AN EXPLICIT FORMULA FOR PBW QUANTIZATION

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Abstract. Let k be a field of characteristic zero, $\mathfrak{g}$ a k-Lie algebra, $e : S\mathfrak{g} \longrightarrow U\mathfrak{g}$ the symmetrization map. The PBW quantization is the one parameter family of associative products:

$$x \star_t y = \sum_{p=0}^{\infty} B_p(x,y)t^p \qquad (t \in k)$$

where B_p is the homogeneous component of degree $-p$ of the map $B : S\mathfrak{g} \otimes_k S\mathfrak{g} \longrightarrow S\mathfrak{g}$, $B(x,y) = e^{-1}(exey)$. In this paper we give an explicit formula for B. As an application, we prove that for each $p \geq 0$, B_p is a bidifferential operator of order $\leq p$.

0. Introduction

We consider (possibly infinite dimensional) Lie algebras over a fixed field k of characteristic zero. Let $\mathfrak{g}$ be a Lie algebra, $S = S\mathfrak{g}$ and $U = U\mathfrak{g}$ the symmetric and universal enveloping algebras, and $\mathcal{F}^0 \subset \mathcal{F}^1 \subset \cdots \subset U$ the coalgebra filtration $\mathcal{F}^n := k + \mathfrak{g} + \mathfrak{g}^2 + \cdots + \mathfrak{g}^n$. Recall that the Poincaré-Birkhoff-Witt isomorphism between S and the associated graded ring $G_{\mathcal{F}}U$ is induced by the symmetrization map $e : S \tilde{\to} U$ defined as

$$e(g_1 \dots g_p) = \frac{1}{p!} \sum_{\sigma \in S_p} g_{\sigma(1)} \dots g_{\sigma(p)} \tag{1}$$

Thus the associative product

$$B : S \otimes S \to S, \qquad B(x,y) = e^{-1}(exey) \tag{2}$$

maps $S^{\leq n} = \sum_{p=0}^{n} S^p$ into itself ($n \geq 0$), whence it can be written as

$$B = \sum_{p=0}^{\infty} B_p \tag{3}$$

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where B_p is homogeneous of degree $-p$. We have

$$B_0(x,y) = xy, \qquad B_1(x,y) = \frac{1}{2}\{x,y\} \tag{4}$$

Here $\{,\}$ is the Poisson bracket induced by the Lie bracket of $\mathfrak{g}$. Thus if $x_1,\dots,x_n$, $y_1,\dots,y_m \in \mathfrak{g}$, then

$$B_1(x,y) = \frac{1}{2}\sum_{i,j} x_1 \dots \overset{i}{\vee} \dots x_n y_1 \dots \overset{j}{\vee} \dots y_m [x_i, y_j]$$

In general we will have

$$B_p(x_1 \dots x_n, y_1 \dots y_m) = \sum_i z_{i,1} \dots z_{i,m+n-p} \tag{5}$$

for some elements $z_{ij} \in \mathfrak{g}$. In this paper we prove a closed formula of the form (5), and an explicit description of the $z_{i,j}$ as Lie monomials in the x_r, y_s (Theorem 1.1). As an application of our formula, we show that B_p is a bidifferential operator of order $\le p$ (Theorem 2.2).

The operators B_p appear naturally in deformation theory and mathematical physics. One considers the family of products

$$x \star_t y = \sum_{p=0}^{\infty} B_p(x,y) t^p \qquad (t \in k)$$

as a one parameter deformation of the usual commutative product $\star_0$ of S into the noncommutative product $\star_1$ of the enveloping algebra. If furthermore $\mathfrak{g}$ is finite dimensional, then S can be regarded as the ring of algebraic functions on the dual $\mathfrak{g}^*$, and the $S_t = (S, \star_t)$ as the rings of functions of a family of noncommutative varieties deforming or "quantizing" the Poisson variety $\mathfrak{g}^*$. We call this the PBW quantization because the B_p are defined by means of the Poincaré-Birkhoff-Witt theorem.

The idea of the proof of Theorem 1.1 is to use a well-known expression of the B_p in terms of the Campbell-Hausdorff series ([1]) in combination with Dynkin's explicit formula for the latter ([3], LA 4.17). Although particular cases of our formula were known ([1],[2]) this paper is to our knowledge the first where it appears in its full generality. An analytic proof of the bidifferentiality of the B_p was given in [1], but no estimate of its order is made there. Our proof of the bidifferentiality is combinatoric and is derived from the explicit formula of Theorem 1.1.

The rest of this paper is organized as follows. The formula for B_p is established in section 1 (theorem 1.1). Section 2 is devoted to the proof of its bidifferentiality (theorem 2.2).

