DOMAINS OF HOLOMORPHY WITH EDGES AND LOWER DIMENSIONAL BOUNDARY SINGULARITIES

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ABSTRACT. Necessary and sufficient geometric conditions are given for domains with regular boundary points and edges to be domains of holomorphy provided the remainder boundary subset is of zero Hausdorff 1-codimensional measure.

1. INTRODUCTION

The positive solution to the classical Levi problem due to OKA [O42, O53], BREMERMANN [B54] and NORGUET [No54] asserts that a domain $\Omega \subset \mathbb{C}^N$ whose boundary is of class C^2 is a domain of holomorphy provided the Levi form of the boundary is everywhere positively semidefinite (see e.g. surveys [S84, Pe94]). In contrast to this, for domains with singularities on the boundary there seems to be a lack of such geometric conditions in the literature.

1.1. **Piecewise smooth domains.**

A natural generalization of smooth domains is given by the class of so-called *piecewise smooth* domains whose boundaries are pieces of hypersurfaces satisfying suitable transversality conditions (see e.g. [SH81, Pi82, Na88, K92, F93, MP94]). However, for the domains of holomorphy, the class of piecewise smooth domains seems to be very restrictive. For instance, an envelope of holomorphy of a domain with real-analytic (even algebraic) boundary does not need to be piecewise smooth as the example

$$\Omega := \{(z,w) \in \mathbb{C}^2 : |w|^2 + (|z|^2 - 1)^2 < 2\}$$

shows. Indeed, the Cauchy formula argument implies that the envelope of holomorphy (and also convex, polynomially convex and rationally convex hulls) of Ω is the union $\Omega \cup \{|z| < 1, |w| < \sqrt{2}\}$.

In this paper we consider a larger class of domains Ω, whose smooth boundary pieces may not be extended to closed smooth hypersurfaces in a neighborhood of $\partial\Omega$. We also allow singular subsets in the boundary that are only controlled to have zero 1-codimensional (with respect to the dimension of the boundary) Hausdorff measure.

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1.2. **The class $L^{2,\infty}$.** For an open subset $U \subset \mathbb{R}^m$ $(m \geq 1)$, denote by $L^{2,\infty}(U)$ the space of real continuous functions h on U that are twice continuously differentiable with bounded derivatives on an open dense subset of U.

We say that a domain $\Omega \subset \mathbb{R}^n$ is *of class* $L^{2,\infty}$ if for every $a \in \partial\Omega$ there exists a system of local (C^2-smooth) coordinates $(x_1, \ldots, x_n) = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ in a neighborhood $U = U' \times I$ of a and a function $h \in L^{2,\infty}(U')$ such that

$$\Omega \cap U = \{(x', x_n) \in U : x_n > h(x')\}. \tag{1}$$

It is easy to see that every domain with piecewise C^2-smooth boundary is of class $L^{2,\infty}$.

1.3. **Regular and edge points.** Given a subset $A \subset \mathbb{R}^n$, we say that a point $a \in A$ is (C^2-)*regular* if $A \cap U_a$ is a smooth hypersurface of class C^2 for some neighborhood $U_a \subset \mathbb{R}^n$ of a. If $a \in A$ is not regular, we call it a (C^1-)*edge point* if there exists a neighborhood U_a of a and a connected closed $(n-2)$-dimensional submanifold $M_a \subset \partial\Omega \cap U_a$ of class C^1, referred to as an *edge at* a, that contains all nonregular points of $A \cap U_a$.

It $\Omega \subset \mathbb{C}^N$ is a domain of holomorphy, it is a standard fact that the Levi form (see §2) at every regular point is positively semidefinite. The classical example of Hartogs

$$\Omega := \{|z| < 1, |w| < 1/2\} \cup \{1/2 < |z| < 1, |w| < 1\} \subset \mathbb{C}^2$$

shows that the converse does not hold in general even for piecewise smooth domains. The edge boundary points of Ω in this example, where all holomorphic functions extend, are precisely those whose tangent cones are not convex. Recall that the *tangent cone* (in the sense of Whitney) of $\Omega \subset \mathbb{C}^N$ at a point $a \in \partial\Omega$, denoted by $T_a\Omega$, is defined to be the set of all possible limits of $t_k(a_k - a) \in \mathbb{C}^N$, where $a_k \in \Omega$ and $t_k \in \mathbb{R}_+$ are sequences with $a_k \to a$ as $k \to \infty$.

