|p\|_E \, e^{n g_\Omega(z,\infty)}, \quad z \in \mathbb{C}, \tag{3.3}$$

where $g_\Omega(z,\infty)$ is the Green function of Ω, with pole at ∞. The following representation for $g_\Omega(z,\infty)$ is found in Theorem III.37 of [13, p. 82]).

$$g_\Omega(z,\infty) = \log \frac{1}{\text{cap}(E)} + \int \log |z-t| \, d\mu_E(t), \quad z \in \mathbb{C}. \tag{3.4}$$

It follows from (3.1)-(3.4) and Fubini's theorem that

$$\begin{aligned} \frac{1}{n} \log \frac{\|q\|_E}{\|p\|_E} &\le \int \log \frac{|p(t)|^{1/n}}{\|p\|_E^{1/n}} \, d\lambda_u(t) \le \int g_\Omega(t,\infty) \, d\lambda_u(t) \\ &= \log \frac{1}{\text{cap}(E)} + \int \int \log |z-t| \, d\lambda_u(t) d\mu_E(z) \\ &= \log \frac{1}{\text{cap}(E)} + \int \log \rho_u(z) \, d\mu_E(z). \end{aligned}$$

Using the definition of $\rho_u(z)$, we obtain from the above estimate that

$$\begin{aligned} \|q\|_E &\le \left(\frac{\max_{u \in \partial E} \exp \left(\int \log \rho_u(z) \, d\mu_E(z) \right)}{\text{cap}(E)} \right)^n \|p\|_E \\ &= \left(\frac{\max_{u \in \partial E} \exp \left(\int_{|z-u| \ge 1} \log |z-u| \, d\mu_E(z) \right)}{\text{cap}(E)} \right)^n \|p\|_E. \end{aligned}$$

Hence

$$C_E \le \frac{\max_{u\in\partial E} \exp\left(\int_{|z-u|\ge 1} \log|z-u|\, d\mu_E(z)\right)}{\text{cap}(E)}. \tag{3.5}$$

In order to prove the inequality opposite to (3.5), we consider the n-th Fekete points $\{a_{k,n}\}_{k=1}^n$ for the set E (cf. [12, p. 152]). Let

$$p_n(z) := \prod_{k=1}^n (z - a_{k,n})$$

be the Fekete polynomial of degree n. Define the normalized counting measures on the Fekete points by

$$\tau_n := \frac{1}{n}\sum_{k=1}^n \delta_{a_{k,n}}, \quad n \in \mathbb{N}.$$

It is known that (see Theorems 5.5.4 and 5.5.2 in [12, pp. 153-155])

$$\lim_{n\to\infty} \|p_n\|_E^{1/n} = \text{cap}(E). \tag{3.6}$$

Furthermore, we have the following weak* convergence of counting measures (cf. [12, p. 159]):

$$\tau_n \xrightarrow{*} \mu_E, \quad \text{as } n \to \infty. \tag{3.7}$$

Let $u \in \partial E$ be a point, where the maximum on the right hand side of (3.5) is attained. Define the factor $q_n(z)$ for $p_n(z)$, with zeros being the n-th Fekete points satisfying $|a_{k,n} - u| \ge 1$. Then we have by (3.7) that

$$\begin{aligned}
\lim_{n\to\infty} \|q_n\|_E^{1/n} &\ge \lim_{n\to\infty} |q_n(u)|^{1/n} = \lim_{n\to\infty} \exp\left(\frac{1}{n}\sum_{|a_{k,n}-u|\ge 1} \log|u - a_{k,n}|\right) \\
&= \exp\left(\lim_{n\to\infty} \int_{|z-u|\ge 1} \log|u-z|\, d\tau_n(z)\right) \\
&= \exp\left(\int_{|z-u|\ge 1} \log|u-z| d\mu_E(z)\right).
\end{aligned}$$

Combining the above inequality with (3.6) and the definition of C_E, we obtain that

$$C_E \ge \lim_{n\to\infty} \frac{\|q_n\|_E^{1/n}}{\|p_n\|_E^{1/n}} \ge \frac{\exp\left(\int_{|z-u|\ge 1} \log|z-u|\, d\mu_E(z)\right)}{\text{cap}(E)}.$$

This shows that (2.1) holds true. Moreover, if $u \in \partial E$ is a regular point for Ω, then we obtain by Theorem III.36 of [13, p. 82]) and (3.4) that

$$\log\frac{1}{\text{cap}(E)} + \int \log|u-t|\, d\mu_E(t) = g_\Omega(u,\infty) = 0.$$

Hence

$$\log\frac{1}{\text{cap}(E)} + \int_{|z-u|\ge 1} \log|u-t|\, d\mu_E(t) = -\int_{|z-u|\le 1} \log|u-t|\, d\mu_E(t),$$

which implies (2.2) by (2.1). $\square$

Proof of Corollary 2.2. It is well known [13, p. 84] that $\mathrm{cap}(\overline{D_r}) = r$ and $d\mu_{\overline{D_r}}(re^{i\theta}) = d\theta/(2\pi)$, where $d\theta$ is the angular measure on ∂D_r. If $r \in (0, 1/2]$ then the numerator of (2.1) is equal to 1, so that

$$C_{\overline{D_r}} = \frac{1}{r}, \quad 0 < r \le 1/2.$$

Assume that $r > 1/2$. We set $z = re^{i\theta}$ and let $u_0 = re^{i\theta_0}$ be a point where the maximum in (2.1) is attained. On writing