1. A formula for B_p

In preparation for theorem 1.1 below, we introduce some notation. Let $n, m \ge 1$, $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_m\}$ two sets of noncommuting indeterminates. If $\alpha \in \{1, \dots, n\}^p$ $(p \ge 1)$ is a multi-index, then we write:

$$|\alpha| = p, \quad \operatorname{Im}\alpha = \{\alpha(1), \dots, \alpha(p)\}, \quad ad(x)^\alpha = ad(x_{\alpha(1)}) \circ \dots \circ ad(x_{\alpha(p)}) \tag{6}$$

The formula of theorem 1.1 below involves the following element of the free Lie algebra on the disjoint union $X \amalg Y$:

$$w(X,Y) = \frac{1}{n+m}(w'(X,Y) + w''(X,Y)) \tag{7}$$

Here w' and w'' are given by the sums (8) and (9); the restrictions in the summation indexes are explained below; see (11), (12).

$$w'(X,Y) = \sum \frac{(-1)^{p+1}}{p} \frac{ad(x)^{\alpha_1} \circ ad(y)^{\beta_1} \circ \cdots \circ ad(x)^{\alpha_p}(y_k)}{|\alpha_1|!|\beta_1|! \dots |\alpha_p|!} \tag{8}$$

$$w''(X,Y) = \sum \frac{(-1)^{p+1}}{p} \frac{ad(x)^{\alpha_1} \circ ad(y)^{\beta_1} \circ \cdots \circ ad(y)^{\beta_{p-1}}(x_k)}{|\alpha_1|!|\beta_1|! \dots |\beta_{p-1}|!} \tag{9}$$

In (8) the sum is taken over arbitrary $p \geq 1$ and all *injective* multi-indices

$$\alpha_i \in \{1, \dots, n\}^{|\alpha_i|}, \qquad \beta_j \in \{1, \dots, m\}^{|\beta_j|} \tag{10}$$

satisfying

$$\begin{gathered} |\alpha_1| + \cdots + |\alpha_p| = n \qquad |\beta_1| + \cdots + |\beta_{p-1}| = m - 1 \\ |\alpha_i| + |\beta_i| \geq 1 \quad (1 \leq i \leq p-1) \qquad |\alpha_p| \geq 1 \\ \bigcup_{i=1}^{p} \operatorname{Im} \alpha_i = \{1, \dots, n\} \qquad \{k\} \cup \bigcup_{i=1}^{p-1} \operatorname{Im} \beta_j = \{1, \dots, m\} \end{gathered} \tag{11}$$

The sum in (9) is also taken over arbitrary $p \geq 1$, and all injective multi-indices (10), but now

$$\begin{gathered} |\alpha_1| + \cdots + |\alpha_{p-1}| = n - 1 \qquad |\beta_1| + \cdots + |\beta_{p-1}| = m \\ |\alpha_i| + |\beta_i| \geq 1 \quad (1 \leq i \leq p-1) \\ \{k\} \cup \bigcup_{i=1}^{p-1} \operatorname{Im} \alpha_i = \{1, \dots, n\} \qquad \bigcup_{i=1}^{p-1} \operatorname{Im} \beta_j = \{1, \dots, m\} \end{gathered} \tag{12}$$

In the theorem below the we consider the element $w(A, B)$ for $A \subset X$, $B \subset Y$. If none of A, B is empty, then $w(A, B)$ is already defined by (7); we further define

$$w(\{a\}, \emptyset) = w(\emptyset, \{a\}) = a \qquad (a \in X \amalg Y) \tag{13}$$

Definition 1.0. Let A_1, A_2 be sets, and $\mathcal{P}(A_i)$ the set of all subsets of A_i $(i = 1, 2)$. A *bipartition* of (A_1, A_2) is a subset $\pi \subset \mathcal{P}(A_1) \times \mathcal{P}(A_2)$ such that the following two conditions are satisfied:

i) If $S, T \in \pi$ are distinct, then $S_i \cap T_i = \emptyset$ $(i = 1, 2)$.

ii) $A_i = \cup_{S \in \pi} S_i$ $(i = 1, 2)$

A bipartition π is called *special* if $w(S, T)$ is defined for all $(S, T) \in \pi$; that is if the following holds

$$(\emptyset, S) \text{ or } (S, \emptyset) \in \pi \Rightarrow \#S = 1 \tag{14}$$

Theorem 1.1. *Let $n, m, p \geq 1$, $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_m\}$ two sets of indeterminates, and B_p the operator of* (3) *for the free Lie algebra on the disjoint union $X \amalg Y$. Then*

$$B_p(x_1 \dots x_n, y_1 \dots y_m) = \sum_{\pi} w(\pi_1^X, \pi_1^Y) \dots w(\pi_{n+m-p}^X, \pi_{n+m-p}^Y) \tag{15}$$

