Cayley Hypersurfaces

Michael Eastwood and Vladimir Ezhov

Abstract: We exhibit a family of homogeneous hypersurfaces in affine space, one in each dimension, generalising the Cayley surface.

The Cayley surface in affine three space is given by

$$x_3 = x_1x_2 - \tfrac{1}{3}{x_1}^3.$$

See, for example, [3, Chapter III §6] for a discussion of its properties. In N-dimensions, we may consider the hypersurface given by

$$\Phi_N(x_1, x_2, \ldots, x_N) \equiv \sum_{d=1}^{N}(-1)^d \tfrac{1}{d} \sum_{i+j+\cdots+m=N} \overbrace{x_i x_j \cdots x_m}^{d} = 0. \tag{1}$$

This is the Cayley surface when $N = 3$. The next few are as follows.

$$x_4 = x_1x_3 + \tfrac{1}{2}{x_2}^2 - {x_1}^2x_2 + \tfrac{1}{4}{x_1}^4$$
$$x_5 = x_1x_4 + x_2x_3 - {x_1}^2x_3 - x_1{x_2}^2 + {x_1}^3x_2 - \tfrac{1}{5}{x_1}^5$$
$$x_6 = x_1x_5 + x_2x_4 + \tfrac{1}{2}{x_3}^2 - {x_1}^2x_4 - 2x_1x_2x_3 - \tfrac{1}{3}{x_2}^3 + {x_1}^3x_3 + \tfrac{3}{2}{x_1}^2{x_2}^2 - {x_1}^4x_2 + \tfrac{1}{6}{x_1}^6.$$

Since the first term in (1) is $-x_N$ and this is the only occurrence of this variable, these hypersurfaces are polynomial graphs over the remaining variables.

The Cayley surface is affine homogeneous. This follows immediately from Φ_3 being annihilated by the following two linearly independent affine vector fields:

$$\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_3} \quad \text{and} \quad \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3}.$$

The hypersurface defined by (1) generalises sufficiently many properties of the Cayley surface that we call it the Cayley hypersurface. That it is affine homogeneous is an immediate consequence of the following:

Proposition 1 *The polynomial $\Phi_N(x_1, x_2, \ldots, x_N)$ is annihilated by the vector fields*

$$X_p \equiv \frac{\partial}{\partial x_p} + \sum_{h=p+1}^{N} x_{h-p}\frac{\partial}{\partial x_h} \tag{2}$$

for $p = 1, 2, \ldots, N-1$.

Proof. We compute

$$\frac{\partial}{\partial x_h}\Phi_N = \sum_{d=1}^{N}(-1)^d \sum_{j+\cdots+m=N-h} \overbrace{x_j \cdots x_m}^{d-1} \quad \text{for } h = 1, 2, \ldots N$$

with the convention that a product with no terms is 1 (so that when $h = N$, this formula gives -1 and, otherwise, we can start the sum at $d = 2$). Therefore,

$$\begin{aligned} X_p\Phi_N &= \sum_{d=2}^{N}(-1)^d \sum_{j+\cdots+m=N-p} \overbrace{x_j \cdots x_m}^{d-1} + \sum_{h=p+1}^{N} x_{h-p} \sum_{d=1}^{N}(-1)^d \sum_{j+\cdots+m=N-h} \overbrace{x_j \cdots x_m}^{d-1} \\ &= \sum_{d=2}^{N}(-1)^d \sum_{j+\cdots+m=N-p} \overbrace{x_j \cdots x_m}^{d-1} + \sum_{d=1}^{N}(-1)^d \sum_{j+\cdots+m=N-p} \overbrace{x_i x_j \cdots x_m}^{d}. \end{aligned}$$

These expressions evidently cancel. □

Proposition 2 *The Cayley hypersurface admits a transitive Abelian group of affine motions.*

Proof. The vector fields (2) commute and so may be exponentiated to the required Abelian group. (In fact, this is how the Cayley surface is defined in [3, p. 93].) □

Proposition 3 *The isotropy algebra of the Cayley hypersurface is generated by*

$$H \equiv \sum_{h=1}^{N} h x_h \frac{\partial}{\partial x_h}. \tag{3}$$

Proof. Each term in Φ_N is weighted homogeneous of weight N if x_h has weight h. It follows that $H\Phi_N = N\Phi_N$ and, in particular, $H\Phi_N|_{\{\Phi_N=0\}} = 0$. We are required to prove that, up to scale, H is the only vector field of the form

$$X = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} x_i \frac{\partial}{\partial x_j}$$

with this property. Since Φ_N is an irreducible polynomial of degree N and $X\Phi_N$ is a polynomial of degree at most N, if $X\Phi_N$ vanishes along $\{\Phi_N = 0\}$, then $X\Phi = c\Phi_N$ for some constant c. Therefore, after adding a suitable multiple of H, it suffices to show that if $X\Phi_N = 0$, then $X = 0$. Notice that Φ_N has the following form

$$\pm \underbrace{{x_1}^N/N}_{\deg=N} \mp \underbrace{{x_1}^{N-2}x_2}_{\deg=N-1} \pm \underbrace{{x_1}^{N-3}x_3 + p(x_1,x_2)}_{\deg=N-2} \mp \underbrace{{x_1}^{N-4}x_4 + q(x_1,x_2,x_3)}_{\deg=N-3} \pm \cdots$$

for suitable polynomials $p(x_1, x_2)$, $q(x_1, x_2, x_3)$, ... By considering firstly the leading term ${x_N}^N/N$, it follows that X cannot have any terms in $\partial/\partial x_1$. Alternatively, it is this consideration which determines which multiple of H to use in the initial modification. Then, by looking at the term of degree $N-1$, we see that X cannot involve $\partial/\partial x_2$. Then the terms of degree $N-2$ dispense with $\partial/\partial x_3$ and so on. Working through Φ_N in this way, it follows that $X = 0$. □

