Twist deformations for generalized Heisenberg algebras

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Abstract

Multidimensional Heisenberg algebras, whose creation a^{+} and annihilation a^{-} operators are the n-dimensional vectors, can be injected into simple Lie algebras g. It is demonstrated that the spectrum of their deformations can be investigated using chains of extended Jordanian twists applied to $U(g)$. In the case of $U(sl(N))$ (for $N > 5$) the two-dimensional Heisenberg subalgebras $\widetilde{\mathcal{H}}$ have nine deformed costructures connected by four "internal" and "external" twists composing the commutative diagram.

1 Introduction

A Hopf algebra $\mathcal{A}(m, \Delta, \epsilon, S)$ can be transformed [1] by an invertible **twisting element** $\mathcal{F} \in \mathcal{A} \otimes \mathcal{A}$, into a **twisted algebra** $\mathcal{A}_{\mathcal{F}}(m, \Delta_{\mathcal{F}}, \epsilon, S_{\mathcal{F}})$, that has the same multiplication and counit but the twisted coproduct and the antipode given by

$$\Delta_{\mathcal{F}}(a) = \mathcal{F}\Delta(a)\mathcal{F}^{-1}, S_{\mathcal{F}}(a) = vS(a)v^{-1}, v = \sum f_i^{(1)} S(f_i^{(2)}), a \in \mathcal{A}. \quad (1)$$

The twisting element has to satisfy the equations

$$\mathcal{F}_{12}(\Delta \otimes id)(\mathcal{F}) = \mathcal{F}_{23}(id \otimes \Delta)(\mathcal{F}), \quad (\epsilon \otimes id)(\mathcal{F}) = (id \otimes \epsilon)(\mathcal{F}) = 1. \quad (2)$$

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Let $\mathcal{A}$ and $\mathcal{B}$ be the universal enveloping algebras: $\mathcal{A} = U(l) \subset \mathcal{B} = U(g)$ with $l \subset g$. If $U(l)$ is the minimal subalgebra on which $\mathcal{F}$ is completely defined as $\mathcal{F} \in U(l) \otimes U(l)$ then l is called the **carrier** algebra for $\mathcal{F}$.

The first and well known example of the twist that was written in the explicit form [2] corresponds to the carrier subalgebra $B(2)$ with generators H and E, $[H, E] = E$. It is called the **Jordanian twist** and has the twisting element

$$\Phi_{\mathcal{J}} = e^{H \otimes \sigma}, \qquad \sigma = \ln(1 + E). \tag{3}$$

For the **extended Jordanian twists** suggested in [3] the carrier subalgebra **L** is four-dimensional:

$$\begin{array}{l} [H, E] = E, \quad [H, A] = \alpha A, \quad [H, B] = \beta B, \\ [A, B] = E, \quad [E, A] = [E, B] = 0, \qquad \alpha + \beta = 1. \end{array} \tag{4}$$

The corresponding twisting element contains the **Jordanian factor**:

$$\mathcal{F}_{\mathcal{E}(\alpha,\beta)} = \Phi_{\mathcal{E}(\alpha,\beta)} \Phi_{\mathcal{J}} \tag{5}$$

and the **extension**

$$\Phi_{\mathcal{E}(\alpha,\beta)} = \exp\{A \otimes B e^{-\beta\sigma}\}. \tag{6}$$

This twist defines the deformed Hopf algebras $\mathbf{L}_{\mathcal{E}(\alpha,\beta)}$ with the costructure

$$\begin{array}{lcl} \Delta_{\mathcal{E}(\alpha,\beta)}(H) & = & H \otimes e^{-\sigma} + 1 \otimes H - A \otimes B e^{-(\beta+1)\sigma}, \\ \Delta_{\mathcal{E}(\alpha,\beta)}(A) & = & A \otimes e^{-\beta\sigma} + 1 \otimes A, \\ \Delta_{\mathcal{E}(\alpha,\beta)}(B) & = & B \otimes e^{\beta\sigma} + e^{\sigma} \otimes B, \\ \Delta_{\mathcal{E}(\alpha,\beta)}(E) & = & E \otimes e^{\sigma} + 1 \otimes E. \end{array} \tag{7}$$

In general the composition of two twists is not a twist. But there are some important examples of the opposite behavior. When **L** is a subalgebra **L** $\subset$g there may exist several pairs of generators of the type (A, B) arranged so that the Jordanian twist can acquire several similar extensions [3]. This demonstrates that some twistings can be applied successively to the initial Hopf algebra even in the case when their carrier subalgebras are nontrivially linked. In the universal enveloping algebras for classical Lie algebras there exists the possibility to construct systematically the special sequences of twists called **chains** [4]:

$$\mathcal{F}_{\mathcal{B}_{p\prec 0}} \equiv \mathcal{F}_{\mathcal{B}_p}\mathcal{F}_{\mathcal{B}_{p-1}} \ldots \mathcal{F}_{\mathcal{B}_0}. \qquad (8)$$