Here w is as defined in (7), *and the sum runs over all special bipartitions $\pi = \{(\pi_1^X, \pi_1^Y), \dots, (\pi_{n+m-p}^X, \pi_{n+m-p}^Y)\}$ of cardinality $m+n-p$ of (X, Y). In particular,*

$$B_{n+m-1}(x_1 \dots x_n, y_1 \dots y_m) = w(X, Y) \tag{16}$$

Proof. Write $\mathfrak{g}$ for the free Lie algebra on $X \amalg Y$. Let $t_1, \dots, t_n$, $u_1, \dots, u_m$ be commuting algebraically independent variables. Put

$$x(t) = \sum_{i=1}^{n} t_i x_i, \qquad y(t) = \sum_{i=1}^{m} u_i y_i$$

One checks that $B(x_1 \dots x_n, y_1 \dots y_m)$ is the coefficient of $t_1 \dots t_n u_1 \dots u_m$ in the product of the exponential series

$$exp(x(t))exp(y(u)) = exp(z(t, u)) \tag{17}$$

Here $z(x(t), y(u))$ is the Campbell-Hausdorff series. Consider the expansion of z as a series in t, u. In order to compute the coefficient of $t_1 \dots t_n u_1 \dots u_m$ in (17), all terms in the expansion of z corresponding to monomials in which any of the t_i, u_j has exponent ≥ 2 may be discarded. Each of the remaining terms is an element of $\mathfrak{g}$ times a monomial of the form:

$$t_A u_B := t_{a_1} \dots t_{a_r} u_{b_1} \dots u_{b_s}$$

for some subsets $A = \{a_1, \dots, a_r\} \subset \{1, \dots, n\}$, $B = \{b_1, \dots, b_s\} \subset \{1, \dots, m\}$. One checks, using Dynkin's formula ([3],LA 4.17), that the coefficient of $t_A u_B$ in z is precisely the element $w(A, B)$. The theorem follows from this and the definition of the symmetrization map (1). □

In the course of the proof of the theorem above we introduced a notation which shall be used often in what follows. If $\mathcal{A}$ is a k-algebra, $a_1, \dots, a_n \in \mathcal{A}$ and $S = \{i_1, \dots, i_r\} \subset \{1, \dots, n\}$ is a subset of r elements, then we write:

$$a_S := a_{i_1} \dots a_{i_r} \tag{18}$$

In particular

$$a_\emptyset = 1$$

2. The bidifferentiality of B_p

Let $q, k \geq 0$; define inductively

$$c_0(q) = 1, \qquad c_k(q) = 1 - \sum_{l=0}^{k-1} c_l(q) \binom{q+k}{k-l} \quad (k \geq 1) \tag{19}$$

Lemma 2.0. *Let $\mathfrak{g}$ be a Lie algebra, $p \geq 1$, $q \geq 0$, $r \geq 1$, $x_1, \dots, x_{p+q}$, $y_1, \dots, y_r \in \mathfrak{g}$ and $c_k(q)$ as in (19) above. Then, with the notation of (18), we have*

$$B_p(x_1 \dots x_{p+q}, y_1 \dots y_r) = \sum_{k=0}^{r-1} c_k(q) (\sum_{\#S=q+k} x_S B_p(x_{S^c}, y_1 \dots y_r)) \tag{20}$$

Here $S \subset \{1, \dots, p+q\}$, and S^c is the complement of S. The symmetric formula holds for $B_p(y_1 \dots y_r, x_1 \dots x_{p+q})$.

Proof. We may assume that the x_i, y_j are indeterminates and that $\mathfrak{g}$ is the free Lie algebra. Apply theorem 1.1 to write the left hand side of (20) as a sum of terms indexed by all special bipartitions π of $(\{1, \dots, p+q\}, \{1, \dots, r\})$ of $q+r$ elements. It follows from the definition of a special bipartition that the number d_π of empty sets in the list $\pi_1^Y, \dots, \pi_{q+r}^Y$ of subsets of $\{1, \dots, r\}$ is at least q and at most $q+r-1$. Write b_k for the sum of those terms whose indexing bipartition has $d_\pi = q+k$ $(0 \leq k \leq r-1)$. By definition,

$$B_p(x_1 \dots x_{p+q}, y_1 \dots y_r) = \sum_{l=0}^{r-1} b_l$$

Moreover, a counting argument shows that for $0 \leq k \leq r-1$

$$\sum_{\#S=q+k} x_S B_p(x_{S^c}, y_1 \dots y_r) = \sum_{i=0}^{r-1-k} \binom{q+k+i}{i} b_{k+i}$$