$$|z - u_0| = 2r \left| \sin \frac{\theta - \theta_0}{2} \right|,$$

we obtain that

$$\begin{aligned} C_{\overline{D_r}} &= \frac{1}{r} \exp \left(\frac{1}{2\pi} \int_{\theta_0 + 2\arcsin\frac{1}{2r}}^{2\pi + \theta_0 - 2\arcsin\frac{1}{2r}} \log \left| 2r \sin \frac{\theta - \theta_0}{2} \right| d\theta \right) \\ &= \frac{1}{r} \exp \left(\frac{1}{2\pi} \int_{2\arcsin\frac{1}{2r} - \pi}^{\pi - 2\arcsin\frac{1}{2r}} \log \left(2r \cos \frac{x}{2} \right) dx \right) \\ &= \frac{1}{r} \exp \left(\frac{1}{\pi} \int_{0}^{\pi - 2\arcsin\frac{1}{2r}} \log \left(2r \cos \frac{x}{2} \right) dx \right), \end{aligned}$$

by the change of variable $\theta - \theta_0 = \pi - x$. □

Proof of Corollary 2.3. Recall that $\mathrm{cap}([-a, a]) = a/2$ (see [13, p. 84]) and

$$d\,\mu_{[-a,a]}(t) = \frac{dt}{\pi\sqrt{a^2 - t^2}}, \quad t \in [-a, a].$$

It follows from (2.1) that

$$C_{[-a,a]} = \frac{2}{a} \exp \left(\max_{u \in [-a,a]} \int_{[-a,a] \setminus (u-1, u+1)} \frac{\log |t - u|}{\pi\sqrt{a^2 - t^2}}\, dt \right). \tag{3.8}$$

If $a \in (0, 1/2]$ then the integral in (3.8) obviously vanishes, so that $C_{[-a,a]} = 2/a$. For $a > 1/2$, let

$$f(u) := \int_{[-a,a] \setminus (u-1, u+1)} \frac{\log |t - u|}{\pi\sqrt{a^2 - t^2}}\, dt. \tag{3.9}$$

One can easily see from (3.9) that

$$f'(u) = \int_{u+1}^{a} \frac{dt}{\pi(u - t)\sqrt{a^2 - t^2}} < 0, \quad u \in [-a, 1 - a],$$

and

$$f'(u) = \int_{-a}^{u-1} \frac{dt}{\pi(u - t)\sqrt{a^2 - t^2}} > 0, \quad u \in [a - 1, a].$$

However, if $u \in (1 - a, a - 1)$ then

$$f'(u) = \int_{u+1}^{a} \frac{dt}{\pi(u - t)\sqrt{a^2 - t^2}} + \int_{-a}^{u-1} \frac{dt}{\pi(u - t)\sqrt{a^2 - t^2}}.$$

It is not difficult to verify directly that

$$\int \frac{dt}{\pi(u - t)\sqrt{a^2 - t^2}} = \frac{1}{\pi\sqrt{a^2 - u^2}} \log \left| \frac{a^2 - ut + \sqrt{a^2 - t^2}\,\sqrt{a^2 - u^2}}{t - u} \right| + C,$$

which implies that

$$f'(u) = \frac{1}{\pi\sqrt{a^2-u^2}} \log\left(\frac{a^2-u^2+u+\sqrt{a^2-(u-1)^2}\,\sqrt{a^2-u^2}}{a^2-u^2-u+\sqrt{a^2-(u+1)^2}\,\sqrt{a^2-u^2}}\right),$$

for $u \in (1-a, a-1)$. Hence

$$f'(u) < 0,\ u \in (1-a, 0), \quad \text{and} \quad f'(u) > 0,\ u \in (0, a-1).$$

Collecting all facts, we obtain that the maximum for $f(u)$ on $[-a, a]$ is attained at the endpoints $u = a$ and $u = -a$, and it is equal to

$$\max_{u\in[-a,a]} f(u) = \int_{1-a}^{a} \frac{\log(t+a)}{\pi\sqrt{a^2-t^2}}\,dt.$$

Thus (2.3) follows from (3.8) and the above equation. □

Proof of Corollary 2.4. Note that the numerator of (2.1) is equal to 1, because $|z-u| \le 1,\ \forall\, z \in E,\ \forall\, u \in \partial E$. Thus (2.5) follows immediately. □

Proof of Corollary 2.5. Observe that $C_E \ge 1$ for any $E \in \mathbb{C}$, so that $C_{\alpha E} \ge 1$. Since E is regular, we use the representation for C_E in (2.2). Let $T : E \to \alpha E$ be the dilation mapping. Then $|Tz - Tu| = \alpha|z-u|,\ z, u \in E$, and $d\mu_{\alpha E}(Tz) = d\mu_E(z)$. This gives that

$$\begin{aligned} C_{\alpha E} &= \max_{Tu\in\partial(\alpha E)} \exp\left(-\int_{|Tz-Tu|\le 1} \log|Tz-Tu|\,d\mu_{\alpha E}(Tz)\right) \\ &= \max_{u\in\partial E} \exp\left(-\int_{|z-u|\le 1/\alpha} \log(\alpha|z-u|)\,d\mu_E(z)\right) \\ &= \max_{u\in\partial E} \exp\left(-\mu_E(\overline{D_{1/\alpha}(u)})\log\alpha - \int_{|z-u|\le 1/\alpha} \log|z-u|\,d\mu_E(z)\right) \\ &< \max_{u\in\partial E} \exp\left(-\int_{|z-u|\le 1/\alpha} \log|z-u|\,d\mu_E(z)\right), \end{aligned}$$

where $\alpha \ge 1$. Using the absolute continuity of the integral, we have that

$$\lim_{\alpha\to+\infty} \int_{|z-u|\le 1/\alpha} \log|z-u|\,d\mu_E(z) = 0,$$

which implies (2.6). □

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