The factors $\mathcal{F}_{\mathcal{B}_k} = \Phi_{\mathcal{E}_k}\Phi_{\mathcal{J}_k}$ of the chain are the twisting elements of the extended Jordanian twists for the initial Hopf algebra $\mathcal{A}_0$. Here the extensions $\{\Phi_{\mathcal{E}_k}, k = 0, \ldots p-1\}$ contain the fixed set of normalized factors $\Phi_{\mathcal{E}(\alpha,\beta)} = \exp\{A \otimes Be^{-\beta\sigma}\}$, the so called **full set**. It was proved that in the classical Lie algebras that conserve symmetric invariant forms such chains can be made maximal and proper. This means that for the algebras $U(A_n)$, $U(B_n)$ and $U(D_n)$ there exist chains $\mathcal{F}_{\mathcal{B}_{p\prec 0}}$ that cannot be reduced to a chain for a simple subalgebra and their full sets of extensions are the maximal sets in the sense described below.

To construct a maximal proper chain for a classical Lie algebra the sequence $\mathcal{A} \equiv \mathcal{A}_0 \supset \mathcal{A}_1 \supset \ldots \supset \mathcal{A}_{p-1} \supset \mathcal{A}_p$ of Hopf subalgebras is to be fixed. For example in the case of $U(sl(N))$ the corresponding sequence is: $U(sl(N)) \supset U(sl(N-2)) \supset \ldots \supset U(sl(N-2k))\ldots$. In each element of the sequence the **initial root** λ_0^k must be chosen. Here $\lambda_0^k = e_1 - e_2$ for each $sl(M)$ (the roots are written in the standard e-basis). For the initial root let us form the set π_k of **constituent roots**,

$$\pi_k = \left\{\lambda', \lambda'' | \lambda' + \lambda'' = \lambda_0^k; \quad \lambda' + \lambda_0^k, \lambda'' + \lambda_0^k \neg \in \Lambda_{\mathcal{A}}\right\} \qquad (9)$$

(here $\Lambda_{\mathcal{A}}$ is the root system of $\mathcal{A}_0$). For each element $\lambda' \in \pi_k$ one can choose such an element $\lambda'' \in \pi_k$ that $\lambda' + \lambda'' = \lambda_0^k$. So, π_k is naturally decomposed as

$$\pi_k = \pi_k' \cup \pi_k'', \qquad \pi_k' = \{\lambda'\}, \quad \pi_k'' = \{\lambda''\}. \qquad (10)$$

In these terms the factors $\mathcal{F}_{\mathcal{B}_k}$ of the chain (8) are fixed as follows:

$$\mathcal{F}_{\mathcal{B}_k} = \Phi_{\mathcal{E}_k}\Phi_{\mathcal{J}_k} \tag{11}$$

with

$$\Phi_{\mathcal{J}_k} = \exp\{H_{\lambda_0^k} \otimes \sigma_0^k\}, \qquad \sigma_0^k = \ln(1 + L_{\lambda_0^k}); \tag{12}$$

$$\Phi_{\mathcal{E}_k} = \prod_{\lambda' \in \pi'_k} \Phi_{\mathcal{E}_{\lambda'}} = \prod_{\lambda' \in \pi'_k} \exp\{L_{\lambda'} \otimes L_{\lambda_0^k - \lambda'} e^{-\frac{1}{2}\sigma_0^k}\} \tag{13}$$

(here L_λ is the generator associated to the root λ).

2 Deformed Heisenberg algebras

One of the characteristic features of the extension $\Phi_{\mathcal{E}}$ in the ordinary extended Jordanian twists is that it connects the Heisenberg subalgebras $\mathcal{H}_{\mathcal{J}}$ and $\mathcal{H}_{\mathcal{E}\mathcal{J}}$ with $[A, B] = E, \alpha + \beta = 1$ and different deformed costructures (see [5]):

$$\begin{array}{c} \Phi_{\mathcal{E}} : \left\{ \begin{array}{lcl} \Delta_{\mathcal{J}}(A) & = & A \otimes e^{\alpha\sigma} + 1 \otimes A, \\ \Delta_{\mathcal{J}}(B) & = & B \otimes e^{\beta\sigma} + 1 \otimes B, \\ \Delta_{\mathcal{J}}(E) & = & E \otimes e^{\sigma} + 1 \otimes E, \end{array} \right\} \longrightarrow \\ \left\{ \begin{array}{lcl} \Delta_{\mathcal{E}\mathcal{J}}(A) & = & A \otimes e^{-\beta\sigma} + 1 \otimes A, \\ \Delta_{\mathcal{E}\mathcal{J}}(B) & = & B \otimes e^{\beta\sigma} + e^{\sigma} \otimes B, \\ \Delta_{\mathcal{E}\mathcal{J}}(E) & = & E \otimes e^{\sigma} + 1 \otimes E. \end{array} \right\} \end{array} \tag{14}$$