1.4. **Main results.** It turns out that, together with the Levi form condition for regular points of $\partial\Omega$, the cone convexity for edges points guarantees that Ω is a domain of holomorphy. No condition on the other points is required provided the set of those points if of Hausdorff $(2N-2)$-dimensional measure zero. More precisely, we have the following result.

Theorem 1.1. *Let $\Omega \subset \mathbb{C}^N$ be a domain of class $L^{2,\infty}$ and $E \subset \partial\Omega$ be a closed subset of Hausdorff $(2N-2)$-dimensional measure zero. Suppose that the following hold:*

(i) *every point of $\partial\Omega \setminus E$ is either regular or an edge point;*

(ii) *at every regular point of $\partial\Omega \setminus E$ the Levi form is positively semi-definite;*

(iii) *at every edge point of $\partial\Omega \setminus E$ the tangent cone of Ω is convex.*

Then Ω is a domain of holomorphy.

By the well-known fact, Ω is a domain of holomorphy if and only if the function $\psi(z) := -\log d(z, \partial\Omega)$ is plurisubharmonic in Ω, where d denotes the euclidean distance (see e.g. Theorems 2.6.5 and 4.2.8 in [H90]). In particular, it follows from Theorem 1.1 for $N \geq 2$ that the tangent cone C_a of Ω at any $a \in \partial\Omega$ cannot be strictly concave (i.e. the interior of C_a cannot contain a hyperplane). Indeed, otherwise the function ψ would be equal to $-\log \|z - a\|$ in an open subset of Ω and hence would not be plurisubharmonic. This shows, on the other hand, that ψ cannot be directly used to prove Theorem 1.1 because in Theorem 1.1 there is no convexity condition on the cone at the points from the "exceptional" subset $E \subset \partial\Omega$. In fact we prove the plurisubharmonicity of the function

$$\phi(z) := -\log(x_{2N} - h(x')) + \lambda\|z\|^2 \tag{2}$$

near the boundary for some $\lambda > 0$ rather than of ψ, where $(x_1, \ldots, x_{2N})$ and h satisfy (1).

The necessity of the convexity condition (iii) depends on the complex geometry of the edges. We show:

Proposition 1.2. *Let $\Omega \subset \mathbb{C}^N$ be a domain of holomorphy and suppose that for an edge point $a \in \partial\Omega$, there exists an edge M_a which is not a complex hypersurface in any neighborhood of a. Then the tangent cone of Ω at a is convex.*

On the other hand, if an edge can be chosen to be a complex hypersurface, the convexity condition (iii) does not need to hold as the example of $\Omega := D \times \mathbb{C} \subset \mathbb{C}^2$ shows with $D \subset \mathbb{C}$ a nonconvex polygon. Therefore we have to distinguish between edge points satisfying the assumptions of Proposition 1.2 that we call *real edge points* and other edge points $a \in \partial\Omega$, where any edge must be locally a complex hypersurface. In the second case a is said to be a *complex edge point.* Then we impose the convexity condition only at *real* edge points. In this more general situation the above function ϕ given by (2) is not always plurisubharmonic. Nevertheless, we obtain the following necessary and sufficient geometric conditions for domains to be domains of holomorphy as a consequence of Theorem 1.1 and Proposition 1.2.