Proposition 4 *The affine normals of the Cayley hypersurface are everywhere parallel to the x_N-axis.*

Proof. At the origin we have

$$x_N = \sum_{i,j} g^{ij} x_i x_j + \sum_{i,j,k} a^{ijk} x_i x_j x_k + \cdots$$

where

$$g^{ij} = \begin{cases} 1 & \text{if } i+j=N \\ 0 & \text{else} \end{cases} \quad \text{and} \quad a^{ijk} = \begin{cases} 1 & \text{if } i+j+k=N \\ 0 & \text{else.} \end{cases} \tag{4}$$

Thus g_{ij}, the inverse of g^{ij}, is given by the same formula and $\sum_{i,j} g_{ij} a^{ijk} = 0$. In fact, all the higher order tensors are trace-free too. This property of the cubic terms characterises the affine normal [4]. Note from (2) that the symmetries X_p all commute with $\partial/\partial x_N$. It follows that they preserve the x_N-direction but, on the other hand, since the affine normal is affinely invariant, these symmetries take one affine normal to another. For the isotropy symmetry (3) we have $[\partial/\partial x_N, H] = N(\partial/\partial x_N)$ which says that the corresponding 1-parameter subgroup simply rescales x_N along its axis. □

We conjecture these various properties are enough to characterise these hypersurfaces:

Conjecture 1 *Suppose Σ is a non-degenerate hypersurface in affine N-space such that:*

- *Σ admits a transitive Abelian group of affine motions*
- *The full symmetry group of Σ has one-dimensional isotropy*
- *The affine normals to Σ are everywhere parallel*
  (i.e. Σ is an 'improper affine hypersphere').

Then Σ is given by (1) *in a suitable affine coördinate system.*

We have verified this for hypersurfaces in four dimensions (as a special case of classifying the homogeneous non-degenerate hypersurfaces with isotropy or classifying those with a transitive Abelian group of affine motions). Details will appear elsewhere. If two-dimensional isotropy is allowed, then another variation on the Cayley surface arises, namely

$$x_4 = x_1 x_3 + \tfrac{1}{2} {x_2}^2 - \tfrac{1}{3} {x_1}^3.$$

This sort of variation is discussed in [1] and [3, pp. 121–122].

In [5], Nomizu and Pinkall give a differential geometric characterisation of the Cayley surface: it is not assumed *a priori* that the surface is homogeneous. It is not clear how to extend this characterisation to Cayley hypersurfaces.

Yet another generalisation of Cayley surface, is suggested by Dillen and Vrancken [2] as a hypersurface with parallel difference tensor together with some genericity condition.

The explicit defining equation (6.3) of [2] very much resembles (1). Curiously, there is a whole family of homogeneous hypersurfaces

$$\sum_{d=1}^{N}(-1)^d \tfrac{1}{d!}\prod_{n=0}^{d-3}[(1-b)n+2]\sum_{i+j+\cdots+m=N}\overbrace{x_i x_j \cdots x_m}^{d} = 0$$

interpolating between them. If $b = 0$ this is (1) and if $b = 1$ it is (6.3) of [2].

The Cayley surfaces are ruled. This property also extends to hypersurfaces:

Proposition 5 *If N is odd, then the Cayley hypersurface is uniquely ruled by $(N-1)/2$-planes. If N is even, then it is uniquely ruled by $(N-2)/2$-planes.*

Proof. If N is odd and we fix $x_1, x_2, \ldots, x_{(N-1)/2}$, then (1) is linear in the remaining variables. If N is even and we fix $x_1, x_2, \ldots, x_{N/2}$, then (1) is linear in the remaining variables. In both cases, uniqueness follows by examining the quadratic terms which are non-degenerate of split signature. These determine the possible directions in which a maximal embedded plane may point, only one of which is consistent with the higher order terms. □

For surfaces, the Pick invariant is precisely the third order obstruction to its being ruled. Though there is no such obstruction in higher dimensions, we have:

Proposition 6 *The Cayley hypersurfaces have vanishing Pick invariant.*

Proof. With reference to (4), the Pick invariant is

$$\sum_{i,j,k,l,m,n} g_{il}g_{jm}g_{kn}a^{ijk}a^{lmn}$$

which evidently vanishes. □

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Department of Pure Mathematics
University of Adelaide
South AUSTRALIA 5005

E-mail: meastwoo@spam.maths.adelaide.edu.au
vezhov@spam.maths.adelaide.edu.au