A chain of twists (8) $\mathcal{F}_{\mathcal{B}_{p\prec 0}} \equiv \mathcal{F}_{\mathcal{B}_p}\mathcal{F}_{\mathcal{B}_{p-1}} \ldots \mathcal{F}_{\mathcal{B}_0} : \mathcal{A} \longrightarrow \mathcal{A}_{\mathcal{B}_{p\prec 0}}$ contains $p + 1$ Jordanian factors $\Phi_{\mathcal{J}_k}$. Their product $\Phi_{\mathcal{J}_{p\prec 0}} \equiv \prod_k \Phi_{\mathcal{J}_{k-1}}$ is also a twisting element for the universal enveloping algebra $\mathcal{A}$, it produces the **multijordanian deformation** $\Phi_{\mathcal{J}_{p\prec 0}} : \mathcal{A} \longrightarrow \mathcal{A}_{\mathcal{J}_{p\prec 0}}$. This means that the product $\widehat{\Phi}_{\mathcal{E}_{p\prec 0}} \equiv \prod_k \widehat{\Phi}_{\mathcal{E}_k}$ with $\widehat{\Phi}_{\mathcal{E}_k} = (\prod_{m \geq k} \Phi_{\mathcal{J}_m})\Phi_{\mathcal{E}_k}(\prod_{m>k} \Phi_{\mathcal{J}_m})^{-1}$ is in turn a twisting element for $\mathcal{A}_{\mathcal{J}_{p\prec 0}}$, $\widehat{\Phi}_{\mathcal{E}_{p\prec 0}} : \mathcal{A}_{\mathcal{J}_{p\prec 0}} \longrightarrow \mathcal{A}_{B_{p\prec 0}}$.

In [6] it was shown that $\widehat{\Phi}_{\mathcal{E}_{p\prec 0}}$ plays the role of an extension for the multijordanian twists $\Phi_{\mathcal{J}_{p\prec 0}}$. It was proved that there are the subalgebras in $\mathcal{A}$ that are carriers for $\widehat{\Phi}_{\mathcal{E}_{p\prec 0}}$ and whose costructures are shifted from one possible deformed "state" to the other by the twists $\widehat{\Phi}_{\mathcal{E}_{p\prec 0}}$.

Here we consider a certain type of such subalgebras, namely the multidimensional Heisenberg subalgebras $\widetilde{\mathcal{H}}(2, N-4)$ in $U\left(sl(N)\right)$ generated by the 2×2-block of central and $2\times(N-4)$ pairs of creation and annihilation operators:

$$\begin{array}{cc} \boxed{\begin{array}{cccc} E_{13} & E_{14} & \ldots & E_{1,N-2} \\ E_{23} & E_{24} & \ldots & E_{2,N-2} \end{array}} & \boxed{\begin{array}{cc} E_{1,N-1} & E_{1N} \\ E_{2,N-1} & E_{2N} \end{array}} \\ & \boxed{\begin{array}{cc} E_{3,N-1} & E_{3N} \\ \vdots & \vdots \\ E_{N-3,N-1} & E_{N-3,N} \\ E_{N-2,N-1} & E_{N-2,N} \end{array}} \end{array} \tag{15}$$

We shall normalize the Cartan elements as $H_{i,k} = 1/2(E_{ii} - E_{kk})$, use the standard $gl(N)$-basis $\{E_{ij}\}_{i,j=1,\ldots N}$, and $\sigma = \ln(1+E)$.