Corollary 1.3. *Let $\Omega \subset \mathbb{C}^N$ be a domain of class $L^{2,\infty}$ and $E \subset \partial\Omega$ be a closed subset of Hausdorff $(2N-2)$-dimensional measure zero. Suppose that the following hold:*

(i) *every point in* $\partial\Omega \setminus E$ *is either regular or an edge point;*
(ii) *for every* $a \in \partial\Omega$ *there exist a neighborhood* U_a *and a complex hypersurface* $N \subset \partial\Omega$ *that contains all complex edge points in* $\partial\Omega \cap U_a$*.*

Then Ω *is a domain of holomorphy if and only if the Levi form at every regular point* $a \in \partial\Omega \setminus E$ *is positively semidefinite and the tangent cone at every real edge point* $a \in \partial\Omega \setminus E$ *is convex.*

Finally we would like to mention that the statements of Theorem 1.1 (and of Corollary 1.3) also hold for relatively compact domains in Stein manifolds. Indeed, in this case Theorem 1.1 implies that the domain is *locally Stein*. Hence it is a domain of holomorphy by a result of Fornaess and Narasimhan ([FN80], Theorem 3.1.1).

2. The Levi form and plurisubharmonicity

Recall that the *Levi form* at a point a of a real function ρ of class C^2 in an open subset of $\mathbb{C}^N$ is the Hermitian form defined in local holomorphic coordinates $z = (z_1, \dots, z_N)$ by

$$L\rho(\xi,\eta) = L\rho(a)(\xi,\eta) := \sum_{k,l} \frac{\partial^2\rho}{\partial z^k \partial \overline{z^l}}(a)\, \xi^k \overline{\eta^l}, \quad \xi, \eta \in \mathbb{C}^N.$$

We write

$$\partial\rho(a)^\perp := \{\xi \in \mathbb{C}^N : \partial\rho(a)(\xi) = 0\}.$$

The *Levi form of a domain* Ω at a regular point $a \subset \partial\Omega$ is the restriction $L\rho|_{\partial\rho^\perp}$, where $d\rho \neq 0$ and Ω is locally given by $\rho < 0$. The norms are defined in the standard way:

$$\|\partial\rho(a)\| := \sup_{\|\xi\|=1} |\partial\rho(a)(\xi)|, \quad \|L\rho(a)\| := \sup_{\|\xi\|=\|\eta\|=1} |L\rho(a)(\xi,\eta)|.$$

Lemma 2.1. *Let* $\rho < 0$ *be a negative function of class* C^2 *in an open subset of* $\mathbb{C}^N$ *and* $\lambda > 0$ *be a constant such that the following holds:*
(i) $L\rho|_{\partial\rho^\perp}$ *is positive definite;*
(ii) $\|L\rho\|^2 \le \lambda(\|\partial\rho\|^2 + \rho\|L\rho\|)$*.*

Then the function $\phi(z) := -\log(-\rho(z)) + \lambda\|z\|^2$ *is plurisubharmonic.*

Proof. We have

$$L\phi(\xi,\eta) = \rho^{-2}\partial\rho(\xi)\overline{\partial\rho(\eta)} - \rho^{-1}L\rho(\xi,\eta) + \lambda\langle\xi,\eta\rangle, \tag{3}$$

where $\langle\xi,\eta\rangle := \xi_1\overline{\eta}_1 + \cdots + \xi_N\overline{\eta}_N$. Every vector $\zeta \in \mathbb{C}^N$ can be written as $\zeta = \zeta_1 + \zeta_2$ with

$$|\partial\rho(a)(\zeta_1)| = \|\partial\rho(a)\| \cdot \|\zeta_1\| \text{ and } \partial\rho(a)(\zeta_2) = 0. \tag{4}$$

Applying (3) to $\xi = \eta = \alpha_1\zeta_1 + \alpha_2\zeta_2$ with ζ_1, ζ_2 satisfying (4) we obtain

$$L\phi(a)(\alpha_1\zeta_1 + \alpha_2\zeta_2, \alpha_1\zeta_1 + \alpha_2\zeta_2) = (\alpha_1, \alpha_2)(A + B)\begin{pmatrix}\overline{\alpha}_1\\ \overline{\alpha}_2\end{pmatrix},$$

where

$$A = \begin{pmatrix}\rho^{-2}\|\partial\rho\|^2 \cdot \|\zeta_1\|^2 - \rho^{-1}L\rho(\zeta_1,\zeta_1) & -\rho^{-1}L\rho(\zeta_1,\zeta_2)\\ -\rho^{-1}L\rho(\zeta_2,\zeta_1) & \lambda\|\zeta_2\|^2\end{pmatrix}$$

and

$$B = \begin{pmatrix}\lambda\|\zeta_1\|^2 & 0\\ 0 & -\rho^{-1}L\rho(\zeta_2,\zeta_2)\end{pmatrix}.$$