Hence

$$\begin{aligned} \sum_{k=0}^{r-1} c_k(q) (\sum_{\#S=q+k} x_S B_p(x_{S^c}, y_1 \dots y_r)) &= \sum_{k=0}^{r-1} \sum_{i=0}^{r-1-k} c_k(q) \binom{q+k+i}{i} b_{k+i} \\ &= \sum_{l=0}^{r-1} (\sum_{k=0}^{l} c_k(q) \binom{q+l}{l-k}) b_l = \sum_{l=0}^{r-1} b_l \quad \text{(by (19))} \\ &= B_p(x_1 \dots x_{p+q}, y_1 \dots y_r) \quad \square \end{aligned}$$

Lemma 2.1. *Let $q \geq 1$, $m \geq 0$. Then:*

$$0 = (-1)^m + \sum_{t} \binom{m+q}{m+t} (-1)^t c_m(t)$$

Proof. Fix $q \geq 1$; the proof is by induction on $m \geq 0$. The case $m = 0$ is immediate. Assume $m \geq 1$ and by induction that the lemma holds for $s < m$. Then

$$\sum_{t=1}^{q} \binom{m+q}{m+t} (-1)^t c_m(t) =$$

$$= \sum_{t=1}^{q} \binom{m+q}{m+t} (-1)^t - \sum_{t=1}^{q} \sum_{s=0}^{m-1} (-1)^t \binom{m+q}{m+t} \binom{t+m}{m-s} c_s(t) \quad \text{(by (19))}$$

$$= (-1)^m \sum_{t=m+1}^{m+q} \binom{m+q}{t} (-1)^t - \sum_{s=0}^{m-1} \sum_{t=1}^{q} (-1)^t \binom{m+q}{m-s} \binom{s+q}{s+t} c_s(t)$$

$$= (-1)^{m+1} (\sum_{t=0}^{m} (-1)^t \binom{m+q}{t}) + \sum_{s=0}^{m-1} (-1)^s \binom{m+q}{m-s} \quad \text{(by inductive assumption)}$$

$$= (-1)^{m+1} + (-1)^{m+1} (\sum_{t=1}^{m} (-1)^t \binom{m+q}{t}) + \sum_{t=1}^{m} (-1)^{m+t} \binom{m+q}{t} = (-1)^{m+1} \quad \square$$

Recall that a k-linear endomorphism F of a commutative associative algebra $\mathcal{A}$ is a *differential operator* of order $\leq p$ if

$$\sum_{S \subset \{1,\dots,p\}} (-1)^S a_{S^c} F(a_S b) = 0 \quad (a_1, \dots, a_p, b \in \mathcal{A}) \tag{21}$$

One checks that if $\mathcal{A}$ is generated as a k-algebra by a set $X \subset \mathcal{A}$ then F satisfies (21) if and only if it satisfies

$$\sum_{S \subset \{1,\dots,p+q\}} (-1)^S x_{S^c} F(x_S) = 0 \quad (x_1, \dots, x_{p+q} \in X, \quad q \geq 0) \tag{22}$$

Theorem 2.2. *Let $\mathfrak{g}$ be a Lie algebra, $S = S\mathfrak{g}$ the symmetric algebra, $a \in S$, $p \geq 1$. Also let $F : S \to S$ be one of $B_p(a,)$ or $B_p(, a)$. Then F is a differential operator of order $\leq p$.*

Proof. It suffices to show the theorem for a homogeneous. Assume $a = y_1 \dots y_r$, $y_i \in \mathfrak{g}$. We shall show that the identity (22) holds for $X = \mathfrak{g}$. Put $K = \{1, \dots, p+q\}$. The left hand side of (22) is the sum of

$$\sum_{l=p-r+1}^{p} \sum_{\#S=l} (-1)^l x_{K \setminus S} F(x_S) \tag{23}$$

and of

$$\sum_{t=1}^{q} \sum_{\#S=p+t} (-1)^{p+t} x_{K \setminus S} F(x_S) =$$

$$\sum_{t=1}^{q} \sum_{\#S=p+t} \sum_{k=0}^{r-1} \sum_{\substack{T \subset S \\ \#T = p-k}} (-1)^{p+t} c_k(t) x_{K \setminus S} x_{S \setminus T} F(x_T) \quad \text{(by Lemma 2.0)}$$

$$= \sum^{p} \sum (\sum_{}^{q} \binom{p+q-l}{p+t-l} (-1)^{p+t} c_{p-l}(t)) x_{K \setminus S} F(x_S)$$

By lemma 2.1, the sum of the last expression with that of (23) is zero. □

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