First we shall study the properties of $\widetilde{\mathcal{H}}_{\mathcal{F}}(2, N-4)$ twisted by the factors of the 2-chain:

$$\mathcal{F}_{\mathcal{B}_{1\prec 0}} = \Phi_{\mathcal{E}_1}\Phi_{\mathcal{J}_1}\Phi_{\mathcal{E}_0}\Phi_{\mathcal{J}_0} \tag{16}$$

with

$$\begin{array}{rcl} \Phi_{\mathcal{J}_{k-1}} & = & \exp(H_{k,N-k+1}\otimes\sigma_{k,N-k+1}), \\ \Phi_{\mathcal{E}_{k-1}} & = & \exp\left(\sum_{s=k+1}^{N-k} E_{k,s}\otimes E_{s,N-k+1}e^{-\frac{1}{2}\sigma_{k,N-k+1}}\right) \\ & = & \prod_{r=k+1}^{N-k}\Phi_{\mathcal{E}_{k-1}(r)}. \end{array} \tag{17}$$

To visualize the resulting deformations the following notations will be used for the costructures:

$$
\begin{array}{lcl}
P^0(L) & \equiv & L\otimes 1 + 1\otimes L, \\
P_i^{\pm}(L) & \equiv & L\otimes e^{\pm\frac{1}{2}\sigma_{i,N+1-i}} + 1\otimes L, \\
R_i(L) & \equiv & L\otimes e^{\frac{1}{2}\sigma_{i,N+1-i}} + e^{\sigma_{i,N+1-i}}\otimes L, \\
T_i(L) & \equiv & L\otimes e^{\sigma_{i,N+1-i}} + 1\otimes L, \\
T^{++}(L) & \equiv & L\otimes e^{\frac{1}{2}\sigma_{1,N}+\frac{1}{2}\sigma_{2,N-1}} + 1\otimes L, \\
T^{\mp\pm}(L) & \equiv & L\otimes e^{\mp\frac{1}{2}\sigma_{1,N}\pm\frac{1}{2}\sigma_{2,N-1}} + 1\otimes L, \\
T_{R_i}(L) & \equiv & L\otimes e^{\frac{1}{2}\sigma_{1,N}+\frac{1}{2}\sigma_{2,N-1}} + e^{\sigma_{i,N+1-i}}\otimes L, \\
S_2^- & \equiv & -E_{2,r}\otimes E_{1,N-1}e^{-\frac{1}{2}\sigma_{2,N-1}}, \\
S_2^+ & \equiv & E_{2,N}\otimes E_{r,N-1}e^{\frac{1}{2}\sigma_{1,N}-\frac{1}{2}\sigma_{2,N-1}}, \\
S_1^- & \equiv & -E_{1,r}\otimes E_{2,N}e^{-\frac{1}{2}\sigma_{1,N}}, \\
S_1^+ & \equiv & E_{1,N-1}\otimes E_{r,N}e^{-\frac{1}{2}\sigma_{1,N}+\frac{1}{2}\sigma_{2,N-1}}, \\
 & & i=1,2; \quad r=3,\ldots,N-2.
\end{array}
\tag{18}
$$

In these terms the transformations performed in $\mathcal{H}$ by the factors of the canonical extended Jordanian twist (use the formulas (4)-(7) with $\alpha = \beta = 1/2$) can be written as

$$
\Phi_{\mathcal{J}} : \left\{ \begin{array}{cc} P^0(A) & P^0(E) \\ & P^0(B) \end{array} \right\} \longrightarrow \left\{ \begin{array}{cc} P^+(A) & T(E) \\ & P^+(B) \end{array} \right\}. \tag{19}
$$

$$
\Phi_{\mathcal{E}} : \left\{ \begin{array}{cc} P^+(A) & T(E) \\ & P^+(B) \end{array} \right\} \longrightarrow \left\{ \begin{array}{cc} P^-(A) & T(E) \\ & R(B) \end{array} \right\}. \tag{20}
$$

Let us perform the 2-Jordanian twisting in $\widetilde{\mathcal{H}}(2, N-4)$, $\Phi_{\mathcal{J}_1\mathcal{J}_2} : \widetilde{\mathcal{H}}(2, N-4) \longrightarrow \widetilde{\mathcal{H}}_{\mathcal{J}_1\mathcal{J}_2}(2, N-4)$. Here the deformed coproducts will be presented by the schemes analogous to (19) and (20). In terms of (18) the costructure of $\widetilde{\mathcal{H}}_{\mathcal{J}_1\mathcal{J}_2}(2, N-4)$ will acquire the following form:

$$
\begin{array}{|cc|}\hline P_1^+(E_{13}) \;\ldots\; P_1^+(E_{1,N-2}) \\ P_2^+(E_{23}) \;\ldots\; P_2^+(E_{2,N-2}) \\ \hline \end{array}
\quad
\begin{array}{|cc|}\hline T^{++}(E_{1,N-1}) & T_1(E_{1N}) \\ T_2(E_{2,N-1}) & T^{++}(E_{2N}) \\ \hline P_2^+(E_{3,N-1}) & P_1^+(E_{3N}) \\ \vdots & \vdots \\ P_2^+(E_{N-2,N-1}) & P_1^+(E_{N-2,N}) \\ \hline \end{array}
\tag{21}
$$