The matrix B is positively semidefinite by (i). It is sufficient to show that A is also positively semidefinite, i.e. $\det A \geq 0$ by Sylvester's criterion. But this follows from (ii):

$$\det A \geq \rho^{-2}(\lambda\|\partial\rho\|^2 + \lambda\rho\|L\rho\| - \|L\rho\|^2)\|\zeta_1\|^2\|\zeta_2\|^2 \geq 0.$$

□

3. Piecewise plurisubharmonicity

3.1. **One-dimensional case.** In the following let $I \subset \mathbb{R}$ denote an open interval and $A \subset I$ a finite subset. If f is continuously differentiable with bounded derivative on $I \setminus A$, then for every $a \in S$ there exist one-sided limits

$$f(a)_- := \lim_{x\to a,\, x<a} f(x) \quad \text{and} \quad f(a)_+ := \lim_{x\to a,\, x>a} f(x).$$

By elementary calculus we have

Lemma 3.1. *Let $A \subset I$ and f be as before and suppose that f has compact support in I. Then*

$$\int_I f' dx + \sum_{a\in A}(f(a)_+ - f(a)_-) = 0.$$

Corollary 3.2. *Let $A \subset I$ be as before, $\phi \in C^0(I)\cap C^2(I\setminus A)\cap L^{2,\infty}(I)$ be arbitrary and $\alpha \in C^2(I)$ have compact support in I. Then*

$$\int_I (\phi\alpha'' - \phi''\alpha)dx = \sum_{a\in A}(\phi'(a)_+ - \phi'(a)_-)\alpha(a). \tag{5}$$

The corollary is obtained by applying Lemma 3.1 to $f := \phi\alpha' - \phi'\alpha$.

3.2. **Higher-dimensional case.** Now consider an open subset $\Omega \subset \mathbb{R}^n$ and let $S \subset \Omega$ be a (locally closed) hypersurface of class C^1. Given a point $a_0 \in S$ we fix a neighborhood $U \subset \Omega$ of a_0 such that $U \setminus S$ has exactly two connected components U_+ and U_-.

Lemma 3.3. *Let $\phi \in C^2(U \setminus S) \cap L^{2,\infty}(U)$ be a function in U. Then for every $v \in \mathbb{R}^n$ the directional derivatives $D_v\phi|_{U_-}$ and $D_v\phi|_{U_+}$ extend Lipschitz-continuously to $U_- \cup S$ and $U_+ \cup S$ respectively with the one-sided limits*

$$D_v\phi(a)_- := \lim_{x\to a,\, x\in U_-} D_v\phi(x) \quad \text{and} \quad D_v\phi(a)_+ := \lim_{x\to a,\, x\in U_+} D_v\phi(x) \tag{6}$$

for $a \in S$. Moreover, if ϕ is in addition continuous on U, one has $D_v\phi(a)_- = D_v\phi(a)_+$ whenever v is tangent to S at a. In particular, the sign of the expression

$$D_v\phi(a)_+ - D_v\phi(a)_- \tag{7}$$

is independent of the choice of a transversal vector v pointing into U_+.

Proof. Boundedness of the first and second derivatives of ϕ on U_- and U_+ implies the one-sided Lipschitz extendibility of ϕ and its first derivatives. In particular, if ϕ is continuous on U, then the restriction $\phi|_S$ coincides with both one-sided limits. Hence for v tangent to S, one has $D_v\phi_+ = D_v(\phi|_S) = D_v\phi_-$ as required. □

We observe that, if we interchange U_- with U_+, the sign of (7) remains the same, because also v (pointing into U_+) changes the sign. This consideration motivates the following definition.

Definition 3.4. *A function $\phi \in C^2(U \setminus S) \cap L^{2,\infty}(U)$, continuous in U, is said to be* transversally convex *at S if the expression* (7) *is non-negative for any $a \in S$ and any v pointing into U_+.*

In the following we write $\mathcal{H}^m$ for the Hausdorff m-dimensional measure.

Proposition 3.5. *Let $\Omega \subset \mathbb{R}^n$ be open, G be closed in Ω with $\mathcal{H}^{n-1}(G) = 0$, $S \subset \Omega$ be a hypersurface of class C^1 with $\overline{S} \setminus S \subset G$ and*

$$\phi \in C^0(\Omega) \cap C^2(\Omega \setminus (S \cup G)) \cap L^{2,\infty}(\Omega)$$

be a function which is transversally convex at S. Then for every non-negative function $\alpha \in C^2(\Omega)$ with compact support in Ω the inequality

$$\int_\Omega \phi \,\frac{\partial^2 \alpha}{\partial x_j^2}\, dx \geq \int_\Omega \frac{\partial^2 \phi}{\partial x_j^2}\, \alpha \, dx \tag{8}$$

holds for every $j = 1, \dots, n$.

Remark. Since all functions in (8) are measurable and bounded, both integrals exist with respect to the Lebesgue measure on $\mathbb{R}^n$.

Proof. Without loss of generality we may assume $j = n$ and, by taking a suitable partition of unity,

$$\operatorname{supp}(\alpha) \subset U' \times I \subset \Omega$$

for an open subset $U' \subset \mathbb{R}^{n-1}$ and an interval $I \subset \mathbb{R}$. We write $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Since $\mathcal{H}^{n-1}(G) = 0$, we see from Fubini's theorem that $G \cap (\{x'\} \times I) = \emptyset$ for all x' outside a zero measure subset $G' \subset U'$. Furthermore, by Sard's theorem applied to the projection $S \mapsto \mathbb{R}^{n-1}$, the vector $v := (0, 1)$ is not tangent to S at the points of $S \cap (\{x'\} \times I)$ for all x' outside a zero measure subset $G'' \subset U'$. Then for $x' \notin G' \cup G''$ the set

$$A_{x'} := \{x_n \in I : (x', x_n) \in (S \cap \operatorname{supp}(\alpha))\}$$

is finite. Thus we can apply Corollary 3.2 to the restriction of ϕ and α to $\{x'\} \times I$. According to Definition 3.4, the right-hand side in (5) is non-negative. Then the required inequality is obtained by integrating over $U' \setminus (G' \cup G'')$ the non-negative left-hand side of (5). □

Corollary 3.6. *Under the assumptions of Proposition* 3.5 *suppose that* $\Omega \subset \mathbb{C}^N$ *and* ϕ *is plurisubharmonic in* $\Omega \setminus (S \cup G)$. *Then* ϕ *is plurisubharmonic in the whole* Ω.

Proof. Given a vector $\xi \in \mathbb{C}^N$ we can find linear complex coordinates $z_k = x_k + iy_k$ $(k = 1, \ldots, N)$ such that $\xi = (0, \ldots, 0, 1)$. Then

$$L\phi(a)(\xi, \xi) = \frac{\partial^2 \phi}{\partial z_N \partial \overline{z_N}}(a)\xi^N \overline{\xi^N} = \frac{1}{2}\left(\frac{\partial^2 \phi}{\partial x_N^2}(a) + \frac{\partial^2 \phi}{\partial y_N^2}(a)\right) \geq 0 \quad (9)$$

for all $a \in \Omega \setminus (S \cup G)$ by the plurisubharmonicity of ϕ there. Since $S \cup G \subset \Omega$ is of zero measure, (9) implies

$$\int_\Omega \frac{\partial^2 \phi}{\partial x_N^2}\, \alpha \, dz\overline{dz} + \int_\Omega \frac{\partial^2 \phi}{\partial y_N^2}\, \alpha \, dz\overline{dz} \geq 0 \quad (10)$$

for every non-negative function $\alpha \in C^2(\Omega)$ with compact support in Ω. By Proposition 3.5,

$$2\int_\Omega \phi \, L\alpha(a)(\xi, \xi) dz\overline{dz} = \int_\Omega \phi \, \frac{\partial^2 \alpha}{\partial x_N^2}\, d\Omega + \int_\Omega \phi \, \frac{\partial^2 \alpha}{\partial y_N^2}\, d\Omega \geq 0. \quad (11)$$