As we have mentioned above any Heisenberg subalgebra with the costructure $\{P^+, T, P^+\}$ can be twisted by the corresponding extension $\Phi_{\mathcal{E}}$. In (21) such are the triples $\left\{P_i^+(E_{ir}), T_i(E_{i,N-i+1}), P_i^+(E_{r,N-i+1})\right\}$. Obviously the deformations performed by the extensions $\Phi_{\mathcal{E}_{i-1}(r)} = \exp\left(E_{i,r} \otimes E_{r,N-i+1} e^{-\frac{1}{2}\sigma_{i,N-i+1}}\right)$ in these triples do not touch the generators other than $\{E_{1,r}, E_{2,r}, E_{r,N-1}, E_{r,N}\}$. Thus to study the deformations induced on $\widetilde{\mathcal{H}}(2, N-4)$ by the chain of twists (16) it is sufficient to focus the attention on the behavior of the subalgebra $\widetilde{\mathcal{H}} \equiv \widetilde{\mathcal{H}}(2,1)$,

$$
\begin{array}{|c|}\hline E_{1r} \\ E_{2r} \\ \hline \end{array}
\quad
\begin{array}{|cc|}\hline E_{1,N-1} & E_{1N} \\ E_{2,N-1} & E_{2N} \\ \hline E_{r,N-1} & E_{rN} \\ \hline \end{array}
\tag{22}
$$

Starting with its 2-Jordanian deformation $\widetilde{\mathcal{H}}_{\mathcal{J}_1\mathcal{J}_0}$ one can apply any of the two extensions: $\Phi_{\mathcal{E}_{i-1}(r)} = \exp\left(E_{i,r} \otimes E_{r,N-i+1} e^{-\frac{1}{2}\sigma_{i,N-i+1}}\right)$ $(i = 1, 2)$. Notice that in the chain (16) the extension $\Phi_{\mathcal{E}_1(r)}$ commutes not only with all the other extension factors $\Phi_{\mathcal{E}_{i-1}(r)}$ but also with $\Phi_{\mathcal{J}_1}$ .We get three twisted "states":

$$
\widetilde{\mathcal{H}}_{\mathcal{E}_0\mathcal{J}_1\mathcal{J}_0} \overset{\Phi_{\mathcal{E}_0(r)}}{\longleftarrow} \widetilde{\mathcal{H}}_{\mathcal{J}_1\mathcal{J}_0} \overset{\Phi_{\mathcal{E}_1(r)}}{\longrightarrow} \widetilde{\mathcal{H}}_{\mathcal{E}_1\mathcal{J}_1\mathcal{J}_0}
$$

The corresponding coalgebras are transformed as follows

$$\begin{array}{c} \left\{ \begin{array}{cc} \boxed{\begin{array}{c} P_1^- \\ P_2^+ + S_1^- \end{array}} & \boxed{\begin{array}{cc} T^{++} & T_1 \\ T_2 & T^{++} \end{array}} \\ & \boxed{\begin{array}{cc} P_2^+ + S_1^+ & R_1 \end{array}} \end{array} \right\} \longleftarrow \left\{ \begin{array}{cc} \boxed{\begin{array}{c} P_1^+ \\ P_2^+ \end{array}} & \boxed{\begin{array}{cc} T^{++} & T_1 \\ T_2 & T^{++} \end{array}} \\ & \boxed{\begin{array}{cc} P_2^+ & P_1^+ \end{array}} \end{array} \right\} \\ \longrightarrow \left\{ \begin{array}{cc} \boxed{\begin{array}{c} P_1^+ + S_2^- \\ P_2^- \end{array}} & \boxed{\begin{array}{cc} T^{++} & T_1 \\ T_2 & T^{++} \end{array}} \\ & \boxed{\begin{array}{cc} R_2 & P_1^+ + S_2^+ \end{array}} \end{array} \right\} \end{array} \tag{23}$$