Since ξ is arbitrary, we conclude that ϕ has a non-negative Levi form in distributional sense. By the continuity of ϕ, the last fact is equivalent to the plurisubharmonicity of ϕ on the whole Ω (see [H90], Theorem 1.6.11). □

4. Proof of Theorem 1.1

Suppose that conditions (i) and (ii) hold. Since a domain in $\mathbb{C}^N$ is a domains of holomorphy if it is pseudoconvex (see [H90], Theorem 4.2.8) and due to the local characterization of pseudoconvexity ([H90], Theorem 2.6.10) it is sufficient to show that every point $x_0 \in \partial\Omega$ has a neighborhood U such that $\Omega \cap U$ is pseudoconvex. We use the standard identification $\mathbb{C}^N \cong \mathbb{R}^n$ with $n := 2N$. Since Ω is of class $L^{2,\infty}$, we can choose a neighborhood $U = U' \times I \subset \mathbb{R}^{n-1} \times \mathbb{R}$ of x_0 and a continuous function $h\colon U' \to I$ of class $L^{2,\infty}$ such that (1) holds. Define a continuous function $\rho\colon U \to \mathbb{R}$ with $\Omega \cap U = \{\rho < 0\}$ by

$$\rho(x', x_n) := h(x') - x_n.$$

Set $\widetilde{\Omega} := \Omega \cap U$ and denote by $\widetilde{\Omega}_{\rm reg}$ the subset of all $x = (x', x_n) \in \widetilde{\Omega}$ such that $(x', h(x')) \in \partial\Omega$ is (C^2-)regular. We wish to apply Lemma 2.1 to the restriction of ρ to $\widetilde{\Omega}_{\rm reg}$. To check the assumption (i) in Lemma 2.1 we observe that

$$L\rho(x', x_n) = L\rho(x', h(x')), \quad \partial\rho(x', x_n) = \partial\rho(x', h(x')) \tag{12}$$

for all $x \in \widetilde{\Omega}_{\rm reg}$. Hence it follows by condition (ii) in Theorem 1.1 that $L\rho(x)|_{\partial\rho(x)^\perp}$ is positively semidefinite whenever $x \in \widetilde{\Omega}_{\rm reg}$. In order to satisfy the assumption (ii) in Lemma 2.1 we shrink the neighborhood U of x_0 such that

$$|\rho(x)|\, \|L\rho(x)\| \le 1/8 \tag{13}$$

holds for all $x \in \widetilde{\Omega}_{\rm reg}$. This is possible because the second derivatives of ρ are bounded on $\widetilde{\Omega}_{\rm reg}$ and ρ is continuous with $\rho(x_0) = 0$. Since $\|\partial\rho(x)\| \ge 1/2$ for $x \in \widetilde{\Omega}_{\rm reg}$, (13) implies

$$\|\partial\rho\|^2 + \rho\|L\rho\| \ge 1/4 - 1/8 = 1/8$$

on $\widetilde{\Omega}_{\rm reg}$ and hence the existence of $\lambda > 0$ such that the assumption (ii) in Lemma 2.1 in satisfied. By Lemma 2.1, the function

$$\phi\colon \widetilde{\Omega} \to \mathbb{R}, \quad \phi(z) := -\log(-\rho(z)) + \lambda\|z\|$$

is plurisubharmonic in $\widetilde{\Omega}_{\rm reg}$.