Now we shall show that among the twists for $U\left(sl(N)\right)$ one can find those that perform further deformations of the states (23). Their carrier subalgebras are not included in the $\widetilde{\mathcal{H}}$, so with respect to $\widetilde{\mathcal{H}}$ these twists must be treated as external. Consider again the chain (16). According to the "matreshka" effect (see [4]) after the first two twists $\Phi_{\mathcal{J}_0}$ and $\Phi_{\mathcal{E}_0}$ the generators in the subalgebra $U_{\mathcal{E}_0\mathcal{J}_0}\left(sl(N-2)\right)$ will acquire trivial coproducts. Consequently when the first three factors $\Phi_{\mathcal{J}_1}\Phi_{\mathcal{E}_0}\Phi_{\mathcal{J}_0}$ are applied the generators $\{E_{2,r}, E_{r,N-1}\}$ return to the state P_2^+. The twisting factor that performs this transition appears when we drag the Jordanian twisting element $\Phi_{\mathcal{J}_1}$ to the right:

$$\begin{array}{c} \Phi_{\mathcal{J}_1}\Phi_{\mathcal{E}_0}\Phi_{\mathcal{J}_0} = \Phi_{\mathcal{J}_1}\Phi_{\mathcal{E}_0(2)}\Phi_{\mathcal{E}_0(N-1)}\Phi_{\mathcal{E}_0(N-2\prec 3)}\Phi_{\mathcal{J}_0} = \\ \Phi_{\mathcal{J}_1}\Phi_{\mathcal{E}_0(2)}\Phi_{\mathcal{E}_0(N-1)}\Phi_{\mathcal{J}_1}^{-1}\Phi_{\mathcal{J}_1}\Phi_{\mathcal{E}_0(N-2\prec 3)}\Phi_{\mathcal{J}_0} = \\ \widetilde{\Phi}_{\mathcal{E}_0(2,N-1)}\Phi_{\mathcal{E}_0(N-2\prec 3)}\Phi_{\mathcal{J}_1\mathcal{J}_0}. \end{array} \tag{24}$$

The factor $\Phi_{\mathcal{E}_0(N-2\prec 3)}$ is a twisting element for $U_{\mathcal{J}_1\mathcal{J}_0}$. The relation (25) signifies that

$$\begin{array}{c} \widetilde{\Phi}_{\mathcal{E}_0(2,N-1)} = \exp\Big(\Big(E_{1,2} + \frac{1}{2}E_{1,N-1} + E_{1,N-1}H_{2,N-1}\Big)\otimes \\ \otimes E_{2,N}e^{-\frac{1}{2}\left(\sigma_{1,N}+\sigma_{2,N-1}\right)} + E_{1,N-1}\otimes E_{N-1,N}e^{-\frac{1}{2}\left(\sigma_{1,N}-\sigma_{2,N-1}\right)}\Big) \end{array} \tag{25}$$

twists the costructure of $U_{\mathcal{E}_0(N-2\prec 3)\mathcal{J}_1\mathcal{J}_0}$. It is important that for the generators in (25) the coproducts in $U_{\mathcal{J}_1\mathcal{J}_0}$ and in $U_{\mathcal{E}_0(N-2\prec 3)\mathcal{J}_1\mathcal{J}_0}$

are the same. This means that the factor $\widetilde{\Phi}_{\mathcal{E}_0(2,N-1)}$ can twist directly the algebra $U_{\mathcal{J}_1\mathcal{J}_0}$ and the subalgebra $\widetilde{\mathcal{H}}_{\mathcal{J}_1\mathcal{J}_0}$ in it. The structure constants of this subalgebra are invariant with respect to the renumbering $(1 \rightleftharpoons 2, N-1 \rightleftharpoons N)$ of the indices of generators. Thus there exists the second external twisting factor

$$\begin{array}{c} \widetilde{\Phi}_{\mathcal{E}_1(2,N-1)} = \exp\Big(\Big(E_{2,1} + \frac{1}{2}E_{2,N} + E_{2,N}H_{1,N}\Big) \otimes \\ \otimes E_{1,N-1}e^{-\frac{1}{2}\left(\sigma_{1,N}+\sigma_{2,N-1}\right)} + E_{2,N} \otimes E_{N,N-1}e^{+\frac{1}{2}\left(\sigma_{1,N}-\sigma_{2,N-1}\right)}\Big) \end{array} \tag{26}$$

that can be applied to $\widetilde{\mathcal{H}}_{\mathcal{J}_1\mathcal{J}_0}$. One can find this twisting element directly considering the chain

$$\mathcal{F}'_{\mathcal{B}_{0\prec 1}} = \Phi'_{\mathcal{E}_0}\Phi_{\mathcal{J}_0}\Phi'_{\mathcal{E}_1}\Phi_{\mathcal{J}_1} \tag{27}$$

where the maximal set of the constituent roots is used in the extension $\Phi'_{\mathcal{E}_1}$ (that is for the root $(e_2 - e_{N-1})$). The algebra $U_{\mathcal{J}_1\mathcal{J}_0} = U_{\mathcal{J}_0\mathcal{J}_1}$ can be considered as an intermediate deformed object for both chains (16) and (27).

When any of the external factors $\widetilde{\Phi}_{\mathcal{E}_0(2,N-1)}$ and $\widetilde{\Phi}_{\mathcal{E}_1(2,N-1)}$ are applied to $\widetilde{\mathcal{H}}_{\mathcal{J}_1\mathcal{J}_0}$ one of the pairs of creation-annihilation generators retains the costructure P^+:

$$\left\{ \boxed{\begin{array}{c} P_1^+ \\ P_2^+ - S_1^- \end{array}} \quad \begin{array}{c} \boxed{\begin{array}{cc} T^{-+} & T_1 \\ T_2 & T_{R_1} \end{array}} \\ \boxed{P_2^+ - S_1^+ \quad P_1^+} \end{array} \right\} \leftarrow \left\{ \boxed{\begin{array}{c} P_1^+ \\ P_2^+ \end{array}} \quad \begin{array}{c} \boxed{\begin{array}{cc} T^{++} & T_1 \\ T_2 & T^{++} \end{array}} \\ \boxed{P_2^+ \quad P_1^+} \end{array} \right\}$$
$$\longrightarrow \left\{ \boxed{\begin{array}{c} P_1^+ - S_2^- \\ P_2^+ \end{array}} \quad \begin{array}{c} \boxed{\begin{array}{cc} T_{R_2} & T_1 \\ T_2 & T^{+-} \end{array}} \\ \boxed{P_2^+ \quad P_1^+ - S_2^+} \end{array} \right\} \tag{28}$$

This shows that to any of them the corresponding extension $\Phi_{\mathcal{E}_{i-1}}$ can be applied. Returning to the sequence (28) we see that these extensions remove the summands of the form $S_i^{\pm}$ so that the alternative extension also becomes applicable. As a result we get for the Heisenberg algebra $\widetilde{\mathcal{H}}$ nine deformed costructures

$$\Delta_{\mathcal{J}_1\mathcal{J}_0}\left(\widetilde{\mathcal{H}}\right)=\left\{\begin{array}{ccc} P_1^{+} & T^{++} & T_1 \\ P_2^{+} & T_2 & T^{++} \\ & P_2^{+} & P_1^{+} \end{array}\right\}$$

$$\Delta_{\widetilde{\mathcal{E}}_0\mathcal{J}_1\mathcal{J}_0}\left(\widetilde{\mathcal{H}}\right)=\left\{\begin{array}{ccc} P_1^{+} & T^{-+} & T_1 \\ P_2^{+}-S_1^{-} & T_2 & T_{R_1} \\ & P_2^{+}-S_1^{+} & P_1^{+} \end{array}\right\}$$

$$\Delta_{\widetilde{\mathcal{E}}_1\mathcal{J}_1\mathcal{J}_0}\left(\widetilde{\mathcal{H}}\right)=\left\{\begin{array}{ccc} P_1^{+}-S_2^{-} & T_{R_2} & T_1 \\ P_2^{+} & T_2 & T^{+-} \\ & P_2^{+} & P_1^{+}-S_2^{+} \end{array}\right\}$$

$$\Delta_{\mathcal{E}_0\mathcal{J}_1\mathcal{J}_0}\left(\widetilde{\mathcal{H}}\right)=\left\{\begin{array}{ccc} P_1^{-} & T^{++} & T_1 \\ P_2^{+}+S_1^{-} & T_2 & T^{++} \\ & P_2^{+}+S_1^{+} & R_1 \end{array}\right\}$$

$$\Delta_{\widetilde{\mathcal{E}}_0\mathcal{E}_0\mathcal{J}_1\mathcal{J}_0}\left(\widetilde{\mathcal{H}}\right)=\left\{\begin{array}{ccc} P_1^{-} & T^{-+} & T_1 \\ P_2^{+} & T_2 & T_{R_1} \\ & P_2^{+} & R_1 \end{array}\right\}$$

$$\Delta_{\mathcal{E}_1\mathcal{E}_0\widetilde{\mathcal{E}}_1\mathcal{J}_1\mathcal{J}_0}\left(\widetilde{\mathcal{H}}\right)=\left\{\begin{array}{ccc} P_1^{-} & T_{R_2} & T_1 \\ P_2^{-}+S_1^{-} & T_2 & T^{+-} \\ & R_2+S_1^{+} & R_1 \end{array}\right\}$$

$$\Delta_{\mathcal{E}_1\mathcal{J}_1\mathcal{J}_0}\left(\widetilde{\mathcal{H}}\right)=\left\{\begin{array}{ccc} P_1^{+}+S_2^{-} & T^{++} & T_1 \\ P_2^{-} & T_2 & T^{++} \\ & R_2 & P_1^{+}+S_2^{+} \end{array}\right\}$$