Next we wish to apply Corollary 3.6 to ϕ in $\widetilde{\Omega}$. For this we first construct a C^1 hypersurface $S \subset \widetilde{\Omega}$ satisfying the assumptions. By using the definition, for every edge point $a \in \partial\Omega \cap U$, we can choose a neighborhood $U_a = U'_a \times I \subset \mathbb{R}^{n-1} \times \mathbb{R}$ with $U'_a \subset \mathbb{R}^{n-1}$ a euclidean ball and a closed $(n-2)$-dimensional real submanifold $M_a \subset \partial\Omega \cap U_a$ of class C^1 such that all nonregular points of $\partial\Omega \cap U_a$ are contained in M_a. Furthermore we can shrink U_a and M_a such that all nonregular points of $\partial\Omega \cap \overline{U_a}$ are contained in $\overline{M_a}$ and $\mathcal{H}^{n-2}(\overline{M_a} \setminus M_a) = 0$. By the choice of the coordinates (x', x_n), the projection

$$M'_a := \{x' \in \mathbb{R}^{n-1} : \exists x_n, (x', x_n) \in M_a\}$$

is a closed submanifold of U'_a. In view of condition (i) any nonregular point of $\partial\Omega \setminus E$ is an edge point. By considering a sequence of all rational points in $\mathbb{C}^N$ we may choose a sequence a_m $(1 \le m < \infty)$ of edge points such that the union $\cup_m (V'_{a_m} \times I)$ covers all nonregular points in $(\partial\Omega \setminus E) \cap U$, where $V'_{a_m} \times I$ is an open neighborhood of a_m and $\overline{V'_{a_m}} \subset U'_{a_m}$. Then by applying Fubini's theorem, we may shrink the balls U'_{a_m} and their submanifolds M'_{a_m} to obtain the additional property

$$\mathcal{H}^{n-2}(\partial U'_{a_m} \cap M'_{a_k}) = 0 \text{ for all } m, k \ge 1. \tag{14}$$

Define $U'_m := \cup_{j \le m} U'_{a_j}$.

We claim that there exist increasing sequences of C^1 submanifolds $S'_m \subset U'_m$ and closed subsets $E'_m \subset U'_m$ with $\mathcal{H}^{n-2}(E'_m) = 0$ such that

- (i) $S'_k \cap U'_m = S'_m$ and $E'_k \cap U'_m = E'_m$ for all $k \ge m$,
- (ii) all nonregular points in $(\partial\Omega \setminus E) \cap (\overline{U'_m} \times I)$ outside $E'_m \times I$ are contained in $\overline{S'_m} \times I$,
- (iii) $\mathcal{H}^{n-2}(\overline{S'_m} \setminus S'_m) = 0$.

We construct the sequences S'_m and E'_m by induction on m. For this we set $S'_1 := M'_{a_1}$, $E'_1 := \emptyset$. If S'_{m-1} and E'_{m-1} are already constructed, define

$$S'_m := S'_{m-1} \cup (M'_{a_m} \cap (U'_{a_m} \setminus \overline{U'_{m-1}}))$$

and $E'_m := E_{m-1} \cup (\overline{M'_{a_m}} \cap \partial U'_{m-1})$. It is easy to see from our construction that (i)-(iii) are satisfied.

Define

$$S := ((\cup_m S'_m) \times I) \cap \widetilde{\Omega} \quad \text{and} \quad F := ((\cup_m E'_m) \times I) \cap \widetilde{\Omega}.$$

Then $\mathcal{H}^{n-1}(F) = 0$ and F is closed in $\widetilde{\Omega}$ by (i). It follows from (i) that S is a C^1 hypersurface in $\widetilde{\Omega}$. Furthermore, $\mathcal{H}^{n-1}(\overline{S} \setminus S) = 0$ by (iii).

Denote by $E' \subset U'$ the projection of the subset $E \subset \partial\Omega$. Since $\partial\Omega \cap U$ is the graph of h, E' is closed in U'. Next we subtract $E' \times I$ from S and denote the remainder again by S. Then for every $x = (x', x_n) \in S$, $(x', h(x')) \in \partial\Omega$ is either C^2-regular or an edge point. Finally we define

$$G := F \cup (\overline{S} \setminus S) \cup ((E' \times I) \cap \widetilde{\Omega}),$$

where the closure of S is taken in $\widetilde{\Omega}$. By (ii), $\widetilde{\Omega} \setminus (S \cup G) \subset \widetilde{\Omega}_{\text{reg}}$ and therefore ϕ is plurisubharmonic in $\widetilde{\Omega} \setminus (S \cup G)$.

In order to apply Corollary 3.6 it remains to show that ϕ is transversally convex at S (see Definition 3.4). But this is a direct consequence of condition (iii) in Theorem 1.1 on the convexity of tangent cones. We conclude that ϕ is plurisubharmonic on $\widetilde{\Omega}$. Then, for a sufficiently small euclidean ball $B(x_0, \varepsilon)$ centered at x_0,

$$\max\big(\phi(z), -\log(\varepsilon - ||z - x_0||)\big)$$

is a plurisubharmonic exhaustion function for $\Omega \cap B(x_0, \varepsilon)$. This shows pseudoconvexity of $\Omega \cap B(x_0, \varepsilon)$ and hence completes the proof.

5. Proof of Proposition 1.2

Let $a \subset \partial\Omega$ be a real edge point satisfying the assumptions of Proposition 1.2. As before we choose a neighborhood $U = U' \times I \subset \mathbb{R}^{n-1} \times \mathbb{R}$ of a and a continuous function $h \colon U' \to I$ of class $L^{2,\infty}$ such that (1) holds. Let $M_a \subset \partial\Omega$ be an edge and denote by M'_a its projection on U'. As in §3.2 choose U' sufficiently small such that M'_a divides U' into two parts U'_- and U'_+. Then the tangent cone is given by

$$T_a\Omega = \{(v', v_n) \in \mathbb{R}^n : v_n \geq D_{v'}h(a)_\pm \text{ for } v' \in T_aU'_\pm\}, \tag{15}$$

where the one-sided limits exist by Lemma 3.3.

We first suppose that M_a is generic at a, i.e. $T_aM_a + iT_aM_a = \mathbb{C}^N$. We prove the statement by contradiction assuming that the tangent cone of Ω at a is not convex. Then there exists a linear disc $A \colon \Delta \to T_a\Omega$, $t \mapsto tv$ with $v \in T_aM_a$ and $A(\partial\Delta \setminus \{-1, 1\})$ in the interior of $T_a\Omega$. Here $\Delta := \{|t| < 1\} \subset \mathbb{C}$. Furthermore, for $\zeta := (0, 1) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $\varepsilon > 0$ sufficiently small, the "deformed disc" $A'(t) := tv + t^2\varepsilon\zeta$ sends the whole boundary $\partial\Delta$ to the interior of $T_a\Omega$. Finally for $\mu > 0$ sufficiently small the "rescaled disc" $A''(t) := a + \mu A'(t)$ sends the boundary $\partial\Delta$ into Ω. Then the Cauchy formula argument for A'' and its translations shows that all holomorphic functions in Ω extend holomorphically to a neighborhood of $a = A''(0)$ which contradicts the assumption that Ω is a domain of holomorphy.

Now consider the general case. Let M_a be an edge at a satisfying the assumptions. Then M_a contains generic points arbitrarily close to a. Hence the tangent cones at those points are convex by the above argument. But then the explicit formula (15) shows that the cone $T_a\Omega$ is also convex completing the proof.

6. Proof of Corollary 1.3

The necessity of the Levi form condition is well-known. The necessity of the convexity follows from Proposition 1.2. For the converse it is sufficient to show the local pseudoconvexity at every point $a \in \partial\Omega$ as in §4. Let $N \subset U_a$ be given by condition (ii). By a coordinate change we may assume that N is locally given by $z_N = 0$. Then it follows from (15) that, if U_a is sufficiently small polydisc centered at a, the intersection $\Omega \cap U_a$ can be mapped via

$$(z_1, \ldots, z_{N-1}, z_N) \mapsto (z_1, \ldots, z_{N-1}, z_N^\alpha)$$

biholomorphically onto a domain satisfying assumptions of Theorem 1.1. Here α is a sufficiently small positive number. The required conclusion follows now from Theorem 1.1.

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