$$\Delta_{\mathcal{E}_1\mathcal{E}_0\widetilde{\mathcal{E}}_0\mathcal{J}_1\mathcal{J}_0}\left(\widetilde{\mathcal{H}}\right)=\left\{\begin{array}{ccc} P_1^{-}+S_2^{-} & T^{-+} & T_1 \\ P_2^{-} & T_2 & T_{R_1} \\ & R_2 & R_1+S_2^{+} \end{array}\right\}$$

$$\Delta_{\mathcal{E}_1\widetilde{\mathcal{E}}_1\mathcal{J}_1\mathcal{J}_0}\left(\widetilde{\mathcal{H}}\right)=\left\{\begin{array}{ccc} P_1^{+} & T_{R_2} & T_1 \\ P_2^{-} & T_2 & T^{+-} \\ & R_2 & P_1^{+} \end{array}\right\}$$

Here is how these costructures are connected by the external and internal extension twists:

$$
\begin{array}{ccccc}
\widetilde{\mathcal{H}}_{\mathcal{E}_1\mathcal{E}_0\widetilde{\mathcal{E}}_0\mathcal{J}_1\mathcal{J}_0} & & & & \\
\uparrow \Phi_{\mathcal{E}_1} & & & & \\
\widetilde{\mathcal{H}}_{\mathcal{E}_0\widetilde{\mathcal{E}}_0\mathcal{J}_1\mathcal{J}_0} & \stackrel{\widetilde{\Phi}_{\mathcal{E}_0}}{\longleftarrow} & \widetilde{\mathcal{H}}_{\mathcal{E}_0\mathcal{J}_1\mathcal{J}_0} & & \\
\uparrow \Phi_{\mathcal{E}_0} & & \uparrow \Phi_{\mathcal{E}_0} & & \\
\widetilde{\mathcal{H}}_{\widetilde{\mathcal{E}}_0\mathcal{J}_1\mathcal{J}_0} & \stackrel{\widetilde{\Phi}_{\mathcal{E}_0}}{\longleftarrow} & \widetilde{\mathcal{H}}_{\mathcal{J}_1\mathcal{J}_0} & \stackrel{\widetilde{\Phi}_{\mathcal{E}_1}}{\longrightarrow} & \widetilde{\mathcal{H}}_{\widetilde{\mathcal{E}}_1\mathcal{J}_1\mathcal{J}_0} \\
 & & \downarrow \Phi_{\mathcal{E}_1} & & \downarrow \Phi_{\mathcal{E}_1} \\
 & & \widetilde{\mathcal{H}}_{\mathcal{E}_1\mathcal{J}_1\mathcal{J}_0} & \stackrel{\widetilde{\Phi}_{\mathcal{E}_1}}{\longrightarrow} & \widetilde{\mathcal{H}}_{\mathcal{E}_1\widetilde{\mathcal{E}}_1\mathcal{J}_1\mathcal{J}_0} \\
 & & & & \downarrow \Phi_{\mathcal{E}_0} \\
 & & & & \widetilde{\mathcal{H}}_{\mathcal{E}_1\mathcal{E}_0\widetilde{\mathcal{E}}_1\mathcal{J}_1\mathcal{J}_0}
\end{array}
\qquad (29)
$$

The vertical arrows are the ordinary extension twists for Heisenberg subalgebras. The horizontal arrows correspond to the external twists borrowed from the chains for $U(sl(N))$. The squares of the diagram are commutative. Its asymmetry is justified by the fact that $\Phi_{\mathcal{E}_i}$ and $\widetilde{\Phi}_{\mathcal{E}_j}$ commute only for $i = j$. At the same time in the columns of the diagram (29) both $\Phi_{\mathcal{E}_i}$ are applicable, one – as a legal extension for the $\mathcal{H}_{\mathcal{J}}$ and the other due to the corresponding "matreshka" effect.

3 Conclusions

We have shown that the multidimensional Heisenberg algebras (of the type (15)) have the fixed spectrum of deformed costructures. The properties of this spectrum is tightly connected with different chains of extended twists that can be applied to the initial universal enveloping algebra. It is obvious that such Heisenberg algebras can be included in universal enveloping algebras other than $U(sl(N))$ and there they can be treated analogously. It is equally obvious that increasing the number of initial Jordanian twists (and enlarging correspondingly the rows and the columns of creation and annihilation operators in the multidimensional Heisenberg subalgebra) one can get more complicated costructures. The main feature here is the appearance of the $S_i^{\pm}$ summands that mix the creation (annihilation) operators from different rows (columns).

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